Superspaces of $(s)$ with basis

by

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Abstract. The sequence space $(s)$ is characterized as the unique nuclear Fréchet space $F$ with basis and a continuous norm such that when $E$ is isomorphic to a subspace of a nuclear Fréchet space $E$ with a basis $(a_n)$, then a subsequence of $(a_n)$ generates a subspace isomorphic to $E$. Variations of this result are considered in the context of stable $D_1$ and $D_2$ spaces.

1. Introduction. A superspace of $(s)$, the space of rapidly decreasing sequences, is a nuclear Fréchet space which contains a subspace isomorphic to $(s)$. The simplest way to construct such a space is to take an arbitrary nuclear Fréchet space $E$ and form the cartesian product $E \times (s)$. We will show that if the superspace is assumed to have a basis, it actually always has this simple form. In fact, the basis of the superspace can be divided into two disjoint subsequences one of which generates a (necessarily complemented) subspace isomorphic to $(s)$. Among nuclear Fréchet spaces with basis and a continuous norm this property of superspaces characterizes $(s)$. We make a generalization to arbitrary stable $D_1$ spaces and consider as a natural dualization a stable $D_2$ space as a quotient space.

2. Preliminaries. If not otherwise stated, the subscripts and superscripts appearing are assumed to run through $\mathbb{N} = \{1, 2, \ldots\}$. The symbol $\mathbb{R}$ stands for the scalar field (real or complex numbers).

We refer to [12], [3] and [8] for the undefined concepts and the basic results used.

Let $E$ be a nuclear Fréchet space and let the topology of $E$ be defined by an increasing sequence $(p_n)$ of seminorms. Suppose the sequence $(a_n)$ of $E$ is a basis, that is, for every $x \in E$ there is a unique sequence $(\xi_n)$ of scalars such that

$$x = \sum_n \xi_n a_n.$$  

By the absolute basis theorem ([12], 10.2.1) $(a_n)$ is absolute, i.e. in (1),

$$\sum_n |\xi_n| p_n(a_n) < \infty.$$
for all $k$. Denote $a_n^k = p_k(e_n)$.

(2) \[ K(a) = \{ \langle \xi, e \rangle \in K^V | \langle \xi, e \rangle_{|k} = \sum |\xi_n|a_n^k < \infty \forall k \} .\]

If the sequence space $K(a)$ is equipped with the topology defined by the seminorms $|.|_k$, then the assignment $a \rightarrow (\xi)$ defines an isomorphism $E \rightarrow K(a)$. The space $K(a)$ is called the Köthe space associated with $E$, $(a_n)$. The infinite matrix $a = (a_n^k)$ is called a Köthe matrix representing $(e_n)$ and it has the properties $0 \leq a_n^k \leq a_n^{k+1}$, $\sum a_n^k > 0$, for each $k$ there is $l$ with $(a_n^k, a_n^k) \in l_1$, (we agree $0/0 = 0$). Conversely, every such matrix defines through (2) a nuclear Fréchet space with $(a_n)$, the sequence of coordinate vectors, as a basis. In particular, $(a) = K(n)$.

Consider another Köthe space

$K(b) = \{ \langle \eta_n \rangle | \sum |\eta_n|b_n^k < \infty \forall k \}$

and let $T: K(b) \rightarrow K(a)$ be a continuous linear mapping. If $T$ is represented by the matrix $(t_{n}^{k})$, that is,

$T\eta_n = \sum t_n^k \eta_n^{k} ,$

then the adjoint $T': K(a) \rightarrow K(b)'$ is represented by the transpose of the matrix $(t_{n}^{k})$,

$T'\eta_n = \sum t_n^k \eta_n^{k} .$

Here $(\eta_n)$ (resp. $(\xi_n)$) is the sequence of coordinate functionals of the coordinate basis $(\eta_n)$ of $K(a)$ (resp. $(\xi_n)$ of $K(b)$). One easily checks that $(\eta_n)$ and $(\xi_n)$ are weak bases. We will use the identifications

$K(a)' = \{ \langle \xi_n \rangle | \exists N, |\xi_n| \leq \xi_N^k \} , \quad K(b)' = \{ \langle \eta_n \rangle | \exists N, |\eta_n| \leq \eta_N^k \} .

Here $C$ denotes a positive constant. Now

$T'\xi_n = \sum \langle \eta_n | t_n^k \rangle \xi_n^k .

In comparing two Köthe matrices $(a_n^k)$ and $(b_n^k)$ we use the following notations: $(a_n^k \leq b_n^k)$ means for all $k$ there is $l$ with $(a_n^k, a_n^l \sim l b_n^k)$ and $(b_n^k \leq a_n^k)$. If $(a_n^k \sim l b_n^k)$ then $K(a) = K(b)$.

