PROBLEM 3. Let
(a) $f, g \in \mathscr{D}'$ and the Δ_m-product $[f \cdot g]$ exist in $\mathscr{D}'$,
(b) $f, g \in \mathscr{S}'$ and the Δ_m-product $[f \cdot g]$ exist in $\mathscr{S}'$.
Does exist then
(a) the Δ-product $[f \cdot g]$ in $\mathscr{D}'$,
(b) the Δ^s-product (or Δ-product, or Δ_m^s-product) $[f \cdot g]$ in $\mathscr{S}'$?

PROBLEM 4. Let $f, g \in \mathscr{S}'$ and let the E_m^s-convolution $[f * g]$ exist in $\mathscr{S}'$. Does exist then the Δ^s-product (or Δ-product) $[\mathscr{F}(f) \cdot \mathscr{F}(g)]$ in $\mathscr{S}'$ or in $\mathscr{D}'$?

References

[1] P. Antosik, J. Mikusiński, R. Sikorski, *Theory of distributions. The sequential approach*, Elsevier–PWN, Amsterdam–Warszawa 1973.
[2] C. Chevalley, *Theory of distributions*, Lectures at Columbia University in 1950–1951.
[3] P. Dierolf, J. Voigt, *Convolution and $\mathscr{S}'$-convolution of distributions*, Collect. Math. 29 (3) (1978), 185–196.
[4] Y. Hirata, H. Ogata, *On the exchange formula for distributions*, J. Sci. Hiroshima Univ. Ser. A 22 (1958), 147–152.
[5] M. Itano, *On the multiplicative products of distributions*, ibid. 29 (1965), 51–74.
[6] A. Kamiński, *On the exchange formula*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 26 (1978), 19–24.
[7] — *Remarks on delta- and unit-sequences*, ibid. 26 (1978), 25–30.
[8] — *Operations on distributions*, [to appear].
[9] S. Łojasiewicz, *Sur la valeur et la limite d'une distribution dans un point*, Studia Math. 16 (1957), 1–36.
[10] J. Mikusiński, *Irregular operations on distributions*, ibid. 20 (1961), 163–169.
[11] — *Criteria of the existence and of the associativity of the product of distributions*, ibid. 21 (1962), 253–259.
[12] — *On the square of the Dirac delta-distribution*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 14 (1966), 511–513.
[13] L. Schwartz, *Produits tensoriels topologiques d'espaces vectoriels topologiques. Espaces vectoriels topologiques nucléaires. Applications*, Séminaire Schwartz, Année 1953–54, Paris 1954.
[14] R. Shiraishi, *On the definition of convolutions for distributions*, J. Sci. Hiroshima Univ. Ser. A 23 (1959), 19–32.
[15] R. Shiraishi, M. Itano, *On the multiplicative products of distributions*, ibid. 28 (1964), 223–235.
[16] V. S. Vladimirov, *The equations of the Mathematical Physics* [in Russian], Nauka, Moscow 1967.

Received January 20, 1981
Revised version March 11, 1981 (1661)

STUDIA MATHEMATICA, T. LXXIV. (1982)

Non-removable ideals in commutative Banach algebras

by

V. MÜLLER (Praha)

Abstract. We show that an ideal I in a commutative Banach algebra with unit is non-removable if and only if it consists of joint topological divisors of zero. This gives the positive answer to the conjecture of Arens and Żelazko. From this it follows also that any finite family of removable ideals is removable.

Introduction. All algebras considered in this paper are assumed to be commutative complex Banach algebras with unit. However, some of these properties (complexity and completeness) are not essential.

We say that an ideal I in a commutative Banach algebra A is *removable* if there exists a superalgebra $B \supset A$ (i.e. B is a commutative Banach algebra and there is an isometric isomorphism $f\colon A \to B$ preserving the unit) such that I is not contained in a proper ideal in B. A family $\{I_j\}_{j \in J}$ of ideals in A is called *removable* if there is a superalgebra $B \supset A$ such that, for each $j \in J$, I_j is not contained in a proper ideal in B. An ideal which is not removable is said to be *non-removable*.

These notions were introduced by Arens in [1] where the following question was also presented: Is every (every finite) family of removable ideals removable?

Removability of ideals was further studied by Arens [2], Żelazko [8], [9], [10] and Bollobás [3]. Bollobás exhibited an example of a non-countable family of removable ideals which is not removable.

