Structure of spaces of holomorphic functions on infinite dimensional polydiscs

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Abstract. Let $A_1(\alpha)$ be a nuclear finite type power series space. We characterize the open polydiscs $D_\alpha$ in $A_1(\alpha)$ for which the space $(H(D_\alpha), \tau_\alpha)$ of all holomorphic functions on $D_\alpha$ under the compact-open topology $\tau_\alpha$ is a power series space. This characterization implies the existence of open polydiscs $D_\alpha$ and $D_\beta$ with finite radii for which $(H(D_\alpha), \tau_\alpha)$ and $(H(D_\beta), \tau_\beta)$ are not isomorphic. Furthermore we give sufficient conditions on nuclear Fréchet spaces $A(P)$ and on $a \in A(P)$ implying that for the open polydisc $D_\alpha$ in $A(P)$ the space $(H(D_\alpha), \tau_\alpha)$ is isomorphic to a subspace (resp. a quotient) of a power series space.

Preface. Let $E^\infty$ be a nuclear Fréchet space and let $U$ be an open subset of $E^\infty$. Then the space $(H(U), \tau_\alpha)$ of all holomorphic functions on $U$, endowed with the compact-open topology $\tau_\alpha$, is a nuclear Fréchet space by the theorem of Boedel and Waelbroek [27]. If $H$ is, moreover, a Köthe sequence space $A(P)$, then Boland and Dineen [3] have shown that the monomials in the coordinate functions form an absolute basis of $(H(U), \tau_\alpha)$ for all open polydiscs $U$ in $A(P)^\infty$, where for $a \in A(P)$, $a \geq 0$ (for convenience), the sets $D_\alpha = \{ y \in A(P)^\infty \mid \sup \| y/a \| < 1 \}$ are called open polydiscs. From this result Börgens, Meise and Vogt [3] have derived that $H(A_1(\alpha), \tau_\alpha)$ is isomorphic to $A_1(\beta(\alpha))$ and that in $A_1(\alpha)$ there exists an open polydisc $D$ with $(H(D), \tau_\alpha) \simeq A_1(\beta(\alpha))$. Furthermore, they have determined $\beta(\alpha)$ up to equivalence for many interesting sequences $\alpha$.

The main result of the present article is the following: Let $A_1(\alpha)$ be nuclear and take $a \in A_1(\alpha)$, $a \geq 0$. Then $(H(D_\alpha), \tau_\alpha)$ is isomorphic to $A_1(\beta(\alpha))$ if and only if $(H(D_\beta), \tau_\beta)$ is isomorphic to a quotient space of a finite type power series space, and this happens if and only if $a > 0$ and $1/a \in A_1(\alpha)$ which is equivalent to $a > 0$ and $\lim (1/a_j) \alpha_j = 0$. This implies in particular that there exist $a, b \in A_1(\alpha)$, $a > 0$, $b > 0$ such that $(H(D_\alpha), \tau_\alpha)$ and $(H(D_\beta), \tau_\beta)$ are not isomorphic as locally convex spaces; a phenomenon which does not occur in the finite dimensional situation.
The proof of this result is based on a linear topological invariant \( (\mathcal{D}) \) which has been used by Wagner [28], to characterize the quotient spaces with a basis of stable power series spaces of finite type.

Knowing this, it was natural to consider also the other linear topological invariants from the structure theory of nuclear Fréchet spaces which have been used to characterize the subspaces and quotients of stable nuclear power series spaces. These characterizations are given in terms of the \( \Lambda_{1} \)- (resp. the \( \Lambda_{2} \)-) and of one of the invariants \( (\mathcal{DN}), (\mathcal{DN}_{1}), (\mathcal{D}) \) and \( (\mathcal{D}) \) (see Dubinsky [11], Vogt [21]-[24], Vogt and Wagner [25], [26] and Wagner [28]). Since optimal theorems on the \( \Lambda_{2} \)-nuclearity of spaces of type \( (\mathcal{DN}), (\mathcal{DN}_{1}) \) have been given by Börgez, Meise and Vogt [4], since \( \Lambda_{2} \) is always stable, the characterization theorems can be applied to \( (\mathcal{DN}), (\mathcal{DN}_{1}) \) as soon as this space has one of these invariants. In this direction we prove the following: If \( A(P) \) is a quotient of \( A_{1}(a) \), then \( (\mathcal{DN}), (\mathcal{DN}_{1}) \) is isomorphic to a quotient space of \( A_{1}(a') \) if and only if it satisfies a certain condition. If \( A(P) \) is a quotient of \( A_{1}(a') \) (resp. a subspace of \( A_{1}(a') \)), then \( (\mathcal{DN}), (\mathcal{DN}_{1}) \) is isomorphic to a quotient of \( A_{1}(a') \) (resp. a subspace of \( A_{1}(a') \)) for all open polyhedra \( D_{n} \) in \( A(P) \). The proofs rely on the fact that for Köthe spaces \( \mathcal{Q} \) the invariants can be expressed in terms of the Köthe set \( Q \). Hence the basis theorem of Boland and Dineen, mentioned above, gives the opportunity to check the invariants by direct calculation. Furthermore we apply an extension theorem of Boland [5], a duality theorem of Boland and Dineen [8] and the isomorphism \( (\mathcal{D}) \) to some point proved by Börgez, Meise and Vogt [3].

Concluding, we briefly indicate the content of the four sections of this article. In the first section we recall some definitions and results and fix the notation. In the second one we state the basis theorem of Boland and Dineen in an appropriate form and provide some lemmas. These are applied in Section 3 to obtain our main result as well as the invariants related with finite type power series spaces. The same topic for infinite type power series spaces is treated in Section 4.

1. Preliminaries. In this section we introduce some notation and conventions used throughout the whole article. We also mention some results which we shall use in the subsequent sections without further references.

Notation. The symbols \( C, \mathbb{R}, \mathbb{R}_{+}, \mathbb{N}, \) and \( \mathbb{N} \), denote the sets of complex, real, non-negative real, natural and non-negative integer numbers, respectively. All locally convex (i.e.) vector spaces \( E \) are assumed to be complex vector spaces and Hausdorff. For an i.e. vector space \( E \), the strong dual is denoted by \( E' \). An i.e. space \( E \) is called a subspace (quotient space) of an i.e. space \( E' \) if \( E \) is isomorphic to a topological subspace (quotient space) of \( F \). The space \( C^{\omega} \) of all complex sequences is denoted by \( \omega \), while \( \varphi \subseteq \omega \) denotes the space of all complex sequences which are eventually zero.