A Köthe matrix $(a_n^k)$ is regular if the sequence $(a_n^k/a_n^{k+1})$ is non-increasing for every $k$. We say that $(\xi_n^k)$ is a $D_1$ matrix if it is regular, $p_k^{\xi} = 1$ and $(\xi_n^k)^{\pi} \leq \xi_n^k$. Correspondingly, $(\xi_n^k)$ is a $D_2$ matrix if it is regular, $q_n^{\xi} > 0$, $\lim q_n^{\xi} = 1$ and $(\xi_n^k) \leq (\xi_n^k)$. The definitions of the types $D_1$ and $D_2$ are equivalent to those in [1].

Let $E$ be a locally convex space and denote by $\Xi_E$ its neighborhood basis at 0 consisting of barrels. Denote by $d_\pi(V, U) (n = 0, 1, 2, \ldots)$ the $n^{\pi}$ Kolmogorov diameter of $V \in \Xi_E$ with respect to $U \in \Xi_E$. Suppose $(p_n^\pi)$ is a $D_1$ matrix. We say that $E$ is $\pi$-nuclear if for every $U$ and $k$ there is $V$ such that $p_n^{\pi}d_n^V(V, U)n \in l_\pi$. If $(q_n^\pi)$ is a $D_2$ matrix, $E$ is said to be $\pi$-nuclear if for every $U$ there is $V$ and $k$ such that $\lim (q_n^\pi)d_n^V(V, U)n \in l_\pi$. The $(\pi, N)$-nuclearity was introduced in [33] with the type $\theta_\pi$ in place of $D_1$. These two types are however equivalent and so are, respectively, $G_1$ and $D_1([14], [16])$. The above definition of $\pi$-nuclearity is consistent with the general concept of $\lambda$-nuclearity ([14], [15]).

A locally convex space $E$ is stable if $E \times E \cong E$. If $(p_n^\pi)$ is a $D_1$ matrix, the stability of $K(p)$ is equivalent to $(p_n^\pi) \leq (p_n^\pi)$ and if $(q_n^\pi)$ is a $D_2$ matrix, the stability of $K(q)$ is equivalent to $(q_n^\pi) \leq (q_n^\pi)$. (see [17]); we also call the Köthe matrix in question stable.

The following is proved in [5]:

**Proposition 2.1.** (i) If $(p_n^\pi)$ is a stable $D_1$ matrix, $K(a)$ is $(\pi, N)$-nuclear and $a_n^\pi = 0$, then there is a strictly increasing sequence $(\xi_n)$ of indices such that $(p_n^\pi) \leq (a_n^\pi)$ and $\sum \xi_n^\pi < \infty$.

(ii) If $(q_n^\pi)$ is a stable $D_2$ matrix, $K(a)$ is $(\pi, N)$-nuclear and $0 < \xi_n^\pi < 1$, then there is a strictly increasing sequence $(\eta_n)$ of indices such that $(a_n^\pi) \leq (q_n^\pi)$ and $\sum \eta_n^\pi < \infty$.

We will also need an important result from combinatorics called Hall's theorem.

**Theorem 2.2.** (Hall) Let $A$ be a set and suppose $(A_i)$ is a family of finite subsets of $A$. There is a system of distinct representatives $a_i \in A_i$, $i \in I$, if and only if the following condition is satisfied; for all distinct $i_1, \ldots, i_k \in I$ the set $A_{i_1} \cup \ldots \cup A_{i_k}$ has at least $k$ elements.

For a proof and a thorough discussion of theorems of this type we refer to [9].

**3. Superspace of $(\xi)$**. We begin with a general result on imbedding one nuclear Fréchet space with basis into another.

**Proposition 3.1.** Let $E$ and $F$ be nuclear Fréchet spaces with bases $(\xi_n)$ and $(\eta_n)$, respectively and suppose $F$ has a continuous norm. If $E$ is isomorphic to a subspace of $F$, then there are representations $(a_n^1)$ and $(b_n^2)$ of $(\xi_n)$ and $(\eta_n)$ respectively, such that for each $k$ there is an injection $n \rightarrow j_n^k$ and scalars $\mu_n^k > 0$ with

\[ b_n^k \leq \mu_n^k a_n^{k+1} , \quad n \in N, \]

\[ \mu_n^k a_n^{k+1} \leq b_n^k , \quad n \in N, \]
Proof. Let \((e_k^0)\) and \((d_k^0)\), \(d_k^0 > 0\), be representations of \((\mathcal{E}_0)\) and \((\mathcal{E}_1)\), respectively, and let \(T^* : K(d) \to K(e)\) be an imbedding (i.e. isomorphism into) represented by a matrix \((t_{ij})\). Set

\[
U_k^* = \{ (\xi, \eta) \in K(e) | \eta_n \leq 0 \} \quad V_k^* = \{ (\xi, \eta) \in K(d) | \sum_{n} |\eta_n| b_k^0 \leq 1 \}.
\]

It is a consequence of nuclearity that the scalar multiples of the neighborhoods \(U_k^*\) form a neighborhood basis of \(0 \in K(e)\). Since \(T^*\) is an imbedding, there are strictly increasing sequences \((\xi_j)\) and \((\lambda_k)\) of indices and decreasing sequences \((\xi_j)\) and \((\lambda_k)\) of positive constants such that

\[
T(D_k \xi_j V_k^*) = T(K(d)) \cap (C_0 U_k^*) \subset T(D_k \xi_j V_k^*)
\]

for all \(k\). Set \(a_k^0 = C_k^{-1} b_k^0\), \(b_k^0 = D_k^{-1} b_k^0\). We may assume that \(a_k^{0+1} / a_k^0 > 2\).

