STUDIA MATHEMATICA, T. LXXIV. (1982)

A dominated ergodic estimate for L_p spaces with weights

by

E. ATENCIA and A. DE LA TORRE (Malaga)

Abstract. In this note we characterize those positive functions w such that the ergodic maximal function associated to an invertible, measure preserving ergodic transformation on a probability space is a bounded operator in $L_p(wd\mu)$.

1. Introduction. Let $(X, \mathfrak{F}, \mu)$ be a non-atomic probability space and let $T\colon X \to X$ be an ergodic, invertible measure preserving transformation.

For each pair of non negative integers n, m we define the operator $T_{n,m}$, acting on measurable functions, as

$$T_{n,m}f(x) = (n+m+1)^{-1}\sum_{i=-n}^{m}|f(T^i x)|.$$

It is well known that in order to study the a.e. convergence of the averages $T_{n,m}$ it is enough to prove a Dominated Ergodic Estimate (D.E.E.) with respect to the measure μ, i.e. if

$$f^*(x) = \sup_{n,m\geqslant 0} T_{n,m}f(x),$$

then there exists a constant, namely $p/(p-1)$, such that

$$\|f^*\|_p \leqslant \frac{p}{p-1}\|f\|_p \quad \text{for all } f \in L_p(d\mu)$$

which certainly holds for $p > 1$ [8].

Our aim is to study the a.e. convergence of $T_{n,m}f$ but with respect to another measure $wd\mu$ where w is a positive integrable function. We are thus led to try and characterize those positive functions w such that the D.E.E. holds but with respect to the measure $wd\mu$. Let us fix $p > 1$. We will say that T *satisfies the* D.E.E. *with respect to the weight* w if

$$\int_X f^{*p} w d\mu \leqslant C_p \int_X |f|^p w d\mu \quad \text{for all } f \text{ in } L_p(w d\mu). \tag{1.1}$$

Our main result is given by the following:

THEOREM. *In the above situation* (1.1) *holds if and only if* w *satisfies the condition*:

(A'_p) *There exists a constant* M *such that for a.e.* x

$$k^{-1}\sum_{i=0}^{k-1} w(T^i x)\cdot\Big[k^{-1}\sum_{i=0}^{k-1}\big(w(T^i x)\big)^{-1/(p-1)}\Big]^{p-1}\leqslant M \tag{1.2}$$

for all positive integers k.

Condition A'_p is nothing but the condition in Theorem 10 of [2] with a constant independent of x and the natural analogue of Muckenhoupt's condition for the Hardy–Littlewood Maximal Operator [4].

Observe that if w satisfies A'_p, we have a D.E.E. for T as an operator from $L_p(wd\mu)$ into $L_p(wd\mu)$. T is obviously a positive operator but it is not, in general, a contraction in $L_p(wd\mu)$ and its powers do not form, in general, a uniformly bounded group of positive operators, i.e we obtain a D.E.E. for an operator T which, even though it separates supports, is not power bounded as in [3].

2. Main results. In this section we will prove our result using the ideas in [1] adapted to our situation.

Our main tool will be the idea of "ergodic rectangle".

DEFINITION. Let B be a subset of X with positive measure and k a positive integer such that

$$(T^iB\cap T^jB)=0,\quad i\neq j,\ 0\leqslant i,\ j\leqslant k-1.$$

Then the set $R=\bigcup_{i=0}^{k-1}T^iB$ will be called an (*ergodic*) *rectangle* with base B and length k.

Obviously $\mu(R)=k\mu(B)$.

In the proof of the theorem will be needed the following two results:

(2.1) PROPOSITION. *Let* k *be a positive integer and let* $A\subset X$ *be a subset of positive measure. Then there exists* $B\subset A$ *such that* B *is base of a rectangle of length* k.

(2.2) LEMMA. X *can be written as a countable union of bases of rectangles of length* k.

Proof of the proposition. First we will consider the case $k=2$. We may assume $\mu(A)<1$. Since T is ergodic, A is not invariant. If $\mu(A\cap TA)=0$, then we choose $B=A$ and we are done.

