$H^m$ is a Grothendieck space

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Abstract. It is shown that a non-weakly compact operator on $H^m$ fixes an $L^m$-copy. In particular, $H^m$ has the Grothendieck property and $L^m$ embeds in any infinite-dimensional complemented subspace of $H^m$.

I. Introduction. This work is a continuation of [3] (cf. also [4]). Let us recall some definitions. $H$ denotes the circle and $m$ its Haar measure. $H_1$ is the space of integrable functions $f$ on $H$ such that $\hat{f}(n) = 0$ for $n \leq 0$. We use the notations $\varphi : L^1 \to L^1/H_1^1$ and $\sigma : L^1/H_1^1 \to L^1$ for the quotient map and the minimum norm lifting, respectively. The duality

$$\langle f, \varphi \rangle = \int f \varphi \, dm$$

identifies the dual $(L^1/H_1^1)^*$ with the space $H^m$ of bounded analytic functions on the unit disc $D$.

It was shown in [3] that $H^m$ has the Dunford--Petitta property (DPP) and $(H^m)^*$ is weakly sequentially complete (WSC). We establish here the Grothendieck property (GP) of $H^m$. Recall that a Banach space $X$ has GP provided weak*-null sequences in $X^*$ are weakly-null, or, equivalently, each operator $T : X \to Y$ is weakly compact. In fact, a stronger result is obtained. If $T : H^m \to X$ is an operator, then $T$ is either weakly compact or there exists a subspace $Z$ of $H^m$, $Z$ isomorphic to $F^*$, on which $T$ induces an isomorphism.

As corollary it follows that $L^m$ embeds in any infinite dimensional complemented subspace of $H^m$, solving one of the questions raised in [15].

II. Operators on $H^m$ and the Grothendieck property. Classical examples of $G$-spaces are the $L^m(\mu)$-spaces. Next result, implying the $G$-property, emphasizes the same behaviour of $H^m$ and $L^m$ in several aspects.

Theorem 1. Assume $T : H^m \to X$ is an operator. If $T$ is not weakly compact, then $T$ is an isomorphism when restricted to a subspace $Z$ of $H^m$, $Z$ isomorphic to $F^*$. 

\[ \text{Received October 12, 1981} \]
It was shown by S.V. Kisliakov [15] and independently by F. Delbaen [7] that non-weakly-compact operators on the disc-algebra \( A \) fix a \( c_0 \)-copy.

The proof of this result makes crucial use of the Riesz-decomposition of \( A^* \)
\[ A^* = M_2(\mathbb{H}) \oplus L^1/\mathbb{H} \],
where \( M_2(\mathbb{H}) \) denotes the space of singular measures on \( \mathbb{H} \).

Such a result does not hold for \((H^\infty)^*\). We will use the more general approach which already enabled us to prove DFP of \( H^\infty \). The technique consists in establishing a finite-dimensional result for \( L^1/\mathbb{H} \), which can therefore be carried over to \((H^\infty)^*\) by arguments of local reflexivity.

The purpose of this section is to state the local \( L^1/\mathbb{H} \)-theorem and derive Th. 1 from it. The proof of the \( L^1/\mathbb{H} \)-result is rather technical and will be presented in the next two sections.

As a formal consequence of the DFP of \( H^\infty \) and Th. 1, we get

**Corollary 2.** \( H^\infty \) embeds in any infinite-dimensional complemented subspace of \( H^\infty \).

It looks reasonable to conjecture that a complemented subspace of \( H^\infty \) is either an \( L^1 \) or an \( H^\infty \)-isomorph.

**Theorem 3.** For each \( \delta > 0 \), there exist \( \delta_1 > 0 \) and a function \( a(n) \) satisfying
\[ \lim a(n) = 0 \]
so that the following property holds:

Let \( f_1, \ldots, f_n \) be disjointly supported functions in \( L^1(\mathbb{H}) \) such that

1. \[ ||f_m|| > \delta \quad (1 \leq m \leq n) \]
2. \[ \sum |f_m| \leq 1 \quad \text{for each } m \]
3. \[ \sum |1 - f_m| < \delta \quad \text{for each } m \]

Then there exist \( H^\infty \)-functions \( \varphi_m, \psi_m \) \((1 \leq m \leq n)\) fulfilling

\[ |\varphi_m| + |\psi_m| < 1 \quad \text{for each } m, \]

\[ \sum |1 - f_m| < \delta \quad \text{for each } m \]

\[ \langle \varphi_m, \psi_m \rangle = \delta \quad \text{for each } m. \]

The reader will find some further comments on Th. 3 in the remarks at the end of this paper.

Our first objective will be to derive from Th. 3 the following result on \((H^\infty)^*\).

**Proposition 1.** There exists \( \tau > 0 \) and \( \alpha > 0 \) and a function \( \beta(n) \) for which \( \lim(\beta(n)/n) = 0 \) such that the following property holds:

Let \( \Phi_1, \ldots, \Phi_n \) be \( n \) elements in the unit ball of \((H^\infty)^*\) and assume

\[ \sum a_n(1 - \Phi_n) < (1 - \tau) \sum |\Phi_n| \quad (a_n \in C). \]

Then there are \( H^\infty \)-functions \( \varphi_m, \psi_m \) \((1 \leq m \leq n)\) satisfying

\[ |\varphi_m| + |\psi_m| < 1 \quad \text{for each } m, \]

\[ \sum |1 - \varphi_m| < \beta(n), \]

\[ \langle \varphi_m, \psi_m \rangle = \alpha \quad \text{for each } m. \]

**Lemma 1.** Prop. 1 holds if one replaces \((H^\infty)^*\) by \( L^1/\mathbb{H} \).

Proof. Taking in Th. 3 \( \delta = 1/2 \) provides some \( \delta_1 > 0 \). Take \( \tau = (1/4)\delta_1 \) and \( \alpha = (1/2)\delta_1 \). Assume \( \varphi_1, \ldots, \varphi_n \) in the unit ball of \( L^1/\mathbb{H} \) satisfying \((*)\). Applying L. Dec's lemma (see [8]) to the minimum norm liftings \( \sigma(\varphi_1), \ldots, \sigma(\varphi_n) \) yields disjoint measurable subsets \( S_m \) of \( \mathbb{H} \) for which

\[ ||\varphi_m|| > (1 - \tau)^2 \]

taking \( f_m = \sigma(\varphi_m)\chi_{S_m}. \)

Now

\[ ||f_m|| > ||\varphi_m|| - ||\sigma(\varphi_m) - f_m||_1 = ||f_m||_1 > 1/2. \]

Thus, applying Th. 3, \( H^\infty \)-functions \( \varphi_m, \psi_m \) \((1 \leq m \leq n)\) are obtained satisfying (ii), (iii), (iv) of Th. 3. Since for each \( m = 1, \ldots, n \)

\[ \langle \varphi_m, \psi_m \rangle < ||\varphi_m - \sigma(\varphi_m)||_1 < 1 - (1 - \tau)^2 < \delta_1/2, \]

we get

\[ ||\varphi_m, \psi_m|| > \alpha. \]

Replacement of \( \varphi_m \) by \( a_n\varphi_m \) for some \( a_n \in C, |a_n| < 1 \), leads to the required conclusion.

Let us next observe that (i), (ii) of Prop. 1 can be reformulated as follows in Banach space language

\[ ||\varphi_m + b\psi_m|| < 1 \quad \text{if } |a|, |b| < 1, \]

\[ \sum a_n(1 - \varphi_n) < \beta(n) \quad \text{whenever } |a_n| < 1. \]

By local reflexivity, it will therefore be enough to obtain \( \varphi_m, \psi_m \) \((1 \leq m \leq n)\) as elements of \((H^\infty)^*\), replacing conditions (i), (ii) by \((i'), (ii')\).

