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H^∞ is a Grothendieck space

by

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Abstract. It is shown that a non-weakly compact operator on H^∞ fixes an l^∞-copy. In particular, H^∞ has the Grothendieck property and l^∞ embeds in any infinite-dimensional complemented subspace of H^∞.

I. Introduction. This work is a continuation of [3] (cf. also [4]). Let us recall some definitions. Π denotes the circle and m its Haar measure. H_0^1 is the space of integrable functions f on Π such that $\hat{f}(n) = 0$ for $n \leqslant 0$. We use the notations $q\colon L^1 \to L^1/H_0^1$ and $\sigma\colon L^1/H_0^1 \to L^1$ for the quotient map and the minimum norm lifting, respectively. The duality

$$\langle f, \varphi\rangle = \int f\varphi\, dm$$

identifies the dual $(L^1/H_0^1)^*$ with the space H^∞ of bounded analytic functions on the unit disc D.

It was shown in [3] that H^∞ has the Dunford–Pettis property (DPP) and $(H^\infty)^*$ is weakly sequentially complete (WSC). We establish here the Grothendieck property (GP) of H^∞. Recall that a Banach space X has GP provided weak*-null sequences in X^* are weakly-null, or, equivalently, each operator $T\colon X \to c_0$ is weakly compact. In fact, a stronger result is obtained. If $T\colon H^\infty \to Y$ is an operator, then T is either weakly compact or there exists a subspace Z of H^∞, Z isomorphic to l^∞, on which T induces an isomorphism.

As corollary it follows that l^∞ embeds in any infinite dimensional complemented subspace of H^∞, solving one of the questions raised in [18].

Latter results where previously announced in [5].

II. Operators on H^∞ and the Grothendieck property. Classical examples of G-spaces are the $L^\infty(\mu)$-spaces. Next result, implying the G-property, emphasizes the same behaviour of H^∞ and L^∞ in several aspects.

THEOREM 1. *Assume $T\colon H^\infty \to Y$ is an operator. If T is not weakly compact, then T is an isomorphism when restricted to a subspace Z of H^∞, Z isomorphic to l^∞.*

It was shown by S.V. Kisliakov [15] and independently by F. Delbaen [7] that non-weakly-compact operators on the disc-algebra A fix a c_0-copy.

The proof of this result makes crucial use of the Riesz-decomposition of A^*

$$A^* = M_s(\Pi) \oplus L^1/H_0^1,$$

where $M_s(\Pi)$ denotes the space of singular measures on Π.

Such a result does not hold for $(H^\infty)^*$. We will use the more general approach which already enabled us to prove DPP of H^∞. The technique consists in establishing a finite-dimensional result for L^1/H_0^1 which can therefore be carried over to $(H^\infty)^*$ by arguments of local reflexivity.

The purpose of this section is to state the local L^1/H_0^1-theorem and derive Th. 1 from it. The proof of the L^1/H_0^1-result is rather technical and will be presented in the next two sections.

As a formal consequence of the DPP of H^∞ and Th. 1, we get

COROLLARY 2. *l^∞ embeds in any infinite-dimensional complemented subspace of H^∞.*

It looks reasonable to conjecture that a complemented subspace of H^∞ is either an l^∞ or an H^∞-isomorph.

THEOREM 3. *For each $\delta > 0$, there exist $\delta_1 > 0$ and a function $a(n)$ satisfying $\lim\limits_{n\to\infty} (a(n)/n) = 0$, so that the following property holds:*

Let $f_1, \ldots, f_n$ be disjointly supported functions in $L^1(\Pi)$ such that

(i) $$1 \geqslant \|f_m\|_1 \geqslant \|q(f_m)\| \geqslant \delta \quad (1 \leqslant m \leqslant n).$$

Then there exist H^∞-functions φ_m, ψ_m $(1 \leqslant m \leqslant n)$ fulfilling

(ii) $$|\varphi_m| + |\psi_m| \leqslant 1 \quad \textit{for each } m,$$

(iii) $$\sum |1-\psi_m| \leqslant a(n),$$

(iv) $$\langle f_m, \varphi_m\rangle = \delta_1 \quad \textit{for each } m.$$

The reader will find some further comments on Th. 3 in the remarks at the end of this paper.

Our first objective will be to derive from Th. 3 the following result on $(H^\infty)^*$.

PROPOSITION 1. *There exists $\tau > 0$ and $\varkappa > 0$ and a function $\beta(n)$ for which $\lim\limits_{n\to\infty} (\beta(n)/n) = 0$ such that the following property holds:*

Let $\Phi_1, \ldots, \Phi_n$ be n elements in the unit ball of $(H^\infty)^*$ and assume

(*) $$\Big\|\sum a_m \Phi_m\Big\| \geqslant (1-\tau)\sum |a_m| \quad (a_m \in \mathbf{C}).$$

Then there are H^∞ functions φ_m, ψ_m $(1 \leqslant m \leqslant n)$ satisfying

(i) $$|\varphi_m| + |\psi_m| \leqslant 1 \quad \textit{for each } m,$$

(ii) $$\sum |1-\psi_m| \leqslant \beta(n),$$

(iii) $$\langle \Phi_m, \varphi_m\rangle = \varkappa \quad \textit{for each } m.$$

LEMMA 1. *Prop. 1 holds if one replaces $(H^\infty)^*$ by L^1/H_0^1.*

Proof. Taking in Th. 3 $\delta = 1/2$ provides some $\delta_1 > 0$. Take $\tau = (1/4)\delta_1$ and $\varkappa = (1/2)\delta_1$. Assume $x_1, \ldots, x_n$ in the unit ball of L^1/H_0^1 satisfying (*).

Applying L. Dor's lemma (see [8]) to the minimum norm liftings $\sigma(x_1), \ldots, \sigma(x_n)$ yields disjoint measurable subsets S_m of Π for which

$$\|f_m\|_1 \geqslant (1-\tau)^2 \quad \text{taking} \quad f_m = \sigma(x_m)\chi_{S_m}.$$

Now

$$\|q(f_m)\| \geqslant \|x_m\| - \|\sigma(x_m) - f_m\|_1 = \|f_m\|_1 > 1/2.$$

Thus, applying Th. 3, H^∞ functions φ_m and ψ_m $(1 \leqslant m \leqslant n)$ are obtained satisfying (ii), (iii), (iv) of Th 3. Since for each $m = 1, \ldots, n$

$$|\langle \varphi_m, x_m\rangle - \langle \varphi_m, f_m\rangle| \leqslant \|f_m - \sigma(x_m)\|_1 \leqslant 1-(1-\tau)^2 < \delta_1/2,$$

we get

$$|\langle \varphi_m, x_m\rangle| > \varkappa.$$

Replacement of φ_m by $a_m\varphi_m$ for some $a_m \in \mathbf{C}$, $|a_m| \leqslant 1$, leads to the required conclusion.

Let us next observe that (i), (ii) of Prop. 1 can be reformulated as follows in Banach space language

(i′) $$\|a\varphi_m + b\psi_m\| \leqslant 1 \quad \text{if} \quad |a|, |b| \leqslant 1,$$

(ii′) $$\Big\|\sum a_m(1-\psi_m)\Big\| \leqslant \beta(n) \quad \text{whenever} \quad |a_m| \leqslant 1.$$

By local reflexivity, it will therefore be enough to obtain φ_m, ψ_m $(1 \leqslant m \leqslant n)$ as elements of $(H^\infty)^{**}$, replacing conditions (i), (ii) by (i′), (ii′).

A simple way to achieve this is using the isometrical embedding of $(H^\infty)^*$ in some ultra-power $B = (L^1/H_0^1)_{\mathscr{U}}$ of L^1/H_0^1. The reader is referred to [13] and [19] for the theory of ultra-products of Banach spaces. We use the notation $\mathbf{1}$ for the element of B^* defined by $\langle \xi, \mathbf{1}\rangle = \lim\limits_{\mathscr{U}} \int \xi_i$, where $\xi = (\xi_i)_{i \in I}$ is an element of B.