If $\mu(A\cap TA)>0$, then $\mu\big(A-(A\cap TA)\big)>0$ since otherwise A would be invariant. So now we pick $B=A-(A\cap TA)$; obviously $\mu(B)>0$, and $B\cap TB=\emptyset$.

The general case follows by applying the same method.

Proof of the lemma. Let $\mathfrak{F}_1=\{B\subset X,\ B$ is base of a rectangle of length $k\}$. Because of the proposition, $\mathfrak{F}_1$ is not empty. Let $\eta_1=\sup_{B\in\mathfrak{F}_1}\mu(B)$. Clearly $0<\eta_1\leqslant 1$. We pick $B^1\in\mathfrak{F}_1$, $\mu(B^1)>\eta_1/2$. Let

$$\mathfrak{F}_2=\{B\in\mathfrak{F}_1\colon B\cap B^1=\emptyset\},\qquad \eta_2=\sup_{B\in\mathfrak{F}_2}\mu(B),$$

and choose $B^2\in\mathfrak{F}_2$, $\mu(B^2)>\eta_2/2$. We proceed by induction and define

$$\mathfrak{F}_n=\Big\{B\in\mathfrak{F}_1\colon B\cap\Big(\bigcup_{i=1}^{n-1}B^i\Big)=\emptyset\Big\},\qquad \eta_n=\sup_{B\in\mathfrak{F}_n}\mu(B)$$

and choose $B^n\in\mathfrak{F}_n$, $\mu(B^n)>\eta_n/2$.

If for some n $\mathfrak{F}_n$ is empty, then $X=\bigcup_{i=1}^{n-1}B^i$ a.e. Indeed, if $X-\bigcup_{i=1}^{n-1}B^i=A$ and $\mu(A)>0$, then by the proposition there is $B\subset A$, $B\in\mathfrak{F}_1$, and obviously $B\cap\big(\bigcup_{i=1}^{n-1}B^i\big)=\emptyset$ against $\mathfrak{F}_n$ being empty.

If no $\mathfrak{F}_n$ is empty, we obtain an infinite pairwise disjoint sequence $B^1,B^2,\ldots,B^n,\ldots$ and we claim that

$$X=\bigcup_n B^n.$$

Let us prove it. First of all note that $\lim_n\mu(B^n)=0$ since the sets are disjoints and $\mu(X)$ is finite. If $X-\bigcup_n B^n=A$ and $\mu(A)>0$, we choose $B\in\mathfrak{F}_1$, $B\subset A$ $\mu(B)=\delta>0$. Then there is n_0 such that $\mu(B_{n_0})<\delta/3$ and observe that $B\in\mathfrak{F}_{n_0}$ which means $\eta_{n_0}\geqslant\delta$ so by the method of choosing B_{n_0} it should be $\mu(B_{n_0})>\delta/2$ against $\mu(B_{n_0})<\delta/3$.

Condition (1.1) *implies* w *satisfies* A'_p: Let k be a non-negative integer and let us fix a rectangle with base B and length k. For each integer n we consider the subset of B

$$B_n=\Big\{x\in B\colon 2^n\leqslant k^{-1}\sum_{i=0}^{k-1}w^{-1/(p-1)}(T^ix)<2^{n+1}\Big\}.$$

Clearly $B=\bigcup_{n\in\mathbf{Z}}B_n$.

Let us fix n and let A be an arbitrary measurable subset of B_n with $\mu(A)>0$. Let $\tilde{R}$ be the rectangle with base A and length k. From the definition of our maximal operator it is obvious that

$$(w^{-1/(p-1)}\chi_{\tilde{R}})^*(T^jx)\geqslant k^{-1}\sum_{i=0}^{k-1}w^{-1/(p-1)}(T^ix)\geqslant 2^n,\qquad x\in A,\ 0\leqslant j<k. \tag{2.3}$$

The last inequality on the right holds since $x \in A \subset B_n$. Raising to the power p, multiplying by $w(T^j x)$, and integrating over A we obtain

$$\int_A (w^{-1/(p-1)}\chi_{\tilde{R}})^{*p}(T^j x) w(T^j x)\,d\mu \geqslant 2^{np} \int_A w(T^j x)\,d\mu .$$