For \( a, b \in \omega \) the set

\[ \mathcal{N}_{a} = \{ \varphi \subseteq \omega | [\varphi]_{a} \leq [\varphi]_{b} \} \]

is called the normal hull of \( a \). For \( a, b \in \omega \) we denote by \( a/b \) the sequence \( c \) which is defined by \( c_{n} = a_{n}/b_{n} \) if \( b_{n} \neq 0 \) and \( c_{n} = 0 \) if \( b_{n} = 0 \). Results from the theory of semi normed spaces which we use without any reference can be found in the books of Köthe [13], Pietsch [16] or Schaefer [20].

1.1. Köthe sequence spaces. We use the Köthe sequence spaces \( A(P) \) as they are defined in Pietsch [16], 6.1.1. Then the Grothendieck–Pietsch criterion (see Pietsch [16], 6.1.2) tells that the nuclearity of \( A(P) \) is characterized by the property: For any \( p \in P \) there exist \( q \in P \) and \( e \in P \) such that \( p \leq e \cdot q \). If \( A(P) \) is nuclear, then a subset \( B \) of \( A(P) \) is bounded if \( B \) is contained in the normal hull of some \( b \in A(P) \). If \( A(P) \) is nuclear and reflexive, then a subset \( K \) of \( A(P)' \) is relatively compact if \( K \) is contained in the normal hull of some \( e \in A(P)' \), where we identify \( A(P)' \) with the linear span of \( \bigcup_{n=1}^{\infty} X_{n} \), in \( \omega \). This description of the relatively compact subsets of \( A(P)' \) also holds if \( A(P) \) is a Mackey space for which \( A(P)' \) is complete (see Börgez, Meise and Vogt [3], 1.1).

1.2. Power series spaces. Let \( a \) be an increasing sequence of positive real numbers with \( \lim a_{n} = \infty \) (\( a \) will be called an exponent sequence). For \( \sum_{n=0}^{\infty} a_{n} < \infty \) we define the power series space

\[ A_{R}(a) = \{ \varphi \subseteq \omega | \tau_{n}(\varphi) = \sum_{n=1}^{\infty} [\varphi]_{a_{n}} < \infty \}

\]

which is given the i.e. topology induced by the semi-norms \( \tau_{n} \). Obviously \( A_{R}(a) \) is a Fréchet space.

\[ A_{R}(a) \] is called power series space of finite type if \( R = \infty \) and of infinite type if \( R = \infty \). For a fixed \( 0 < R < \infty \) all the spaces \( A_{R}(a) \) are isomorphic. A power series space of infinite type cannot be isomorphic to a power series space of finite type.

For \( R = 1 \) and \( R = \infty \) the identity \( A_{R}(a) = A_{R}(a) \) holds if \( a \) and \( \alpha \) are equivalent in the following sense: There exists \( D > 1 \) such that

\[ \left| \frac{1}{D} a_{n} \leq a_{n} \leq D a_{n} \right| \] for all \( n \in \omega \).

The nuclearity of \( A_{R}(a) \) is equivalent to \( \sup \{ \ln(n+1)/[\alpha_{n}] < \infty \} \), while the nuclearity of \( A_{R}(a) \) is equivalent to \( \lim \{ \ln(n+1)/[\alpha_{n}] = 0 \}. \)
An exponent sequence \( a \) is called *stable* if \( \sup \{a_n/a_m < \infty \}, \) which is equivalent to the isomorphism \( A(a) \times A(a) \cong A(a) \) (see Dubinsky and Ramanaiah [12], 2.10).

If \( A(a) \) is nuclear (resp. if \( A(a) = \text{core} \)) then one can define the class of \( A(a) \) (resp. \( A(a) \)); resp. \( A(a) \))-nuclear i.e. spaces, as it was done by Dubinsky and Ramanaiah [12], Robinson [19] and Ramanaiah and Terriglong [18]. Since the definitions of these classes are a bit involved, we do not give them but refer to the articles cited above. A brief introduction which suffices for our purposes, is given in Börgens, Meise and Vogt [4], 1.4.

For stable nuclear power series spaces \( A(a), E = 1, \infty, \) the subspaces and quotient spaces have been characterized by Vogt and Wagner (see [24], [26] and also Dubinsky [11], where further references are given). They have shown that a Fréchet space \( E \) is isomorphic to a subspace (resp. a quotient space of \( A(a) \)) if \( E \) is a \( A(a) \)-nuclear and has property (DN) (resp. (DD)). The same characterization holds true for \( A(a) \), provided that “\( A(a) \)-nuclearity” is replaced by \( A(a) \)-nuclearity and that (DN) (resp. (DD)) is replaced by (DN) (resp. (DD)). Since we do not want to give the definition of these properties, we just indicate that they are linear topological invariants and that (DN) and (DD) are inherited by topological linear subspaces, whereas (DD) is inherited by separated quotient spaces. For Köthe sequence spaces \( A(P) \) this properties can be expressed in terms of the Köthe set \( P \). This form has been introduced by Dragilev [10].

1.3. Analytic functions. Let \( E \) be an i.e. space and let \( Q \neq V \) be an open subset of \( E \). A function \( f: Q \rightarrow \mathbb{C} \) is called

(a) *G-analytic* if for any \( a, b \in E \) the function \( a \mapsto f(a + b) \) is a holomorphic function in one variable on its natural domain of definition,

(b) *hypanalytic* if \( f \in G \)-analytic and continuous on any compact subset of \( Q \),

(c) *holomorphic* if \( f \in G \)-analytic and continuous on \( Q \),

\( H(a) \) (resp. \( H(D) \)) denotes the vector space of all hypanalytic (resp. holomorphic) functions on \( Q \). The compact-open topology on \( H(a) \) and \( H(Q) \) is denoted by \( \tau_c \).

For further details concerning analytic functions on i.e. spaces we refer to the book of Dineen [9].

2. Some fundamental lemmas and results. In this section we introduce some more notation and give a sequence space representation of the space of hypanalytic functions on open polydiscs in the dual of a reflexive nuclear Köthe space \( A(P) \). From this we draw a corollary and then we provide several lemmas which will be applied in the subsequent sections.

2.1. Notation and remarks.

(a) Let \( A(P) \) be a nuclear Köthe space. For any \( a \in A(P) \) with \( a \geq 0 \) the set

\[ D_a := \{ a \in A(P)^+ : \sup_{n \in \mathbb{N}} |a_n| < 1 \} \]

is an open subset of \( A(P)^+ \), called an open polydisc. For a given open polydisc \( D_a \) we call \( r_j \) the \( j \)-th radius of \( D_a \) where \( r_j \) is defined as \( 1/a_j \) for \( a_j > 0 \) and as \( \infty \) for \( a_j = 0 \).