LEMMA 2. *Prop. 1 holds, replacing $(H^\infty)^*$ by B, H^∞ by B^* and* (i), (ii) *by* (i′), (ii′) *(substituting $\mathbf{1}$ to the 1-function).*

Proof. The argument is completely straightforward. Fix some $0 < \varrho < 1$ and assume $\xi(1), \ldots, \xi(n)$ in the unit ball of B satisfying

$$\left\|\sum a_m \xi(m)\right\| \geqslant (1-\varrho\tau) \sum |a_m| \quad (a_m \in \boldsymbol{C}).$$

It follows from the definition of the norm on B that there is some element U in the ultra-filter $\mathscr{U}$ such that for $i \in U$ the L^1/H^1_0-elements

$$\xi_i(1), \ldots, \xi_i(n)$$

behave almost isometrically to

$$\xi(1), \ldots, \xi(n).$$

In particular, we can assume

$$\lambda \sum |a_m| \geqslant \left\|\sum a_m \xi_i(m)\right\| \geqslant \lambda^{-1}(1-\varrho\tau) \sum |a_m| \quad (a_m \in \boldsymbol{C}),$$

where $\lambda^{-1}(1-\varrho\tau) > 1-\tau$.

If we fix $i \in U$ and define

$$x_m = \lambda^{-1}\xi_i(m) \quad (1 \leqslant m \leqslant n),$$

application of Lemma 1 gives H^∞-functions $\varphi_i(m)$ and $\psi_i(m)$ $(1 \leqslant m \leqslant n)$ satisfying (i), (ii) of Prop. 1 and

$$\langle \xi_i(m), \varphi_i(m) \rangle = \lambda \langle x_m, \varphi_i(m)\rangle = \lambda\varkappa.$$

Next, define for each $m = 1, \ldots, n$ the following elements of B^*

$$\langle \xi, \varphi_m\rangle = \lim_{\mathscr{U}} \langle \xi_i, \varphi_i(m)\rangle$$

and

$$\langle \xi, \psi_m\rangle = \lim_{\mathscr{U}} \langle \xi_i, \psi_i(m)\rangle.$$

One verifies immediately (i′), (ii′). Moreover,

$$\langle \xi(m), \varphi_m\rangle = \lambda\varkappa \quad (1 \leqslant m \leqslant n).$$

This proves the lemma.

It remain now to restrict the elements φ_m, ψ_m of B^* obtained in Lemma 2 to $(H^\infty)^*$ in order to obtain $(H^\infty)^{**}$ elements satisfying (i′), (ii′). The only thing to notice here is that, from the embedding properties of $(H^\infty)^*$ in B, the restriction of $\mathbf{1}$ to $(H^\infty)^*$ is $1 \in H^\infty$. This completes the proof of Prop. 1.

We now turn back to Th. 1. Assume $T\colon H^\infty \to Y$ is a non-weakly compact operator. Then the set

$$K = \{T^*(y^*);\ y^* \in Y^* \text{ and } \|y^*\| \leqslant 1\}$$

is not weakly compact and therefore, since $(H^\infty)^*$ is WSC, not weakly conditionally compact. Applying the James-regularization principle for l^1-sequences, it is possible to construct in some multiple of K an infinite sequence $(\varphi_m)_{m=1,2,\ldots}$ satisfying the hypothesis of Prop. 1

LEMMA 3. *There exist $\varepsilon > 0$ and a sequence $(\eta_r)_{r=1,2,\ldots}$ of H^∞-functions so that*

(i) $\sum\limits_{m=1}^{\infty} |\eta_r| \leqslant 1$,

(ii) $\langle \Phi_r, \eta_r\rangle = \varepsilon$, *where (Φ_r) is a subsequence of (Φ_m).*

The extraction of the l^∞-subsequence (η'_r) of (η_r) such that T induces an isomorphism on the w^*-closure of $\operatorname{span}[\eta'_r;\ r = 1, 2, \ldots]$ is then done using the standard procedure (see [17] for instance).

Proof of Lemma 3. Fix a sequence (ε_r) of positive numbers and positive integers (N_r) such that $\beta(N_r) < \varepsilon_r N_r$. If $\psi \in H^\infty$ and $\Phi \in (H^\infty)^*$, define $\psi\Phi \in (H^\infty)^*$ by $\langle \psi\Phi, \varphi\rangle = \langle \Phi, \varphi\psi\rangle$. Notice also that if ψ_m $(1 \leqslant m \leqslant n)$ are H^∞-functions satisfying (ii) of Prop. 1 and $\Phi \in (H^\infty)^*$, then

$$\|\Phi - \psi_m\Phi\| \leqslant (\beta(n)/n)\|\Phi\| \quad \text{for some } m = 1, \ldots, n.$$

We now make the following construction.

Defining $D_0 = \boldsymbol{N}$ and fixing the first N_1 elements $\Phi_1, \Phi_2, \ldots, \Phi_{N_1}$ of D_0, we apply Prop. 1. The preceding observation allows us to fix some $m_1 = 1, \ldots, N_1$ for which

$$\|\Phi_m - \psi_{m_1}\Phi_m\| < \varepsilon_1$$

holds for all m in an infinite subset D_1 of D_0. Also

$$|\varphi_{m_1}| + |\psi_{m_1}| \leqslant 1$$

and

$$\langle \Phi_{m_1}, \varphi_{m_1}\rangle = \varkappa.$$

Starting again with the N_2 first elements of D_1 yields some $m_2 \in D_1$, $m_2 > m_1$, H^∞-functions $\varphi_{m_2}, \psi_{m_2}$ and an infinite subset D_2 of D_1 so that

$$\|\Phi_m - \psi_{m_2}\Phi_m\| < \varepsilon_2 \quad \text{for} \quad m \in D_2,$$

$$|\varphi_{m_2}| + |\psi_{m_2}| \leqslant 1, \quad \langle \Phi_{m_2}, \varphi_{m_2}\rangle = \varkappa.$$

Continuing in this way, a subsequence (Φ_{m_r}) of (Φ_m) is obtained. Define for each r the H^∞-function

$$\eta_r = \psi_{m_1}\psi_{m_2} \cdots \psi_{m_{r-1}}\varphi_{m_r}.$$

Since at each step

$$|\varphi_{m_r}|+|\psi_{m_r}|\leqslant 1,$$

then η_r satisfy (i) of Lemma 3. Now for $m\in D_{r-1}$, one has

$$\|\Phi_m-(\psi_{m_1}\ldots\psi_{m_{r-1}})\Phi_m\|\leqslant\|\Phi_m-\psi_{m_1}\Phi_m\|+\ldots+\|\Phi_m-\psi_{m_{r-1}}\Phi_m\|<\sum_{s=1}^{r-1}\varepsilon_s.$$

Therefore

$$|\langle\Phi_{m_r},\eta_r\rangle|>|\langle\Phi_{m_r},\varphi_{m_r}\rangle|-\sum\varepsilon_s=\varkappa-\sum\varepsilon_s.$$

So we just have to fix $\varepsilon=\varkappa/2$ and a sequence (ε_s) with $\sum\varepsilon_s=\varepsilon$.

III. Some preliminary lemmas. In this section we will give several lemmas which will be used in the proof of Th. 3. Some of them appear also in [3], but we repeat them here for the sake of completeness. We denote by $\mathscr{H}$ the Hilbert-transform. H^∞ is seen as subalgebra of $L^\infty(\Pi)$.

LEMMA 4. *Assume a in $L^\infty(\Pi)$ such that $0\leqslant a\leqslant 1$ and $\log(1-a)$ is integrable. Then there is an H^∞-function f satisfying*

(i) $$|f|=1-a\quad\text{on}\quad\partial D,$$

(ii) $$\|(1-t)-f\|_2\leqslant\|a-t\|_2+\big\|\log\big((1-a)/(1-t)\big)\big\|_2\quad\text{whenever}\quad 0\leqslant t\leqslant 1.$$

Proof. Since $\log(1-a)$ is in $L^1(\Pi)$, we can consider the function f defined by

$$f(z)=\exp\Big\{\int_\Pi\log(1-a)\big((e^{i\theta}+z)/(e^{i\theta}-z)\big)m(d\theta)\Big\}$$

for $z\in D$. Then f has boundary value

$$f=(1-a)e^{i\tau},\quad\text{where}\quad\tau=\mathscr{H}\big(\log(1-a)\big).$$

Now

$$|(1-t)-f|\leqslant|a-t|+|1-e^{i\tau}|=|a-t|+2|\sin(\tau/2)|\leqslant|a-t|+|\tau|,$$

implying

$$\|(1-t)-f\|_2\leqslant\|a-t\|_2+\|\tau\|_2.$$

Since

$$\mathscr{H}\big(\log(1-a)\big)=\mathscr{H}\Big(\log\big((1-a)/(1-t)\big)\Big),$$

it follows that

$$\|\tau\|_2\leqslant\big\|\log\big((1-a)/(1-t)\big)\big\|_2,$$

providing the required estimate.