Let \(U_k = C_k U_k^* V_k = D_k V_k^*\). Then

\[
U_k^* = \{ (\xi, \eta) \in K(e) | \sum_{n} |\eta_n| a_k^0 \leq 1 \} \quad V_k^* = \{ (\xi, \eta) \in K(d) | \eta_n \leq b_k^0 \}.
\]

By (3),

\[
T(V_k^*) = T(K(b)) \cup U_k \subset T(V_k^*).
\]

Polarizing the right side of (4) we get

\[
T(K(b)) \cup U_k \subset T(V_k^*)
\]

which by the surjectivity of \(T^*\) gives

\[
V_k^* = T((K(b)) \cup U_k)^c.
\]

It then follows from the Hahn–Banach theorem that

\[
V_k^* \subset T(U_k^*).
\]

From (4) we also obtain

\[
\sum_{n} |\eta_n| b_k^0 \leq \sup_{x} \sum_{n} t_{mn} |x_n| |\eta_n| \leq \sum_{n} |\eta_n| b_k^{0+1}
\]

for all \((\eta_n) \in K(b)\). Setting \((\eta_n) = \xi_j\) we get

\[
b_k^0 \leq \sup_{n} |\eta_n| b_k^{0+1}.
\]

Fix now \(k\) and define

\[
I_k = \{ (\xi) \in \mathbb{K} | b_k^0 \leq 2 |\lambda_k| |\xi| \}.
\]

By (6) the set \(I_k\) is non-empty and since \(\lim_{n \to \infty} |\lambda_n| b_k^0 = 0\) and \(b_k^0 > 0\), \(I_k\) is finite. We will show that the family \((I_k)_{k \in \mathbb{N}}\) satisfies the condition of Theorem 2.2.
Corollary 3.2. Suppose a nuclear Fréchet space $E$ with a basis $(y_n)$ and a continuous norm is isomorphic to a subspace of a nuclear Fréchet space $F$ with a basis $(x_n)$. Then there are representations $(a^k_n)$ and $(b^l_n)$ of $(a_n)$ and $(y_n)$, respectively, and injections $w_j: F_j \to F_k$, $k \geq 2$.

$$(7) \quad \frac{a^{k-1}_n}{a^{k+1}_n} \leq \frac{b^l_n}{b^{l+1}_n} \leq \frac{a^{k+1}_n}{a^{k-1}_n}, \quad n \in N, \quad k, l \geq 2.$$ 

Proof. Using the notations of Proposition 3.1, we obtain

$$\frac{a^{k-1}_n}{a^{k+1}_n} = \frac{\mu_n a^{k-1}_n}{\mu_n a^{k+1}_n} \leq \frac{b^l_n}{b^{l+1}_n} \leq \frac{\mu_n b^{l+1}_n}{\mu_n b^l_n} = \frac{a^{k+1}_n}{a^{k-1}_n} \quad \text{for} \quad n \in N, \quad k, l \geq 2.$$ 

Formula (7) could be regarded as a refinement of Duhinský's fundamental inequality for subspaces [(3), III (1.3)] in that now the sequences $(\xi^n_k)$, known to have no repetitions. For a similar result in a different context see [7], Lemma 5.

Theorem 3.3. Let $E$ be a nuclear Fréchet space with a basis $(y_n)$. If $(a)$ is isomorphic to a subspace of $E$, then a subsequence of $(a_n)$ generates a subspace isomorphic to $(a)$.

Proof. Let $(a^k_n)$ and $(b^l_n)$ be representations of $(a_n)$ and the coordinate basis of $(a)$, respectively, as in Proposition 3.1. Since $(\xi^n_k) \sim (\xi^n_k)$, there is $\xi^0_n$ such that $b^{l+1}_n \geq 1$ for sufficiently large $n$. Set $\xi^0_n = \xi^0_k = 0$. By (i) of Proposition 3.1, $\mu_n a^{k+1}_n \geq 1$ for large $n$. Since $w_j: F_j \to F_k$ is injective, $K(\mu_n a^{k+1}_n)$ is nuclear. Now $K(|w|, N)$-nuclearity is just ordinary nuclearity so that we can apply (i) of Proposition 3.1 to find a strictly increasing sequence $(\xi^n_k)$ with $(\xi^n_k) \leq (\mu_n a^{k+1}_n)$; $\xi^n_k \leq a^{k+1}_n$. Thus,

$$(a^n) \leq (\mu_n a^{k+1}_n) \sim (\xi^n_k) \leq (a^n),$$

where the second estimate follows from (ii) of Proposition 3.1. Hence, $(a) = K(\mu_n a^{k+1}_n) \sim \text{sp}(a_n)$.