W. Żelazko introduced the following definition: We say that an ideal $I \subset A$ *consists of joint topological divisors of zero* if $\inf_{\substack{x \in A \\ |x|=1}} \sum_{i=1}^{n} |s_i x| = 0$ for every finite family $s_1, \ldots, s_n \in I$. We denote this shortly $I \in \mathfrak{l}(A)$.

It is easy to see that if $I \in \mathfrak{l}(A)$ then it is non-removable. Żelazko [7], [9] conjectured that the converse statement is also true, i.e. that I is non-removable if and only if $I \in \mathfrak{l}(A)$. However, the same question was presented (in an equivalent formulation) in the original paper of Arens [1].

The answer has been known in some special cases. In the case of principal ideals the conjecture turns into the theorem of Arens: An element (in a commutative Banach algebra) is permanently singular if and only if it is a topological divisor of zero. In [10] it was proved that $I \in l(A)$ implies that I can be removed in some locally convex extension of A.

We intend to improve these results and to give a positive answer to the Arens–Żelazko conjecture in general. As easy consequences this yields that every finite family of removable ideals is removable and it also gives positive answers to several other questions of Żelazko [9] and Arens [2].

The paper is divided into three sections. In the first section some combinatorial identities and estimates needed later are proved. In Section II the main theorem—an equivalent formulation of the Arens–Żelazko conjecture—is proved. Some consequences of it are given in Section III.

I. Denote by $\boldsymbol{N}$ the set of non-negative integers. Let $n \geqslant 1$ and $\boldsymbol{l}, \boldsymbol{j} \in \boldsymbol{N}^n$, $\boldsymbol{l} = (l_1, \ldots, l_n)$, $\boldsymbol{j} = (j_1, \ldots, j_n)$. We shall use the notation $|\boldsymbol{l}| = \sum_{t=1}^{n} l_t$ and $\boldsymbol{l} \leqslant \boldsymbol{j}$ if $l_t \leqslant j_t$ for every $t = 1, \ldots, n$.

For $m \geqslant 1$ and $\boldsymbol{l} = (l_1, \ldots, l_n) \in \boldsymbol{N}^n$ $(n \geqslant 1)$, define the numbers

$$a_{m,\boldsymbol{l}} = \binom{|\boldsymbol{l}|+m-1}{m-1} \frac{|\boldsymbol{l}|!}{l_1! \ldots l_n!}.$$

Notice that

$$a_{m,\boldsymbol{l}} = \binom{|\boldsymbol{l}|+m-1}{m-1} \cdot a_{1,\boldsymbol{l}}.$$

We shall prove several lemmas concerning these numbers which will be used in the following section.

LEMMA 1. $a_{1,\boldsymbol{l}} = \sum_{\substack{\boldsymbol{j} \leqslant \boldsymbol{l} \\ |\boldsymbol{j}| = |\boldsymbol{l}|-1}} a_{1,\boldsymbol{j}}$ *for* $\boldsymbol{l} \in \boldsymbol{N}^n$, $|\boldsymbol{l}| \geqslant 1$.

Proof. Denote

$$R = \sum_{\substack{\boldsymbol{j} \leqslant \boldsymbol{l} \\ |\boldsymbol{j}| = |\boldsymbol{l}|-1}} a_{1,\boldsymbol{j}} = \sum_{\substack{1 \leqslant r \leqslant n \\ l_r \neq 0}} \frac{(|\boldsymbol{l}|-1)!}{l_1! \ldots (l_r-1)! \ldots l_n!}.$$

Then $l_1! \ldots l_n! R = (|\boldsymbol{l}|-1)! \cdot \sum_{\substack{1 \leqslant r \leqslant n \\ l_r \neq 0}} l_r = |\boldsymbol{l}|! = a_{1,\boldsymbol{l}} \cdot l_1! \ldots l_n!$, hence $a_{1,\boldsymbol{l}} = R$.

LEMMA 2. $\sum_{\substack{\boldsymbol{l} \in \boldsymbol{N}^n \\ |\boldsymbol{l}| = s}} a_{1,\boldsymbol{l}} = n^s$ *for* $s \in \boldsymbol{N}$.

Proof. The proof follows from the identity $(x_1 + \ldots + x_n)^s = \sum_{\substack{\boldsymbol{l} \in \boldsymbol{N}^n \\ |\boldsymbol{l}| = s}} a_{1,\boldsymbol{l}} x_1^{l_1} \ldots x_n^{l_n}$ by putting $x_1 = \ldots = x_n = 1$.