LEMMA 5. *Let A be a measurable subset of Π and $0<\varepsilon<1/e$. Then there exist H^∞-functions φ and ψ such that*

(i) $$|\varphi|+|\psi|\leqslant 1,$$

(ii) $$|\varphi(z)-1/3|\leqslant\varepsilon/3\quad\text{for}\quad z\in A,$$

(iii) $$|\psi(z)|\leqslant\varepsilon\quad\text{for}\quad z\in A,$$

(iv) $$\|\varphi\|_2\leqslant\big(\log(1/\varepsilon)\big)m(A)^{1/2},$$

(v) $$\|1-\psi\|_2\leqslant 6\big(\log(1/\varepsilon)\big)m(A)^{1/2}.$$

Proof. Take $\varrho=1-\varepsilon$. Application of Lemma 4 with $a=\varrho\chi_A$ yields f in H^∞ such that $(t=0)$

$$|f|=1-\varrho\chi_A\quad\text{on}\quad\partial D$$

and

$$\|1-f\|_2\leqslant m(A)^{1/2}+\big(\log(1/\varepsilon)\big)m(A)^{1/2}.$$

Thus $|f(z)|<\varepsilon$ on A. The function $\varphi=(1/3)(1-f)$ satisfies (ii) and (iv). Remark that $|\varphi|\leqslant 2/3$. Apply again Lemma 4 taking now $a=|\varphi|$. We obtain an H^∞-function g satisfying $(t=0)$

$$|g|=1-|\varphi|\quad\text{on}\quad\partial D$$

and

$$\begin{aligned}\|1-g\|_2&\leqslant\|\varphi\|_2+\|\log(1-|\varphi|)\|_2\\&\leqslant\|\varphi\|_2+3\|\varphi\|_2\leqslant 4\big(\log(1/\varepsilon)\big)m(A)^{1/2}.\end{aligned}$$

Define $\psi=gf$. Then (i), (iii) obviously hold. Moreover

$$\|1-\psi\|_2\leqslant\|1-f\|_2+\|1-g\|_2\leqslant 6\big(\log(1/\varepsilon)\big)m(A)^{1/2},$$

completing the proof.

In the following lemma, we make a careful analysis of a well-known construction in peak-set-theory. This result is important in order to realize condition (iii) of Th. 3 and was not used in [3].

LEMMA 6. *Given $0<\tau\leqslant 1/e$ there is a constant $C_\tau<\infty$ such that if (S_i) is a sequence of measurable subsets of Π and (ε_i) a sequence in $[0,1]$, there are H^∞-functions f and g satisfying*

(i) $$|f|+|g|\leqslant 1,$$

(ii) $$|f(z)-1|\leqslant\varepsilon_i\quad\text{if}\quad z\in S_i,$$

(iii) $$\|f-\tau\|_2^2 \leqslant C_\tau \sum \varepsilon_i^{-2} m(S_i),$$

(iv) $$\|g-(1-\tau)\|_2^2 \leqslant C_\tau \sum \varepsilon_i^{-2} m(S_i).$$

(*The constant* C_τ *is of order* $(\log(1/\tau))^4$.)

Proof. We start by recalling the following elementary estimates for $z, w \in C$

$$|\exp z - 1| \leqslant e^{|z|}-1 \quad \text{and} \quad |\exp z - \exp w| \leqslant e^{\max(\operatorname{Re} z, \operatorname{Re} w)}|z-w|.$$

If $\sum \varepsilon_i^{-2} m(S_i) = \infty$, we just have to take $f=1$, $g=0$. So assume $\sum \varepsilon_i^{-2} m(S_i) < \infty$. It is easily seen that we can assume the sets S_i to be mutually disjoint. Consider the harmonic function u on D with boundary value

$$u = -2\sum \varepsilon_i^{-1}\chi_i - \delta,$$

where χ_i denotes the characteristic function of S_i and $\tau = e^{-1/\delta}$. Notice that $u \leqslant -\delta$. Let v be the conjugate of u and define

$$f = \exp\big(1/(u+iv)\big).$$

Then f is analytic on D and since

$$|f| = \exp\big(u/(u^2+v^2)\big) < 1,$$

f is an H^∞-function.

We first verify (ii) and (iii).

(ii): $$|f-1| \leqslant \exp(1/\sqrt{u^2+v^2})-1 \leqslant (1/\sqrt{u^2+v^2})\exp(1/\sqrt{u^2+v^2}) \leqslant (1/|u|)\exp(1/|u|)$$

and hence

$$|f(z)-1| \leqslant (\varepsilon_i/2)\exp(\varepsilon_i/2) \leqslant \varepsilon_i \quad \text{for} \quad z \in S_i.$$

(iii): $$|f-\tau| = \big|\exp\big(1/(u+iv)\big) - \exp(-1/\delta)\big| \leqslant |1/(u+iv)+1/\delta| \leqslant \delta^{-2}(|u+\delta|+|v|).$$

Therefore

$$\|f-\tau\|_2^2 \leqslant 2\delta^{-4}(\|u+\delta\|_2^2+\|v\|_2^2) \leqslant 4\delta^{-4}\|u+\delta\|_2^2 = 16\,\delta^{-4}\sum \varepsilon_i^{-2} m(S_i).$$

Our next goal is to estimate

$$\big\|\log\big((1-|f|)/(1-\tau)\big)\big\|_2.$$

For $\lambda \geqslant 5$, one has

$$\{\theta \in \Pi;\ 1/(1-|f(e^{i\theta})|) \geqslant \lambda\} \subset \{|u|/(u^2+v^2) \leqslant -\log(1-1/\lambda)\} \subset \{|u| \geqslant \lambda/4\} \cup \{v^2 \geqslant \delta\,\lambda/4\}.$$

It follows from the definition of u and the fact that $\lambda/4 > 1 \geqslant \delta$, that

$$\{|u| \geqslant \lambda/4\} \subset \bigcup_i S_i.$$

By Tchebycheff's inequality, we find following weak-type estimation

$$m[(1-|f|)^{-1} \geqslant \lambda] \leqslant 4\lambda^{-1}\sum_i \int_{S_i} |u| + 4\delta^{-1}\lambda^{-1}\int v^2 \leqslant 12\lambda^{-1}\sum_i \varepsilon_i^{-1} m(S_i) + 4\delta^{-1}\lambda^{-1}\|u+\delta\|_2^2 \leqslant 28\delta^{-1}\lambda^{-1}\sum_i \varepsilon_i^{-2} m(S_i).$$

Write

$$(1-|f|)/(1-\tau) = 1-(|f|-\tau)/(1-\tau).$$

Since

$$|\log(1-x)| \leqslant 7|x| \quad \text{for} \quad -\infty < x \leqslant 4/5,$$

$$\int_{[|f|\leqslant 4/5]} \log^2\big((1-|f|)/(1-\tau)\big) \leqslant 49(1-\tau)^{-2}\|f-\tau\|_2^2 \leqslant 784(1-\tau)^{-2}\delta^{-4}\sum \varepsilon_i^{-2} m(S_i).$$

On the other hand, applying the weak-type inequality

$$\int_{[|f|>4/5]} \log^2\big((1-|f|)/(1-\tau)\big) \leqslant 2\int_{[|f|>4/5]} \log^2\big(1/(1-|f|)\big) + 2\log^2(1-\tau)\,m[|f|>4/5] \leqslant 4\log^2 5\ m[(1-|f|)^{-1}>5] + 4\int_5^\infty m[(1-|f|)^{-1}>\lambda](\log\lambda/\lambda)\,d\lambda \leqslant 120\ \delta^{-1}\sum \varepsilon_i^{-2} m(S_i).$$

Combining inequalities

$$\big\|\log\big((1-|f|)/(1-\tau)\big)\big\|_2^2 \leqslant 2^{12}\delta^{-4}\sum \varepsilon_i^{-2} m(S_i).$$

Since in particular $\log(1-|f|)$ is integrable on Π, we may apply Lemma 4 taking $a=|f|$ and $t=\tau$. Thus an H^∞-function g is obtained satisfying

(i) and, since

$$\|g-(1-\tau)\|_2^2 \leqslant 2\|f-\tau\|_2^2 + 2\left\|\log\left((1-|f|)/(1-\tau)\right)\right\|_2^2,$$

also (iv).