(b) We put

\[ M := \{ m \in \mathbb{N} \mid m_j = 0 \text{ for almost all } j \in \mathbb{N} \} \]

and define for any \( a \in M \) and any \( m \in M \) the \( m \)-th power of \( a \) as

\[ a^m := \prod_j a_j^m. \]

(c) If \( \alpha \) is an exponent sequence with \( \sup \{ \ln(n+1)/\alpha_n < \infty \} \), we put

\[ (a|m) = \sum_{n \in \mathbb{N}} a_n m_n. \]

Then we define the exponent sequence \( \beta = \beta(a) \) as the increasing arrangement of the family \( \{(a|m)\}_{m \in M} \) and we fix a bijection \( b: \mathbb{N} \rightarrow M \) with the property \( \beta_n = (a|b(n)) \) for any \( n \in \mathbb{N} \). We remark that \( \beta(a) \) has been determined — up to equivalence — for a large number of sequences \( a \) which are of importance in analysis in Börgens, Meise and Vogt [9]. Explicit formulas and examples are given in Section 5 of [3]. Moreover, it was shown in [3], 5.4 (b), that \( \beta(a) \) is always a stable sequence.

(d) If \( D_a \) is an open polydisc in \( A(P)^+ \) and \( f \in H(\varphi \cap D_a) \), then for any \( m = (m_1, \ldots, m_n, 0, \ldots) \in M \) the \( m \)-th Taylor coefficient of \( f \) (with respect to the origin) is given by

\[ a_m(f) = \left( \frac{1}{2\pi} \right)^n \int_{|z_1| = r_1} \cdots \int_{|z_n| = r_n} \frac{f(z_1, \ldots, z_n, 0, \ldots)}{z_1^{m_1} \cdots z_n^{m_n}} \, dz_1 \cdots dz_n, \]

where \( 0 < r_j < 1/a_j \) for \( 1 \leq j \leq n \) are arbitrarily chosen real numbers.

Since \( A(P) \) is assumed to be nuclear, \( \varphi \cap D_a \) is sequentially dense in \( D_a \) with respect to the topology \( \tau(\varphi \cap A(P)) \). Hence \( f \) is uniquely determined by the family of its Taylor coefficients.

The importance of nuclear sequence spaces in connection with the Taylor expansion by monomials in the coordinate functions was demonstrated by Boland and Dineen [8]. The following theorem is essentially due to them [8], Thm. 11. We repeat it here because it is fundamental for our results; its proof is given by an easy modification of the proof of Börgens, Meise and Vogt [3], Thm. 2.1.
2.2. Theorem. Let \( A(P) \) be a nuclear and reflexive and let \( D_0 \) be an open polydisc in \( A(P)_0 \). Let \( Q \) be a subset of \( A(P)_0 \) consisting of non-negative sequences, such that \( (N_q \cap Q) \) is a fundamental system for the compact subsets of \( D_0 \).

(a) For \( f \in H(q \cap D_0) \) let \( (a_{m,n})_{m,n} \) denote the Taylor coefficients of \( f \), defined as in 2.1 (d). Equivalent are:

1. \( f = g \in H(q \cap D_0) \) for some \( g \in H(q \cap D_0) \);
2. \( f|_{(N_q \cap Q)} \) is bounded for any \( q \in Q \);
3. \( \sup_{m,n} |x_m| q^n < \infty \) for any \( q \in Q \);
4. \( \sum_{m,n} |x_m| q^n < \infty \) for any \( q \in Q \).

(b) The mapping \( T: H(q \cap D_0), \tau \rightarrow A(M, Q^M), T(f) = (a_{m,n})_{m,n} \), is a topological isomorphism, where \( Q^M = ((x_m)_{m,n} : q \in Q) \). The space \( (M, Q^M) \) is nuclear.

Remark. (a) Since \( A(P) \) is reflexive and nuclear, it follows from 1.1 that there exists a set \( Q \) having the properties required in 2.2.

(b) In 2.2 the reflexivity of \( A(P) \) is only used to get a convenient formulation. If \( A(P) \) is assumed to be only nuclear, then 2.2 holds if \( H(q \cap D_0) \) is replaced by the space of all \( G \)-analytic functions on \( D_0 \) which are bounded on the equicontinuous subsets of \( D_0 \) endowed with the topology of uniform convergence on these sets.

The following corollary of 2.2 has already been stated in Börgens, Meise and Vogt [3], 2.5.

2.3. Corollary. Let \( A_1(a) \) be nuclear and let \( 1 \in A_1(a) \) denote the sequence identically 1. Then \( (H(D_0), \tau) \) is nuclear and isomorphic to \( A_1(1) \) by the mapping

\[ T: f \mapsto (a_{m,n}(f))_{m,n} \].

Proof. It is easily seen that \( Q = \{(x^r)_{m,n} : 0 < r < 1\} \) is a fundamental system for the compact subsets of \( D_0 \). Since any hyperanalytic function on a \( GF(N) \)-space is already continuous, we get from 2.2 that \( (H(D_0), \tau) \) is isomorphic to \( A_1(1) \). However, this space is isomorphic to \( A_1(1) \) by the mapping

\[ (a_{m,n})_{m,n} \mapsto (a_{m,n})_{m,n} \].

In the subsequent sections we shall use the sequence space representation given in 2.2 for a number of computations. For this purpose it will be useful to have a convenient description for the systems \( Q = Q(P, a) \) appearing in 2.2. Therefore we now indicate how such systems can be obtained for nuclear Fréchet spaces \( A(P) \).
have \( q_k < q_{k+1} \) for all \( j \in \mathbb{N} \) and all \( k \in \mathbb{N} \). Since \( q \) is increasing, this follows from (a) and (a).

The following two lemmas will be needed in the next sections.

2.2. Lemma. Let \( E \) be an l.c. space and let \( F \) be a nuclear Fréchet space.

(a) If \( E \) is a subspace of \( F \), then \( (H(E), \tau_1) \) is a subspace of \( (H(F), \tau_1) \).

(b) If \( E \) is a quotient space of \( F \), then \( (H(E), \tau_1) \) is a quotient space of \( (H(F), \tau_1) \).