A simple way to achieve this is using the isometrical embedding of \((H^\infty)^*\) in some ultra-power \( B = (L^1/\mathbb{H})_0' \) of \( L^1/\mathbb{H} \). The reader is referred to [13] and [19] for the theory of ultra-products of Banach spaces. We use the notation \( \mathcal{I} \) for the element of \( B^* \) defined by \( \langle \xi, \mathcal{I} \rangle = \lim \int \xi_n \), where \( \xi_n \) is \( (x_n=0) \) is an element of \( B \).
Proof. The argument is completely straightforward. Fix some \( 0 < \varepsilon < 1 \) and assume \( \xi(1), \ldots, \xi(n) \) in the unit ball of \( B \) satisfying

\[
\lVert \sum a_m \xi(m) \rVert \geq (1 - \varepsilon^2) \sum |a_m| \quad (a_m \in C).
\]

It follows from the definition of the norm on \( B \) that there is some element \( U \) in the ultra-filter \( \mathcal{U} \) such that for \( i \in U \) the \( L^2 \)-elements

\[
\xi_i(1), \ldots, \xi_i(n)
\]

behave almost isometrically to

\[
\xi(1), \ldots, \xi(n).
\]

In particular, we can assume

\[
\sum |\alpha_m| \geq \sum |a_m| \xi_i(m) \geq \lambda^{-1}(1 - \varepsilon^2) \sum |a_m| \quad (a_m \in C),
\]

where \( \lambda^{-1}(1 - \varepsilon^2) > 1 - \varepsilon. \)

If we fix \( i \in U \) and define

\[
\alpha_m = \lambda^{-1} \xi_i(m) \quad (1 \leq m \leq n),
\]

application of Lemma 1 gives \( H^m \)-functions \( \phi_i(m) \) and \( \psi_i(m) \) \((1 \leq m \leq n)\) satisfying (i), (ii) of Prop. 1 and

\[
\langle \xi_i(m), \phi_i(m) \rangle = \lambda \langle \alpha_m, \phi_i(m) \rangle = \lambda \varepsilon.
\]

Next, define for each \( m = 1, \ldots, n \) the following elements of \( B^{*} \)

\[
\langle \xi_i(m), \phi_i(m) \rangle = \lim \langle \xi_i(m), \psi_i(m) \rangle
\]

and

\[
\langle \xi_i(m), \psi_i(m) \rangle = \lim \langle \xi_i(m), \psi_i(m) \rangle.
\]

One verifies immediately (i'), (ii'). Moreover,

\[
\langle \xi_i(m), \psi_i(m) \rangle = \lambda \varepsilon \quad (1 \leq m \leq n).
\]

This proves the lemma.

It remain now to restrict the elements \( \phi_i, \psi_i \) of \( B^{*} \) obtained in Lemma 2 to \( (H^m)^{*} \) in order to obtain \( (H^m)^{*} \)-elements satisfying (i'), (ii'). The only thing to notice here is that, from the embedding properties of \( (H^m)^{*} \) in \( B \), the restriction of \( 1 \) to \( (H^m)^{*} \) is 1 in \( H^m \). This completes the proof of Prop. 1.

We now turn back to Th. 1. Assume \( \mathcal{T} : H^m \to Y \) is a non-weakly compact operator. Then the set

\[
K = \{ y^{*} \in Y^{*}; \ y^{*} \in Y^{*} \text{ and } \| y^{*} \| \leq 1 \}
\]

is not weakly compact and therefore, since \( (H^m)^{*} \) is WSC, not weakly conditionally compact. Applying the James-regularization principle for \( p \)-sequences, it is possible to construct in some multiple of \( K \) an infinite sequence \( (\phi_m)_{m \geq 1} \ldots \) satisfying the hypothesis of Prop. 1.

**Lemma 3.** There exist \( \varepsilon > 0 \) and a sequence \( (\eta_r)_{r=1,2,\ldots} \) of \( H^m \)-functions so that

\[
(i) \sum_{m=1}^{\infty} |\eta_r| \leq \varepsilon,
\]

(ii) \( \langle \phi_r, \eta_r \rangle = \varepsilon \), where \( \phi_r \) is a subsequence of \( (\phi_m) \).

The extraction of the \( l^m \)-subsequence \( \phi_r \) of \( \eta_r \) such that \( I \) induces an isomorphism on the \( w^* \)-closure of \( \text{span} \{ \eta_r; r = 1, 2, \ldots \} \) is then done using the standard procedure (see [17] for instance).

Proof of Lemma 3. Fix a sequence \( (\alpha_r) \) of positive numbers and positive integers \( \{ N_r \} \) such that \( \beta(N_r) < \varepsilon N_r \), \( \nu \in H^m \) and \( \phi \in (H^m)^{*} \), define \( \psi \in (H^m)^{*} \) by \( \langle \psi, \phi \rangle = \langle \theta, \nu \rangle \).

Noticing also that if \( \eta_m (1 \leq m \leq n) \) are \( H^m \)-functions satisfying (ii) of Prop. 1 and \( \phi \in (H^m)^{*} \), then

\[
\| \phi - \eta_m \| \| \phi \| \leq \varepsilon \| \phi \| \quad \| \phi \| \quad \text{for some } m = 1, \ldots, n
\]

We now make the following construction.

Defining \( D_k = N \) and fixing the first \( N_k \) elements \( \Phi_1, \Phi_2, \ldots, \Phi_{N_k} \) of \( D_k \), we apply Prop. 1. The preceding observation allows us to fix some \( m_1, m_2, \ldots, m_k \) for which

\[
\| \phi_{m_1} - \phi_{m_2} \| < \varepsilon
\]

holds for all \( m \) in an infinite subset \( D_k \) of \( D_k \). Also

\[
|\eta_{m_1}| + |\eta_{m_2}| \leq \varepsilon
\]

and

\[
\langle \phi_{m_1}, \eta_{m_2} \rangle = \lambda.
\]

Starting again with the \( N_k \) first elements of \( D_k \) yields some \( m_1 \in D_1, m_2 > m_1, H^m \)-functions \( \eta_{m_1}, \eta_{m_2} \) and an infinite subset \( D_1 \) of \( D_k \) so that

\[
\| \phi - \eta_{m_1} \| < \varepsilon \quad \text{for } m \in D_1,
\]

\[
|\eta_{m_1}| + |\eta_{m_2}| \leq \varepsilon \quad \langle \phi_{m_1}, \eta_{m_2} \rangle = \lambda.
\]

Continuing in this way, a subsequence \( (\phi_m) \) of \( (\phi_m) \) is obtained. Define for each \( r \) the \( H^m \)-function

\[
\eta_r = \eta_{m_1} \eta_{m_2} \ldots \eta_{m_r}. 
\]
Since at each step
\[ |v_{mx}| + |v_{m^*}| < 1, \]
then \( \eta_r \) satisfy (i) of Lemma 3. Now for \( m \in D_{\leq 1} \), one has
\[ \|\Phi_m - (v_{mx} \cdots v_{m^*_1}) \Phi_m\| < \|\Phi_m - v_{mx} \Phi_m\| + \cdots + \|\Phi_m - v_{m^*_1} \Phi_m\| < \sum_{i=1}^\infty \varepsilon_i. \]
Therefore
\[ \langle \Phi_m, \eta_r \rangle > \langle \Phi_m, v_{mn} \rangle - \sum_{i=1}^\infty \varepsilon_i = x - \sum \varepsilon_i. \]
So we just have to fix \( \varepsilon = \frac{x}{2} \) and a sequence \( (\varepsilon_i) \) with \( \sum \varepsilon_i = x. \)

III. Some preliminary lemmas. In this section we will give several lemmas which will be used in the proof of Th. 3. Some of them appear also in [3], but we repeat them here for the sake of completeness. We denote by \( \mathcal{H} \) the Hilbert-transform. \( H^m \) is seen as subalgebras of \( L^m(\Omega) \).