LEMMA 7. *Assume (A_m) to be a sequence of disjoint sets in Π. Let for each m a sequence $(B_{m,k})$ of disjoint subsets of Π be given and let (S_i) be a sequence of sets in Π. Take $\varepsilon > 0$ and $(\varkappa_k)$, (ε_i) sequences in $]0, 1]$.*

Then there exists for each m H^∞-functions φ_m and ψ_m satisfying

(i) $$|\varphi_m| + |\psi_m| \leqslant 1,$$

(ii) $$|\varphi_m| \leqslant \varkappa_k \quad \text{on} \quad B_{m,k},$$

(iii) $$|1-\psi_m| \leqslant \varepsilon_i \quad \text{on} \quad S_i,$$

(iv) $$\|\varphi_m\|_1 \leqslant C_1 \varepsilon^{-1} m(A_m),$$

(v) $$\sum_m \int_{A_m} |\gamma_1 - \varphi_m| \leqslant \varepsilon \sum m(A_m) + C_1 \sum_{m,k} \varkappa_k^{-1} m(B_{m,k}) + C_1 \sum \varepsilon_i^{-2} m(S_i),$$

(vi) $$\|1-\psi_m\|_2^2 \leqslant C_1 \varepsilon^{-1} m(A_m) + C_1 \sum \varepsilon_i^{-2} m(S_i).$$

For any sequence of disjoint subsets Ω_m of Π

(vii) $$\sum_m \int_{\Omega_m} |1-\psi_m|^2 \leqslant C_1 \varepsilon^{-1} \sum m(A_m) + C_1 \sum \varepsilon_i^{-2} m(S_i)$$

$\gamma_1 > 0$ and $C_1 < \infty$ denote numerical constants).

Proof. We assume $\sum_{m,k} \varkappa_k^{-1} m(B_{m,k}) < \infty$ since otherwise $\varphi_m = 0$, $\psi_m = 1$ satisfy.

Fixing m and applying Lemma 4, an H^∞-function η_m is obtained satisfying

(viii) $$|\eta_m| = 1 - \sum_k (1-\varkappa_k)\chi_{B_{m,k}} \quad \text{on} \quad \partial D$$

and $(t = 0)$

(ix) $$\|1-\eta_m\|_2^2 \leqslant 2\sum m(B_{m,k}) + 2\sum \log^2(1/\varkappa_k) m(B_{m,k}) \leqslant \text{const} \sum \varkappa_k^{-1} m(B_{m,k}).$$

We also obtain from Lemma 5 H^∞-functions φ_m', ψ_m' such that

(x) $$|\varphi_m'| + |\psi_m'| \leqslant 1,$$

(xi) $$|\varphi_m'(z) - 1/3| \leqslant \varepsilon/3 \quad \text{for} \quad z \in A_m,$$

(xii) $$\|\varphi_m'\|_2^2 \leqslant \text{const}\, \varepsilon^{-1} m(A_m),$$

(xiii) $$\|1-\psi_m'\|_2^2 \leqslant \text{const}\, \varepsilon^{-1} m(A_m).$$

Finally, application of Lemma 6 to the sequence (S_i), taking $\tau = 1/e$, provides H^∞-functions f and g fulfilling

(xiv) $$|f| + |g| \leqslant 1,$$

(xv) $$|f(z) - 1| < \varepsilon_i/2 \quad \text{for} \quad z \in S_i,$$

(xvi) $$\|f - 1/e\|_2^2 \leqslant \text{const} \sum \varepsilon_i^{-2} m(S_i),$$

(xvii) $$\|g - (1-1/e)\|_2^2 \leqslant \text{const} \sum \varepsilon_i^{-2} m(S_i).$$

Define

$$\varphi_m'' = g(\varphi_m')^2 \eta_m \quad \text{and} \quad \psi_m = f + g\psi_m'.$$

Then clearly

(xviii) $$|\varphi_m''| + |\psi_m| \leqslant 1,$$

(xix) $$|\varphi_m''| \leqslant \varkappa_k \quad \text{on} \quad B_{m,k}.$$

Since

$$|1-\psi_m| \leqslant |1-f| + |g| \leqslant 2|1-f|,$$

(iii) follows from (xv). From (xii), we get

(xx) $$\|\varphi_m''\|_1 \leqslant \text{const}\ \varepsilon^{-1} m(A_m).$$

Combining (ix), (xi) and (xvii), we see that

(xxi) $$\sum_m \int_{A_m} |(1/9)(1-1/e) - \varphi_m''|^2 \leqslant \varepsilon \sum m(A_m) + \text{const} \sum_{m,k} \varkappa_k^{-1} m(B_{m,k}) + \text{const} \sum \varepsilon_i^{-2} m(S_i),$$

using the fact that the sets A_m are mutually disjoint. Define

$$\varphi_m = (1/2)[(2/9)(1-1/e) - \varphi_m'']\varphi_m'' \quad \text{and} \quad \gamma_1 = (1/2.9^2)(1-1/e)^2.$$

Then (xxi) implies (v) and, since $|\varphi_m| \leqslant |\varphi_m''|$, also (i), (ii), (iv) follow from (xviii), (xix), (xx), respectively.

Let us verify (vi) and (vii). Since

$$|1-\psi_m| \leqslant |1/e - f| + |(1-1/e) - g| + |1-\psi_m'|,$$

the required inequalities are deduced from (xiii), (xvi), (xvii).

IV. Proof of Theorem 3. We will use a decomposition procedure for the functions f_m. Our first lemma solves the problem in the case the functions f_m are L^1-normalized characteristic functions of disjoint subsets of Π.

However, in order to make the result applyable in the general situation, additional conditions must be added.

LEMMA 8. *Assume $(A_m)_{1\leqslant m\leqslant n}$, $(\Omega_m)_{1\leqslant m\leqslant n}$ to be finite sequences of disjoint sets and (S_i) a sequence of sets. Let for each m, $(B_{m,k})$ be a sequence of disjoint sets. Let $\varepsilon>0$, $\eta>0$ and $(\varkappa_k)$, (ε_i) sequences in $]0,1]$. Then there exist for each m H^∞-functions φ_m and ψ_m satisfying*

(i) $$|\varphi_m|+|\psi_m|\leqslant 1,$$

(ii) $$|\varphi_m|<\varkappa_k \quad \text{on} \quad B_{m,k},$$

(iii) $$|1-\psi_m|<\varepsilon_i \quad \text{on} \quad S_i,$$

(iv) $$\sum|1-\psi_m|<\eta n,$$

(v) $$\|\varphi_m\|_1\leqslant C_2\varepsilon^{-1}m(A_m),$$

(vi) $$\sum\int_{A_m}|\varphi_m-\gamma_1|\leqslant\big(\varepsilon+\xi(\eta)\varepsilon^{-2}n^{-1/2}\big)\sum m(A_m)+ +C_2\varepsilon^{-1}\sum_{m,k}\varkappa_k^{-1}m(B_{m,k})+\xi(\eta)\varepsilon^{-1}\sum\varepsilon_i^{-2}m(S_i),$$

(vii) $$\sum\|1-\psi_m\|_2^2\leqslant\xi(\eta)\varepsilon^{-1}\sum m(A_m)+n\xi(\eta)\sum\varepsilon_i^{-2}m(S_i),$$

(viii) $$\sum\int_{\Omega_m}|1-\psi_m|^2\leqslant C_2\varepsilon^{-1}\sum m(A_m)+C_2\sum\varepsilon_i^{-2}m(S_i).$$

Here γ_1 is the constant of Lemma 7, $C_2<\infty$ is a numerical constant and $\xi(\eta)$ is a function depending on η.

