STUDIA MATHEMATICA, T. LXXIV. (1982)

Properties of the L function

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Abstract. We study the function $L(t) = \inf\limits_{0<x<1} (x^p + t(1-x)^q)$ and related functions, which arise in the theory of interpolation spaces.

0. Introduction. In the theory of interpolation spaces, more precisely in connection with the L method (cf. e.g. [5], [3], [2]) one is interested in the infimum of the expression $x^p + ty^q$ under the condition $x+y = 1$, $x > 0$, $y > 0$, where p, q, t are given numbers > 0. Keeping p and q fixed let us denote this infimum by $L(t)$. Most results in the literature use the crude approximations $L(t) \approx \min(1, t)$ (cf. however Sparr [7], notably Lemma 3.4, pp. 237–238). We are here interested in *exact* results. (In particular, in all of our discussion we take $p > 1$, $q > 1$.) In particular, we are concerned in evaluating integrals of the form $\psi(z) = \int\limits_0^\infty L(zt)\,dk(t)$, where k is a given positive measure on $(0, \infty)$, the most important case being $dk(t) = t^{-\eta}\frac{dt}{t}$, $0 < \eta < 1$. (Such functions often arise as "interpolation functions".) In fact our central result around which most of the discussion revolves is the formula

$$\int\limits_0^\infty t^{-\eta} L(t)\frac{dt}{t} = \frac{p^{1-\eta}q^{\eta}}{(1-\eta)\eta} B\big((1-\eta)(p-1)+1, \eta(q-1)+1\big),$$

where $B(a, b)$ denotes the Euler beta function. (In this simple case it suffices to consider the case $z = 1$, since $\psi(z) = \psi(1)z^{\eta}$ by a change of variables.) Actually the most straightforward approach to (1) is via Sparr's formula [7]. We also show that $L(t)$ has an analytic continuation to the sector $|\arg t| < \pi\min(p-1, q-1)$. It follows that the function $\psi(z)$ can be continued to the same sector, too. Such a result is of interest also because in the special case $p = q = 2$ we have essentially the Stieltjes transform. We also consider, in particular, the generalization to several variables, that is the infimum of the expression $\sum\limits_{i=0}^{n} t_i x_i^{p_i}$. Here some new rather unexpected complications arise. For instance Sparr's formula [7] does not seem to be applicable any more.

We remark that the theory of interpolation spaces will not be touched upon from now on; it just served as a general motivation.

The paper is organized as follows.

After a preparatory Sec. 1 in Sec. 2 we give several useful representations for the function $L(t)$. Then we turn (Sec. 3) to the problem of evaluation of integrals. In Sec. 4 we consider the generalization, where the functions x^p and y^q are replaced by more general functions $F(x)$ and $G(y)$. Sec. 5 is concerned with the analyticity of $L(t)$. Finally, Sec. 6 is devoted to an extension to the case of several variables.

1. Simplest properties. Let us thus set for $0 < t < \infty$

$$L(t) = \inf_{\substack{x+y=1\\ x>0,y>0}} (x^p + ty^q), \tag{1}$$

where we always assume that $p, q > 1$. (If $p \leqslant 1$, $q \leqslant 1$ it is easy to see—and well known—that $L(t) = \min(1, t)$. We have not at all investigated the cases $p \leqslant 1$, $q > 1$ and $p \geqslant 1$, $q < 1$.) From (1) follows at once that L is a concave function. From the concavity again follows (draw a picture!) at once that we must have $L(t) \geqslant L(1)\min(1, t)$. On the other hand, taking x or y in (1) close to 0 we see that $L(t) \leqslant \min(1, t)$. Thus we have the inequality

$$c_1 \min(1, t) \leqslant L(t) \leqslant c_2 \min(1, t) \tag{2}$$

with $c_1 = L(1)$, $c_2 = 1$. It is clear that $c_1 = L(1)$ is the best constant (for we have equality if $t = 1$). It is also easy to see (Sec. 5) that $c_2 = 1$ is best possible, too.

If $p = q$ one can show (the case of equality of Hölder's inequality in two variables) that

$$L(t) = \frac{1}{(1+t^{-1/(p-1)})^{p-1}}$$

and, in particular, if $p = 2$

$$L(t) = \frac{t}{1+t}.$$

Returning to the general case we see that the inequality $x^p + ty^q \geqslant L(t)$, as well as its extension to several variables (see Sec. 6) may be conceived as a generalization of Hölder's inequality.

Also for some other values of p and q (see Sec. 2) rather explicit expressions for $L(t)$ are available.

To get more symmetric formulas one can put also

$$L(s, t) = \inf_{\substack{x+y=1\\ x>0,y>0}} (sx^p + ty^q) \tag{1'}$$

with $0 < s < \infty$, too. Clearly $L(s, t)$ is concave and, in addition, homogeneous of degree 1. The two L's are related by the formula $L(s, t) = sL(t/s)$.