**Lemma 4.** Assume \( a \in L^m(\Omega) \) such that \( 0 \leq a \leq 1 \) and \( \log(1 - a) \) is integrable. Then there is an \( H^m \)-function \( f \) satisfying

(i) \[ |f| \leq 1 - a \quad \text{on} \quad \mathbb{D}, \]
(ii) \[ \| (1 - t - f) \|_2 \leq \| (1 - t - a) + \| \log(1 - a)/(1 - t) \|_2 \| \text{whenever} \quad 0 \leq t \leq 1. \]

**Proof.** Since \( \log(1 - a) \) is in \( L^1(\Omega) \), we can consider the function \( f \) defined by
\[ f(z) = \exp \left( \int_{\mathbb{D}} \log((1 - a)(e^{\theta} + z))/(e^{\theta} - z) m(d\theta) \right) \]
for \( z \in \mathbb{D} \). Then \( f \) has boundary value
\[ f = (1 - a)e^{it} \quad \text{where} \quad t = \mathcal{H} \log(1 - a). \]

Now
\[ |(1 - t - f)| \leq |a - t + |1 - \varepsilon^2| = |a - t + |1 - \varepsilon^2| \leq |a - t| + |t|, \]
implying
\[ \| (1 - t - f) \|_2 \leq \| a - t \|_2 + |t|_2. \]
Since
\[ \mathcal{H} \log(1 - a) = \mathcal{H} \log((1 - a)/(1 - t)), \]
it follows that
\[ \mathcal{H} \log(1 - a) = \mathcal{H} \log((1 - a)/(1 - t)), \]
providing the required estimate.

**Lemma 5.** Let \( A \) be a measurable subset of \( \Omega \) and \( 0 < \varepsilon < 1/e \). Then there exist \( H^m \)-functions \( \varphi \) and \( \psi \) such that

(i) \[ |\varphi| + |\psi| \leq 1, \]
(ii) \[ |\varphi(z) - 1/3| \leq \varepsilon/3 \quad \text{for} \quad z \in A, \]
(iii) \[ |\psi(z)| \leq \varepsilon \quad \text{for} \quad z \in A, \]
(iv) \[ \|\varphi\|_2 \leq 4 \log(1/e) \|\psi\|_2. \]

**Proof.** Take \( \varepsilon = 1 - \varepsilon \). Application of Lemma 4 with \( a = \varepsilon \mathcal{X}_D \) yields \( f \) in \( H^m \) such that \( (t = 0) \)
\[ |f| = 1 - \varepsilon \mathcal{X}_D \quad \text{on} \quad \mathbb{D}, \]
and
\[ \| f \|_2 \leq m(\mathcal{X}_D)^{1/2} + \log(1/e) m(\Omega)^2. \]
Thus \( |f(z)| < \varepsilon \) on \( A \). The function \( \varphi = (1/3)(1 - f) \) satisfies (ii) and (iv). Remark that \( |\varphi| \leq 2/3 \). Apply again Lemma 4 taking now \( a = |\psi| \). We obtain an \( H^m \)-function \( g \) satisfying \( (t = 0) \)
\[ |\psi| = 1 - |\varphi| \quad \text{on} \quad \mathbb{D}, \]
and
\[ \| f - g \|_2 \leq m(\mathcal{X}_D)^{1/2} + \log(1/e) m(\Omega)^2. \]
Defining \( \psi = g \) we see that (i), (ii) obviously hold. Moreover
\[ \| f - g \|_2 \leq m(\mathcal{X}_D)^{1/2} + \log(1/e) m(\Omega)^2, \]
completing the proof.

In the following lemma, we make a careful analysis of a well-known construction in peak-set theory. This result is important in order to realize condition (iii) of Th. 3 and was not used in [3].

**Lemma 6.** Given \( 0 < \tau \leq 1/e \) there is a constant \( C_\tau < \infty \) such that if \( (S_\tau) \) is a sequence of measurable subsets of \( \Omega \) and \( (\varepsilon_\tau) \) a sequence in \( (0, 1] \), there are \( H^m \)-functions \( f \) and \( g \) satisfying

(i) \[ |f| + |g| \leq 1, \]
(ii) \[ f(z) = 1 \quad \text{if} \quad z \in S_\tau, \]
(iii) \[ |f(z) - 1| \leq \varepsilon_\tau \quad \text{if} \quad z \in S_\tau. \]
For \(\lambda \geq 5\), one has
\[
\{ \theta \in \mathcal{H}; \ 1/(1 - |f(\theta)|) \geq \lambda \} \subset \{ |u|/(u^4 + v^4) \leq -\log(1 - \lambda) \} \subset \{ |u| \geq \lambda/4 \} \cup \{ v^4 \geq \delta^4 \lambda^4 \}.
\]
It follows from the definition of \(\delta\) and the fact that \(\lambda/4 > 1 \geq \delta\), that
\[
\{ |u| \geq \lambda/4 \} = \bigcup S_i.
\]
By Tchebycheff's inequality, we find following weak-type estimation
\[
m \{ |f| \geq \lambda \} \leq 4\lambda^{-1} \sum_i m_i \{ |u| + \delta^{-1} \lambda^{-1} \int \varepsilon^2 \}
\leq 19\lambda^{-1} \sum_i \varepsilon_i^{-2} m_i \{ |u| + \delta^{-1} \lambda^{-1} |u| + \delta_i^2 \}
\leq 28\delta^{-1} \lambda^{-1} \sum_i \varepsilon_i^{-2} m_i \{ |S_i| \}.
\]
Write
\[
(1 - |f|)/(1 - \tau) = 1 - (1 - |f|)/(1 - \tau) = 1 - (|f| - \tau)/(1 - \tau).
\]
Since
\[
\log(1 - u) < 7u \quad \text{for} \quad -\infty < u < 4/5,
\]
\[
\int_{|u| < 4/5} \log^4(1 - |f|)/(1 - \tau) \leq 40(1 - \tau)^{-4} |f - \tau|^4
\leq 784(1 - \tau)^{-4} \delta^{-4} \sum_i \varepsilon_i^{-2} m_i \{ S_i \}.
\]
On the other hand, applying the weak-type inequality
\[
\int_{|u| > 4/5} \log^4(1 - |f|)/(1 - \tau) < 2 \int_{|u| > 4/5} \log^4(1 - |f|) +
+ 2\log^4(1 - \tau) m \{ |f| > 4/5 \} \leq 4\log^4(45) m \{ |f| > 4/5 \} +
+ 4 \int_\mathcal{H} m \{ |f| > 4/5 \} > \lambda \log(\lambda/\delta) d\mathcal{H} \leq 120 \delta^{-1} \sum_i \varepsilon_i^{-2} m_i \{ S_i \}.
\]
Combining inequalities
\[
\| \log(1 - |f|)/(1 - \tau) \|_4 \leq 21\delta^{-1} \sum_i \varepsilon_i^{-2} m_i \{ S_i \}.
\]
Since in particular \(\log(1 - |f|)\) is integrable on \(\mathcal{H}\), we may apply Lemma 4 taking \(a = |f|\) and \(\tau = \tau\). Thus an \(H^\infty\)-function \(g\) is obtained satisfying
(i) and, since
\[ \|y_1 - (1 - \tau)\| \leq 2 |1 - \tau| + 2 \|\log (1 - |f|) (1 - |\tau|)\|, \]
also (iv).