Adding up in j from 0 to $k-1$ and keeping in mind that $\mu(T^i A \cap T^j A) = 0$, $0 \leqslant i, j \leqslant k-1$, we have

$$2^{np} \int_{\tilde{R}} w(y)\,d\mu \leqslant \int_{\tilde{R}} (w^{-1/(p-1)}\chi_{\tilde{R}})^{*p}(y) w(y)\,d\mu .$$

But using (1.1) the last term is majorized by

$$C_p \int_X (w^{-1/(p-1)}(y))^p \chi_{\tilde{R}}(y) w(y)\,d\mu = C_p \int_{\tilde{R}} w^{-1/(p-1)}\,d\mu ,$$

i.e.

$$2^{np} \int_{\tilde{R}} w\,d\mu \leqslant C_p \int_{\tilde{R}} w^{-1/(p-1)}\,d\mu \tag{2.4}$$

or

$$2^{(n+1)p} \int_{\tilde{R}} w\,d\mu \Big(\int_{\tilde{R}} w^{-1/(p-1)}\,d\mu\Big)^{-1} \leqslant C_p 2^p . \tag{2.5}$$

On the other hand, we have

$$\mu(A)^{-1} \int_A k^{-1} \sum_{i=0}^{k-1} w^{-1/(p-1)}(T^i x)\,d\mu \leqslant 2^{n+1}$$

and raising to the power p and using (2.5) we get

$$\Big(\mu(A)^{-1} \int_A k^{-1} \sum_{i=0}^{k-1} w^{-1/(p-1)}(T^i x)\,d\mu\Big)^p \int_{\tilde{R}} w\,d\mu \Big(\int_{\tilde{R}} w^{-1/(p-1)}\,d\mu\Big)^{-1}$$
$$\leqslant 2^{(n+1)p} \int_{\tilde{R}} w\,d\mu \Big(\int_{\tilde{R}} w^{-1(p-1)}\,d\mu\Big)^{-1} \leqslant C_p 2^p$$

or

$$\Big(\mu(\tilde{R})^{-1} \int_{\tilde{R}} w^{-1/(p-1)}\,d\mu\Big)^p \int_{\tilde{R}} w\,d\mu \Big(\int_{\tilde{R}} w^{-1/(p-1)}\,d\mu\Big)^{-1} \leqslant C_p 2^p$$

which can be written as

$$\mu(\tilde{R})^{-1} \int_{\tilde{R}} w\,d\mu \Big(\mu(\tilde{R})^{-1} \int_{\tilde{R}} w^{-1/(p-1)}\,d\mu\Big)^{p-1} \leqslant C_p 2^p . \tag{A_p}$$

We call it A_p because it looks like condition A_p in [4] but with the special rectangles $\tilde{R}$ instead of the cubes of the classical case.

Write A_p as

$$\mu(A)^{-1} \int_A k^{-1} \sum_{i=0}^{k-1} w(T^i x)\,d\mu \Big(\mu(A)^{-1} \int_A k^{-1} \sum_{i=0}^{k-1} w^{-1/(p-1)}(T^i x)\,d\mu\Big)^{p-1}$$
$$\leqslant C_p 2^p .$$

Since this holds for every A, arbitrary measurable subset of positive measure of B_n, we easily obtain A_p' for almost all x in B_n. A straightforward application of the proposition and lemma gives A_p'.

Before proving the converse we will state some results that will be needed in the proof. These results are a discrete version of the Calderón–Zygmund decomposition [6] and of some results in [1]. The proof follows the same pattern as in [1] and we will include it only to make the article selfcontained.

Calderón–Zygmund decomposition. *Let us fix the integers* $0, 1, 2, \ldots, k-1$ *and let* λ *be a real number such that*

$$\lambda > k^{-1} \sum_{i=0}^{k-1} w(T^i x)$$

where x *is a fix point of* X. *Then for the set of integers* $0, 1, 2, \ldots, k-1$ *we can choose a (possibly empty) family of disjoint subsets* $I_1, \ldots, I_l$ *each of them made up of consecutive integers and such that the following holds:*

(a) *For each* I_i, $i = 1, \ldots, l$

$$\lambda < \frac{1}{|I_i|} \sum_{j \in I_i} w(T^j x) \leqslant 3\lambda$$

where $|I_i|$ *denotes the number of integers in* I_i.