2. Some representation for $L(t)$. If the minimum in Sec. 1, (1) is attained for some set of values x, y then by differentiation of $x^p + t(1-x)^q$ with respect to x we see that we must have $px^{p-1} = tq(1-x)^{q-1}$ or

$$t = \frac{px^{p-1}}{q(1-x)^{q-1}} \left(= \frac{px^{p-1}}{qy^{q-1}}\right). \tag{1}$$

It is easy to see that this equation has a unique solution x for each t and that the solution really corresponds to a minimum.

Formally we could also have proceeded by Lagrange's method. In the symmetric formulation (see Sec. 1, (1′)) this leads to the relations

$$\begin{aligned} spx^{p-1} &= \lambda, \\ tqy^{q-1} &= \lambda \end{aligned} \tag{2}$$

which of course reduce to (1) if $s = 1$. It turns out to be advantageous to express everything in terms of the Lagrange parameter λ. If we eliminate λ between the two equations (2) we get

$$\left(\frac{\lambda}{sp}\right)^{1/(p-1)} + \left(\frac{\lambda}{tq}\right)^{1/(q-1)} = 1. \tag{3}$$

For λ fixed we may conceive this as a kind of "generalized hyperbola" in the s, t-plane. (If $p = g = 2$ it is a hyperbola.) Using (2) we further get the formula

$$L(s, t) = \lambda\left(\frac{1}{p}x + \frac{1}{q}y\right) \tag{4}$$

where thus x and y are determined by (2), λ by (3). (Notice that if $p = q$ then $L(s, t) = \lambda/p$. Notice likewise that λ is homogeneous of degree 1 in s and t while x and y are homogeneous of degree 0.) By elimination of x and y we can write (4) also as

$$L(s, t) = \lambda\left(\frac{1}{p}\left(\frac{\lambda}{sp}\right)^{1/(p-1)} + \frac{1}{q}\left(\frac{\lambda}{tq}\right)^{1/(q-1)}\right) \tag{4'}$$

which reiterates the fact that L is entirely determined by λ.

Later on (Sec. 5) it will be advantageous to make a change of variables writing $w = \lambda/p$, $z = (p/tq)^{1/(q-1)}$. Then (3) with $s = 1$ takes the form

$$w^a + zw^b = 1, \tag{3'}$$

where we have further put $a = 1/(p-1)$, $b = 1/(q-1)$. In particular, if $(q-1)/(p-1) = 2$ or $a = 2b$ we have a quadratic equation for w^b. In this case we thus find

$$w^b = \frac{1}{\frac{z}{2}+\sqrt{1+\frac{z^2}{4}}}$$

which gives

(5) $$\lambda = \frac{p}{\left(\frac{1}{2}\left(\frac{p}{tq}\right)^{1/q-1}+\sqrt{1+\frac{1}{4}\left(\frac{p}{tq}\right)^{1/q-1}}\right)^{q-1}}$$

$$((q-1)/(p-1) = 2,\ s = 1).$$

By duality we can similarly find an explicit expression for λ (and L) if $(p-1)/(q-1) = 2$.

Another useful representation of L is obtained by noticing that L being a concave function is the *Legendre transform* of a convex curve in the positive quadrant. Let the curve be given in Cartesian coordinates α, β by the equation $\alpha = \varphi(\beta)$, where φ thus is a decreasing positive convex function. Then one has the defining relations

(6) $$L(t) = \varphi(\beta)+t\beta, \qquad t = -\varphi'(\beta).$$

Geometrically this means that $\alpha+t\beta = L(t)$ is the equation of the tangent of the curve $\alpha = \varphi(\beta)$ at the particular point $(\varphi(\beta), \beta)$ as pictured by the figure below (Fig. 1).

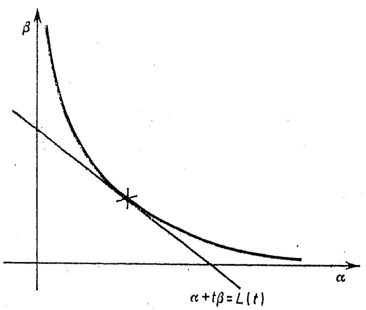

Fig. 1

(6) means further that the expression $\varphi(\beta)+t\beta$ is minimized for the particular value of β such that $t = -\varphi'(\beta)$. Therefore we must have $x^p = \varphi(\beta)$, $y^q = \beta$. In other words, $\varphi(\beta) = (1-\beta^{1/q})^p$ and our curve has the equation $a^{1/p}+\beta^{1/q} = 1$.

If we differentiate relation (6) we get

$$L'(t)\,dt = \varphi'(\beta)\,d\beta+dt\beta+td\beta = \varphi'(\beta)\,d\beta+dt\beta-\varphi'(\beta)\,d\beta = \beta dt.$$

Therefore we end up with the important relation

(7) $$\beta = L'(t)$$

complementing relation (6).