**Lemma 7.** Assume \((A_m)\) to be a sequence of disjoint sets in \(\Pi\). Let for each \(m\) a sequence \((B_{m,k})\) of disjoint subsets of \(\Pi\) be given and let \((S_i)\) be a sequence of sets in \(\Pi\). Take \(\varepsilon > 0\) and \((\eta_m, \varepsilon)\) sequences in \([0, 1]\).

Then there exists for each \(m\) \(H^\infty\)-functions \(\psi_m\) and \(\varphi_m\) satisfying

(i) \[ |\varphi_m| + |\psi_m| \leq 1, \]
(ii) \[ |\varphi_m| \leq \varepsilon \] on \( B_{m,k} \),
(iii) \[ |1 - \psi_m| \leq \varepsilon \] on \( S_i \),
(iv) \[ \|\varphi_m\| \leq C_\varepsilon \varepsilon^{-1} m(A_m), \]
(v) \[ \sum m(B_{m,k}) \leq C_\varepsilon \varepsilon^{-1} m(B_{m,k}) + C_\varepsilon \sum \varepsilon_i m(S_i), \]
(vi) \[ \sum m(B_{m,k}) \leq C_\varepsilon \varepsilon^{-1} m(A_m) + C_\varepsilon \sum \varepsilon_i m(S_i). \]

For any sequence of disjoint subsets \((A_m)\) of \(\Pi\)

(vii) \[ \sum m(B_{m,k}) \leq C_\varepsilon \varepsilon^{-1} \sum m(B_{m,k}) + C_\varepsilon \sum \varepsilon_i m(S_i), \]

\(\gamma > 0\) and \(C_\varepsilon < \infty\) denote numerical constants.

Proof. We assume \(\sum \varepsilon_i m(B_{m,k}) < \infty\) since otherwise \(\varphi_m = 0, \psi_m = 1\) satisfy.

Fixing \(m\) and applying Lemma 4, an \(H^\infty\)-function \(\eta_m\) is obtained satisfying

(viii) \[ |\psi_m| = 1 - \sum \frac{1}{1 - \varepsilon} \eta_m \] on \( \partial \mathcal{D} \)
and \((t = 0)\)

(ix) \[ \|1 - \psi_m\| \leq 2 \sum m(B_{m,k}) + 2 \sum \log(1/\varepsilon) m(B_{m,k}) \]
\[ \leq C_\varepsilon \sum \varepsilon_i m(B_{m,k}). \]

We also obtain from Lemma 5 \(H^\infty\)-functions \(\varphi'_m, \psi'_m\) such that

(x) \[ |\varphi'_m| + |\psi'_m| \leq 1, \]
(xi) \[ |\psi'_m(x) - 1/|x| \| \leq \frac{\varepsilon}{3} \] for \( x \in A_m \),
(xii) \[ \|\psi'_m\| \leq C_\varepsilon \varepsilon^{-1} m(A_m), \]
(xiii) \[ \|1 - \psi'_m\| \leq C_\varepsilon \varepsilon^{-1} m(A_m). \]

Finally, application of Lemma 6 to the sequence \((S_i)\), taking \(\tau = 1/\varepsilon\), provides \(H^\infty\)-functions \(f\) and \(g\) fulfilling

(xiv) \[ |f| + |g| \leq 1, \]
(xv) \[ |f(z) - 1| \leq \varepsilon/2 \] for \( z \in S_i, \)
(xvi) \[ \|f - 1/|z|\| \leq \text{const} \sum \varepsilon_i m(S_i), \]
(xvii) \[ \|g - (1 - 1/|z|\| \leq \text{const} \sum \varepsilon_i m(S_i). \]

Define

\[ \varphi''_m = \varphi'_{\varepsilon} \psi_{\varepsilon} \quad \text{and} \quad \psi''_m = f + \varphi_{\varepsilon}. \]

Then clearly

(xviii) \[ |\varphi''_m| + |\psi''_m| \leq 1, \]
(xix) \[ |\psi''_m| \leq \varepsilon \] on \( B_{m,k} \).

Since

\[ |1 - \psi''_m| \leq |1 - f| + |g| \leq 2 |1 - f|, \]
(iii) follows from (xv). From (xii), we get

(xx) \[ \|\psi''_m\| \leq \text{const} \varepsilon^{-1} m(A_m). \]

Combining (ix), (xi) and (xvii), we see that

(xxii) \[ \sum m(B_{m,k}) \leq \|1/2 |1 - (1-|e| - \varphi''_m |^2 \|^2 \]
\[ \leq \varepsilon \sum m(B_{m,k}) + \text{const} \sum \varepsilon_i m(B_{m,k}) + \text{const} \sum \varepsilon_i m(S_i), \]
using the fact that the sets \(A_m\) are mutually disjoint. Define

\[ \varphi_m = (1/2) [2|/2 |1 - (1-|e| - \varphi''_m |^2 \|^2 \]
\[ = \text{const} \varepsilon^{-1} m(A_m). \]

Then (xxii) implies (vy) and, since \(|\varphi_m| \leq |\varphi''_m|\), also (i), (iii), (iv) follow from (xvii), (xxiv), (xxv), respectively.

Let us verify (vy) and (vii). Since

\[ |1 - \psi''_m| \leq |1 - f| + |1 - (1 - |e| - g| + 1 - \psi''_m|, \]
the required inequalities are deduced from (xiii), (xvii), (xvii).

**IV. Proof of Theorem 3.** We will use a decomposition procedure for the functions \(f_m\). Our first lemma solves the problem in the case the func-
tions \( f_n \) are \( L^1 \)-normalized characteristic functions of disjoint subsets of \( \mathbb{U} \).

However, in order to make the result applicable in the general situation, additional conditions must be added.\(^8\)

**Lemma 8.** Assume \((A_\delta)_{\delta \in \mathbb{C}^+} \subseteq \mathbb{C}^+ \times \mathbb{C}^+\) to be finite sequences of disjoint sets and \((S_\delta)_{\delta \in \mathbb{C}^+}\) a sequence of sets. Let for each \( m \), \((B_{m,\delta})_\delta \) be a sequence of disjoint sets. Let \( \varepsilon > 0 \), \( \eta > 0 \) and \((\eta_1), (\eta_2)\) sequences in \([0, 1]\). Then there exist for each \( m \) \( H^m\)-functions \( \gamma_m \) and \( \nu_m \), satisfying

(i) \( |\gamma_m| + |\nu_m| \leq 1 \),

(ii) \( |\gamma_m| < \eta_0 \) on \( B_{m,\delta} \),

(iii) \( |1 - \gamma_m| \leq \varepsilon \) on \( S_\delta \),

(iv) \( \sum m(\nu_m \leq \gamma_m \leq 1) \),

(v) \( \sum \int_{A_m} |\nu_m - \gamma_m| \leq \varepsilon \),

(vi) \( \sum \int_{A_m} \nu_m \leq \varepsilon \sum m(A_m) \).

Here \( \gamma_m \) is the constant of Lemma 7, \( C_8 < \infty \) is a numerical constant and \( \varepsilon(\eta) \) is a function depending on \( \eta \).

**Proof.** We first partition \([1, n]\) into sets \( \mathcal{M}, \mathcal{N} \) taking

\[ \mathcal{M} = \{ m ; m(A_m) > n^{-1/2} \sum m(A_m) \} \quad \text{and} \quad \mathcal{N} = \{ 1, \ldots, n \} \setminus \mathcal{M}. \]

Notice that card \( \mathcal{M} \leq n^{1/2} \). Let us first deal with the small set \( \mathcal{M} \). Application of Lemma 7 yields \( H^m\)-functions \((\phi_m)_{m \in \mathcal{M}}\) \((\nu_m)_{m \in \mathcal{M}}\) satisfying (i), (ii), (iii), (v) and

(ix) \( \sum \int_{A_m} |\phi_m - \nu_m| \leq \varepsilon \sum m(A_m) \),

(x) \( |\phi_m| \leq \varepsilon^{-1} m(B_m) \),

(xi) \( \sum \int_{A_m} |\phi_m - \nu_m| \leq \varepsilon^{-1} m(\nu_m) \).