(b) *If* $j \notin \bigcup_{i=1}^{l} I_i$, $0 \leqslant j \leqslant k-1$, *then* $w(T^j x) \leqslant \lambda$.

Proof. Let us call a set of consecutive integers an interval. Split $0, 1, \ldots, k-1$ into two disjoint intervals I_1, I_2 where $I_1 = 0, 1, \ldots, [(k-1)/2]$. Now consider

$$\frac{1}{|I_i|} \sum_{j \in I_i} w(T^j x), \qquad i = 1, 2 .$$

If this average is bigger than λ, we select this interval and we have

$$\frac{1}{|I_i|} \sum_{j \in I_i} w(T^j x) \leqslant \frac{k}{|I_i|} \frac{1}{k} \sum_{j=0}^{k-1} w(T^j x) \leqslant 3\lambda .$$

If this average is not bigger than λ, we repeat the process. This process will finish in a finite number of steps. The chosen intervals satisfy (a) and if an integer r is left out, then obviously

$$w(T^r x) \leqslant \lambda.$$

In what follows we will often use for the averages the notation established in the introduction. In particular remember that

$$k^{-1}\sum_{i=0}^{k-1} w(T^i x) = T_{0,k-1}w(x).$$

(2.6) LEMMA. *Let w satisfy A'_p; then there exist positive constants α, β depending only on the constant M of condition A'_p such that if*

$$E = \Big\{i:\ 0 \leqslant i \leqslant k-1:\ w(T^i x) > \beta k^{-1}\sum_{j=0}^{k-1} w(T^j x)\Big\},$$

then $\# E > \alpha k$ ($\# E$ is the number of integers in E).

Proof. Observe that for any positive β if $E' = \{0, 1, \ldots, k-1\} - E$, then

(2.7) $$\beta^{-1}(k^{-1}\# E')^{p-1} \leqslant T_{0,k-1}w(x)\Big(k^{-1}\sum_{i\in E'} w(T^i x)^{-1/(p-1)}\Big)^{p-1}$$

this is because in E' is $w(T^i x)^{-1} \geqslant (\beta T_{0,k-1}w(x))^{-1}$. But the last term in (2.7) is, obviously, dominated by

$$T_{0,k-1}w(x)\big(T_{0,k-1}w^{-1/(p-1)}(x)\big)^{p-1} \leqslant M \quad \text{since } A'_p.$$

Choose $\beta < M^{-1}$, $\alpha = 1-(M\beta)^{1/(p-1)}$ and the lemma is proved.

NOTE. *If instead of $0 \leqslant i \leqslant k-1$ we start with any other interval I, then we have*

$$\#\Big(\Big\{i \in I:\ w(T^i x) > \beta |I|^{-1}\sum_{j\in I} w(T^j x)\Big\}\Big) > \alpha |I|.$$

The Calderón–Zygmund decomposition and the preceding lemma allow us to prove, in our context, the "reverse Hölder inequality".

(2.8) LEMMA. *Let w satisfy A'_p, $1 < p < \infty$; then there exist positive constants C, δ such that*

$$\Big(k^{-1}\sum_{j=0}^{k-1}\big(w(T^j x)\big)^{1+\delta}\Big)^{1/(1+\delta)} \leqslant C\, k^{-1}\sum_{j=0}^{k-1} w(T^j x)$$

for every k and x.

Proof. Let λ be a positive number such that

$$\lambda > T_{0,k-1}w(x).$$

We want to estimate $\sum w(T^i x)$ extended to those i's, $0 \leqslant i \leqslant k$, where $w(T^i x) > \lambda$. Using the Calderón–Zygmund decomposition for this λ, we have a family of disjoint intervals I_j satisfying (a) and (b) of the said decomposition, so

$$A(\lambda) \equiv \{i:\ 0 \leqslant i < k:\ w(T^i x) > \lambda\} \subset \bigcup_j I_j.$$