3. Evaluation of some integrals. Let us begin by the simplest case, viz. the integral

$$I = \int_0^\infty t^{-\eta}L(t)\frac{dt}{t}, \qquad 0<\eta<1.$$

Integrating by parts and using Sec. 2, (7), we find

$$I = \frac{1}{\eta}\int_0^\infty t^{-\eta}L'(t)\,dt = \frac{1}{\eta}\int_0^\infty t^{-\eta}y^q\,dt.$$

(It is easy to convince oneself that the end-point terms vanish.) Integrating by parts once more and using Sec. 2, (1), we obtain

$$I = -\frac{1}{\eta(1-\eta)}\int_{-1}^0 t^{1-\eta}qy^{q-1}\,dy = \frac{1}{\eta(1-\eta)}\,q\int_0^1\left(\frac{px^{p-1}}{qy^{q-1}}\right)^{1-\eta}y^{q-1}\,dy$$
$$= \frac{p^{1-\eta}q^\eta}{\eta(1-\eta)}\int_0^1 x^{(p-1)(1-\eta)}(1-x)^{(q-1)\eta}\,dx.$$

If we finally invoke the definition of Euler's beta function we see that we have proved the formula [1]

(1) $$\int_0^\infty t^{-\eta}L(t)\frac{dt}{t} = \frac{p^{1-\eta}q^\eta}{\eta(1-\eta)}B\big((p-1)(1-\eta)+1,\ (q-1)\eta+1\big).$$

Alternatively, by the available functional equations for the beta function [2],

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[1] Recall that $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$.

[2] Viz. $B(a+1, b) = \frac{a}{a+b}B(a, b)$, $B(a, b+1) = \frac{b}{a+b}B(a, b)$, which is an easy consequence of the formula $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, where again $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\,dx$.

we can write this as

$$(1') \qquad \int_0^\infty t^{-\eta} L(t) \frac{dt}{t} = \frac{p^{1-\eta} q^\eta (p-1)(q-1)}{P(P-1)} B\big((p-1)(1-\eta), (q-1)\eta\big)$$

with $P = p(1-\eta)+q\eta$.

We could here also have used Sparr's formula ([7], Lemma 3.4, pp. 237–238) to the effect that

$$(2) \qquad L(t) = \int_0^1 \min\left(\frac{px^{p-1}}{q(1-x)^{q-1}}, t\right) q(1-x)^{q-1} dx.$$

Formula (2) again is based on the observation that any concave function L can be obtained as the superposition of functions $\min(\tau, t)$. More precisely, if $L(t) = o\big(\max(1, t)\big)$ as $t \to 0$ or ∞ then we have

$$(3) \qquad L(t) = \int_0^\infty \min(\tau, t)\, dl(\tau),$$

in fact with $dl = -L''(t)dt$. If we use (3) in conjunction with Sec. 2, (1) and (7), we readily get (2). Again (1) follows at once from (2) just interchanging the order of integration. We see thus that the two partial integration sin our first attack on (1) are hidden in formula (2).

We can now easily extend the previous result, viz. formula (1) to the case of the general integral $\int_0^\infty L(t)\, dk(t)$, where k is a given positive measure on $(0, \infty)$ such that $\int_0^\infty \min(1, t)\, dk(t) < \infty$. If we associate with k the concave function

$$K(t) = \int_0^\infty \min(\tau, t)\, dk(t)$$

we see that at the heart of the matter is the formula

$$(4) \qquad \int_0^\infty L(t)\, dk(t) = \int_0^\infty K(t)\, dl(\tau).$$

This is a kind of duality for concave functions. In particular, if $dk = t^{-\eta} \dfrac{dt}{t}$ we have $K(t) = \dfrac{t^{1-\eta}}{\eta(1-\eta)}$ so (4) gives back (1). In the general case we can write the result

$$(5) \qquad \int_0^\infty L(t)\, dk(t) = \int_0^1 K\big(q(1-x)^{q-1}, px^{p-1}\big) dx,$$

where $K(s, t) = sK(t/s)$, the function of two variables corresponding to $K(t)$.

We conclude this section by indicating a quite different approach. *It is the method that will be used later* (Sec. 6) *in the case of several variables.* The idea is to write our integral as

$$(6) \qquad \int_C \frac{L(s, t)}{s^{1-\eta} t^\eta} \left(-\frac{ds}{s} + \frac{dt}{t}\right).$$

Since the integrand in (6) is a closed[3] differential form, we have by Stokes' theorem a great freedom in the choice of the "contour of integration" C. Thus taking $s = 1$, the integral (6) reduces to the one in formula (1). The natural choice, however, is to take C to be the curve $\lambda = 1$ (with the proper orientation) which by Sec. 2, (3) is the generalized hyperbola $\left(\dfrac{1}{sp}\right)^{1/(p-1)} + \left(\dfrac{1}{tq}\right)^{1/(q-1)} = 1$. (That the infinity does not give any contribution requires a proof but we omit the details.) By taking logarithmic derivatives in Sec. 2, (2), we get

$$\frac{ds}{s} + (p-1)\frac{dx}{x} = 0, \qquad \frac{dt}{t} + (q-1)\frac{dy}{y} = 0$$

so that

$$-\frac{ds}{s} + \frac{dt}{t} = (p-1)\frac{dx}{x} - (q-1)\frac{dy}{y} \qquad \text{if} \quad \lambda = 1.$$