Denote by \( d \) a positive integer (depending on \( \eta \)) which will be fixed later.

The set \( \mathcal{N} \) will be partitioned into subsets \( \mathcal{N}_\delta \), card \( \mathcal{N}_\delta = d \), and a “negligible” remainder \( \mathcal{N}_\text{rem} \).

To each \( \delta \), we will associate systems \((\mu_{m,\delta})_{0 \leq m \leq s}, (\nu_{m,\delta})_{0 \leq m \leq s}\) of \( H^m\)-functions fulfilling (i), (ii), (iii), (v) and moreover

(xii) \( \sum_{\delta \in \mathcal{N}_\delta} |1 - \nu_{m,\delta}| \leq \varepsilon \),

(xiii) \( \sum_{\delta \in \mathcal{N}_\delta} |1 - \nu_{m,\delta}| \leq \varepsilon \sum m(A_{m,\delta}) + \sum \varepsilon^{-1} m(S_\delta) \),

(xiv) \( \sum_{\delta \in \mathcal{N}_\delta} \sum_{m \leq s} \| \mu_{m,\delta} \| \leq (2C_8)^{1/2} \varepsilon^{-1} \sum m(A_{m,\delta}) + \sum \varepsilon^{-1} m(S_\delta) \),

(xv) \( \sum_{\delta \in \mathcal{N}_\delta} \sum_{m \leq s} \| \nu_{m,\delta} \| \leq \varepsilon \sum m(A_{m,\delta}) + \sum \varepsilon^{-1} m(S_\delta) \).

The negligibility of \( \mathcal{N}_\text{rem} \) is in the sense that

(xvi) \( \sum_{\delta \in \mathcal{N}_\text{rem}} m(A_{m,\delta}) \leq \varepsilon \),

where we define for simplicity

\[ \Theta = 8 \varepsilon^{-1} C_8 \sum m(B_{m,\delta}) + 8 \varepsilon^{-1} (2C_8)^{1/2} \varepsilon^{-1} \left( \sum m(A_{m,\delta}) + \sum \varepsilon^{-1} m(S_\delta) \right). \]

Suppose \( N_1, N_2, \ldots, N_r \) are already obtained. Define \( N_{r+1} = N_r \cup \ldots \cup N_r \). If \( \sum m(A_m) \leq \varepsilon \), take \( N_{r+1} = N_r \) and define for \( m \in N_{r+1} \)

\[ \gamma_m = 0 \quad \text{and} \quad \nu_m = 1. \]

Then, obviously,

(xvii) \( \sum_{\delta \in \mathcal{N}_\text{rem}} \int_{A_m} |\nu_m - \gamma_m| \leq 2 \Theta. \)

If \( \sum m(A_m) > \varepsilon \), then we can proceed to the extraction of a subset \( N_{r+1} \subset N_r \). Suppose we have already obtained \( m_1, m_2, \ldots, m_r \), \( r < d \) in \( N_r \), such that following condition is satisfied:

(xviii) \( \| \nu_{m,\delta} \| \leq (2C_8)^{1/2} \varepsilon^{-1} \sum m(A_{m,\delta}) + \sum \varepsilon^{-1} m(S_\delta) \)

for \( s = 1, \ldots, r \). Define the set

\[ U_r = \{ \theta \in \mathcal{P} ; \sum_{i=1}^r |1 - \nu_{m_i}(e^{i\theta})|^2 \geq 1 \}, \]

for \( m_i \in N_r \).
for which, by (xviii),

\[(xix)\] \[m(U_r) \leq (2C_r)^{d/2-r} \left( \varepsilon^{-1} \sum_{i=1}^r m(A_{m_i}) + \sum_{i \neq 1} \varepsilon_i^{-1} m(S_i) \right).\]

Apply then again Lemma 7 considering the sets \((A_{m_n})_{m \in \mathbb{N}^+}\), where \(N^+ = N^+ \setminus \{m_1, \ldots, m_r\}\), and adding the set \(U_r\) to the sequence of the \((S_i)\) to which we associate the value \(d^{-1/2}\). \(H^m\)-functions \((\varphi_{m_n})_{m \in \mathbb{N}^+}\) and \((\varphi_{m_n})_{m \in \mathbb{N}^+}\) are obtained satisfying in addition to (i), (ii), (iii), (v).

\[(xx)\] \[|1 - \varphi_m| < d^{-1/2}\] on \(U_r\),

\[(xxi)\] \[\sum_{m \in \mathbb{N}^+} \int |\varphi_1 - \varphi_m| \leq (\varepsilon/4) \sum_{m \in \mathbb{N}^+} m(A_m) + C_1 \sum_{m \in \mathbb{N}^+} \varepsilon_1^{-1} m(B_{m_1}) + C_1 \sum_{m \in \mathbb{N}^+} \varepsilon_i^{-1} m(S_i) + 2(C_r \varepsilon)^{d/2} \left( \varepsilon^{-1} \sum_{i=1}^r m(A_i) + \sum_{i \neq 1} \varepsilon_i^{-1} m(S_i) \right),\]

\[(xxii)\] \[|1 - \varphi_m|^2 \leq 4C_r \varepsilon^{-1} m(A_m) \left( (2C_r \varepsilon)^{d/2} \left( \varepsilon^{-1} \sum_{i=1}^r m(A_i) + \sum_{i \neq 1} \varepsilon_i^{-1} m(S_i) \right) \right),\]

\[(xxiii)\] \[\sum_{m \in \mathbb{N}^+} \int |1 - \varphi_m|^2 \leq 4C_r \varepsilon^{-1} \sum_{m \in \mathbb{N}^+} m(A_m) + (2C_r \varepsilon)^{d/2} \left( \varepsilon^{-1} \sum_{i=1}^r m(A_i) + \sum_{i \neq 1} \varepsilon_i^{-1} m(S_i) \right),\]

Since now

\[\sum_{i=1}^r m(A_i) \leq dm^{-1/2} \sum_{m \in \mathbb{N}^+} m(A_m),\]

we find

\[C_1 \sum_{m \in \mathbb{N}^+} \varepsilon_1^{-1} m(B_{m_1}) + (2C_r \varepsilon)^{d/2} \left( \varepsilon^{-1} \sum_{i=1}^r m(A_i) + \sum_{i \neq 1} \varepsilon_i^{-1} m(S_i) \right) \leq (1/8) \varepsilon_0\theta.\]

By hypothesis

\[\theta = \sum_{m \in \mathbb{N}^+} m(A_m) \leq \sum_{m \in \mathbb{N}^+} m(A_m) + (1/2) \theta.\]

Thus we deduce from (xxii) and (xxiii)

\[(xxiv)\] \[\sum_{m \in \mathbb{N}^+} \int |1 - \varphi_m|^2 \leq 4C_r \varepsilon^{-1} \sum_{m \in \mathbb{N}^+} m(A_m),\]

\[(xxv)\] \[\sum_{m \in \mathbb{N}^+} \int |1 - \varphi_m|^2 \leq 5C_r \varepsilon^{-1} \sum_{m \in \mathbb{N}^+} m(A_m).\]

So we can choose \(m_n+1 \in N^+\) fulfilling (xxiii) and (xxv). From (xxiii), it is clear that (xxviii) will hold for \(s = r + 1\). Summation of (xxviii) provides inequality (xiv) for the system \((\varphi_{m_n})_{m \in \mathbb{N}^+}\).