LEMMA 3. $\sum_{m=1}^{k} \sum_{\substack{\boldsymbol{l} \in \boldsymbol{N}^n \\ |\boldsymbol{l}| \leqslant k}} a_{m,\boldsymbol{l}} \leqslant 8^k n^k$ *for* $k \in \boldsymbol{N}$.

Proof. We have

$$\sum_{m=1}^{k} \sum_{|\boldsymbol{l}| \leqslant k} a_{m,\boldsymbol{l}} = \sum_{m=1}^{k} \sum_{|\boldsymbol{l}| \leqslant k} \binom{|\boldsymbol{l}|+m-1}{m-1} \cdot a_{1,\boldsymbol{l}} \leqslant \sum_{|\boldsymbol{l}| \leqslant k} a_{1,\boldsymbol{l}} \cdot \sum_{m=1}^{k} 2^{|\boldsymbol{l}|+m-1}$$

$$\leqslant 2^{2k} \sum_{s=0}^{k} \sum_{|\boldsymbol{l}|=s} a_{1,\boldsymbol{l}} = 4^k \sum_{s=0}^{k} n^s \leqslant 4^k (k+1) n^k \leqslant 8^k n^k.$$

LEMMA 4. $a_{m,\boldsymbol{l}} = \sum_{\boldsymbol{j} \leqslant \boldsymbol{l}} a_{m-1,\boldsymbol{j}} \cdot a_{1,\boldsymbol{l}-\boldsymbol{j}}$ *for* $m \geqslant 2$, $\boldsymbol{l} \in \boldsymbol{N}^n$.

Proof. $a_{1,\boldsymbol{l}}$ is the number of ways how to order n elements $x_1, \ldots, x_n$ into a sequence of length $|\boldsymbol{l}|$ in which every element x_i occurs exactly l_i times (permutations with repetition). $a_{m,\boldsymbol{l}}$ is the number of ways how to divide these sequences into m (possibly empty) subsequences (combinations with repetition), i.e. in how many ways it is possible to form m sequences $s_1, \ldots, s_m$ from elements $x_1, \ldots, x_n$ such that x_i occurs in them altogether l_i times (s_1, s_2 and s_2, s_1 are counted two times).

The right hand side of the considered equality is the same number obtained in the other way: for $\boldsymbol{j} \leqslant \boldsymbol{l}$, $a_{m-1,\boldsymbol{j}} \cdot a_{1,\boldsymbol{l}-\boldsymbol{j}}$ is the number of ways how to form those m subsequences in such a way that x_i occurs in the initial $m-1$ subsequences exactly j_i times (for each i).

II. Let A be a commutative Banach algebra with unit, $n \geqslant 1$, $v, u_1, \ldots, u_n \in A$. As in [11] we say that v is *dominated by* $u_1, \ldots, u_n$ and write $v \prec u_1, \ldots, u_n$ if there exists a constant $K \geqslant 0$ such that $|vx| \leqslant K \cdot \sum_{i=1}^{n} |u_i x|$ for each $x \in A$.

THEOREM 1. *Let A be a commutative Banach algebra with unit* 1, $n \geqslant 1$, $u_1, \ldots, u_n \in A$, $|u_1| = \ldots = |u_n| = 1$ *and* $1 \prec u_1, \ldots, u_n$. *Then there exists a commutative Banach algebra $B \supset A$ and $b_1, \ldots, b_n \in B$ such that* $1 = \sum_{i=1}^{n} u_i b_i$.

Proof. We may assume $n \geqslant 2$ as for $n = 1$ Theorem 1 is the result of Arens [1]. (The proof of Theorem 1 is much simpler in the case $n = 2$ than in general; in this case Section I is reduced to the well-known properties of binomial coefficients.)

Put $R = 128 K^{4n} n^{4n+2}$, where K is the constant from the definition of domination ($|x| \leqslant K \cdot \sum_{i=1}^{n} |u_i x|$).