Using also Sec. 2, (4), and (2) once more, we arrive at the integral

$$p^{1-\eta} q^\eta \int x^{(1-\eta)(p-1)} y^{\eta(q-1)} \left(\frac{x}{p} + \frac{y}{q}\right)\left((p-1)\frac{dx}{x} - (q-1)\frac{dy}{y}\right)$$

which again can be expressed as the sum of four beta integrals. The result

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(3) Indeed if $\chi(s, t)$ is any homogeneous of degree 0 function one has

$$d\left(\chi(s, t)\left(-\frac{ds}{s} + \frac{dt}{t}\right)\right) = \left(\frac{\partial \chi}{\partial s} ds + \frac{\partial \chi}{\partial t} dt\right)\left(-\frac{ds}{s} + \frac{dt}{t}\right)$$
$$= \left(s\frac{\partial \chi}{\partial s} + t\frac{\partial \chi}{\partial t}\right)\frac{ds}{s}\frac{dt}{t} = 0,$$

where we have utilized Euler's theorem. (Alternatively one can use polar coordinates $s = \varrho\cos\theta$, $t = \varrho\sin\theta$ which gives

$$\int_C \chi(s, t)\left(-\frac{ds}{s} + \frac{dt}{t}\right) = \int_0^{\pi/2} \chi(\cos\theta, \sin\theta) \frac{d\theta}{\cos\theta\sin\theta}.)$$

therefore comes in the form (with $\theta = 1-\eta$)

$$p^\theta q^\eta \Big\{ \frac{p-1}{p} B\big((p-1)\theta+1, (q-1)\eta+1\big) + \frac{q-1}{p} B\big((p-1)\theta+2,$$

$$(q-1)\eta\big) + \frac{p-1}{q} B\big((p-1)\theta, (q-1)\eta+2\big) + \frac{q-1}{q} B\big((p-1)\theta+1,$$

$$(q-1)\eta+1\big)\Big\}.$$

The two inner terms we here transform as

$$\frac{(p-1)\theta+1}{(q-1)\eta} B\big((p-1)\theta+1, (q-1)\eta+1\big)$$

and

$$\frac{(q-1)\eta+1}{(p-1)\theta} B\big((p-1)\theta+1, (q-1)\eta+1\big),$$

respectively. Hence we get the common beta factor $B\big((p-1)\theta+1, (q-1)\eta+1\big)$ with the coefficient

$$\frac{p-1}{p} + \frac{(p-1)\theta+1}{p\eta} + \frac{(q-1)\eta+1}{q\theta} + \frac{q-1}{q} = \frac{1}{\eta} + \frac{1}{\theta} = \frac{1}{\eta\theta}$$

since $\theta+\eta = 1$, and we end up with the same result, viz. formula (1).

In a similar way we can treat the case of a general measure k. Assume that k is absolutely continuous, $dk = K(t)\dfrac{dt}{t}$, one then has to consider the line integral

(6′)
$$\int_C \frac{L(s,t)}{K(s,t)}\Big(-\frac{ds}{s}+\frac{dt}{t}\Big),$$

where $K(s,t) = sK(t/s)$. (The integrand in (6′) is closed since $\dfrac{L(s,t)}{K(s,t)}$ is homogeneous of degree 0.) We won't enter into the details.

4. A more general case. We now quickly outline the extension of the results of Sec. 1–3 — as far as this is possible — to the more general case where we have more general convex functions $F(x)$ and $G(y)$ in place of x^p and y^q. *Thus in this section we put*

(1)
$$L(t) = \inf_{\substack{x+y=1\\ 0<x<1,\,0<y<1}} \big(F(x)+tG(y)\big)$$

and use $L(s,t)$ in a similar fashion (cf. Sec. 1, (1) and (1′)). We do not discuss the precise assumptions of F and G needed.