Since, by construction

\[\sum_{m \in \mathbb{N}^+} |1 - \varphi_m|^2 \leq 6\]

since \(|\varphi_m|^2 \leq 1\) for each \(m\). Thus the family \((\varphi_{m_n})_{m \in \mathbb{N}^+}\) satisfies (xxiii).

This completes the construction. It remains to choose the integer \(d\) and precise the function \(\xi(\eta)\).

First, one has by (xxii)

\[\sum_{m \in \mathbb{N}^+} |1 - \varphi_m|^2 \leq \sum_{m \in \mathbb{N}^+} |1 - \varphi_m| + \sum_{m \in \mathbb{N}^+} |1 - \varphi_m| + \sum_{m \in \mathbb{N}^+} |1 - \varphi_m| \]

\[\leq 2m^{-1/2} + 3d^{-1/2} m/d\]

\[= (2m^{-1/2} + 3d^{-1/2} d).\]

Thus it suffices to take \(d \sim \eta^{1/2}\), assuming a large enough with respect to \(\eta\). If this is not the case, it will follow from the definition of \(\xi(\eta)\) that \(\varphi_m = 0, \varphi_m = 1\) already satisfy the lemma.

Define

\[\xi(\eta) = (4C_r \varepsilon)^{d/2}.\]

Then (vi) follows from (ix), (xxiii), (xxvii), (vii) follows from (x) and (xiv), (viii) follows from (xi) and (xv).

This completes the proof.

Remark. The function \(\xi(\eta)\) obtained by preceding estimations is of the form \(\eta^{-\alpha_0 \mathrm{const} \cdot \eta^{1/2}}\). Taking the first term of the right hand side in inequality (vi) in account, it is clear that the lemma will only be useful for \(\eta > (\log n)^{-1/2+4}\).
LEMMA 9. Assume $\varepsilon > 0$, $\delta > 0$, $\eta > 0$ and $n$ a positive integer satisfying the inequalities
\[ \delta < (1/4) \varepsilon^2 (\eta^{-1})^2 \quad \text{and} \quad n = 2^{1/2} (\eta^{-1})^2. \]
Let $(A_{n,k})_{k=0}^n$ be a system of disjoint subsets of $\Omega$ such that
\[ m(A_{n,k}) \leq \delta m(A_{n,k+1}), \quad \text{where} \quad A_{n,k} = \bigcup_{m=1}^n A_{n,m}. \]
Then there exists a system of $H^m$-functions $(\varphi_{m,k})$, $(\psi_{m,k})$ fulfilling
\begin{enumerate}[(i)]
  \item $|\varphi_{m,k}| + |\psi_{m,k}| \leq 1$,
  \item $|\varphi_{m,k}| \leq \varepsilon^{-1}$ on $A_{n,k}$ for $k > 1$,
  \item $|\varphi_{m,k}| \leq \varepsilon^{-1} m(A_{n,k})$,
  \item $\sum_{m} \int_{A_{n,m}} |\varphi_{m,k} - \varphi| \leq \varepsilon m(A_{n,k})$,
  \item $\sum_{m} \int_{A_{n,m}} |\varphi_{m,k} - \varphi| \leq \varepsilon m(A_{n,k})$,
  \item $\sum_{m} \int_{A_{n,m}} |\psi_{m,k}| \leq \varepsilon m(A_{n,k})$, taking $A_{n,k} = \bigcup_{m=1}^n A_{n,m}$.
\end{enumerate}
(C is again a numerical constant.)

Proof. We construct the $H^m$-functions by induction on $k$. Let us define for convenience
\[ U_{n,k} = \sum_{m} \int_{A_{n,m}} |\varphi_{m,k} - \varphi| \leq \varepsilon^{-1} m(A_{n,k+1}). \]
and
\[ \psi = \sum_{m} |\varphi_{m,k}|. \]
Then, by Cauchy–Schwarz and Teichmüller
\begin{enumerate}[(vii)]
  \item $\sum_{m} \int_{A_{n,m}} |\varphi_{m,k}| \leq \varepsilon m(A_{n,k+1})^2$.
\end{enumerate}

Step 1. Application of Lemma 8 to the sets $(A_{n,k})$ gives $H^m$-functions satisfying (i), (iii), (v) and
\begin{enumerate}[(ix)]
  \item $\sum_{m} \int_{A_{n,m}} |\varphi_{m,k} - \varphi| \leq (1 + \frac{1}{2} \varepsilon (\eta^{-1})^2 m(A_{n,1})$,
\end{enumerate}
and
\begin{enumerate}[(x)]
  \item $\sum_{m} \int_{A_{n,m}} |\varphi_{m,k} - \varphi| \leq C \varepsilon m(A_{n,1})$ (take $Q_m = A_{n,m}$),
\end{enumerate}
and
\begin{enumerate}[(xi)]
  \item $\gamma_k \leq \varepsilon (\eta^{-1})^2 m(A_{n,1})$.
\end{enumerate}

Inductive step. Assume the construction done up to level $k$. We apply Lemma 8 in the following situation:
\[ A_{n,k} = A_{n,k+1}, \quad Q_m = A_{n,k}, \quad B_k = U_{k+1,k}, \quad (i \leq k), \]
\[ B_{n,k} = A_{n,k}, \quad (i \leq k), \quad \varepsilon_k = \varepsilon_k \leq \varepsilon^{k+1} \leq \varepsilon^{k-1} \leq (k+1). \]
This gives $H^m$-functions $(\varphi_{m,k+1})$, $(\psi_{m,k+1})$ fulfilling (i), (ii), (iii), (v), (vi) (replacing $k$ by $k+1$) and, from (vii)
\begin{enumerate}[(xii)]
  \item $\sum_{m} \int_{A_{n,k+1}} |\varphi_{m,k+1}| \leq (1 + \frac{1}{2} \varepsilon (\eta^{-1})^2 m(A_{n,1}) + C \varepsilon m(A_{n,1})$ \[+ \varepsilon (\eta^{-1})^2 m(A_{n,1})^2 + \varepsilon (\eta^{-1})^2 m(A_{n,1}) \]
\end{enumerate}

($C$ is again a numerical constant).

Let us next estimate $\varphi_k$ using (xi) and the recursive inequality (x). Define for convenience
\[ I_k = \varphi_k + \varepsilon^{-1} \varphi_{k-1} + \varepsilon^{-1} \varphi_{k-2} + \ldots + \varepsilon^{-1} \varphi_0. \]

Reformulating (xi) and (xiv),
\[ \gamma_{k+1} \leq \varepsilon (\eta^{-1})^2 m(A_{n,1}) + \varepsilon (\eta^{-1})^2 I_k. \]

and hence
\begin{enumerate}[(xvi)]
  \item $I_{k+1} \leq \varepsilon (\eta^{-1})^2 m(A_{n,1}) + (1 + \varepsilon (\eta^{-1})^2) I_k$.
\end{enumerate}

taking $\varepsilon_k = 2 \varepsilon (\eta^{-1})^2$. Iteration leads to the inequality
\[ I_k \leq \varepsilon_k m(A_{n,1}) + \varepsilon_k m(A_{n,1}) + \ldots + \varepsilon_k m(A_{n,1}) \]
and from the hypothesis on the $A_{n,1}$
\[ I_k \leq \varepsilon_k (1/(1 - \delta \varepsilon_k)) m(A_{n,1}) \leq 2 \varepsilon_k m(A_{n,1}) \]
and
since \( \delta t_{1,\alpha} < 1/2 \). Thus we have in particular

(iv) \( \nu_0 \leq 2t_{1,\alpha}m(A[k]) \).

By the choice of \( n \), (ix) implies (iv) for \( k = 1 \). In general, we get from (xii), (xv) and the hypothesis on the sets \( A[k] \)

\[ \sum_{m} \int_{d_{m,b+1}} |\psi_{m,b+1} - \gamma_m| \leq 2m(A[k+1]) + (C_{\psi} \psi_{m,b+1} |A[k]|) + \]

\[ + (2 \xi(n) \psi_{m,b+1} m(A[k])) \]

leading again to (iv).