Now

$$\begin{aligned}
\sum_{i\in A(\lambda)} w(T^i x) &\leqslant \sum_j \sum_{i\in I_j} w(T^i x) \leqslant \sum_j 3\lambda |I_j| \\
&\leqslant 3\lambda \sum_j \alpha^{-1}\#\Big\{h \in I_j:\ w(T^h x) > \beta |I_j|^{-1}\sum_{i\in I_j} w(T^i x)\Big\} \\
&\leqslant 3\lambda\alpha^{-1}\sum_j \#\{h \in I_j:\ w(T^h x) > \beta\lambda\} \\
&= 3\lambda\alpha^{-1}\#\{h \in \bigcup I_j:\ w(T^h x) > \beta\lambda\} \leqslant 3\lambda\alpha^{-1}\# A(\beta\lambda).
\end{aligned}$$

In other words, for any $\lambda > T_{0,k-1}w(x)$ we have

$$\sum_{i\in A(\lambda)} w(T^i x) \leqslant C\lambda\, \# A(\beta\lambda).$$

Multiplying by $\lambda^{\delta-1}$ ($\delta > 0$) and integrating we obtain

$$\begin{aligned}
\int_{T_{0,k-1}w(x)}^{\infty} \lambda^{\delta-1}\sum_{i\in A(\lambda)} w(T^i x)\, d\lambda &\leqslant C\int_{T_{0,k-1}w(x)}^{\infty} \lambda^{\delta}\,\# A(\beta\lambda)\, d\lambda \\
&\leqslant C\sum_{i=0}^{k-1}\int_0^{w(T^i x)/\beta} \lambda^{\delta}\, d\lambda \\
&= C(1+\delta)^{-1}\beta^{-\delta-1}\sum_{i=0}^{k-1}\big(w(T^i x)\big)^{1+\delta} \\
&= C'(1+\delta)^{-1}\sum_{i=0}^{k-1}\big(w(T^i x)\big)^{1+\delta}.
\end{aligned}$$

The first term of this inequality can be written as

(2.9) $$\sum_{i\in A(\mathfrak{I})} w(T^i x)\int_{T_{0,k-1}w(x)}^{w(T^i x)} \lambda^{\delta-1}\, d\lambda = \delta^{-1}\sum_{i\in A(\mathfrak{I})} w(T^i x)\big[\big(w(T^i x)\big)^{\delta} - \big(T_{0,k-1}w(x)\big)^{\delta}\big]$$

where $\mathfrak{I}$ in $A(\mathfrak{I})$ is $T_{0,k-1}w(x)$. Now if $0 \leqslant j \leqslant k-1$, $j \notin A(\mathfrak{I})$, then $w(T^j x))^{\delta} - (T_{0,k-1}w(x))^{\delta}$ is non-positive. Therefore the last term in (2.9) is not less than

$$\delta^{-1}\sum_{i=0}^{k-1} w(T^i x)\left[(w(T^i x))^{\delta}-(T_{0,k-1}w(x))^{\delta}\right];$$

so we obtain

$$\delta\left(\delta^{-1}-C'(1+\delta)^{-1}\right)k^{-1}\sum_{j=0}^{k-1}\left(w(T^j x)\right)^{1+\delta}\leqslant\left(k^{-1}\sum_{j=0}^{k-1}w(T^j x)\right)^{1+\delta}$$

and the lemma follows by choosing δ small enough to make

$$\delta^{-1}-C'(1+\delta)^{-1}>0.$$

(2.10) LEMMA. *Let w satisfy A'_p, then there exists $\varepsilon>0$ so that w satisfy $A'_{p-\varepsilon}$.*

Proof. Check first that if w satisfies A'_p, then $v=w^{-1/(p-1)}$ satisfies A'_q with $p^{-1}+q^{-1}=1$. Applying now the preceding lemma to v we have for some $\delta>0$

$$\left(T_{0,k-1}v^{1+\delta}(x)\right)^{1/(1+\delta)}\leqslant C\,T_{0,k-1}v(x);$$

replacing v by $w^{-1/(p-1)}$ and taking $\varepsilon=(p-1)\delta(1+\delta)^{-1}$ we have

$$k^{-1}\sum_{j=0}^{k-1}w(T^j x)\left(k^{-1}\sum_{j=0}^{k-1}\left(w(T^j x)\right)^{-1/(p-\varepsilon-1)}\right)^{p-\varepsilon-1}$$

$$\leqslant C^{p-1}k^{-1}\sum_{j=0}^{k-1}w(T^j x)\left(k^{-1}\sum_{j=0}^{k-1}\left(w(T^j x)\right)^{-1/(p-1)}\right)^{p-1}\leqslant C^{p-1}M.$$