Proof. (a): Let \( \bar{E} \) denote the completion of \( E \). Then it follows from Köthe [15], §29.6 (1) and §27.2 (5), that \( E' = \bar{E}' \). Hence it follows from the Hahn–Banach theorem that the restriction map \( \pi: E' \rightarrow \bar{E}' \) is surjective and open. If one defines

\[
\tilde{\pi}: (H(E'), \tau_1) \rightarrow (H(F'), \tau_1)
\]

by

\[
\tilde{\pi}(f) = f \circ \pi,
\]

then \( \tilde{\pi} \) is an injective topological homomorphism since any compact set in \( E' \) is the \( \pi \)-image of a compact set in \( F' \).

(b): Let \( \pi: F \rightarrow E \) denote the quotient map. Then \( E' \) can be regarded as a subspace of \( F' \) by means of the adjoint \( \pi' \) of \( \pi \) and hence the extension theorem of Bohr [5], 3.1, tells that

\[
\varrho: (H(F'), \tau_1) \rightarrow (H(E'), \tau_1),
\]

\[
\varrho(f) = f \circ \pi',
\]

is surjective and open.

The proof of the following lemma is an easy consequence of the Cauchy inequalities (see e.g. Aron and Schettner [1], Thm. 2.3, (c) \( \Rightarrow \) (d)).

2.3. Lemma. Let \( E \) be a Fréchet–Montel space and let \( U \neq \emptyset \) be an open subset of \( E \). Then \( E \) is a complemented subspace of \( (H(U), \tau_1) \).

Remark. From 2.6 and the inheritance properties of \( (DN), (DN), (Q) \) and \( (Q) \) it easily follows that a necessary condition for \( (H(U), \tau_1) \) having one of these properties is that \( E \) has the corresponding one.

3. Subspaces and quotients of finite type power series spaces. This section in which we deal with the properties \( (Q) \) and \( (DN) \) contains the main result of this article, namely a characterization of those open polydiscs \( D_\alpha \) in a nuclear space \( A_\alpha \) for which \( (H(D_\alpha), \tau_1) \) is a power series space. The proof of the main theorem is prepared by the following two lemmas.

3.1. Lemma. Let \( A_\alpha \) be nuclear. A diagonal map \( D: A_\alpha \rightarrow A_\alpha \), \( D = (d_\alpha)_{\alpha \in \mathbb{N}} \), is an automorphism iff \( d \in A_\alpha \) and \( 1/d \in A_\alpha \).

Proof. Since \( A_\alpha \) is nuclear, we have \( 1 \in A_\alpha \). Hence \( d = D(1) \in A_\alpha \) and \( 1/d = D^{-1}(1) \in A_\alpha \) if \( D \) is an automorphism. In order to prove the converse implication it suffices to show that \( D \) is continuous for any \( d \in A_\alpha \). However, this is a consequence of the following estimate which holds for all \( x \in A_\alpha \) and all \( r \) with \( 0 < r < 1/2 \):

\[
\|D(x)\| = \sum_{k=1}^{\infty} |d_k| r^{2k} \leq (\sup_{k \in \mathbb{N}} |d_k| r^{2k}) \sum_{k=1}^{\infty} |x_k| r^{2k} \leq \|x\| r^{2/3}.\]

3.2. Lemma. Let \( A(P) \) be a nuclear Fréchet space, where \( P = \{(p_i)_{i \in \mathbb{N}}: h \in \mathbb{N}\} \). For \( a \in A(P), a \geq 0 \), the following are equivalent:

1. \( (H(D_h), \tau_1) \) has \( (Q) \);
2. For any \( s \in \mathbb{N} \) there exists \( t > s \) and \( j \in \mathbb{N} \) such that

\[
q_{j,s} \leq \frac{a_j}{a_t} \quad \text{for all } j \geq j.
\]

Proof. We may assume that \( P \) satisfies 2.4 (a) and (b). Then we have by 2.4 and 2.2 that \( (H(D_h), \tau_1) \) is isomorphic to \( A(M, Q^M) \), where \( Q \) is defined as in 2.4.

(1) \( \Rightarrow \) (2): First we show that (1) implies \( a_j \neq 0 \) for all \( j \in \mathbb{N} \). In order to prove this we put \( D_j = (x \in C) : |x| < r_j \) where \( r_j = 1/a_j \). Then it is easy to see that \( (H(D_j)) \simeq A_{\alpha_j}(h) \) is a quotient space of \( (H(D_h), \tau_1) \). Hence (1) implies that \( A_{\alpha_j}(h) \) has \( (Q) \). By Wagner [28], 1.11, this shows \( r_j < \infty \), i.e. \( a_j > 0 \).

Then we remark that property \( (Q) \) of \( A(M, Q^M) \) implies by Wagner [28], 1.11, that the following holds true:

3. For any \( s \in \mathbb{N} \) there exists \( t > s \) such that for any \( h \in \mathbb{N} \) there exists \( C > 0 \) such that

\[
q_{j,s} \leq C q_{j,t} \quad \text{for all } s \in \mathbb{N}.
\]

By choosing \( m = n_0 \), \( C_j = (A_{\alpha_j}(h)_{h \in \mathbb{N}}) \) and taking \( s \)-th roots we get from (3) by going to the limit \( s \rightarrow \infty \):

4. For any \( s \in \mathbb{N} \) there exists \( t > s \) such that for any \( h \in \mathbb{N} \)

\[
q_{j,s} \leq \frac{a_j}{a_t} \quad \text{for all } j \geq j.
\]

In order to see that (4) implies (2) we choose \( j \geq n_1 \) arbitrarily. Then the definition of \( Q \) in 2.4 and (4) imply that for any \( k < j < n_0 \) we have

\[
q_{j,s} \leq \frac{a_j}{a_k} \quad \text{for all } k \geq k.
\]

Because of \( \lim_{k \rightarrow \infty} a_k = 1 \), \( \lim_{k \rightarrow \infty} n_k = \infty \) and \( a_j \neq 0 \) this implies

\[
q_{j,s} \leq \frac{a_j}{a_t} \quad \text{for all } j \geq j.
\]

hence (2) holds.
(2) $\Rightarrow$ (1): As the proof of 2.4 shows, we may assume that the sequence $(n_k)_{k\in\mathbb{N}}$ in 2.4 (i) is constructed in such a way that $n_k \geq j_k$ for all $s \in \mathbb{N}$. Then we get from (2):

(5) For any $s \in \mathbb{N}$ there exists $t > s$ such that

$$p_{j,s} < \frac{a}{p_{j,t}}$$

for all $j \geq n_s$.

Now we show that (5) can be used to prove that $Q$ satisfies (4). This implies that $Q^N$ satisfies (3), which gives by Wagner [28], 1.10, that $A(M, Q^N)$ has (D), i.e., that (1) holds.