Remark 3.4. The previous theorem shows in particular that if $T: (a) \to E$ is an imbedding, then $T(\bar{z})(a)$ is isomorphic to a complemented subspace of $E$. It may however happen that $T(\bar{z})$ itself is not complemented. For the details of the following example we refer to ([10], Chapter 7).

*Added in proof: Kudinov has recently reported a result essentially equivalent to Proposition 3.1.

Denote by $C^r([−1, 1])$ the space of infinitely differentiable functions on the closed interval $[−1, 1]$ which vanish together with all their derivatives at the endpoints. When equipped with the topology of uniform convergence in all derivatives on $[−1, 1]$ this space is isomorphic to $(a)$. By E. Borel's theorem the mapping

$$S: C^r([−1, 1]) \to \mathbb{K}^N, \quad S(f) = (f^{(m)}(0))_0$$

is surjective; of course it is continuous and linear. There is a natural isomorphism

$$S^{-1}(0) \cong C^r([−1, 0]) \times C^r([0, 1]).$$

Also, $C^r([−1, 0]) \cong C^r([0, 1]) \cong (a)$ so that $S^{-1}(0) \cong (a)$. If $S^{-1}(0)$ were complemented, $C^r([−1, 1]) = S^{-1}(0) \oplus G$, then $S(G)$ would be an isomorphism $G \to \mathbb{K}^N$ (by the open mapping theorem) which is absurd since $G$ has a continuous norm but $\mathbb{K}^N$ has not.

Next we show that the property of superspaces of $(a)$ demonstrated in Theorem 3.3 in fact characterizes $(a)$ among nuclear Fréchet spaces with basis and a continuous norm.

Proposition 3.5. Suppose the nuclear Fréchet space $E$ with basis and a continuous norm has the following property: if $F$ is isomorphic to a subspace of a nuclear Fréchet space $E$ with a basis $(a_n)$, then a subsequence of $(a_n)$ generates a subspace isomorphic to $F$. Then $E$ is isomorphic to $(a)$.

Proof. By (2) or (6) $F$ is isomorphic to a subspace of $(a)^N$. A basis of $(a)^N$ is given by the family $(\varepsilon_{mn})$, $\varepsilon_{mn} = (0, \ldots, 0, \varepsilon_m, 0, \ldots)$, where $\varepsilon_m$ appears in the $m$th place. Let

$$V_b = \{(\varepsilon_{mn}) \in (a)^N \mid \sum_{n=1}^{\infty} |\varepsilon_{mn}|^2 \leq 1\}$$

and set $W_b = (V_b)^b \times (a)^N$. Then $(W_b)$ is a neighborhood basis of $0 \in (a)^N$ and it gives rise to a representation $(a^N_{mn})$ of $(a)^N$.

$$(8) \quad a^N_{mn} = \begin{cases} a^n_m, & 1 \leq m \leq k, n \in N, \\ 0, & m > k. \end{cases}$$

By hypothesis, there is an injection $\varepsilon_i \mapsto (\varepsilon_{mn})$, $(\varepsilon_{mn})$ such that $K(a^N_{mn}) \cong F$. Since $F$ has a continuous norm, there is $k_b$ with $a^N_{mn} > 0$ for all $i$. This implies by (8) that $m \leq k_b$. Hence $F \cong K(\varepsilon_{mn})$, where each value $\varepsilon_{mn}$ is repeated at most $k_b$ times. It then follows that $F \cong a_{kn}$, where $\varepsilon = (a_n)$ is a nuclear exponent sequence of infinite type, i.e. $\sup_{n} \log n/\varepsilon_n < \infty$. It remains to be shown that $\sup_{n} \varepsilon_n/\log n < \infty$. 