The following maximal function appears in a natural way associated to the weight w

$$\mathfrak{M}_w f(x)=\sup_{n,m>0}\frac{\sum_{i=-n}^{m}|f(T^i x)|\,w(T^i x)}{\sum_{i=-n}^{m}w(T^i x)}.$$

As we will see this maximal function controls f^*. Indeed, if $p^{-1}+q^{-1}=1$, we have

$$(n+m+1)^{-1}\sum_{i=-n}^{m}|f(T^i x)|$$

$$=(n+m+1)^{-1}\sum_{i=-n}^{m}|f(T^i x)|\,w^{1/p}(T^i x)\,w^{-1/p}(T^i x)$$

$$\leqslant\left((n+m+1)^{-1}\sum_{-n}^{m}|f|^p(T^i x)\,w(T^i x)\right)^{1/p}\left((n+m+1)^{-1}\sum_{-n}^{m}w^{-q/p}(T^i x)\right)^{1/q}$$

$$\leqslant M\left(\frac{(n+m+1)^{-1}\sum_{i=-n}^{m}|f|^p(T^i x)\,w(T^i x)}{(n+m+1)^{-1}\sum_{i=-n}^{m}w(T^i x)}\right)^{1/p}\leqslant M\left(\mathfrak{M}_w f^p(x)\right)^{1/p}.$$

The next to the last inequality is because w satisfies A'_p. Taking sups over n and m, we obtain

$$f^*(x)\leqslant M(\mathfrak{M}_w f^p)^{1/p}. \tag{2.11}$$

Since w satisfies also A'_s for some s such that $1<s<p$, we also have

$$f^*(x)\leqslant M(\mathfrak{M}_w f^s)^{1/s}.$$

Observe that

$$\int_X (f^*(x))^p w(x)\,d\mu\leqslant M\int_X(\mathfrak{M}_w f^s(x))^{p/s}w(x)\,d\mu$$

where $p/s>1$; so if we prove that the maximal operator $\mathfrak{M}_w$ is bounded in $L_r(w\,d\mu)$ for all $r>1$ we will have

$$\int_X(\mathfrak{M}_w f^s)^{p/s}w\,d\mu\leqslant MC\int_X|f|^p w\,d\mu \tag{2.12}$$

and we will be done. Since $\mathfrak{M}_w$ is obviously bounded in L^∞, it will be enough to prove weak type (1,1) and use the Marcinkiewicz interpolation theorem.

(2.13) THEOREM. *The maximal operator with weight defined by*

$$\mathfrak{N}_w f(x)=\sup_{k>0}\frac{\sum_{i=0}^{k-1}f(T^i x)\,w(T^i x)}{\sum_{i=0}^{k-1}w(T^i x)} \tag{2.14}$$

is of weak type (1,1) *with respect to the measure $w\,d\mu$.*

Proof. We may assume f is non-negative. Let λ be a positive number bigger than

$$\nu=\frac{\int_X fw\,d\mu}{\int_X w\,d\mu}$$

and let

$$O_\lambda=\{x\in X\colon\ \mathfrak{N}_w f(x)>\lambda\}.$$

The set O_λ is, clearly, measurable. For any $x\in X$ we consider the orbit of x in O_λ, that we denote by J_x, i.e.

$$J_x=\{T^k x\in O_\lambda,\ k\in Z\}.$$

We associate, in a natural way, to the orbit of x in O_λ, J_x, the subset of the integers given by

$$\{k\colon\ T^k x\in O_\lambda\}$$

that we can express as a countable union of disjoint intervals $\bigcup_i I_i^x$.