In order to prove (4) we first remark that (5) implies $a_j \neq 0$ for all $j \in \mathbb{N}$. Then, for a given $s \in \mathbb{N}$, we choose $t > s$ such that (5) holds. Now we remark that for any $l$ with $j \geq n_l$ we get from 2.4 (a) that $p_{j,l} \leq \frac{a_l}{a_j}$. Hence $q_{j,l} \leq \frac{a_l}{a_j}$ for all $j \in \mathbb{N}$ and all $l \in \mathbb{N}$ by the definition of $Q$. Using this and 2.4 (b) we get for $j \geq n_l$ and $k \in \mathbb{N}$

$$q_{j,k} q_{j,s} \leq \frac{q_k}{a_j} \frac{q_s}{a_j} \leq \frac{q_{j-1}}{a_j} \leq \left( \frac{q_s}{a_j} \right)^2 = q_{j,s}^2.$$

If $j \geq n_l$, then we get by the same arguments and (5) that for any $k \in \mathbb{N}$ we have

$$q_{j,k} q_{j,s} \leq \frac{1}{a_j} p_{j,k} = p_{j,s} = q_{j,s}^2.$$

This shows that $Q$ satisfies (4), which completes the proof.

Remark (a) If $A$ is $P$ and $Q$ has a continuous norm, we may assume that $p_{j,s} > 0$ for all $j$ and $s \in \mathbb{N}$. Then 3.3 (2) is equivalent to

(2') For $s > 0$ and for any $s \in \mathbb{N}$ there exists $t > s$ such that

$$\liminf_{j \to \infty} (a_j p_{j,t}^2/p_{j,s}) \geq 1.$$

(b) In the proof of 3.2 we have shown that (1) implies $a_j > 0$ for all $j \in \mathbb{N}$. Hence it follows from the definition of $Q$ that $A(P) = A(Q)$. But then (4) in connection with Wagner [28], 1.11, proves that (1) is equivalent to $A = A(D)$, which has already been remarked to be a consequence of 2.6.

In 3.3 we have seen that for any nuclear space $A$ we have

$$(H(D), \tau_a) = A \beta(a)$$

for the open polydisc $D_a$ in $A$. The following theorem gives a characterization of all polydiscs $D_a$ in $A$ for which $(H(D), \tau_a)$ is isomorphic to $A \beta(a)$ and shows that this property also characterizes the polydiscs $D_a$ for which $(H(D), \tau_a)$ has (D).

3.3. Theorem. Let $A$ be a nuclear and let $a \in A$ satisfy

Then the following are equivalent:

1. $a > 0$ and $\lim_{j \to \infty} a_j = 0$;
2. $a > 0$ and $1/a \in A$;
3. $(H(D), \tau_a)$ is a polydisc space;
4. $(H(D), \tau_a)$ is a polydisc space of finite type;
5. $(H(D), \tau_a)$ is a polydisc space of finite type;
6. $(H(D), \tau_a)$ is a polydisc space of finite type;
7. $(H(D), \tau_a)$ has (D).

Proof. (3) $\Rightarrow$ (2): For an arbitrary $r$ with $0 < r < 1$, we put $s = \ln(1/r) > 0$. Then there exists $J = J(s)$ such that $|a_j| \cdot |s| 

(2) $\Rightarrow$ (3): Since $a$ and $1/a$ belong to $A$, we get from Lemma 3.1 that the diagonal map $A: A = A(a) \to \tau_a$, $A: \pi \mapsto (\pi \pi)$, is an automorphism of $A(a)$. Obviously we have $A(D) = D_a$. Hence $A$ induces an isomorphism between $(H(D), \tau_a)$ and $(H(D), \tau_a)$. Because of 2.3 this implies that (3) holds.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (6) hold trivially.

(4) $\Rightarrow$ (5): By 2.6 $(H(D), \tau_a)$ has a subspace which is isomorphic to $A(a)$. Hence (4) implies that (3) holds.

(6) $\Rightarrow$ (7): This follows from the remark that any polydisc space of finite type has (D) and that $(D)$ is inherited by quotient spaces (see Wagner [28], 1.11, 1.2).

(7) $\Rightarrow$ (1): We choose a strictly increasing sequence $\sigma$ in $(0, 1)$ with $\lim_{\sigma \to 0} = 1$. Then $P = \{(\sigma_k(e_k, k) \in \mathbb{N})$ satisfies $2.4$ (a) and (3) and we have $A(P) = A(a)$. Hence we can apply 3.2, which shows that $a > 0$. Since $a \in A(a)$ implies $\lim_{j \to \infty} a_j = 0$, it suffices to show $\liminf_{j \to \infty} a_j = 0$.

This follows from 3.2 (2), which implies that for any $s \in \mathbb{N}$ there exist $t = t(s) > s$ and $j_0 \in \mathbb{N}$ such that

$$a_j \geq \frac{p_{j,s}}{p_{j,t}} = \left( \frac{q_j}{q_t} \right)^2$$

for all $j \geq j_0$.

and consequently

$$\liminf_{j \to \infty} a_j \geq \liminf_{j \to \infty} a_j = 0.$$

3.4. Corollary. Let $A$ be nuclear and let $a$, $b \in A$ satisfy $a > 0$ and $b \geq 0$. If $1/a \in A$ and $1/b \in A$, then $(H(D), \tau_a)$ and $(H(D), \tau_b)$ are not isomorphic.
Remark. (a) Theorem 3.3 holds if $A_1(a)$ is replaced by $A_2(a)$, $0 < R < \infty$, provided that (1) (resp. (2)) is replaced by (1)' (resp. (2)');

(1)' $a > 0$ and $\lim_{q \to a^+} q_n a_n = \ln R$;

(2)' $a > 0$ and $1/a \in A_2(a)$.

(b) Theorem 3.3 is optimal in the following sense: If $E$ is a nuclear Fréchet space for which there exists an open subset $U$ in $E'$ such that $(H(U), \tau_U)$ is isomorphic to a power series space and contains a bounded function which is not constant, then $E$ is isomorphic to a power series space of finite type. This is obtained in the following way: As the proof of Vogt [21], 2.6, shows, $(H(U), \tau_U)$ does not have (DN), hence $(H(U), \tau_U)$ is isomorphic to some $A_1(\gamma)$ which is nuclear by the theorem of Dolan and Waerstedt [19], from 2.6 we get that $E$ is isomorphic to a subspace and a quotient space of $A_1(\gamma)$. Hence $E$ is isomorphic to some nuclear $A_1(\delta)$ by a result of Mityagin [15] (see also Vogt [24], 1.6).