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We assume that $\sup_n a_n / \log n = \infty$ and show that this leads to a contradiction. Choose a strictly increasing sequence $(n_i)$ of indices such that $n_i = 1$ and
\[
(9) \quad a_{n_{i+1}} < (1/i) a_{n_i}, \quad i \geq 2,
\]
and
\[
(10) \quad \rho \log n_i \leq a_{n_i}, \quad i \in \mathbb{N}.
\]
Then define $m_i = \min \{ n_i (1/i) a_{n_i} \leq a_{n_i} \}$. Note that $m_1 = 1$ and by (9) and the definition of $m_i$,
\[
(11) \quad n_{i+1} + 1 < m_i \leq n_i, \quad i \geq 2
\]
\[
(12) \quad a_{m_{i+1}} < (1/i) a_{n_i} \leq a_{n_{i+1}}, \quad i \in \mathbb{N}.
\]
Set $N_1 = \bigcup_{i=1}^{\infty} \{ n \in \mathbb{N} \mid n \leq n_i \}$ and $N_2 = \mathbb{N} \setminus N_1$. Note that by (11), $N_2$ is infinite. Denote by $(a^1_n)$ and $(a^2_n)$ the subsequence of a corresponding to $N_1$ and $N_2$, respectively, that is,
\[
a^1_i = a_{n_1}, a_{n_2}, \ldots, a_{n_i}; \quad a^2_i = a_{m_1}, a_{m_2}, \ldots, a_{m_i}, a_{m_{i+1}}, \ldots,
\]
\[
a^1_i = a_{n_1}, a_{n_2}, \ldots, a_{n_i}; \quad a^2_i = a_{m_1}, a_{m_2}, \ldots, a_{m_i} a_{m_{i+1}}, \ldots
\]
Trivially, $A_n(a) \sim A_n(a^1) \times A_n(a^2)$. Now $a^1_n$ is of finite type. In fact, for any $n$, $a^1_n = a_{n+i}$ for some $i$ and $j \geq 0$, $n \leq m_i + j \leq n_i$, so that by (12) and (10),
\[
\frac{a_{n_i}}{\log n} \geq \frac{a_{n_i}}{\log n_i} \geq \frac{a_{n_i}}{i} \geq \frac{1}{i} \rho \log n_i = 1.
\]
Consequently, $\lim n a^1_n / \log n = \infty$. By ([3], III (2.4.3)) there is a stable finite type exponent sequence $(\beta_n)$ such that $\sup_n \beta_n < \infty$ so that by
\[
(33), \text{III (2.4.4)} \text{ there is an embedding } A_n(a) \rightarrow A_n(\beta).
\]
It follows that $A_n(a) \sim E$ is isomorphic to a subspace of $A_n(\beta) \times A_n(\alpha^2)$. By the hypothesis, a subsequence of the basis of $A_n(\beta) \times A_n(\alpha^2)$ (the union of the coordinate bases of $A_n(\beta)$ and $A_n(\alpha^2)$) generates a subspace isomorphic to $A_n(a)$. This subspace contains only finitely many basis vectors of $A_n(\beta)$ since otherwise $A_n(a)$ would contain a subspace isomorphic to a finite type power series space and this is impossible ([33], III (2.4.3)). Thus $A_n(a)$ is isomorphic to a subspace of $E \times A_n(\alpha^2)$, where $E$ is a finite dimensional subspace of $A_n(\beta)$. If $\dim E = n_0$, we obtain by computing the Kolmogorov diameters both in $A_n(a)$ and $E \times A_n(\alpha^2)$ and using ([33], I (6.2.2)) that $a^2_n \leq C a_{n+n_0}, \quad n \in \mathbb{N}$, where $C$ is a constant. But suppose $i \geq n_0 + 2$ and that $a^2_n = a_{n+i+1}$. Since $n \leq m_i + i - 1$ we get by (12),
\[
a^2_n = a^2_{n+i+1} \leq a_{n+i} \leq a_{n+i-1} \leq a_{m_i} \leq a_{n_i - i}, \quad i \geq 2.
\]
Thus we arrive at the contradiction $\sup_n a^2_n / a_{n+i} = \infty$. ■

Remark 3.6. Proposition 3.5 holds if we only assume that $F$ is isomorphic to a complemented subspace of any of its superspaces. In fact, if $F$ is isomorphic to a complemented subspace of $(a')^\gamma$, we can use ([11], Theorem 2.2) to obtain a mapping $i \rightarrow (m(i), n(i))$ (not necessarily injective) such that $F \cong E(F(n(i)))$ and, as before, $m(i) \leq n_i$. By the nuclearity of $F$ each value $n(i)$ occurs only for finitely many different $i$, so that $F \cong A_n(a)$. We conclude that $A_n(a)$ is isomorphic to a complemented subspace of $A_n(\beta) \times A_n(\alpha^2)$. By ([11], Proposition III.1.6) there is $n_0$ and a continuous linear surjection $A_n(a) \rightarrow A_n(a^1)$. The contradiction follows from ([33], I (6.2.3)).

4. A generalization and a dualization. We get immediately the following generalization of Theorem 3.3.

Theorem 4.1. Let $(p^i_n)$ be a stable $D_n$ matrix and suppose $E$ is a $K(p, N)$-nuclear Fréchet space with a basis $(a_n)$. If $K(p)$ is isomorphic to a subspace of $E$, then a subsequence of $(a_n)$ generates a subspace isomorphic to $K(p)$.