Let is prove that, for almost all x, no I_i^x has infinite number of integers. The individual ergodic theorem tells us that

$$\lim_{k\to\infty} k^{-1}\sum_{i=0}^{k-1} f(T^i x)w(T^i x)\Big(k^{-1}\sum_{i=0}^{k-1} w(T^i x)\Big)^{-1} = \int_X fw\,d\mu\Big(\int_X w\,d\mu\Big)^{-1} \text{ a.e.}$$

If for some i $I_i^x = \{l, l+1, l+2, \ldots\}$, then, by the above-mentioned theorem, we have

$$\lim_{k\to\infty}\sum_{i=0}^{k-1} f(T^i T^l x)w(T^i T^l x)\Big(\sum_{i=0}^{k-1} w(T^i T^l x)\Big)^{-1} = \nu.$$

Thus, being $\lambda > \nu$, there exists a positive integer K such that

$$\sum_{i=0}^{k-1} f(T^i T^l x)w(T^i T^l x) < \lambda\sum_{i=0}^{k-1} w(T^i T^l x) \quad (k \geqslant K). \tag{2.15}$$

Clearly, by the definitions of $\mathfrak{N}_w$ and O_λ, there exists r verifying

$$\sum_{l}^{l+r} f(T^j x)w(T^j x) > \lambda\sum_{l}^{l+r} w(T^j x) \quad (T^l x \in O_\lambda)$$

where, by (2.15), $r < K$. Considering now $T^{l+r+1}(x)$ that belongs to O_λ there exists $r_1 \geqslant r+1$ such that

$$\sum_{l+r+1}^{l+r_1} f(T^j x)w(T^j x) > \lambda\sum_{l+r+1}^{l+r_1} w(T^j x)$$

and applying the same process we obtain a sum of the type

$$\sum_{l}^{s} f(T^j x)w(T^j x) > \lambda\sum_{l}^{s} w(T^j x)$$

where $s > K$, in contradiction with (2.15). Therefore, there are not intervals of infinite lenght.

Choose now an interval I_i^x, namely

$$I_i^x = \{l, l+1, \ldots, l+m\}$$

which means that $T^l x, \ldots, T^{l+m}x \in O_\lambda$ and $T^{l+m+1}x$, $T^{l-1}x \notin O_\lambda$. The aim is to prove

$$\sum_{j=0}^{m} f(T^{j+l}x)w(T^{j+l}x) \geqslant \lambda\sum_{j=0}^{m} w(T^{j+l}x). \tag{2.16}$$

If this were not the case, then for every positive integer r we would have

$$\sum_{j=0}^{m+r} f(T^{j+l}x)w(T^{j+l}x) = \sum_{j=0}^{m} f(T^{j+l}x)w(T^{j+l}x) + \sum_{j=m+1}^{m+r} f(T^{j+l}x)w(T^{j+l}x)$$
$$< \lambda\sum_{j=0}^{m} w(T^{j+l}x) + \lambda\sum_{j=m+1}^{m+r} w(T^{j+l}x)$$

where we have used that $T^{l+m+1}x \notin O_\lambda$, so for every positive integer r we would have

$$\sum_{j=0}^{m+r} f(T^{j+l}x)w(T^{j+l}x) < \lambda\sum_{j=0}^{m+r} w(T^{j+l}x).$$

On the other hand, for some h, clearly not bigger than m it should be

$$\sum_{j=0}^{h} f(T^{j+l}x)w(T^{j+l}x) > \lambda\sum_{j=0}^{h} w(T^{j+l}x);$$

if we call

$$\bar{h} = \max\Big\{h \leqslant m\colon \sum_{0}^{h} f(T^{j+l}x)w(T^{j+l}x) > \lambda\sum_{0}^{h} w(T^{j+l}x)\Big\},$$

then we claim that $\bar{h} = m$.

Suppose that $\bar{h} < m$. Then $T^{l+\bar{h}+1}x \in O_\lambda$, fact that implies the existence of a $t \geqslant \bar{h}+1$ such that

$$\sum_{\bar{h}+1}^{t} f(T^{j+l}x)w(T^{j+l}x) > \lambda\sum_{\bar{h}+1}^{t} w(T^{j+l}x), \tag{2.17}$$

and

$$\sum_{0}^{\bar{h}} f(T^{j+l}x)w(T^{j+l}x) > \lambda\sum_{0}^{\bar{h}} w(T^{j+l}x) \tag{2.18}$$

adding up (2.17) and (2.18) we obtain a contradiction with the assumption that $\bar{h}$ was the considered maximum. Therefore $\bar{h} = m$, and the inequality (2.16) follows.