Proof. We first partition $\{1,\ldots,n\}$ into sets M, N taking
$$M=\{m;\ m(A_m)>n^{-1/2}\sum m(A_m)\}\quad\text{and}\quad N=\{1,\ldots,n\}\setminus M.$$
Notice that $\operatorname{card}M<n^{1/2}$. Let us first deal with the small set M. Application of Lemma 7 yields H^∞-functions $(\varphi_m)_{m\in M}$, $(\psi_m)_{m\in M}$ satisfying (i), (ii), (iii), (v) and

(ix) $$\sum_M\int_{A_m}|\gamma_1-\varphi_m|\leqslant\varepsilon\sum_M m(A_m)+C_1\sum_{m\in M,k}\varkappa_k^{-1}m(B_{m,k})+C_1\sum\varepsilon_i^{-2}m(S_i),$$

(x) $$\|1-\psi_m\|_2^2\leqslant C_1\varepsilon^{-1}m(A_m)+C_1\sum\varepsilon_i^{-2}m(S_i),$$

(xi) $$\sum_M\int_{\Omega_m}|1-\psi_m|^2\leqslant C_1\varepsilon^{-1}\sum_M m(A_m)+C_1\sum\varepsilon_i^{-2}m(S_i).$$

Denote by d a positive integer (depending on η) which will be fixed later.

The set N will be partitioned into subsets N_α, $\operatorname{card}N_\alpha=d$, and a "negligible" remainder N_{rem}.

To each α, we will associate systems $(\varphi_m)_{m\in N_\alpha}$ and $(\psi_m)_{m\in N_\alpha}$ of H^∞-functions fulfilling (i), (ii), (iii), (v) and moreover

(xii) $$\sum_{N_\alpha}|1-\psi_m|\leqslant 3d^{1/2},$$

(xiii) $$\int_{A_m}|\varphi_m-\gamma_1|\leqslant\varepsilon m(A_m)\quad\text{for}\quad m\in N_\alpha,$$

(xiv) $$\sum_{N_\alpha}\|1-\psi_m\|_2^2\leqslant(2C_1d^3)^d\Big(\varepsilon^{-1}\sum_{N_\alpha}m(A_m)+\sum\varepsilon_i^{-2}m(S_i)\Big),$$

(xv) $$\int_{\Omega_m}|1-\psi_m|^2\leqslant 10\ C_1\varepsilon^{-1}m(A_m)\quad\text{for}\quad m\in N_\alpha.$$

The neglegability of N_{rem} is in the sense that

(xvi) $$\sum_{N_{\text{rem}}}m(A_m)\leqslant\Theta,$$

where we define for simplicity
$$\Theta=8\ \varepsilon^{-1}C_1\sum_{m,k}\varkappa_k^{-1}m(B_{m,k})+8\varepsilon^{-1}(2C_1d^3)^{d-1}\Big\{\varepsilon^{-1}dn^{-1/2}\sum m(A_m)+ +\sum\varepsilon_i^{-2}m(S_i)\Big\}.$$

Suppose $N_1,N_2,\ldots,N_\alpha$ are already obtained. Define $N'=N\setminus(N_1\cup N_2\cup\ldots\cup N_\alpha)$. If $\sum_{N'}m(A_m)\leqslant\Theta$, take $N_{\text{rem}}=N'$ and define for $m\in N_{\text{rem}}$
$$\varphi_m=0\quad\text{and}\quad\psi_m=1.$$
Then, obviously,

(xvii) $$\sum_{N_{\text{rem}}}\int_{A_m}|\varphi_m-\gamma_1|\leqslant 2\Theta.$$

If $\sum_{N'}m(A_m)>\Theta$, then we can proceed to the extraction of a subset $N_{\alpha+1}\subset N'$. Suppose we have already obtained $m_1,m_2,\ldots,m_r$ $(r<d)$ in N', such that following condition is satisfied:

(xviii) $$\|1-\psi_{m_s}\|_2^2\leqslant(2C_1d^3)^s\Big(\varepsilon^{-1}\sum_{t=1}^{s}m(A_{m_t})+\sum\varepsilon_i^{-2}m(S_i)\Big)$$

for $s=1,\ldots,r$. Define the set
$$U_r=\Big\{\theta\in\Pi;\sum_{s=1}^{r}|1-\psi_{m_s}(e^{i\theta})|^2\geqslant 1\Big\},$$

for which, by (xviii),

$$m(U_r) \leqslant (2C_1)^r d^{2r+1}\Big(\varepsilon^{-1}\sum_{t=1}^{r} m(A_{m_t}) + \sum \varepsilon_i^{-2} m(S_i)\Big). \tag{xix}$$

Apply then again Lemma 7 considering the sets $(A_m)_{m\in N_r'}$, where $N_r' = N'\setminus\{m_1, \ldots, m_r\}$, and adding the set U_r to the sequence of the (S_i) to which we associate the value $d^{-1/2}$. H^∞-functions $(\varphi_m)_{m\in N_r'}$ and $(\psi_m)_{m\in N_r'}$ are obtained satisfying in addition to (i), (ii), (iii), (v).

$$|1-\psi_m| < d^{-1/2} \quad \text{on} \quad U_r, \tag{xx}$$

$$\sum_{m\in N_r'}\int_{A_m} |\gamma_1-\varphi_m| \leqslant (\varepsilon/4)\sum_{N_r'} m(A_m) + C_1\sum_{N_r',k} \varkappa_k^{-1} m(B_{m,k}) + \tag{xxi}$$
$$+C_1\sum \varepsilon_i^{-2} m(S_i) + 2^r(C_1 d^2)^{r+1}\Big(\varepsilon^{-1}\sum_{t=1}^{r} m(A_{m_t}) + \sum \varepsilon_i^{-2} m(S_i)\Big),$$

$$\|1-\psi_m\|_2^2 \leqslant 4C_1\varepsilon^{-1} m(A_m) + (2C_1 d^2)^{r+1}\Big(\varepsilon^{-1}\sum_{t=1}^{r} m(A_{m_t}) + \tag{xxii}$$
$$+\sum \varepsilon_i^{-2} m(S_i)\Big),$$

$$\sum_{m\in N_r'}\int_{\Omega_m} |1-\psi_m|^2 \leqslant 4C_1\varepsilon^{-1}\sum_{N_r'} m(A_m) + (2C_1 d^2)^{r+1}\Big(\varepsilon^{-1}\sum_{t=1}^{r} m(A_{m_t}) + \tag{xxiii}$$
$$+\sum \varepsilon_i^{-2} m(S_i)\Big).$$

Since now

$$\sum_{t=1}^{r} m(A_{m_t}) \leqslant d n^{-1/2}\sum m(A_m),$$

we find

$$C_1\sum_{m,k}\varkappa_k^{-1} m(B_{m,k}) + (2C_1 d^2)^{r+1}\Big(\varepsilon^{-1}\sum_{t=1}^{r} m(A_{m_t}) + \sum \varepsilon_i^{-2} m(S_i)\Big) \leqslant (1/8)\varepsilon\Theta.$$

By hypothesis

$$\Theta < \sum_{N'} m(A_m) \leqslant \sum_{N_r'} m(A_m) + (1/2)\Theta.$$

Thus we deduce from (xxi) and (xxiii)

$$\sum_{N_r'}\int_{A_m} |\gamma_1-\varphi_m| \leqslant \varepsilon/2\sum_{N_r'} m(A_m), \tag{xxiv}$$

$$\sum_{N_r'}\int_{\Omega_m} |1-\psi_m|^2 \leqslant 5C_1\varepsilon^{-1}\sum_{N_r'} m(A_m). \tag{xxv}$$

So we can choose $m_{r+1}\in N_r'$ fulfilling (xiii) and (xv). From (xxii), it is clear that (xviii) will hold for $s = r+1$. Summation of (xviii) provides inequality (xiv) for the system $(\psi_m)_{m\in N_{a+1}}$.