The verification of (vii) from (x) and (xiii) is analogous.

The next lemma, which is the final step in the proof of Th. 3 uses a "decomposition" technique for functions which was also applied in [3], [4].

Lemma 10. Fix \( r > 0 \) and let \( n \geq \left( C_2 \xi(r/3)\right)^{1/6} \) be a positive integer. Assume \( (f_{m,b})_{b \in [n]} \) positive, disjointly supported integrable functions on \( \mathbb{R} \).

Then there exists \( H^{n} \)-functions \( \psi_{m,b} \) \( m \) so that

(i) \( \sum_{m} \int_{0}^{\psi_{m,b} - \gamma_m} \leq r \sum_{m} \int f_m \),

(ii) \( \sum_{m} |1 - \psi_{m,b}| \leq 1 \),

(iii) \( |\psi_{m,b} + |\gamma_m| \leq 1 \) for each \( m \).

Proof. Define for convenience

\( \eta = \gamma/3 \), \( M = C_2 \xi(\eta) \), \( \psi = M^{-2} \), \( \delta = 1/4C_2 M^{-2} \), \( \delta = 11 \).

For \( -\infty < k < \infty \), take \( A_{m,b} = [4M_k^2 < f_m < 4M_{k+1}] \). Define further for \( a = 0, 1, 2, \ldots, \delta^{-1} \)

\[ A_{\alpha}[\varepsilon] = A_{\alpha}[\varepsilon] \quad \text{and} \quad A_{\alpha} = \sum_{n=1}^{\infty} A_{\alpha}[\varepsilon] \).

For fixed \( \alpha \), we introduce the sequence (which may depend on \( \alpha \))

\[ h_1 > h_2 > \ldots > h_k \]

of integers, where

(iv) \( m(A[k]) < \delta m(A[k+1]) \),

(v) \( m(A[k]) < \delta^{-1} m(A[k+1]) \) for \( k > h_k > h_k + 1 \)

(approximating the \( s_{m,b} \), we can restrict \( k \) to a bounded interval \([a_1, h_1]\)).

Define further

\[ O = O[\varepsilon] = [h_1^2, a = 1, 2, \ldots, a]. \]

For each \( m \), let

\[ A_{\alpha}[\varepsilon] = \bigcup_{\alpha} A_{\alpha}[\varepsilon] \quad \text{and} \quad B[\alpha] = \bigcup_{\beta \neq \alpha} A_{\beta}[\varepsilon]. \]

First, using (v), we find

\[ \sum_{m} \int f_m = \sum_{\alpha} \sum_{m} \int f_m \]

\[ \leq M \sum_{\alpha} \sum_{m} \int m(A_{\alpha}[\varepsilon]) \]

\[ \leq M \sum_{\alpha} \sum_{m} \int m(A_{\alpha}[\varepsilon]) \]

\[ \leq 2 \delta^{-1} \sum_{\alpha} \sum_{m} \int m(A_{\alpha}[\varepsilon]) \]

Hence

\[ \sum_{m} \int f_m \leq 2C_2 \delta^{-1} \sum_{m} \int f_m. \]

Since now \( \alpha, \beta, \eta \) and \( m \) satisfy the conditions of Lemma 9 there are \( H^{r} \)-functions \( \psi[\varepsilon][m] \), \( \psi[\varepsilon][m] \) satisfying (i), (ii), (iii), (iv), (v), (vi), (vii) of Lemma 9 with respect to the sets \( A_{\alpha}[\varepsilon], B_{\alpha}[\varepsilon] \).

For fixed \( m \), let

\[ \psi[\varepsilon][m] = \psi[\varepsilon][m+1] + \psi[\varepsilon][m+1] + \psi[\varepsilon][m+1] + \psi[\varepsilon][m+1] + \psi[\varepsilon][m+1] + \psi[\varepsilon][m+1] \]

and

\[ \psi[\varepsilon][m] = \psi[\varepsilon][m] + \psi[\varepsilon][m] + \psi[\varepsilon][m] + \psi[\varepsilon][m] + \psi[\varepsilon][m] + \psi[\varepsilon][m] \]

Then

(vii) \( |\psi[\varepsilon][m]| + |\psi[\varepsilon][m]| \leq 1 \).

Let us estimate

\[ I[\varepsilon] = \sum_{m} \int_{[\varepsilon]-\gamma_m} |f_m| \]

Write

\[ \sum_{m} \int_{[\varepsilon]-\gamma_m} |f_m|. \]
Then
\[ \int_{\mathcal{D}(\mathcal{K}, h_0)} |\gamma_1 - \varphi| + \int_{\mathcal{D}(\mathcal{K}, h_0)} |\gamma_2 - \varphi| \cdots + \int_{\mathcal{D}(\mathcal{K}, h_0)} |\gamma_n - \varphi| \leq \sum_{m=1}^{M} m(A(c)_{m_k}) + \sum_{m=1}^{M} m(A(c)_{m_k}) \langle \varphi \rangle_{2} \langle \varphi \rangle_{2} \langle \varphi \rangle_{2} \ldots \langle \varphi \rangle_{2} + \sum_{m=1}^{M} m(A(c)_{m_k}) \langle \psi \rangle_{2} \langle \psi \rangle_{2} \langle \psi \rangle_{2} \langle \psi \rangle_{2} \ldots \langle \psi \rangle_{2}.
\]

Taking (ii) of Lemma 9 in account. The first term on the right is dominated by
\[ \int_{\mathcal{D}(\mathcal{K}, h_0)} |\gamma_1 - \varphi| + \int_{\mathcal{D}(\mathcal{K}, h_0)} |\gamma_2 - \varphi| \cdots + \int_{\mathcal{D}(\mathcal{K}, h_0)} |\gamma_n - \varphi| \leq \sum_{m=1}^{M} m(A(c)_{m_k}) + \sum_{m=1}^{M} m(A(c)_{m_k}) \langle \varphi \rangle_{2} \langle \varphi \rangle_{2} \langle \varphi \rangle_{2} \langle \varphi \rangle_{2} \ldots \langle \varphi \rangle_{2}.
\]

It follows by (iv), (vii) of Lemma 9 and Cauchy-Schwartz that
\[ \sum_{m=1}^{M} \int_{\mathcal{D}(\mathcal{K}, h_0)} |\gamma_1 - \varphi| \cdots + \int_{\mathcal{D}(\mathcal{K}, h_0)} |\gamma_n - \varphi| \leq 3eM(A(c)_{m_k}) + \sum_{m=1}^{M} m(A(c)_{m_k}) \langle \varphi \rangle_{2} \langle \varphi \rangle_{2} \langle \varphi \rangle_{2} \langle \varphi \rangle_{2} \ldots \langle \varphi \rangle_{2}.
\]

and hence
\[ \sum_{m=1}^{M} I(c) \leq 15C_0M^{-1} \sum_{m=1}^{M} f_m.
\]

Using the same technique as in [3] let us introduce the $H^m$-functions
\[ \varphi_m = 2^{-m} \left\{ \gamma - \frac{1}{2^{m}} \sum_{c \in \mathcal{D}(\mathcal{K}, h_0)} |\gamma_1 - \varphi| \right\}.
\]

and
\[ \psi_m = (1, \ldots, \frac{10}{11}) \sum_{c \in \mathcal{D}(\mathcal{K}, h_0)} |\varphi|.
\]

for $m = 1, \ldots, n$. Since $|\varphi_m| \leq (1, \frac{10}{11}) \sum |\varphi|$, (iii) follows from (vi). Further
\[ \sum |\varphi_m| \leq (1, \frac{10}{11}) \sum |\varphi| \leq \sup |\varphi_m| \sum |\varphi|.
\]