Let C be the l^1 algebra over A and adjoined elements $b_1, \ldots, b_n$ such that $|b_1| = \ldots = |b_n| = R$, i.e. elements of C are of the form $x = \sum_{\boldsymbol{i} \in \boldsymbol{N}^n} a_{\boldsymbol{i}} \boldsymbol{b}^{\boldsymbol{i}}$ such that $|x| = \sum_{\boldsymbol{i} \in \boldsymbol{N}^n} |a_{\boldsymbol{i}}| R^{|\boldsymbol{i}|} < \infty$ (where $a_{\boldsymbol{i}} \in A$; $\boldsymbol{b}^{\boldsymbol{i}}$ stands for $b_1^{i_1} \ldots b_n^{i_n}$).

Multiplication in A is defined by

$$\Big(\sum_{i\in N^n} a_i b^i\Big)\Big(\sum_{j\in N^n} a_j b^j\Big) = \sum_{k\in N^n}\Big(\sum_{i+j=k} a_i a_j\Big) b^k.$$

Let I be the closed ideal in C generated by $z = 1-\sum_{t=1}^{n} u_t b_t$. Denote $B = C/I$. Obviously we have $1 = \sum_{t=1}^{n} \bar{u}_t \bar{b}_t$ in B (where $\bar{x} = x+I$ for $x \in C$). It is sufficient to prove that A is a subalgebra of B, i.e. $|a|_A = |\bar{a}|_B$ for every $a \in A$.

Let $a \in A$. Then

$$|\bar{a}|_B = \inf_{x\in C} |a+zx|_C = \inf_{a_i\in A} \Big|a+z\sum_{i\in N^n} a_i b^i\Big|_C.$$

So we are to prove $|a+z\sum_{a_i\in A} a_i b^i|_C \geqslant |a|_A$ for every choice of $a_i \in A$ $(i \in N^n)$ such that $\sum_{i\in N^n} a_i b^i \in C$. Obviously it is sufficient to prove the last inequality in the case that only finite number of a_i's are non-zero. For such a_i's, we have

$$\Big|a+z\sum_{i\in N^n} a_i b^i\Big|_C = \Big|a+\Big(1-\sum_{t=1}^{n} b_t u_t\Big)\sum_{i\in N^n} a_i b^i\Big|_C$$
$$= \Big|a+a_0+\sum_{|i|\geqslant 1} b^i f_i\Big|_C = |a+a_0| + \sum_{|i|\geqslant 1} R^{|i|}|f_i|$$
$$\geqslant |a|-|a_0| + \sum_{|i|\geqslant 1} R^{|i|}|f_i|$$

where

$$f_i = a_{i_1\ldots i_n} - \sum_{\substack{1\leqslant r\leqslant n\\ i_r\neq 0}} a_{i_1\ldots i_r-1\ldots i_n} u_r \qquad (|i|\geqslant 1).$$

So it is sufficient to show $|a_0| \leqslant \sum_{|i|\geqslant 1} R^{|i|}|f_i|$. We may assume $|a_0| = 1$.

Suppose on the contrary that there exist elements $a_i \in A$, $i \in N^n$ only finite number of them being non-zero, $|a_0| = 1$ and $\sum_{|i|\geqslant 1} R^{|i|}|f_i| < 1$. Then $|f_i| \leqslant R^{-|i|}$.

We shall need the following lemma:

LEMMA 5. *Let* $i \in N^{n-1}$, $i_n \in N$, $|i|+i_n = k \geqslant 1$. *Then*

$$|a_{i,i_n} u_n^{2k}| \leqslant \sum_{l\leqslant i} \alpha_{k,l} |a_{i-l,i_n+|l|+k}| + \sum_{m=1}^{k}\sum_{l\leqslant i} \alpha_{m,l} R^{-(k+m)}$$

(*where* $\alpha_{m,l}$ *are the numbers defined in the previous section*).

Proof. In order to simplify the notation we write $u = (u_1, \ldots, u_{n-1})$ and, for $j \in N^{n-1}$, $j \leqslant i$, $j_n \leqslant i_n+k$ and $|j|+j_n \geqslant 1$,

$$d_{j,j_n} = a_{j,j_n} u^{i-j} u_n^{2k+i_n-j_n},$$
$$g_{j,j_n} = f_{j,j_n} u^{i-j} u_n^{2k+i_n-j_n}.$$

Then

(1)
$$g_{j,j_n} = d_{j,j_n} - \sum_{\substack{r\leqslant j,\ r_n\leqslant j_n\\ |r|+r_n=|j|+j_n-1}} d_{r,r_n}.$$