Now the values of x and y at which the minimum is attained are determined by the formulas (cf. Sec. 2, (2))

(2)
$$sf(x) = \lambda,\qquad tf(y) = \lambda,$$

where we have put $f = F'$, $g = G'$. Here λ for given s and t can be found by the formula (cf. Sec. 2, (3))

(3)
$$f^{-1}(\lambda/s)+g^{-1}(\lambda/t) = 1.$$

We have thus (cf. Sec. 2, (4) and (4′))

(4)
$$L(s,t) = \lambda\Big(\frac{F(x)}{f(x)}+\frac{G(y)}{g(y)}\Big) \quad\text{with}\quad x = f^{-1}(\lambda/s),\ y = g^{-1}(\lambda/t).$$

Finally $L(t)$ is the Legendre transform of the curve in the positive α, β-quadrant with equation $F(\alpha)+G(\beta) = 1$, the above minimizing values of x and y being determined by $x = F(\alpha)$, $y = G(\beta)$. Also to the Sparr formula (Sec. 3, (2)) there corresponds the more general formula

(5)
$$L(t) = \int_0^1 \min\Big(\frac{f(x)}{g(y)}, t\Big) g(y)\,dx.$$

All our methods for evaluating the integral $\int_0^\infty t^{-\eta} L(t)\dfrac{dt}{t}$ (Sec. 3) are now available. The quickest is perhaps after all the use of (5) and we get

(6)
$$\int_0^\infty t^{-\eta}L(t)\frac{dt}{t} = \frac{1}{\eta(1-\eta)}\int_0^1 (f(x))^{1-\eta}(g(1-x))^\eta\,dx.$$

(It is readily checked that (6) gives back Sec. 2, (1) if $F(x) = x^p$, $G(y) = y^q$.) In the same way we can treat the case if a general measure k and we find

(7)
$$\int_0^\infty L(t)\,dk(t) = \int_0^1 K\big(g(1-x), f(x)\big)\,dx$$

with $K(s,t)$ as in Sec. 3, (5).

More interesting is the application of the line integral of Sec. 3, (6). Using (2) and (4) we get the integral (with, as in Sec. 3, $\theta = 1-\eta$)

$$\int (f(x))^\theta (g(y))^\eta \Big(\frac{F(x)}{f(x)}+\frac{G(y)}{g(y)}\Big)\Big(\frac{f'(x)}{f(x)}dx - \frac{g'(y)}{g(y)}dy\Big)$$

which we have to confront with the previous integral

$$J = \int (f(x))^\theta (g(y))^\eta\,dx.$$

But the latter we can integrate by parts in two different ways: on one hand we get

$$J=\int F'f^{\theta-1}g^{\eta}dx = -(\theta-1)\int Ff^{\theta-2}f'g^{\eta}+\eta\int Ff^{\theta-1}g^{\eta-1}g'$$

$$=\eta\Big(\int Ff^{\theta-2}f'g^{\eta}+\int Ff^{\theta-1}g^{\eta-1}g'\Big)$$

and on the other hand, by a similar calculation,

$$J=\theta\Big(\int Gf^{\theta-1}g^{\eta-1}f'+\int Gf^{\theta}g^{\eta-2}g'\Big).$$

(We have omitted to indicate the variables and the range of integration.) If we multiply the first expression by θ, the second by η and form the sum, the previous result (6) follows.

In the same way we can derive (7) but again we abstain from entering into the details of the calculation.

5. Analyticity. We return to the function $L(t)$ of Sec. 1, (1). In view of Sec. 2, (2)—(4) it is clear that $L(t)$ is analytic in t. We wish to say something more precise about the region of analyticity.

First we investigate, however, the behavior near $t=0$ or $t=\infty$. It is sufficient to consider the function $w=\lambda/p$ of the variable $z=(p/tq)^{1/(q-1)}$ defined by Sec. 2, (3′). It is clear (by the implicit function theorem) that there is locally a unique analytic solution which takes the value 1 at $z=0$. We wish to determine the coefficients of the Taylor development

(1) $$w=1+c_1z+c_2z^2+\ldots$$

By Cauchy's formula we have

$$c_n=\frac{1}{2\pi i}\int\frac{w}{z^{n+1}}dz\qquad(n=0,1,\ldots),$$

where we integrate (in the positive sense) around a small circle with center at the origin. Making a change of variables $z=\dfrac{1-w^b}{w^a}$, $dz=-aw^{-a-1}+(a-b)w^{b-a-1}$, we get

$$c_n=\frac{1}{2\pi i}\int\frac{w}{\left(\dfrac{1-w^b}{w^a}\right)^{n+1}}\left(-aw^{-a-1}+(a-b)w^{b-a-1}\right)dw$$

$$=\frac{(-1)^n}{2\pi i}\int\frac{w^{an}}{(w^b-1)^{n+1}}\left(a+(b-a)w^b\right)dw.$$

Setting again $w^b=\omega$ we get

$$c_n=\frac{(-1)^n}{2\pi ib}\int\frac{1}{(\omega-1)^{n+1}}\left(a\omega^{(an+1)/b-1}+(b-a)\omega^{(an+1)/b}\right)d\omega.$$

If we here apply the residue theorem we find

PROPOSITION 1. *The Taylor coefficients in* (1) *are for* $n\geq 1$ *given by*

(2) $$c_n=\frac{(-1)^n}{n!b}\left(\frac{an+1}{b}-1\right)\left(\frac{an+1}{b}-2\right)\ldots\left(\frac{an+1}{b}-(n-1)\right)$$