Since, by construction

$$\Big[\sum_{s=1}^{r} |1-\psi_{m_s}|^2 \geqslant 1\Big] \subset [|1-\psi_{m_{r+1}}|^2 < d^{-1}],$$

one has

$$\sum_{N_{a+1}} |1-\psi_m|^2 \leqslant 6$$

since $\|\psi_m\|_\infty \leqslant 1$ for each m. Thus the family $(\psi_m)_{m\in N_{a+1}}$ satisfies (xii).

This completes the construction. It remains to choose the integer d and precise the function $\xi(\eta)$.

First, one has by (xii)

$$\sum |1-\psi_m| \leqslant \sum_{M} |1-\psi_m| + \sum_{a}\sum_{N_a} |1-\psi_m| + \sum_{N_{\mathrm{rem}}} |1-\psi_m|$$
$$\leqslant 2n^{1/2} + 3d^{1/2}(n/d)$$
$$= (2n^{-1/2} + 3d^{-1/2})n.$$

Thus it suffices to take $d \sim \eta^{-2}$, assuming n large enough with respect to η. If this is not the case, it will follow from the definition of $\xi(\eta)$ that $\varphi_m = 0$, $\psi_m = 1$ already satisfy the lemma.

Define

$$\xi(\eta) = (4C_1 d^3)^{d+2}.$$

Then (vi) follows from (ix), (xiii), (xvii), (vii) follows from (x) and (xiv), (viii) follows from (xi) and (xv).

This completes the proof.

Remark. The function $\xi(\eta)$ obtained by preceding estimations is of the form $\eta^{-\mathrm{const}\,\eta^{-2}}$. Taking the first term of the right hand side in inequality (vi) in account, it is clear that the lemma will only be useful for $\eta > (\log n)^{-1/2+\delta}$.

LEMMA 9. *Assume $\varepsilon > 0$, $\delta > 0$, $\eta > 0$ and n a positive integer satisfying the inequalities*

$$\delta < (1/4)\varepsilon^4 \xi(\eta)^{-1} \quad \text{and} \quad n > \varepsilon^{-6}\xi(\eta)^2.$$

Let $(A_{m,k})_{\substack{1\leqslant m\leqslant n\\ 1\leqslant k\leqslant K}}$ be a system of disjoint subsets of Π such that

$$m(A[k]) \leqslant \delta m(A[k+1]), \quad \text{where} \quad A[k] = \bigcup_{m=1}^{n} A_{m,k}.$$

Then there exists a system of H^∞-functions $(\varphi_{m,k})$, $(\psi_{m,k})$ fulfilling

(i) $$|\varphi_{m,k}| + |\psi_{m,k}| \leqslant 1,$$

(ii) $$|\varphi_{m,k}| < \varepsilon^{k-l} \quad \text{on} \quad A_{m,l} \quad \text{for} \quad k > l,$$

(iii) $$\|\varphi_{m,k}\|_1 \leqslant C_3 \varepsilon^{-1} m(A_{m,k}),$$

(iv) $$\sum_m \int_{A_{m,k}} |\varphi_{m,k} - \gamma_1| \leqslant 3\varepsilon m(A[k]),$$

(v) $$\sum_m |1-\psi_{m,k}| \leqslant \eta n,$$

(vi) $$\Big[\sum_m |1-\psi_{m,l}| > \varepsilon^{k-l} n\Big] \subset \bigcap_m [|1-\psi_{m,k}| \leqslant \varepsilon^{k-l}] \quad \text{for} \quad k > l,$$

(vii) $$\sum_m \int_{A_m} |1-\psi_{m,k}|^2 \leqslant C_3 \varepsilon^{-1} m(A[k]), \quad \text{taking} \quad A_m = \bigcup_k A_{m,k}.$$

(C_3 is again a numerical constant).

Proof. We construct the H^∞-functions by induction on k. Let us define for convenience

$$U_{k,l} = \Big[\sum_m |1-\psi_{m,l}| > \varepsilon^{k-l} n\Big]$$

and

$$\nu_k = \sum_m \|1-\psi_{m,k}\|_2^2.$$

Then, by Cauchy–Schwartz and Tchebycheff

(viii) $$m(U_{kl}) \leqslant \varepsilon^{2(l-k)} n^{-1} \nu_l.$$

Step 1. Application of Lemma 8 to the sets $(A_{m,1})$ gives H^∞-functions satisfying (i), (iii), (v) and

(ix) $$\sum \int_{A_{m,1}} |\varphi_{m,1} - \gamma_1| \leqslant (\varepsilon + \xi(\eta)\varepsilon^{-2} n^{-1/2}) m(A[1]),$$

(x) $$\sum \int_{A_m} |1-\psi_{m,1}|^2 \leqslant C_2 \varepsilon^{-1} m(A[1]) \quad (\text{take } \Omega_m = A_m),$$

(xi) $$\nu_1 \leqslant \xi(\eta)\varepsilon^{-1} m(A[1]).$$

Inductive step. Assume the construction done up to level k. We apply Lemma 8 in the following situation:

$$A_m = A_{m,k+1}, \quad \Omega_m = A_m, \quad S_l = U_{k+1,l} \quad (l \leqslant k),$$
$$B_{m,l} = A_{m,l} \quad (l \leqslant k), \quad \varepsilon_l = \varkappa_l = \varepsilon^{k+1-l} \quad (l \leqslant k).$$

This gives H^∞-functions $(\varphi_{m,k+1})$, $(\psi_{m,k+1})$ fulfilling (i), (ii), (iii), (v), (vi) (replacing k by $k+1$) and, from (viii)

(xii) $$\sum_m \int_{A_{m,k+1}} |\varphi_{m,k+1} - \gamma_1| \leqslant \big(\varepsilon + \xi(\eta)\varepsilon^{-2} n^{-1/2}\big) m(A[k+1]) + C_2\varepsilon^{-1} \sum_{l\leqslant k} \varepsilon^{l-k-1} m(A[l]) + \xi(\eta)\varepsilon^{-1} \sum_{l\leqslant k} \varepsilon^{4(l-k-1)} n^{-1} \nu_l,$$

(xiii) $$\sum_m \int_{A_m} |1-\psi_{m,k+1}|^2 \leqslant C_2 \varepsilon^{-1} m(A[k+1]) + C_2 \sum_{l\leqslant k} \varepsilon^{4(l-k-1)} n^{-1} \nu_1,$$

(xiv) $$\sum_m \|1-\psi_{m,k+1}\|_2^2 \leqslant \xi(\eta)\varepsilon^{-1} m(A[k+1]) + \xi(\eta) \sum_{l\leqslant k} \varepsilon^{4(l-k-1)} \nu_l.$$

Let us next estimate ν_k, using (xi) and the recursive inequality (xiv). Define for convenience

$$\Gamma_k = \nu_k + \varepsilon^{-4}\nu_{k-1} + \varepsilon^{-8}\nu_{k-2} + \ldots + \varepsilon^{-4(k-1)}\nu_1.$$

Reformulating (xi) and (xiv),

$$\nu_{k+1} \leqslant \xi(\eta)\varepsilon^{-1} m(A[k+1]) + \xi(\eta)\varepsilon^{-4}\Gamma_k$$

and hence

$$\Gamma_{k+1} \leqslant \xi(\eta)\varepsilon^{-1} m(A[k+1]) + \big(1+\xi(\eta)\big)\varepsilon^{-4}\Gamma_k \leqslant \xi_{\eta,\varepsilon}(m(A[k+1]) + \Gamma_k)$$

taking $\xi_{\eta,\varepsilon} = 2\,\xi(\eta)\varepsilon^{-4}$. Iteration leads to the inequality

$$\Gamma_k \leqslant \xi_{\eta,\varepsilon} m(A[k]) + \xi_{\eta,\varepsilon}^2 m(A[k-1]) + \ldots + \xi_{\eta,\varepsilon}^k m(A[1])$$

and from the hypothesis on the $A[k]$

$$\Gamma_k \leqslant \xi_{\eta,\varepsilon}\big(1/(1-\delta\xi_{\eta,\varepsilon})\big)\, m(A[k]) \leqslant 2\xi_{\eta,\varepsilon} m(A[k])$$

since $\delta\xi_{\eta,\varepsilon} < 1/2$. Thus we have in particular

(xv) $$\nu_k \leqslant 2\xi_{\eta,\varepsilon} m(A[k]).$$

By the choice of n, (ix) implies (iv) for $k=1$. In general, we get from (xii), (xv) and the hypothesis on the sets $A[k]$

$$\sum_m \int_{A_{m,k+1}} |\varphi_{m,k+1}-\gamma_1| \leqslant 2\varepsilon m(A[k+1]) + \big(C_2\varepsilon^{-2}/(1-\varepsilon^{-1}\delta)\big)\, m(A[k]) + \\ + \big(2\xi(\eta)\,\xi_{\eta,\varepsilon}\varepsilon^{-5}/n(1-\varepsilon^{-4}\delta)\big)\, m(A[k])$$

leading again to (iv).