Call now

$$B_i = \{x \in O_\lambda\colon x, \ldots, T^{i-1}x \in O_\lambda, T^i x \notin O_\lambda, T^{-1}x \notin O_\lambda\}.$$

B_i is clearly measurable.

Let R_i be defined by

$$R_i = B_i \cup TB_i \cup \ldots \cup T^{i-1}B_i.$$

It is obvious that $T^m B_i \cap T^n B_i = \emptyset$, $0 \leqslant n$, $m \leqslant i-1$, $n \neq m$ and that $O_\lambda = \bigcup_i R_i$.

Consider now $x \in B_i$:

$$\sum_{j=0}^{i-1} w(T^j x) \leqslant \lambda^{-1}\sum_{j=0}^{i-1} f(T^j x)w(T^j x).$$

Therefore

$$\int_{R_i} w\,d\mu \leqslant \lambda^{-1}\int_{R_i} fw\,d\mu$$

and summing over i we have

$$\int_{O_\lambda} w\,d\mu \leqslant \lambda^{-1} \int_{O_\lambda} fw\,d\mu$$

and the theorem is proved.

To prove that $\mathfrak{M}_w$ is also of weak type (1,1) we just observe that $\mathfrak{M}_w f$ is dominated by $\mathfrak{N}_{w,T^{-1}} f + \mathfrak{N}_{w,T} f$ where $\mathfrak{N}_{w,T} f(x)$ is what we called $\mathfrak{N}_w f(x)$ while $\mathfrak{N}\, f_{w,T^{-1}} f(x)$ is the corresponding operator defined as $\mathfrak{N}_{w,T} f(x)$ but using T^{-1} instead T.

3. Final remarks. If w is constant, then clearly satisfies A'_p for any p. But apart from this trivial case the natural question is if there exists a non-constant w satisfying A'_p. In the case of the Hardy–Littlewood maximal function in R^n Stein provides [5] an example: $|x|^\alpha$, $-1 < \alpha < p-1$. Unfortunately this does not make any sense in our context. There is another way of producing good weights. It is to find a function w such that w^* is essentially dominated by Cw.

In [7] it is proved that if f is any function in $L_1(X)$ and we construct

$$w = (f^*)^{1/2},$$

then

$$w^*(x) \leqslant Cw(x) \tag{2.19}$$

for any x where C is an universal constant.

Since $T_{0,k} w(x) \leqslant w^*(T^i x)$, $0 \leqslant i \leqslant k-1$, then by (2.19) we have

$$T_{0,k} w(x) \leqslant C\ w(T^i x), \qquad 0 \leqslant i \leqslant k-1$$

and this certainly implies A'_p for any $p > 1$ with $M = C$. Observe also that A'_p implies easily that $w(Tx) \leqslant Cw(x)$ where C depends only on the constant in A'_p. This means what we said in the introduction, that the operator

$$f \to Tf \qquad (Tf(x) = f(Tx))$$

maps $L^p(w\,d\mu)$ into $L^p(w\,d\mu)$ but, since C is in general bigger than 1, this is neither a contraction nor a uniformly bounded group of operators.

References

[1] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integral*, Studia Math. 51 (1974), 241–250.

[2] R. Hunt, B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for the conjugate and Hilbert Transform*, Trans. Amer. Math. Soc. 176 (1973), 227–251.

[3] C. H. Kan, *Ergodic properties of Lamperti operators*, Canad. J. Math. 30 (1978), 1206–1214.

[4] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[5] E. M. Stein, *Note on singular integrals*, Proc. Amer. Math. Soc. 8 (1952), 250–254.

[6] — *Singular integrals and differentiability properties of functions*, Princeton University Press, 1980.

[7] A. de la Torre, *B.M.O. and the ergodic maximal function* (to appear).

[8] N. Wiener, *The ergodic theorem*, Duke Math. J. 5 (1939), 1–18.

DEPARTAMENTO DE ANALISIS MATEMATICO
FACULTAD DE CIENCIAS
UNIVERSIDAD DE MALAGA, SPAIN

Received January 5, 1981 (1658)