(*n factors*)

or equivalently

(2′) $$c_n=\frac{(-1)^n}{n!b}\,\frac{\Gamma\left(\dfrac{an+1}{b}\right)}{\Gamma\left(\dfrac{(a-b)n+1}{b}+1\right)}.\ \blacksquare$$

Again from (2′) and Stirling's formula we see that the radius of convergence of series (1) is

(3) $$R=\begin{cases}\left|1-\dfrac{b}{a}\right|^{a/b-1} & \text{if}\quad a\neq b,\\ 1 & \text{f}\quad a=b.\end{cases}$$

Remark (historical). According to Hardy [4], pp. 194—195, the formula (2) was found by Ramanujan using one of his general integral formulas ([4], p. 185, formula (A)). Hardy says however: "I do not know whether this formula is new, and I have not attempted to find conditions under which the analysis is justified". We have then filled in this small gap (?) in our knowledge, at least in so far that we do have a proof of (2). (Our proof is of course subsumed in the Lagrange—Bürmann formula.) Note that if a and b are integers then Sec. 2, (3′) is an algebraic equation, known as the *trinomial equation*. It has been much studied in the past. Hardy ([2], p. 210) quotes Birkeland [1] referring also to additional references stated there.

Returning to $L(t)$ we see (use Sec. 2, (4′)) that this function near $t=0$ has an expansion which begins with

(4) $$L(t)=t\left(1+\left(\frac{q}{p}-q\right)\left(\frac{qt}{p}\right)^{1/(p-1)}+\ldots\right)$$

the dots of course representing higher order terms in $z=qt/p$. (Here the first few terms could also have been recaptured using directly Sec. 2, (3) or (3′). If t is near 0 we must have $\lambda\approx tq$ in the first approximation; so setting $\lambda = tq(1+\varepsilon)$ we get from Sec. 2, (3) (with $s = 1$) the equation

$$\left(\frac{tq(1+\varepsilon)}{p}\right)^{1/(p-1)} + (1+\varepsilon)^{1/(q-1)} = 1$$

which readily gives the second approximation $\lambda \approx tq\left(1-(q-1)\left(\frac{qt}{p}\right)^{1/(p-1)}\right)$ in accordance with (4).) In the same way we have near $t = \infty$

$$L(t) = 1 + \left(\frac{p}{q} - p\right)\left(\frac{pt}{q}\right)^{1/(q-1)} + \ldots, \tag{5}$$

where now the dots indicate higher terms in $\left(\frac{pt}{q}\right)^{1/(q-1)}$.

Incidentally using (4) and (5) we can now fill in the gap in Sec. 1 regarding the inequality $L(t) \leqslant \min(1, t)$, because (4) and (5) imply

$$\lim_{t\to 0} \frac{L(t)}{t} = 1 \quad \text{and} \quad \lim_{t\to\infty} L(t) = 1,$$

respectively.

To proceed further we introduce a uniforming parameter s defined by $w^a = e^s$ and we also put $c = b/a$. Then the equation Sec. 2, (3′) can be written as

$$ze^{cs} + e^s = 1$$

or, solving out for z,

$$z = e^{-cs}(1-e^s). \tag{6}$$

This is essentially a *periodic* equation; if we replace s by $s+2\pi i$ then z changes only by a phase factor. This means that we can fix attention to the strip $-\pi \leqslant \operatorname{Im} s \leqslant \pi$. Let us write in polar coordinates $z = re^{i\theta}$. If $\operatorname{Im} s = \pm\pi$ then we have $\theta = \pm\pi c$. We wish to determine for which θ in the interval $[-\pi c, \pi c]$ we can solve (6) for any given value of r in $(0, \infty)$. To this end we first locate the critical points. $\frac{dz}{ds} = 0$ gives $-ce^{-cs} - (1-c)e^{-cs+s} = 0$ or $e^s = \frac{c}{c-1}$ (if $c \neq 1$). Disregarding the trivial case $c = 1$ we have thus to distinguish the two cases $c > 1$ and $c < 1$.

Case 1. $c > 1$. Then we have one critical point only, situated on the positive real axis. We look at the curves $\theta = \text{const.}$ contained in the open strip $-\pi < \operatorname{Im} s < \pi$. If such a curve, starting far out to the left ($\operatorname{Re} s$ negative), where it is approximately a straight line parallel to the real axis, does not pass through the critical point, it can either go to the point 0 or far out to the right ($\operatorname{Re} s$ positive), for r must be a *decreasing* function along the curve. These two categories are by continuity reasons

separated by the two curves $\theta = \pm\pi$ through the critical point (see Fig. 2 below). We conclude that equation (6), where $z = re^{i\theta}$, has a unique solution for any r if $|\theta| < \pi$.

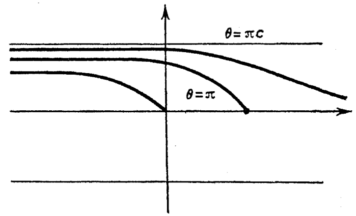

Fig. 2

Case 2. $c < 1$. Now there are two critical points symmetrically situated on the bordering lines $\operatorname{Im} s = \pm\pi$. By a similar argument as in Case 1 we see that now any line $\theta = \text{const.}$ in the open strip $-\pi < \operatorname{Im} s < \pi$ must hit the point 0 (see Fig. 3).