The verification of (vii) from (x) and (xiii) is analogue.

The next lemma, which is the final step in the proof of Th. 3 uses a "decomposition" technique for functions which was also applied in [3], [4].

LEMMA 10. *Fix $\tau > 0$ and let $n \geqslant \big(C_3\xi(\tau/3)\big)^{18}$ be a positive integer. Assume $(f_m)_{1\leqslant m\leqslant n}$ positive, disjointly supported integrable functions on Π. Then there exists H^∞-functions φ_m, ψ_m so that*

(i) $$\sum \int f_m|\gamma_2-\varphi_m| \leqslant \tau \sum \int f_m,$$

(ii) $$\sum |1-\psi_m| \leqslant \tau n,$$

(iii) $$|\varphi_m| + |\psi_m| \leqslant 1 \quad \text{for each } m.$$

Proof. Define for convenience

$$\eta = \tau/3, \quad M = C_3^1\xi(\eta), \quad \varepsilon = M^{-2}, \quad \delta = (1/4C_3)M^{-9}, \quad d = 11.$$

For $-\infty < k < \infty$, take $A_{m,k} = [M^k \leqslant f_m < M^{k+1}]$. Define further for $c = 0, 1, 2, \ldots, d-1$

$$A[c]_{m,k} = A_{m,dk+c} \quad \text{and} \quad A[c]_k = \bigcup_{m=1}^n A[c]_{m,k}.$$

For fixed c, we introduce the sequence (which may depend on c)

$$k_1 > k_2 > \ldots > k_r$$

of integers, where

(iv) $$m(A[c]_{k_s}) < \delta m(A[c]_{k_{s+1}}),$$

(v) $$m(A[c]_k) \leqslant \delta^{-1} m(A[c]_{k_s}) \quad \text{for} \quad k_s > k > k_{k_{s+1}}$$

(approximating the f_m, we can restrict k to a bounded interval $[-k_1, k_1]$). Define further

$$O = O[c] = \{k_s;\ s = 1, 2, \ldots, r\}.$$

For each m, let

$$A[c]_m = \bigcup_k A[c]_{m,k} \quad \text{and} \quad B[c]_m = \bigcup_{k\notin O} A[c]_{m,k}.$$

First, using (v), we find

$$\begin{aligned}\sum_m \int_{B[c]_m} f_m &= \sum_{k\notin O}\sum_m \int_{A[c]_{m,k}} f_m \\ &\leqslant M \sum_{k\notin O} M^{dk+c} m(A[c]_k) \\ &= M \sum_{s=1}^{r} \sum_{k_s>k>k_{s+1}} M^{dk+c} m(A[c]_k) \\ &\leqslant 2M\delta^{-1} \sum_{s=1}^{r} M^{d(k_s-1)+c} m(A[c]_{k_s}) \\ &\leqslant 2M^{1-d}\delta^{-1} \sum_m \int_{A[c]_m} f_m.\end{aligned}$$

Hence

(vi) $$\sum_m \int_{B[c]_m} f_m \leqslant 8C_3M^{-1} \sum_m \int_{A[c]_m} f_m.$$

Since now ε, δ, η and n satisfy the conditions of Lemma 9 there are H^∞-functions $\varphi[c]_{m,s}$, $\psi[c]_{m,s}$ satisfying (i), (ii), (iii), (iv), (v), (vi), (vii) of Lemma 9 with respect to the sets $A[c]_{m,k_s}$.

For fixed m, let

$$\varphi[c]_m = \varphi[c]_{m,1} + \varphi[c]_{m,2}\psi[c]_{m,1} + \ldots + \varphi[c]_{m,r}\psi[c]_{m,1}\ldots\psi[c]_{m,r-1}$$

and

$$\psi[c]_m = \psi[c]_{m,1}\psi[c]_{m,2}\ldots\psi[c]_{m,r}.$$

Then

(vii) $$|\varphi[c]_m| + |\psi[c]_m| \leqslant 1.$$

Let us estimate

$$I[c] = \sum_m \int_{A[c]_m} |\gamma_1 - \varphi[c]_m| f_m.$$

Write

$$\sum_m \int_{A[c]_m} = \sum_m \int_{B[c]_m} + \sum_m \sum_{k\notin O} \int_{A[c]_{m,k}}.$$

Then

$$\int_{A[c]_{m,k_s}} |\gamma_1-\varphi[c]_m| \leqslant \int_{A[c]_{m,k_s}} |\gamma_1-\varphi[c]_{m,s}\psi[c]_{m,1}\ \ldots\ \psi[c]_{m,s-1}|+$$
$$+\int_{A[c]_{m,k_s}} \{|\varphi[c]_{m,1}|+\ldots+|\varphi[c]_{m,s-1}|\}+$$
$$+m(A[c]_{m,k_s})(\varepsilon+\varepsilon^2+\ldots)$$

taking (ii) of Lemma 9 in account. The first term on the right is dominated by

$$\int_{A[c]_{m,k_s}} |\gamma_1-\varphi[c]_{m,s}|+\int_{A[c]_{m,k_s}} (|1-\psi[c]_{m,1}|+$$
$$+|1-\psi[c]_{m,2}|+\ldots+|1-\psi[c]_{m,r-1}|).$$

It follows by (iv), (vii) of Lemma 9 and Cauchy–Schwartz that

$$\sum_m \int_{A[c]_{m,k_s}} |\gamma_1-\varphi[c]_{m,s}\psi[c]_{m,1}\ldots\psi[c]_{m,s-1}|$$
$$\leqslant 3\varepsilon m(A[c]_{k_s})+\sum_{t=1}^{s-1}\sum_m m(A[c]_{m,k_s})^{1/2}\Big\{\int_{A[c]_m}|1-\psi[c]_{m,t}|^2\Big\}^{1/2}$$
$$\leqslant 3\varepsilon m(A[c]_{k_s})+\sum_{t=1}^{s-1} m(A[c]_{k_s})^{1/2}C_3^{1/2}\varepsilon^{-1/2}m(A[c]_{k_t})^{1/2}$$
$$\leqslant\Big\{3\varepsilon+C_3^{1/2}\varepsilon^{-1/2}\sum_{t=1}^{s-1}\delta^{1/2(s-t)}\Big\}m(A[c]_{k_s})$$
$$\leqslant 4\varepsilon m(A[c]_{k_s}).$$

By (iii) of Lemma 9, we find for the second term the estimation

$$C_3\varepsilon^{-1}\{m(A[c]_{m,k_1})+m(A[c]_{m,k_2})+\ldots+m(A[c]_{m,k_{s-1}})\}$$

and after summation over m

$$C_3\varepsilon^{-1}(\delta^{s-1}+\delta^{s-2}+\ldots+\delta)m(A[c]_{k_s})\leqslant\varepsilon m(A[c]_{k_s}).$$

Consequently

$$\sum_m\int_{A[c]_{m,k_s}}|\gamma_1-\varphi[c]_m|f_m\leqslant 7\varepsilon M^{dk_s+1}m(A[c]_{k_s})\leqslant 7\varepsilon M\sum_m\int_{A[c]_{m,k_s}} f_m.$$

So using also previous estimate (vi)

$$I[c]\leqslant(8C_3M^{-1}+7\varepsilon M)\sum_m\int_{A[c]_m} f_m\leqslant 15C_3M^{-1}\sum_m\int_{A[c]_m} f_m$$

and hence

(viii) $$\sum_c I[c]\leqslant 15C_3M^{-1}\sum_m\int f_m.$$

Using the same technique as in [3] let us introduce the H^∞-functions

$$\varphi_m=2^{-14}\Big\{\gamma_1^{11}-\prod_{c=0}^{10}[\gamma_1-\varphi[c]_m]\Big\}$$

and

$$\psi_m=(1/11)\sum_{c=0}^{10}\psi[c]_m$$

for $m=1,\ldots,n$. Since $|\varphi_m|\leqslant(1/11)\sum_{c=0}^{10}|\varphi[c]_m|$. (iii) follows from (vi). Further