The following relation holds

(2)
$$d_{j,j_n} = \sum_{l\leqslant j} (-1)^{|l|}\alpha_{1,l} d_{j-l,j_n+|l|+1} + \sum_{l\leqslant j} (-1)^{|l|+1}\alpha_{1,l} g_{j-l,j_n+|l|+1}.$$

Indeed, substitute (1) into the right side. Then, for $r \leqslant j$, the coefficient at $d_{j-r,j_n+|r|+1}$ equals to zero evidently; the coefficient at $d_{j-r,j_n+|r|}$ is equal to

$$-(-1)^{|r|+1}\alpha_{1,r} - (-1)^{|r|}\sum_{\substack{r'\leqslant r\\ |r'|=|r|-1}} \alpha_{1,r'} = (-1)^{|r|}\Big[\alpha_{1,r} - \sum_{\substack{r'\leqslant r\\ |r'|=|r|+1}} \alpha_{1,r'}\Big].$$

However, the last term is equal to 0 for $r \neq 0$ by Lemma 1 and to 1 for $r = 0$.

From (2) we get

(3)
$$|d_{j,j_n}| \leqslant \sum_{l\leqslant j} \alpha_{1,l} |d_{j-l,j_n+|l|+1}| + \sum_{l\leqslant j} \alpha_{1,l} R^{-(|j|+j_n+1)}.$$

Now we shall prove

(4)
$$|d_{i,i_n}| \leqslant \sum_{l\leqslant i} \alpha_{m,l} |d_{i-l,i_n+|l|+m}| + \sum_{m'=1}^{m}\sum_{l\leqslant i} \alpha_{m',l} R^{-(k+m')}$$

for $m = 1, 2, \ldots, k$.

For $m = 1$, the statement follows from (3). Suppose (4) holds for some $m < k$, and prove it for $m+1$. By (3) we have

$$|d_{i-l,i_n+|l|+m}| \leqslant \sum_{l'\leqslant i-l} \alpha_{1,l'} |d_{i-l-l',i_n+|l|+|l'|+m+1}| + \sum_{l'\leqslant i-l} \alpha_{1,l'} R^{-(k+m+1)}$$

for each $l \leqslant i$.

By substitution into the induction assumption (4) we get

$$|d_{i,i_n}| \leqslant \sum_{l\leqslant i} \beta_{m+1,l} |d_{i-l,i_n+|l|+1+m}| + \sum_{m'=1}^{m}\sum_{l\leqslant i} \alpha_{m',l} R^{-(k+m')} + \sum_{l\leqslant i} \beta_{m+1,l} R^{-(k+m+1)}$$

where

$$\beta_{m+1,l} = \sum_{l'\leqslant l} \alpha_{m,l'}\alpha_{1,l-l'} = \alpha_{m+1,l}$$

by Lemma 4. This completes the induction step and proves relation (4). Relation (4) for $m = k$ implies easily the statement of Lemma 5.

Proof of Theorem 1. Denote (for $k = 1, 2, \ldots$)

$$s_k = \max\{|a_{\boldsymbol{i}}|,\ \boldsymbol{i} \in \boldsymbol{N}^n,\ |\boldsymbol{i}| = k\},$$

$$s'_k = \max\{|a_{\boldsymbol{i}}\boldsymbol{u}^{\boldsymbol{j}}|,\ \boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{N}^n,\ |\boldsymbol{i}| = k,\ |\boldsymbol{j}| = 2nk\}$$

where $\boldsymbol{u} = (u_1, \ldots, u_n)$.

Let $\boldsymbol{i} \in \boldsymbol{N}^n$, $|\boldsymbol{i}| = k$. Using the domination property $|x| \leqslant K \sum_{t=1}^{n} |u_t x|$ ($x \in A$) we can prove easily by induction $|a_{\boldsymbol{i}}| \leqslant K^{2nk} \sum_{\substack{\boldsymbol{j} \in \boldsymbol{N}^n \\ |\boldsymbol{j}| = 2nk}} \alpha_{1,\boldsymbol{j}} |a_{\boldsymbol{i}}\boldsymbol{u}^{\boldsymbol{j}}|$, hence

$$s_k \leqslant K^{2nk} n^{2nk} s'_k. \tag{5}$$

Lemma 5 implies

$$|a_{\boldsymbol{i}} u_n^{2k}| \leqslant \sum_{\substack{\boldsymbol{l} \in \boldsymbol{N}^{n-1} \\ |\boldsymbol{l}| \leqslant k}} \alpha_{k,\boldsymbol{l}} s_{2k} + \sum_{m=1}^{k} \sum_{\substack{\boldsymbol{l} \in \boldsymbol{N}^{n-1} \\ |\boldsymbol{l}| \leqslant k}} \alpha_{m,\boldsymbol{l}} R^{-(k+m)} \leqslant 8^k (n-1)^k s_{2k} + $$
$$+ 8^k (n-1)^k R^{-(k+1)} \leqslant 8^k n^k s_{2k} + 8^k n^k R^{-(k+1)}.$$