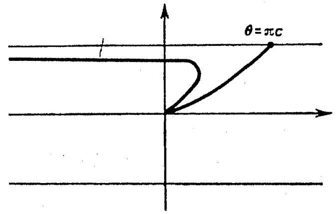

Fig. 3

Altogether we have now established

PROPOSITION 2. *If* $|\arg z| < \pi \min(1, c)$ *equation* (6) *has a unique solution* s *with* $|\operatorname{Im} s| < \pi$. ∎

For our function $L(t)$ this means that it can be analytically continued to the sector $|\arg t| < \pi\min(p-1, q-1)$, as a simple calculations reveals. (We consider of course such a sector as a Riemann surface, so that the same point may cover several points of the complex plane.)

Notice that from this and Sec. 2, (3′) — which is of course valid in the complex, too — follows that we have the inequality $|L(t)| \leqslant C\min(1, |t|)$ in any sector $|\arg t| < \pi\min(p-1, q-1) - \varepsilon$ $(\varepsilon > 0)$. This again implies regarding the function $\varphi(z) = \int_0^\infty L(zt)\,dk(t)$ $k <$ positive measure on

$(0, \infty)$ with $\int_0^\infty \min(1, t)\,dk(t) < \infty$ that the function in question is analytic in $|\arg t| < \pi \min(p-1, q-1)$.

6. Several variables. As is customary in interpolation theory we let the number of variables be $n+1$, denoting the later by $t_0, \ldots, t_n$ each of the t_i ranging in the interval $(0, \infty)$. As a generalization of $L(s, t)$ (Sec. 1, (1′)) we now consider the function

$$L(t_0, \ldots, t_n) = \inf_{\substack{\Sigma x_i = 1 \\ 0 < x_i < 1}} \sum t_i x_i^{p_i}, \tag{1}$$

where the p_i are given numbers > 1. Imposing the restriction $t_0 = 1$ we get the corresponding non-homogeneous version $L(t_1, \ldots, t_n)$ analogous to $L(t)$ (Sec. 1, (1)). (In this section the symbol $\sum$ is used to denote summation over the range $0, \ldots, n$, $\sum'$ over the range $1, \ldots, n$.) Again we have $L(t_0, \ldots, t_n) = t_0 L\left(\frac{t_1}{t_0}, \ldots, \frac{t_n}{t_0}\right)$.

As an obvious generalization of Sec. 2, (2), (3) and (4), we now have

$$t_i p_i x_i^{p_i - 1} = \lambda \quad (i = 0, 1, \ldots, n), \tag{2}$$

$$\sum \left(\frac{\lambda}{t_i p_i}\right)^{1/(p_i - 1)} = 1, \tag{3}$$

$$L(t_0, \ldots, t_n) = \lambda \sum \frac{x_i}{p_i}. \tag{4}$$

Also $L(t_0, \ldots, t_n)$ can be realized as the Legendre transform of a convex hypersurface with the equation in non-homogeneous parametric form $a_0 = \varphi(a_1, \ldots, a_n) = (1 - \sum' a_i^{1/p_i})^{p_0}$ or in homogeneous form $\sum a_i^{1/p_i} = 1$. Corresponding to Sec. 2, (6) and (7) we have the formulae

$$L(t_1, \ldots, t_n) = \varphi(a_1, \ldots, a_n) + \sum' t_i a_i, \quad t_i = -\frac{\partial \varphi}{\partial a_i} \quad (i = 1, \ldots, n) \tag{5}$$

and

$$a_i = \frac{\partial L}{\partial t_i} \quad (i = 1, \ldots, n). \tag{6}$$

As for regularity it follows readily from (2)–(4) that $L(t_0, \ldots, t_n)$ can be continued analytically to complex values of $t_0, \ldots, t_n$ but we have not even an idea of the size of the region of analyticity.

We can however say something on the behaviour of $L(t_0, \ldots, t_n)$ near a "boundary point". For example, consider the case $(t_0, \ldots, t_n)$ close to the point $(0, t_1^*, \ldots, t_n^*)$, where we assume that $t_i^* \neq 0$ $(i = 1, \ldots, n)$. Then from (3) we see that in the first approximation we must have $\lambda \approx t_0 p_0$. Put therefore $\lambda = t_0 p_0 \mu$. Then (3) takes the form

$$\mu^{1/(p_0 - 1)} + \sum' (t_0 p_0)^{1/(p_i - 1)} \left(\frac{\mu}{t_i p_i}\right)^{1/(p_i - 1)} = 1.$$