Proof. Let $(a^1_n)$ and $(a^2_n)$ be representations of $(a_n)$ and the coordinate basis of $K(p)$, respectively as in Proposition 3.1. As in Theorem 3.3, we find $\tilde{a}_n$, an injection $n \rightarrow j$, and scalars $\mu_n > 0$ such that $\mu_n a^1_n \geq 1$ for large $n$. Since $K(\mu_n a^1_n)$ is $K(p, N)$-nuclear, there is a strictly increasing sequence $(\mu_j)$ with $\mu_j \geq (\mu_n a^1_n)$, $\mu_j \leq (\mu_n a^1_n)$. Thus,
\[
(p^i_n) \leq (\mu_n a^1_n) \leq (\mu_j) \sim (p^i_n) \leq (p^i_n),
\]
where in the second estimate (ii) of Proposition 3.1 was used and the last one follows from the stability of $(p^i_n)$ and the fact that $(p^i_n)$ is non-decreasing.

To obtain a dualization of the previous theorem we first prove the quotient space analogue of Proposition 3.1.

Proposition 4.2. Let $E$ and $F$ be nuclear Fréchet spaces with bases $(a_n)$ and $(y_n)$, respectively, and suppose $E$ has a continuous norm. If $F$ is isomorphic to a quotient space of $E$, then there are representations $(a_n)$ and $(b_n)$ of $(a_n)$ and $(y_n)$, respectively, such that for each $k$ there is an injection $n \rightarrow m_k$ and scalars $\gamma^k > 0$ with
\[
(i) \quad b_n \leq \gamma^k a^k_n, \quad n \in N,
\]
\[
(ii) \quad b_n \leq \gamma^k a^k_n, \quad n \in N.
\]

Proof. Let $(a^1_n)$ and $(a^2_n)$ be representations of $(a_n)$ and $(y_n)$, respectively, and let $T : K(a) \rightarrow K(d)$ be a continuous linear surjection.
with a representing matrix \((t_n)\). Set
\[
U_k^* = \{ (\xi_n) \in K(c) : \sum_n |\xi_n|/a_n^k \leq 1 \}, \quad V_k^* = \{ (\eta_n) \in K(d) : \sup_n |\eta_n|/a_n^k \leq 1 \}.
\]

The nuclearity of \(K(d)\) implies that the scalar multiples of the neighborhoods \(V_k^*\) form a neighborhood basis of \(0 \in K(d)\). Since \(T\) is open and continuous, there are strictly increasing sequences \((\lambda_k)\) and \((\lambda_k')\) of indices and decreasing sequences \((\delta_k)\) and \((\delta_k')\) of positive constants such that
\[
T(C_{\lambda_k+1} U_{\lambda_k}^*) \subset D_1 V_{\lambda_k}^* = T(C_{\lambda_k} U_{\lambda_k}^*)
\]
for all \(k\). Set \(a_n^k = C_{\lambda_k}^{-1} \delta_k^2, b_n^k = D_{\lambda_k}^{-1} \delta_k^2\). We assume that \(\lambda_k^2 |b_n^k| > 2\). Let \(U_k = C_k U_k^*\), \(V_k = D_k V_k^*\). By (13),
\[
T(U_{\lambda_k+1}) \subset V_k = T(U_k).
\]

For every \(i\), \((1/b_i^k) \xi_n \in V_k\) so that by the right side of (14) there is \((\xi_n) \in U_k\) with \(T(\xi_n) = (1/b_i^k) \xi_n\). Thus,
\[
1/b_i^k = \sum_n t_n \xi_n.
\]

Choose \(k_0\) with \(\lim_{n \to \infty} a_n^{k_0} = \infty\). Fix \(k \geq k_0\) and define
\[
N_k = \{ n : a_n^{k} < 2|t_n|/a_n^{k_0} \}.
\]

We show that \(N_k\) is non-empty and finite. By the left side of (14),
\[
\sup_n \left( \sum_n t_n \xi_n \right) b_i^k \leq \sup_n \left| \xi_n \right| a_n^{k+1}
\]
which, by setting \((\xi_n) = (\xi_k, k = 1, \ldots, n)\), gives
\[
|a_n^{k_0}| b_i^k \leq \sup_i |a_i^{k_0}| \leq a_n^{k_0}
\]
for all \(i\) and \(n\). Since \(b_i^k > 0\), \(t_n \neq 0\) implies \(a_n^{k_0} > a_n^{k} > 0\). On the other hand, since \(T\) is surjective, there are indices \(n\) for which \(t_n \neq 0\). By summing over these \(n\) we get from (15),
\[
1/b_i^k \leq \sup_n \left| \xi_n \right| a_n^{k_0} \leq \sup_n \left( \sum_n t_n \xi_n \right) a_n^{k_0} \leq \sup_i \left| \xi_n \right| a_n^{k_0}
\]

since \((\xi_n) \in U_{\lambda_k+1}\). Thus, for some \(n\), \(|a_n^{k_0}| a_n^{k} > 1/2b_i^k\). If \(t_n \neq 0\) for infinitely many \(n\), then \(1/b_i^k\) implies
\[
\lim_{i \to \infty} (a_i^{k}/|t_n|) = \lim_{i \to \infty} (|t_n|/a_i^{k}) = \infty
\]
so that \(N_k\) is finite.