As the situation is symmetric in the indices, the same estimate holds also for $|a_{\boldsymbol{i}} u_t^{2k}|$, $t = 1, 2, \ldots, n$. Let $\boldsymbol{j} \in \boldsymbol{N}^n$, $|\boldsymbol{j}| = 2nk$. Then $j_t \geqslant 2k$ for some t and $|a_{\boldsymbol{i}}\boldsymbol{u}^{\boldsymbol{j}}| \leqslant 8^k n^k s_{2k} + 8^k n^k R^{-(k+1)}$ hence $s'_k \leqslant 8^k n^k s_{2k} + 8^k n^k R^{-(k+1)}$. Together with (5) this gives

$$s_k \leqslant K^{2nk} n^{2nk} 8^k n^k s_{2k} + K^{2nk} n^{2nk} 8^k n^k R^{-(k+1)} = R_1^k s_{2k} + R_1^k R^{-(k+1)}$$

where $R_1 = K^{2n} n^{2n} 8n$. For $k = 1$, we have

$$s_1 \leqslant R_1 R^{-2} + R_1 s_2 \leqslant R_1 R^{-2} + R_1 R_1^2 R^{-3} + R_1 R_1^2 s_4 \leqslant \ldots$$
$$\leqslant \sum_{r'=1}^{r} R_1^{2^{r'}-1} R^{-2^{r'-1}-1} + R_1^{2^r-1} \cdot s_{2^r}.$$

As $s_{2^r} = 0$ for r large enough (only finite number of $a_{\boldsymbol{i}}$'s are non-zero) and $R = 2R_1^2$

$$s_1 \leqslant R_1^{-1} R^{-1} \sum_{r'=1}^{\infty} (R_1^2 R^{-1})^{2^{r'-1}} \leqslant R_1^{-1} R^{-1} \sum_{p=0}^{\infty} (R_1^2 R^{-1})^p$$
$$= R_1^{-1} R^{-1} \sum_{p=0}^{\infty} 2^{-p} = 2R_1^{-1} R^{-1}.$$

Hence

$$1 = |a_{\boldsymbol{0}}| \leqslant K \sum_{t=1}^{n} |a_{\boldsymbol{0}} u_t| \leqslant K \sum_{t=1}^{n} (|a_{\boldsymbol{0}} u_t - a_{\underbrace{0\ldots}_{t-1}1\ldots0}| + |a_{\underbrace{0\ldots}_{t-1}1\ldots0}|)$$
$$\leqslant KnR^{-1} + Kns_1 < 1,$$

a contradiction.

III. Theorem 2. *Let A be a commutative Banach algebra with unit, $I \subset A$ an ideal. Then I is non-removable if and only if I consists of joint topological divisors of zero.*

Proof. $\Leftarrow$: Suppose there exists a superalgebra $B \supset A$, $n \in \boldsymbol{N}$ and $b_1, \ldots, b_n \in B$, $i_1, \ldots, i_n \in I$ such that $1 = \sum_{j=1}^{n} b_j i_j$. Then $|x| = |\sum_{j=1}^{n} x b_j i_j| \leqslant \max_{1 \leqslant j \leqslant n} |b_j| \cdot \sum_{j=1}^{n} |x i_j|$ for every $x \in A$ and so the ideal I does not consist of joint topological divisors of zero.

$\Leftarrow$: Let $I \notin \mathfrak{l}(A)$, i.e. there exist $i_1, \ldots, i_n \in I$ such that $\sum_{j=1}^{n} |x i_j| \geqslant \varepsilon \cdot |x|$ for some $\varepsilon > 0$ and for every $x \in A$. By Theorem 1 there exist a superalgebra $B \supset A$ and $b_1, \ldots, b_n \in B$ such that $\sum_{j=1}^{n} b_j i_j = 1$, hence the ideal I is removable.