Let us verify (i), taking $\gamma_1 = 2^{-14} \psi_{11}$. Then
\[ \sum f_m |\gamma_1 - \varphi_m| = 2^{-14} \sum f_m |\gamma_1 - \varphi| \leq 2^{-14} \sum f_m \psi_{11} \leq 2^{-14} \sum f_m \leq C_1 M^{-1} \sum f_m.
\]

In order to verify (ii), fix $c = 0, \ldots, 10$ and evaluate $\sum |\varphi_m|$. By definition of $\varphi_m$, \[ \sum m |\varphi_m| = \sum m |\varphi_m| = \sum m |\varphi_m| \leq \sum m |\varphi_m| \leq \eta_m.
\]

Now for each $s = 1, \ldots, r$, we get that
\[ \sum |\varphi_m| \leq \eta_m.
\]

Moreover for $s < t < \eta$, by (vi) of Lemma 9,
\[ \sum |\varphi_m| \geq d^{-s} \eta \rightarrow \sum |\varphi_m| \geq d^{-w} \eta.
\]
It is an elementary exercise to verify that (**) imply that
\[ \sum_{m} \sum_{n} |1 - \psi(m,n) + \psi(n,m)| \ll \eta + \varepsilon + \cdots = \varepsilon^{-1} \eta. \]
This completes the proof of the lemma.

Proof of Theorem 3. If \( f_1, \ldots, f_n \) are disjointly supported functions in \( L^2(\mathbb{N}) \) satisfying (i) of Th. 3, there are \( H^\infty \)-functions \( g_m \) \( (1 \leq m \leq n) \) satisfying
\[ \|g_m\|_2 \leq 1 \quad \text{and} \quad \langle f_m, g_m \rangle = \delta. \]
For \( \varepsilon > 0 \) (which we fix later) and \( n \gg (C_2 \varepsilon^{2/3})^{1/2} \), let \( \phi_m = \psi_m \) be the \( H^\infty \)-functions obtained in previous lemma, replacing \( f_m \) by \( |f_m| \).
Since
\[ \sum_{m} |\phi_m| |\gamma_0 - \gamma_m| \leq \varepsilon_n, \]
we see that \( \text{card}(N) \leq \varepsilon^{1/2} \eta \)

\[ N = \{ m = 1, \ldots, n; \sum_{m \in N} |\gamma_0 - \gamma_m| > \varepsilon \}. \]
Take \( \psi_m = \phi_m \| \phi_m \|_2, \psi_m = \psi_m \) if \( m \in N \)
and \( \psi_m = 0, \psi_m = 0 \) if \( m \notin N \). Then
\[ \sum_{m \notin N} |1 - \psi_m| \leq \sum_{m \in N} |1 - \psi_m| + \text{card}(N) \leq (\varepsilon + \varepsilon^{1/2} \eta)n. \]
If \( m \notin N \), then
\[ |\langle f_m, \psi_m \rangle - \psi_m \langle f_m, g_m \rangle| \leq \sum_{m \notin N} |1 - \psi_m| |\gamma_m - \gamma_0| \ll \varepsilon^{1/2} \eta. \]
Taking \( \eta < \delta/4 \), we can put \( \delta_1 = \delta/2 \). For \( \alpha(n) \), take \( 2 \varepsilon^{1/2} \eta \), where \( \varepsilon \) must be large enough to ensure the inequality \( n \gg (C_2 \varepsilon^{2/3})^{1/4} \).

V. Remarks.
1. The disjointness hypothesis for the functions \( f_m \) in Th. 3 can be replaced by a weaker hypothesis, i.e.
\[ \| \sum_{m \notin N} |f_m| \|_2 \leq B \]
where \( A_m = \sup |f_m| \) and \( B \) is some constant.

2. In fact, Th. 3 can be combined with results of [3] as follows. Given \( \delta > 0 \), there exist \( \delta_1 > 0 \) and a function \( \alpha(n) \) s.t. \( \alpha(n)/n \to 0 \) so that the following holds:
If \( f_1, \ldots, f_n \) in \( L^2(\mathbb{N}) \) are \( \delta_1 \)-Banachmacher \( P \), i.e. if
\[ \| \sum_{m \in N} |f_m| \|_2 \to 0 \quad \text{as} \quad n \to \infty, \]
and if
\[ \|f_n\|_2 \gg (1 - \delta)n \quad (1 \leq n \leq n) \]
(in particular, if the \( f_n \) are minimum norm liftings), then there are \( H^\infty \)-functions \( \varphi_1, \ldots, \varphi_n \) and \( \varphi_{m+1}, \ldots, \varphi_n \) satisfying following properties
(i) \( |\varphi_1| + |\varphi_2| \ll 1 \) \( (1 \leq k \leq n), \)
(ii) \( \sum |\varphi_k| \leq 1, \)
(iii) \( \sum |\varphi_k| \leq 1, \)
(iv) \( \langle \varphi_n \| \varphi \rangle = \delta \|f_n\|_2. \]
3. Our methods provide estimates of the form \( \alpha(n)/n < (\log n)^{-1+\varepsilon}. \), Is it possible to replace \( \alpha(n) \) by a constant?

References

H\(p\) estimates for weakly strongly singular integral operators on spaces of homogeneous type

by

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Abstract. Let \(X\) be a normalized homogeneous space. We define \"weakly strongly singular\" operators on \(X\) and study the action of the \"convolution\" operator induced by this kernel on the atomic Hardy space \(H^p(X)\), with \(0 < p < 1\). A boundedness result is obtained. These operators are analogues of the weakly strongly operators on \(\mathbb{R}^n\) studied by C. L. Fefferman and E. M. Stein in [6].

1. Introduction. In this paper we study a generalization of convolution operators induced by weakly strongly singular integral kernels. Examples of these kernels, in the case of \(\mathbb{R}^n\) are given by

\[
k(x) = |x|^{-\beta} \psi(x) \exp(|x|^\alpha),
\]

where \(0 < \alpha < 1\), \(\beta > 0\) and \(\psi\) is a \(C^\infty\) function on \(\mathbb{R}^n\), which vanishes near zero and equals 1 outside a bounded set (see [5]). The \(L^p\) theory, \(1 < p < +\infty\), for operators obtained by convolution with kernels \(k(x)\), has been studied by I. I. Hirschmann [7], S. Wainger [12], C. L. Fefferman [5], C. L. Fefferman and E. M. Stein [6], J. E. Björk [1] and P. Sjölin [11].

Also in [6], C. L. Fefferman and E. M. Stein obtain boundedness results for \(H^p(\mathbb{R}^n)\), \(1 > p > p_0(\alpha, \beta, n) > 1/2\). Estimates including the limiting case \(p = p_0(\alpha, \beta, n)\) were obtained by R. R. Coifman in [2] when \(n = 1\). Here we consider a generalization of these kernels and the action of the induced operators on \(\mathbb{R}^n\) spaces, \(p \leq 1\), defined in terms of atoms on spaces of homogeneous type. First we define what we mean by a weakly strongly singular kernel on spaces of homogeneous type. In Theorem 3 we prove that the operator \(K\) induced by this kernel maps atoms into elements of \(H^p(\mathbb{R}^n)\), \(p \leq 1\). In the proof of this theorem we extend some techniques used by R. A. Madsen and C. Segovia in [9]. The extension of the operator to the whole space \(H^p\) requires the introduction of an auxiliary operator, namely \(X^\#\), acting on the space \(\text{Lip}(1/p-1)\) of classes of Lipschitz functions. This operator is an adaptation of the operator \(X^\#\) considered in [9].