(c) If $A_1(a)$ is nuclear and if the open polydiscs $D_{\alpha}$ and $D_{\eta}$ in $A_1(a)$ are such that $\{H(D_{\alpha}), \tau_D\}$ and $\{H(D_{\eta}), \tau_D\}$ have $(\mathcal{D})$, then they are not only isomorphic but isometric as spaces with respect to the topological phenomena. This follows from the proof of 3.3 since it shows the existence of an automorphism $A$ of $A_1(a)$ with $A(D_{\alpha}) = D_{\eta}$. This can also be derived from Meise and Vogt [14], where a classification of the algebra isomorphisms between $(H(D_{\alpha}), \tau_D)$ and $(H(D_{\beta}), \tau_D)$ is given.

(d) It is an obvious consequence of 3.3 that for any open subset $U$ of $A_1(a)$ which is holomorphically equivalent to $D_{\eta}$, the space $(H(D_{\eta}), \tau_D)$ is isomorphic to $A_1(\beta(a))$. Furthermore, it follows from 3.3 that for any analytic subvariety $V$ of $D_{\eta}$ (whatever the right definition will be) for which $g: (H(D_{\eta}), \tau_D) \to (H(V), \tau_V)$, $g$ is the restriction, is surjective and open, $(H(V), \tau_V)$ has $(\mathcal{D})$. Up to now nothing is known in this situation, except if $V$ is the intersection of $D_{\alpha}$ with a closed hyperplane (see Raboin [17], Cor. 3 of Thm. 3).

3.5. Proposition. Let $A(P)$ be a quotient of a nuclear space $A_1(a)$, where $P = \{p_{a,y} \mid a, y \in N\}$. For $a \in A(P)$, $a \geq 0$, the following are equivalent:

(1) $(H(D_{\alpha}), \tau_D)$ is a quotient of $A_1(\beta(a))$;

(2) $a$ satisfies 3.2 (2).

There exists $b \in A(P)$ satisfying condition (2).

Proof. Since $A(P)$ is a quotient of $A_1(a)$, it is $A_1(a)$-nuclear. Hence it follows from Börgers, Meise, and Vogt [1], Thm. 4.1, that $(H(D_{\alpha}), \tau_D)$ is $A_1(\beta(a))$-nuclear. Since $\beta(a)$ is stable by Börgers, Meise, and Vogt [3], 3.4 (b), we get from Wagner [28], Thm. 2.5, that $(H(D_{\alpha}), \tau_D)$ satisfies (1) iff it has $(\mathcal{D})$. By 3.2.3 this is equivalent to (2).

In order to show the existence of some $b \in A(P)$ satisfying (2), we remark that $A(P)$ has $(\mathcal{D})$. Hence we may assume by Wagner [28], 1.11,

(3) $p_{a,y}, p_{a+1,y} \leq b^{p_{a+1,y}+1}$ for all $y \in N$, all $a \in N$ and all $b \in N$.

For any $k \in N$ there exists $q \in N$ such that $p_{a,y} > 0$, hence (3) implies that

$$\sigma_q = \sup_{y \in N} p_{a,y} \leq b^{p_{a,y}+1} < \infty$$

and that $\sigma_q > 0$. Moreover, (3) implies that $b = 1/\sigma$ satisfies

$$p_{a,y} \leq b^{p_{a,y}+1} \text{ for all } a \in N \text{ and all } y \in N.$$
(1) Since $\lim (p_{j,1}^{+}) = 1$ and since $\varepsilon_{0} < \varepsilon_{k+1}$, there exists $\varepsilon_{i} > 0$

such that for all $s$ with $0 < s \leq \varepsilon_{i}$ we have $q_{j}^{+} p_{j,1}^{+} \leq (q_{i}^{+} / \varepsilon_{i})$. Hence we have for $0 < s \leq \varepsilon_{i}$ and all $j$ with $\varepsilon_{j} \\ {\varepsilon_{0}}$ \leq \frac{p_{j,1}^{+}}{q_{j}^{+}} \leq \frac{p_{j,1}^{+}}{q_{i}^{+}}.

(2) Since $A(P)$ is Hausdorff, (2) implies that $p_{j,1}^{+} = 0$ for all $j \in N$. Hence we have $\lim (p_{j,1}^{+}) = 1$ for all $j \in N$ and all $j$ with $\varepsilon_{j} < \varepsilon_{k+1}$, this implies that there exists $\varepsilon_{k} > 0$ such that for all $s$ with $0 < s \leq \varepsilon_{k}$ and all $j$ with $\varepsilon_{j} < \varepsilon_{k}$ and $\varepsilon_{j} \neq 0$ we have

$\frac{p_{j,1}^{+}}{q_{j}^{+}} \leq \frac{p_{j,1}^{+}}{q_{i}^{+}}.

(3) Since $\lim (p_{j,1}^{+}) = 1$ for all $j \in N$ and since $\sup_{s \in N} p_{j,1} < \varepsilon_{0}$

then we have $\varepsilon_{0} > 0$ such that for all $s$ with $0 < s \leq \varepsilon_{0}$ and all $j$ with $\varepsilon_{j} \neq 0$ and $\varepsilon_{j} < \varepsilon_{k+1}$ we have

$p_{j,1}^{+} < \frac{p_{j,1}^{+}}{q_{j}^{+}}.

(4) From (3) we get that the existence of $\varepsilon_{i}$ satisfying $p_{j,1}^{+} < (p_{j,1}^{+})^{+}$ for all $j \in N$. Since $p_{j,1}^{+} \geq 1$ by (a), we even have for all $0 < s \leq \varepsilon_{i}$ and all $j \in N$

$p_{j,1}^{+} \leq \frac{p_{j,1}^{+}}{q_{j}^{+}}.

If we choose $s = \min (\varepsilon_{0}, \varepsilon_{i}, \varepsilon_{k})$, then it follows from the definition of $Q$ and (1)-(4) that

$q_{j}^{+} \leq q_{j}^{+} \frac{p_{j,1}^{+}}{q_{j}^{+}}$ for all $j \in N$.

Since $k$ was arbitrary, we have shown that for any $k \in N$ there exist $s > 0$

such that

$q_{j}^{+} \leq \frac{p_{j,1}^{+}}{q_{j}^{+}}$ for all $m \in M$.

Because of Vogt [23], 4.1, (6) implies that $A(M, Q^{m})$ and consequently $\hat{A}$ has (DN), since $A(P)$ is a quotient of $A_{\alpha}(o)$, it is $A_{\alpha}(o)$-nuclear. Hence

$\hat{A}$ is $A_{\alpha}(o)$-nuclear by Börgens, Meise and Vogt [4], Thm. 4.1. Since $\beta(o)$ is stable, an application of Vogt [23], Satz 3.2, gives that

$\hat{A}$ is a subspace of $A_{\alpha}(o)$.