From this equation and the implicit function theorem we see that μ is an analytic function in the $2n$ arguments $t_0^{1/(p_i - 1)}$ and t_i $(i = 1, \ldots, n)$. In particular we then have a development

$$L(t_0, \ldots, t_n) = t_0 + \sum' c_i t_0^{1 + 1/(p_i - 1)} + \sum' c_{ik} t_0^{1 + 1/(p_i - 1) + 1/(p_k - 1)} + \ldots,$$

where the coefficients $c_i, c_{ik}, \ldots$ thus are analytic functions of $(t_i, \ldots, t_n)$ near $(t_1^*, \ldots, t_n^*)$. We have made no attempts to determine the coefficients. (Perhaps the reference [1] would be of interest here.) The case of a boundary point such as $(0, 0, t_2^*, \ldots, t_n^*)$ with $t_i^* \neq 0$ $(i = 2, \ldots, n)$ is much more complicated since we as a first approximation have to use the L function in *two* (homogeneous) variables.

Finally we consider the problem of evaluating integrals. The methods of Sec. 3 based on the Legendre transform in one form or other does not seem to generalize to higher dimensions. In particular, we know of no workable analogue of Sparr's formula Sec. 3, (2). (This formula expresses also the circumstance that the positive concave functions form a cone and that the extremal functions of that cone are essentially the functions $\min(\tau, t)$. In higher dimensions we still have a cone but *we do not know what are the extremal functions*.) But the approach based on the line integral of Sec. 3, (6) does generalize!

The higher dimensional analogue of that integral is

$$\int_H \frac{L(t_0, \ldots, t_n)}{t_0^{\theta_0} \ldots t_n^{\theta_n}} \sum (-1)^i \prod_{k \neq i} \frac{dt_k}{t_k}, \tag{7}$$

where H is a suitable hypersurface in the positive $(n+1)$-tant, the θ_i being parameters subject to the conditions $\sum \theta_i = 1$, $\theta_i > 0$ $(i = 0, 1, \ldots, n)$. (Integrals of this type are considered in Sparr [6] in the context of interpolation of several spaces.) Since the integrand in (7) is a closed n-form, we can choose H as we want. Thus taking $t_0 = 1$ we get the integral (analogous to the one of Sec. 3, (1))

$$\int \frac{L(t_1, \ldots, t_n)}{t_1^{\theta_1} \ldots t_n^{\theta_n}} \frac{dt_1}{t_1} \ldots \frac{dt_n}{t_n}. \tag{8}$$

But the natural choice is again the hypersurface $\lambda = 1$. Using (2)–(4) we then get the integral

$$\prod p_i^{\theta_i} \int \prod x_i^{(p_i - 1)\theta_i} \sum_j \frac{1}{p_j} x_j \sum_k (-1)^{n+k} \prod_{i \neq k} (p_i - 1) \frac{dx_i}{x_i}$$

which is again —apart from a common factor —the sum of $(n+1)^2$ integrals of the type

$$\frac{1}{p_j(p_k-1)}\int\prod x_i^{(p_i-1)\theta_i}x_j\prod_{i\neq k}\frac{dx_i}{x_i}.$$

We here readily recognize a beta function in $n+1$ arguments; at the ith position there sits $(p_i-1)\theta_i+\delta_{ij}+\delta_{ik}$. Using the available functional equations we can write this beta function as([4])

$$\frac{(p_j-1)\theta_j(p_k-1)\theta_k+\delta_{jk}(p_j-1)\theta_k}{\left(\sum(p_i-1)\theta_i+1\right)\sum(p_i-1)\theta_i}B\big((p_0-1)\theta_0,\ldots,(p_n-1)\theta_i\big).$$

Thus collecting everything and writing $P=\sum\theta_i p_i$ we get the end result

$$(9)\qquad \int\frac{L(t_1,\ldots,t_n)}{t_1^{\theta_1}\ldots t_n^{\theta_n}}\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n} = \frac{\prod p_i^{\theta_i}\prod(p_i-1)}{P(P-1)}B(p_0-1)\theta_0,\ldots,(p_n-1)\theta_n.$$

We see that (9) reduces to Sec. 3, (1′) if $n=1$. By contrast there does not seem to be any direct analogue of Sec. 3, (1). This again seems to reflect the circumstance that the approaches based on the Legendre transform do not generalize, at least not that easily.

In the same way we can bundle the obvious generalizations of (7) or (8), for instance the case when $x_i^{p_i}$ in definition (1) is replaced by a more general convex function $F(x_i)$. But since we have not obtained any particularly nice end formula we refrain from entering into the details.

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([4]) We set
$$B(a_0,\ldots,a_n)=\int\left(1-\sum{}' x_i\right)^{a_0-1}\prod{}' x_i^{a_i-1}\prod{}' dx_i;$$
as in the case $n=1$ one can show that
$$B(a_0,\ldots,a_n)=\frac{\Gamma(a_0)\ldots\Gamma(a_n)}{\Gamma(a_0+\ldots+a_n)}.$$

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Received October 20, 1980 (1646)