The set of all maximal ideals of a commutative Banach algebra A which are non-removable is called a *cortex* of A (it corresponds to the set of all multiplicative functionals on A which can be extended to any superalgebra $B \supset A$).

Corollary. *A maximal ideal I in a commutative Banach algebra A with unit belongs to the cortex of A if and only if it consists of joint topological divisors of zero.*

Theorem 3. *Every non-removable ideal is contained in some element of cortex* (see [8], Problem 1).

Proof. The proof follows from the theorem of Słodkowski ([6]): every ideal $I \in \mathfrak{l}(A)$ is contained in some maximal ideal $J \in \mathfrak{l}(A)$.

Theorem 4. *A finite family of removable ideals is removable.*

Proof. Let $I_1, \ldots, I_k$ be a set of removable ideals. Then in I_j ($j = 1, \ldots, k$) there exist elements $a_{j,1}, \ldots, a_{j,n_j}$ such that $1 \prec a_{j,1}, \ldots, a_{j,n_j}$. Let

$$S = \Big\{s_{\boldsymbol{t}},\ \boldsymbol{t} \in \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \times \ldots \times \{1, \ldots, n_k\} = T,\ s_{\boldsymbol{t}} = \prod_{j=1}^{k} a_{j,t_j}\Big\}.$$

It is easy to prove by induction on k that $1 \prec S$. By Theorem 1 there exist a superalgebra $B \supset A$ and $b_{\boldsymbol{t}}$, $\boldsymbol{t} \in T$ such that $1 = \sum_{\boldsymbol{t} \in T} b_{\boldsymbol{t}} s_{\boldsymbol{t}}$. This means that $I_j \supset S$ is removed in B for $j = 1, \ldots, k$.

Remark 1. Arens [2] calls a finite set $x_1, \ldots, x_n \in A$ *subregular* if there are a superalgebra $B \supset A$ and $b_1, \ldots, b_n \in B$ such that $1 = \sum_{i=1}^{n} b_i x_i$. He also asks whether the product of two subregular systems is again subregular. The proof of Theorem 4 gives the positive answer to this question.

Remark 2. For $n = 2$ Theorem 1 says that if $u_1, u_2 \in A$, $|u_1| = |u_2|$

$= 1$ and $|x| \leqslant |u_1 x| + |u_2 x|$ then the superalgebra B and $b_1, b_2 \in B$ can be chosen so that $|b_1|, |b_2| \leqslant 2^{17}$. A natural question arises what are the least norms of b_1, b_2 in general. The construction giving $|b_1|, |b_2| \leqslant 2^{17}$ can be easily improved (we do not use the best estimates in the proof of Theorem 1). On the other hand, it is not possible to find $B, b_1, b_2 \in B$ in general such that $|b_1| = |b_2| = 1$ as was shown by Bollobás [4].

References

[1] R. Arens, *Extensions of Banach algebras*, Pacific J. Math. 10 (1960), 1–16.
[2] — *Ideals in Banach algebra extensions*, Studia Math. 31 (1968), 29–34.
[3] B. Bollobás, *Adjoining inverses to commutative Banach algebras*, Trans. Amer. Math. Soc. 181 (1973), 165–179.
[4] — *Normally subregular systems in normed algebras*, Studia Math. 49 (1974), 263–266.
[5] V. Müller, *On domination and extensions of Banach algebras*, ibid. 73 (1982), 75–80.
[6] Z. Słodkowski, *On ideals consisting of joint topological divisors of zero*, ibid. 48 (1973), 83–88.
[7] W. Żelazko, *Banach algebras*, Elsevier–PWN, 1973.
[8] — *Concerning a problem of Arens on removable ideals in Banach algebras*, Collect. Math. 30 (1974), 127–131.
[9] — *On some classes of ideals in commutative Banach algebras*, Rend. Sem. Mat. Fis. Milano 48 (in print).
[10] — *A characterization of* LC-*non-removable ideals in commutative Banach algebras*, Pacific J. Math. 87 (1) (1981), 241–247.
[11] — *On domination and separation of ideals in commutative Banach algebras*, Studia Math. 71 (1981), 179–189.

MATHEMATICAL INSTITUTE ČSAV
Žitná 25, 115 67 Praha 1

Received November 29, 1980
Revised version March 17, 1981 (1655)