From 3.6 we get the following corollary which also shows that 3.6 is in a sense optimal.

3.7. Corollary. An l.c. space $E$ is a subspace of a nuclear power series space of finite type iff

$\hat{A}(E_{a})$, $\tau_{a}$ has the property.

Proof: If $E$ is a subspace of some nuclear space $A_{\alpha}(o)$, then by 2.1 (a) $\hat{A}(E_{a}), \tau_{a}$ is a subspace of $\hat{A}(A_{\alpha}(o)), \tau_{a}$, which is a subspace of $\hat{A}(\beta(o))$ by 3.6. If $\hat{A}(E_{a}), \tau_{a}$ is a subspace of some nuclear space $A_{\alpha}(o')$, then $E$ is a subspace of $A_{\alpha}(o')$ by 2.6.

4. Subspaces and quotients of infinite type power series spaces. This section contains some results on subspaces and quotient spaces of power series spaces of infinite type. The proofs are more easy than in Section 3 since $\hat{A}(A_{\alpha}(o)), \tau_{a}$ is isomorphic to $\hat{A}_{\alpha}(\beta(o))$ for any nuclear space $A_{\alpha}(o)$.

4.1. Proposition. Let $A_{\alpha}(o)$ be nuclear and let $E$ be an l.c. space.

(a) If $E$ is a subspace of $A_{\alpha}(o)$, then $\hat{A}(E_{a}), \tau_{a}$ is a subspace of $\hat{A}(\beta(o))$.

(b) If $E$ is a quotient space of $A_{\alpha}(o)$, then $\hat{A}(E_{a}), \tau_{a}$ is a quotient space of $\hat{A}(\beta(o))$.

Proof. (a): Let $A_{\alpha}(o)$ be a subspace of $A_{\alpha}(o)$, then $\hat{A}(E_{a}), \tau_{a}$ is a subspace of $\hat{A}(A_{\alpha}(o)), \tau_{a}$ by 2.1 (a). Hence the result follows from Börgens, Meise and Vogt [3], Thm. 2.1, which tells that $\hat{A}(A_{\alpha}(o)), \tau_{a}$ is isomorphic to $\hat{A}(\beta(o))$.

(b) The same arguments as in part (a) apply if 2.5 (a) is replaced by 2.5 (b).

4.2. Corollary. Let $E$ be a Fréchet-Montel space.

(a) $E$ is a subspace of $s$ iff $\hat{A}(E_{a}), \tau_{a}$ is a subspace of $s$.

(b) $E$ is a quotient space of $s$ iff $\hat{A}(E_{a}), \tau_{a}$ is a quotient space of $s$.

Proof. The "if" part follows from 4.1 since for $s = \min (\alpha = (\alpha + 1))_{\alpha}$ the sequence $\beta(o)$ is equivalent to $A_{\alpha}(o)$ by Börgens, Meise and Vogt [3], Thm. 2.4. The "only if" part follows easily from 2.6.

Remark. Because of the characterization of the subspaces (resp. quotient spaces) of $s$ given by Vogt [21] (resp. Vogt and Wagner [23]), Corollary 4.2 tells that a nuclear Fréchet space $E$ has (DN) (resp. (L)) if $\hat{A}(E_{a}), \tau_{a}$ has (DN) (resp. (L)). Because of Vogt [21], Satz 3.6, it cannot be expected that (DN) holds for many open subsets of $E_{a}$. Indeed, it follows from this result (resp. from Vogt [21], 2.4) and the preceding remark that for a nuclear Fréchet space $A_{\alpha}(o)$ the space $\hat{A}(E_{a}), \tau_{a}$ has (DN) if $\hat{A}(E_{a}), \tau_{a}$ is (DN). We shall show now that more can be obtained for the property (L), since the dual form of (L) can be localized in a sense.

In order to do this we recall the definition of the space $H(K)$ of germs of holomorphic functions on a compact subset $E$ of a metrizable l.c. space $E$. We choose a decreasing open neighborhood basis $(U_{\alpha})_{\alpha}$ of $E$ and let $H^{+}(E_{a})$ denote the space of all bounded holomorphic functions
on $U_n$ endowed with the sup-norm. Then $(H^n(U_n), r_{\text{max}})_{n \in \mathbb{N}}$ is an inductive system if $r_{\text{max}}: H^n(U_n) \to H^n(U_n)$ denotes the restriction map for $n \geqslant \infty$. The inductive limit of this system is denoted by $\mathcal{H}(K)$. We remark that $\mathcal{H}(K)$ does not depend on the particular choice of the neighbourhood basis $(U_n)_{n \in \mathbb{N}}$.

4.3. Proposition. Let $E$ be an l.c. space.

(a) If $E$ is a quotient space of $s$, then $\mathcal{H}(K)_s$ is a quotient space of $s$ for any compact set $K \neq \emptyset$ in $E$.

(b) If $E$ is a Fréchet–Schwartz space and if there exists a compact set $K \neq \emptyset$ in $E$ for which $\mathcal{H}(K)_s$ is a quotient space of $s$, then $E$ is a quotient space of $s$.

Proof. (a) If $E$ is a quotient of $s$, then $\mathcal{H}(K)_s$ is a quotient of $s$ by 4.2 (b). Since $E$ is nuclear, $\mathcal{H}(E)(0)$ is a (DFN)-space by Bierstedt and Meise [2], Thm. 7. From this and the duality result of Boland [5], Thm. 1 and Remark 2 (a), we get that $\mathcal{H}(E)(0)_s$ is isomorphic to $\mathcal{H}(K)_s$, $\tau_0$. Then Sets 1.8 and Lemma 2.1 of Vogt and Wagner [231] imply that $\mathcal{H}(K)_s$ has the following property:

(*) For any $g \in N$ there exists $g \in N$ such that for any $k \in N$ there exist $g \in N$ and $r \geqslant 0$ such that for all $r > 0$ and any $f \in \mathcal{H}(K)_s$,

\[
\sup_{s \in V_f} |f(s)| \leqslant C \left( \sup_{s \in V_f} |f(s)| + \frac{1}{r} \sup_{s \in V_f} |f(s)| \right),
\]

where $(V_f)_{f \in N}$ is a decreasing absolutely convex neighbourhood basis of zero in $E$ and where the supremum is allowed to be infinite.