By deleting the \(k - 1\) first rows from the matrices \((a_i^k)\) and \((b_i^k)\) we can assume that \(k_0 = 1\).

We show that the family \((N_i)_{i \in \mathbb{N}}\) satisfies the condition of Theorem 2.2. Consider a union \(N = N_1 \cup \cdots \cup N_{m+1}\) where \(t_1, \ldots, t_{m+1}\) are distinct and \(m \geq 1\). We assume that \(N = \{ (\eta_1, \ldots, \eta_m) \} \times \{ \eta_{m+1} \} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}\), with \(|\eta_{m+1}| \leq m\) and show that this leads to a contradiction. Denote
\[
D = \{ \xi = (\zeta_1, \ldots, \zeta_{m+1}) \in K_{m+1} : |\zeta_i| a_i^{m+1} \leq 1 \}.
\]

Since \(a_i^{m+1} > 0\), \(D\) is a compact neighborhood of \(0 \in K_{m+1}\). Pick \(\varepsilon \in D\) and define \((\eta_i) \in V_k\) by
\[
\eta_i = \begin{cases} \varepsilon_i, & i \in \{1, \ldots, m\}, \\ 0, & i \in \{m+1\}. \end{cases}
\]

Thus, by (14) there is \((\xi_n) \in U_k\) with \((\eta_n) = T(\xi_n)\) so that
\[
a_n \equiv n_\varepsilon = \sum_n t_n \varepsilon_n \leq \sum_n t_n \eta_n + \sum_n t_n \eta_{m+1} \varepsilon_{m+1} = \varepsilon_1, \ldots, \varepsilon_m, \sum_n t_n \eta_{m+1} \varepsilon_{m+1}.
\]

Denote \(u_{m+1} = (t_{m+1, 1}, \ldots, t_{m+1, m}) \in K_{m+1}\), \(p = 1, \ldots, m\), \(q_1 = \sum_n t_n \varepsilon_n\) \(w = (w_{m+1}, \ldots, w_{m+1}) \in K_{m+1}\). Then
\[
\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{m+1}) = \sum_{p=1}^{m+1} \varepsilon_{p} u_{m+1} + w.
\]

Here
\[
|w_{m+1}| a_{m+1}^k \leq \sum_{p=1}^{m+1} |w_{m+1}| |\varepsilon_p| a_p^{m+1} \leq \sum_{p=1}^{m+1} |w_{m+1}| |\varepsilon_p| a_p^{m+1} \leq \varepsilon_{m+1} \frac{1}{2}
\]
since \((\varepsilon_i) \in U_{m+1}\). Hence, \(w \in D\) and because \(\varepsilon\) was arbitrary, \(D = \text{sp} \{ (\xi_n) \} \subset K_{m+1}\). As in the proof of Proposition 3.1, this is impossible since \(q \leq m\).

By Theorem 2.2 we can find distinct representatives \(v_i^{N} \in N_i, i \in N\). By the definition of \(N\),
\[
v_i^{N} a_i^{k_0} < 2 b_i^{k_0} \leq b_i^{k+1}, \quad i \in N,
\]

where we defined \(v_i^k = 1/(t_n^{m+1})\) and used the condition \(b_i^{k+1}/b_i^{k} > 2\). This proves (i). For (ii) we use (16) with \((\xi_i) \in v_i^{N} \),
\[
(1/v_i^k) b_i^k = |t_n^{m+1}| b_i^k \leq \sup_i |t_n^{m+1}| b_i^k \leq a_i^{k_0} a_i^{k} = \infty, \quad i \in N.
\]

**Corollary 4.3.** Suppose a nuclear Fréchet space \(F\) with a basis \((y_i)\) and a continuous norm is isomorphic to a quotient space of a nuclear Fréchet.
space $E$ with a basis $(a_n)$. Then there are representations $(b_k^0)$ and $(b_k^1)$ of $(a_n)$ and $(y_n)$, respectively, and injections $m \to m_n^k$, $k \in N$, such that

$$(18)\quad a_k^{n+1} = b_k^n \leq \frac{a_k^{n+1}}{a_k^n}, \quad n \in N, k, l \geq 2.$$ 

Proof. We use the same notation as in Proposition 4.2 except that the injection corresponding to $k$ is denoted by $m_n^k$. Then

$$\frac{b_k^n}{a_k^n} = \frac{a_k^{n+1}}{a_k^n} \leq \frac{a_k^{n+1}}{a_k^n} = m_n^k \quad \text{for} \quad n \in N, k, l \geq 2.$$ 

Formula (18) should be interpreted as Dubinsky's fundamental inequality for quotient spaces ([3], IV (1.4)).

Theorem 4.4. Let $(g_n^k)$ be a stable $D_k$-matrix and suppose $E$ is a $K(q)$-nuclear Fréchet space with a basis $(a_n)$. If $K(q)$ is isomorphic to a quotient space of $E$, then a subsequence of $(a_n)$ generates a subspace isomorphic to $K(q)$.