(ix) $$\sum|1-\psi_m|\leqslant\sum(1/11)\sum_{c=0}^{10}|1-\psi[c]_m|\leqslant\sup_c\sum_m|1-\psi[c]_m|.$$

Let us verify (i), taking $\gamma_2=2^{-14}\gamma_1^{11}$.

$$\sum\int f_m|\gamma_2-\varphi_m| =2^{-14}\sum_m\int f_m\Pi_c|\gamma_1-\varphi[c]_m|$$
$$\leqslant 2^{-4}\sum_{m,c}\int_{A[c]_m} f_m|\gamma_1-\varphi[c]_m|$$
$$=2^{-4}\sum_c I[c]$$
$$\leqslant C_3M^{-1}\sum_m\int f_m.$$

In order to verify (ii), fix $c=0,\ldots,10$ and evaluate $\sum_m|1-\psi[c]_m|$. By definition of $\psi[c]_m$,

$$\sum_m|1-\psi[c]_m|=\sum_{s=1}^{r}\sum_m|1-\psi[c]_{m,s}|.$$

Now for each $s=1,\ldots,r$, we get that

(*) $$\sum_m|1-\psi[c]_{m,s}|\leqslant\eta n.$$

Moreover for $s<t\leqslant r$, by (vi) of Lemma 9,

(**) $$\sum|1-\psi[c]_{m,s}|>\varepsilon^{t-s}n\Rightarrow\sum|1-\psi[c]_{m,t}|\leqslant\varepsilon^{t-s}n.$$

It is an elementary exercise to verify that (*) and (**) imply that

$$\sum_{s=1}^{r}\sum_{m}|1-\psi[c]_{m,s}| \leqslant (\eta+\varepsilon+\varepsilon^2+\ldots+\varepsilon^{r-1})n.$$

This completes the proof of the lemma.

Proof of Theorem 3. If $f_1, \ldots, f_n$ are disjointly supported functions in $L^1(\Pi)$ satisfying (i) of Th. 3, there are H^∞-functions g_m $(1 \leqslant m \leqslant n)$ satisfying

$$\|g_m\|_\infty \leqslant 1 \quad \text{and} \quad \langle f_m, g_m\rangle = \delta.$$

For $\tau > 0$ (which we fix later) and $n \geqslant \big(C_3\xi(\tau/3)\big)^{18}$, let φ'_m, ψ'_m be the H^∞-functions obtained in previous lemma, replacing f_m by $|f_m|$. Since

$$\sum\int|f_m||\gamma_2-\varphi'_m| \leqslant \tau n,$$

we see that

$$\operatorname{card}(N) \leqslant \tau^{1/2}n$$

defining

$$N = \Big\{m=1, \ldots, n; \int|f_m|\ |\gamma_2-\varphi'_m| > \tau^{1/2}\Big\}.$$

Take

$$\varphi_m = \varphi'_m g_m, \quad \psi_m = \psi'_m \quad \text{if} \quad m \notin N$$

and

$$\varphi_m = g_m, \quad \psi_m = 0 \quad \text{if} \quad m \in N.$$

Then

$$\sum|1-\psi_m| \leqslant \sum_{m\notin N}|1-\psi'_m| + \operatorname{card}(N) \leqslant (\tau+\tau^{1/2})n.$$

If $m \notin N$, then

$$|\langle f_m, \varphi_m\rangle - \gamma_2\langle f_m, g_m\rangle| \leqslant \int|f_m|\ |g_m|\ |\varphi'_m-\gamma_2| \leqslant \tau^{1/2}.$$

Taking $\tau < \delta^2/4$, we can put $\delta_1 = \delta/2$. For $\alpha(n)$, take $2\tau^{1/2}n$, where τ must be large enough to ensure the inequality $n \geqslant \big(C_3\xi(\tau/3)\big)^{18}$.

V. Remarks.

1. The disjointness hypothesis for the functions f_m in Th. 3 can be replaced by a weaker hypothesis, i.e.

$$\Big\|\sum\chi_{A_m}\Big\|_\infty \leqslant B$$

where $A_m = \operatorname{supp} f_m$ and B is some constant.

2. In fact, Th. 3 can be combined with results of [3] as follows. Given $\delta > 0$, there exist $\delta_1 > 0$ and a function $\alpha(n)$ s.t. $\alpha(n)/n \xrightarrow[n\to\infty]{} 0$ so that the following holds:

If $f_1, \ldots, f_n$ in $L^1(\Pi)$ are δ-Rademacher l^1, i.e. if

$$\int\Big\|\sum\varepsilon_k c_k f_k\Big\|_1 d\varepsilon \geqslant \delta\sum|c_k|\,\|f_k\|_1$$

and if

$$\|q(f_k)\| \geqslant (1-\delta_1)\|f_k\|_1 \quad (1 \leqslant k \leqslant n)$$

(in particular, if the f_k are minimum norm liftings), then there are H^∞-functions $\varphi_1, \ldots, \varphi_n$ and $\psi_1, \ldots, \psi_n$ satisfying following properties

(i) $$|\varphi_k|+|\psi_k| \leqslant 1 \quad (1 \leqslant k \leqslant n),$$

(ii) $$\sum|\varphi_k| \leqslant 1,$$

(iii) $$\sum|1-\psi_k| \leqslant \alpha(n),$$

(iv) $$\langle f_k, \varphi_k\rangle = \delta_1\|f_k\|_1.$$

3. Our methods provides estimates of the form $\alpha(n)/n < (\log n)^{-1/2+\delta}$. Is it possible to replace $\alpha(n)$ by a constant?

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H^p estimates for weakly strongly singular integral operators on spaces of homogeneous type

by

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Abstract. Let X be a normalized homogeneous space. We define "weakly strongly" singular kernel on $X \times X$, and we study the action of the "convolution" operator induced by this kernel on the atomic Hardy spaces $H^p(X)$, with $0 < p \leqslant 1$. A boundedness result is obtained. These operators are analogues of the weakly strongly operators on $\boldsymbol{R}^n$ studied by C. L. Fefferman and E. M. Stein in [6].

1. Introduction. In this paper we study a generalization of convolution operators induced by weakly strongly singular integral kernels. Examples of these kernels, in the case of R^n are given by

$$k(x) = |x|^{-\beta}\psi(x)\exp i|x|^{a},$$

where $0 < a < 1$, $\beta > 0$ and ψ is a C^∞ function on $\boldsymbol{R}^n$, which vanishes near zero and equals 1 outside a bounded set (see [5], page 21). The L^p theory, $1 < p < +\infty$, for operators obtained by convolution with kernels $k(x)$, has been studied by I. I. Hirschmann [7], S. Wainger [12], C. L. Fefferman [5], C. L. Fefferman and E. M. Stein [6], J. E. Björk [1] and P. Sjölin [11].

Also in [6], C. L. Fefferman and E. M. Stein obtain boundedness results for $H^p(\boldsymbol{R}^n)$, $1 \geqslant p > p_0(a, \beta, n) > 1/2$. Estimates including the limiting case $p = p_0(a, \beta, n)$ were obtained by R. R. Coifman in [2] when $n = 1$.

Here we consider a generalization of these kernels and the action of the induced operators on H^p spaces, $p \leqslant 1$, defined in terms of atoms on spaces of homogeneous type. First we define what we mean by a weakly strongly singular kernel on spaces of homogeneous type. In Theorem 3 we prove that the operator K induced by this kernel maps atoms into elements of H^p, $p \leqslant 1$. In the proof of this theorem we extend some techniques used by R. A. Macías and C. Segovia in [9]. The extension of the operator to the whole space H^p requires the introduction of an auxiliary operator, namely $K^{\#}$, acting on the space $\mathrm{Lip}(1/p-1)$ of classes of Lipschitz functions. This operator is an adaptation of the operator $K^{\#}$ considered in [9].