Now let $K \neq \emptyset$ be a compact subset of $E$ and define $U = E + V_f$ for all $f \in N$. Then we have $\mathcal{H}(K) = \text{ind} \mathcal{H}(U)$, and it follows from (*) that $\mathcal{H}(K)$ satisfies (*) if we replace in (*a) $f$ by $U$. Since $E$ is nuclear, it follows from Bierstedt and Meise [2], Thm. 7, that $\mathcal{H}(K)$ is a (DFN)-space. Hence (*) implies by Vogt and Wagner [231], 2.1, that $\mathcal{H}(K)_s$ has $(\omega)$. Since $\mathcal{H}(K)_s$ is nuclear, this proves by Vogt and Wagner [231], 2.8, that $\mathcal{H}(K)_s$ is a quotient of $s$.

(b) Because of Bierstedt and Meise [2], Prop. 10, $\mathcal{H}_0$ is a complemented subspace of $\mathcal{H}(K)$. Hence $E = \mathcal{H}_0 + \mathcal{H}(K)$ is a complemented subspace of $\mathcal{H}(K)_s$ and consequently a quotient space of $s$.

4.4. Corollary. Let $\mathcal{H}(K)_s$ be nuclear and let $E$ be a quotient space of $\mathcal{H}(K)_s$.

(a) $\mathcal{H}(K)_s$ is a quotient space of $\mathcal{H}(K)_s$ for any compact set $K$ in $E$.

(b) If $E = \mathcal{H}(P)$, then $\mathcal{H}(D_a)_{\tau_0}$ is a quotient space of $\mathcal{H}(K)$ for any open polydisc $D_a$ in $\mathcal{H}(P)$. 

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**Structure of spaces of holomorphic functions**

Proof. (a) If $A_s(\alpha)$ is a quotient of $s$, we get from 4.3 (a) and Vogt and Wagner [231], 1.8, that $\mathcal{H}(K)_s$ has $(\omega)$ for any compact set $K$ in $E$. Since $A_s(\alpha)$ is $A(\alpha)$-nuclear (see Ramanujan and Terzić [18], 2.13), we get from Börger, Meise and Vogt [4], Thm. 4.4, that $\mathcal{H}(K)_s$ is $A(\alpha)$-nuclear. Hence Vogt and Wagner [231], 1.8, implies that $\mathcal{H}(K)_s$ is a quotient space of $A_s(\alpha)$.

(b) If $D_a$ is any open polydisc in $\mathcal{H}(P)_s$, then it follows from the duality theorem of Boland and Dinenc [8], Thm. 20, that $\mathcal{H}(D_a)_{\tau_0} = \mathcal{H}(D_a)_{\tau_0}$, where $D_a^N$ is the multiplicative polar of $D_a$, which is a compact subset of $E$. Hence we get from (a) that $\mathcal{H}(D_a)_{\tau_0}$ is a quotient of $A_s(\alpha)$.

4.5. Corollary. If $A(P)$ is a quotient space of $s$, then for any $a \in A(P)$, $a \neq 0$, there is a closed ideal $I$ in the l.c. algebra $\mathcal{A}(a)$ such that $(\mathcal{A}(a) / I) / I$ is isomorphic to $(\mathcal{H}(D_a)_{\tau_0}) / I_c$ as an l.c. algebra.

Proof. By 4.4 (b) there exists a continuous linear surjection $\pi: s \to (\mathcal{H}(D_a)_{\tau_0}) / I_c$. Its transpose $\pi^*: (\mathcal{H}(D_a)_{\tau_0}) / I_c \to s$ is continuous and linear. Hence the mapping $\pi: (\mathcal{H}(D_a)_{\tau_0}) / I_c \to s$ defined by $\pi(f) = f \circ \pi$, where $\pi$ denotes the evaluation at the point $a$, is a holomorphic mapping, i.e. continuous and weakly Gâteaux-analytic. Since the composition of holomorphic mappings is holomorphic again, $\pi$ induces $\Phi: (\mathcal{H}(D_a)_{\tau_0}) / I_c \to (\mathcal{H}(E)_{\tau_0}) / I_c$ by the definition $\Phi(f) = \pi(f)$. It is easy to see that $\Phi$ is a continuous algebra homomorphism. In order to show that $\Phi$ is surjective, we choose $g \in (\mathcal{H}(E)_{\tau_0}) / I_c$ arbitrarly. Since $\pi$ is surjective, there exists $f \in s = (s) / I_c \to (\mathcal{H}(E)_{\tau_0}) / I_c = \mathcal{A}(a)$, which is equivalent to $g = \pi(f) = f \circ \pi$. Hence $\Phi$ is surjective and the result follows from the open mapping theorem for Fréchet spaces.

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**References**


[7] — Studia Math. 71(1)
Norm inequalities relating singular integrals and the maximal function

by

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Abstract. We prove that if the weighted $L^p$ norms $(1 < p < \infty)$ of the Riesz transforms are bounded by the weighted $L^p$ norm of the maximal function, then the weight function satisfies the $C_p$ condition of B. Muckenhoupt. Conversely we show that if the weight function satisfies the $C_p$ condition for some $q > p$, then the weighted $L^p$ norm of any standard singular integral is bounded by the weighted $L^p$ norm of the maximal function.

§ 1. Introduction. We consider the problem of characterizing the non-negative weights $w$ for which $(1 < p < \infty)$

\[ \int |Tf|^p w \leq C \int |f|^p w \]

for all appropriate $f$.

where $Tf = \mathcal{K}f$ is a singular integral in $\mathbb{R}^n$ with kernel $\mathcal{K}$ satisfying the standard conditions

(i) $\mathcal{K}|_\mathbb{R}^n < C$,

(ii) $|\mathcal{K}(0)| \leq C |\sigma|^{-n}$,

(iii) $|\mathcal{K}(x) - \mathcal{K}(x-y)| \leq C|y||x|^{-n-1}$ for $|y| < |x|/2$.

R. Coifman and C. Fefferman have shown ([1], Theorem XIII) that (1) holds for $1 < p < \infty$ provided the weight $w$ satisfies the $A_p$ condition. B. Muckenhoupt has shown ([1], Theorem 2.1) that in the case when $\mathcal{T}$ is the Hilbert transform, inequality (1) does not imply that $w$ satisfies the $A_p$ condition. He has derived ([7], Theorem 1.2) the following necessary condition for (1) (with $\mathcal{T}$ the Hilbert transform) which has been conjectured to be sufficient.

($C_p$) There are positive constants $C$, $\varepsilon$ such that

\[ \int \mathcal{E} \leq C |\mathcal{E}| |Q|^{-1} \int |\mathcal{M}_x|^p w \]

whenever $E$ is a subset of a cube $Q \subset \mathbb{R}^n$.

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