Proof. Let $(a_n^k)$ and $(b_n^k)$ be representations of $(a_n)$ and the coordinate basis of $K(q)$, respectively, as in Proposition 4.2. Since $g_n^k = 0$, the equivalence $(b_n^k) \sim (g_n^k)$ implies $g_n^k = 0$ for all $k$. By injectivity, $m_n^k = 0$. Thus, by (i) of Proposition 4.2 we can select indices $1 = n_1 < n_2 < \ldots$

$$(19)\quad m_n = m_n^k, \quad n_2 < n < n_{k+1}, k \in N,$$

and

$$(20)\quad M_2 = \{m_n^k : n_2 < n < n_{k+1}, k \in N\},$$

then $M_2 \cap M_2 = \emptyset$ for $l \not\in \{k, k+1, k+2\}$.

Let $m_n = m_n^k = v_n^k$ for $n_2 < n < n_{k+1}, k \in N$. Note that by (20), each value in the sequence $(m_n)$ occurs for at most two different values of $n$. Also, by (16),

$$n \leq n < n_{k+1}, k \in N.$$ 

(19) $g_n^k \sim (b_n^k) \leq (v_n^k m_n^k).$

Further, for a fixed $l$ and $k < n < n_{k+1}, k \geq 1$,

$$(21)\quad v_n^k m_n^k \leq v_n^k a_n^k = v_n^k.$$ 

so that $\lim v_n^k a_n^k = 0$. Using the stability of $(g_n^k)$ it is easy to see that the space $K(q_m^k)$ is $K(q)$-nuclear. Hence we can apply (ii) of Proposition 2.1 to find a strictly increasing sequence $(t_n^k)$ with $(v_n^k m_n^k) \leq (g_n^k)$, sup $t_n^k < \infty$. Using (21) we obtain

$$(g_n^k) \sim (t_n^k) \leq (v_n^k m_n^k) \leq (g_n^k),$$

where the first estimate follows from the stability of $(g_n^k)$ and the fact that $(g_n^k)$ is non-increasing. The proof is completed by deleting possible repetitions (at most two of each) from $(m_n^k)$ by passing to a subsequence $(m_n^k)$ with $n > 2n_2$.

$$(g_n^k) \leq (g_n^k) \leq (v_n^k m_n^k) \leq (g_n^k),$$

Thus, $K(q_m^k) = K(q_m^k) \simeq K(q_m^k)$.}

References

A reverse maximal ergodic theorem

by

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Abstract. A reverse maximal ergodic theorem is proved for a d-parameter discrete semigroup $\{T_t; g \in \mathbb{Z}_d^d\}$ of measure preserving transformations on a σ-finite measure space $(X, \mathcal{F}, \mu)$ which is ergodic in the sense that if $E \in \mathcal{F}$ with $\mu(E) = \mu(X)$, then $\mu(E) = 0$ or $\infty$. A continuous version follows from standard approximation arguments.

1. Introduction. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and $(T_t; g \in \mathbb{Z}_d^d)$ a d-parameter discrete semigroup of measure preserving transformations on $(X, \mathcal{F}, \mu)$. For $0 \leq f \in L_1(\mu) + L_\infty(\mu)$, the maximal function $f^*$ is defined by

$$f^*(a) = \sup_{n \geq 1} \sum_{g \in \mathbb{Z}_d^d} f(T_g a) \quad \text{where} \quad V_n = \{0, \ldots, n-1\}^d.$$  

It is then known (cf. [11], [4], [1]) that the maximal inequality holds:

$$\mu\{f^* > a\} \leq \frac{1}{R_d} \int_{\{x > a\}} f \, d\mu \quad \text{for any} \quad a > 0 \quad (1)$$

where $R_d$ is a constant dependent only on the dimension $d$.

The purpose of this paper is to show that a reverse maximal inequality holds provided that the semigroup $(T_t; g \in \mathbb{Z}_d^d)$ is ergodic in the sense that if $E \in \mathcal{F}$ with $\mu(E) = \mu(X)$, then $\mu(E) = 0$ or $\infty$. Here it should be noted that N. Dang-Ngoc [2] has shown a similar inequality for an ergodic d-parameter group $(T_t; g \in \mathbb{Z}_d^d)$ of measure preserving transformations on a probability measure space. However, the maximal function $f^*$ he considered is defined by

$$f^*(a) = \sup_{n \geq 1} \sum_{g \in \mathbb{Z}_d^d} f(T_g a) \quad \text{where} \quad W_n = \{-n+1, \ldots, n-1\}^d,$$

and he remarked that his argument is not modified if $f^*$ is replaced by $f^*$. Nevertheless, we shall modify his argument to prove our result. For the particular case $(T_t; g \in \mathbb{Z}_d^d)$ where $T$ is conservative and ergodic in the usual sense, the inequality was already obtained by Derriennic [3] in a