One also uses the fact that by extending a $W_0^m L_M(\Omega)$ function by zero outside Ω, one gets a $W^m L_M(\boldsymbol{R}^N)$ function. This allows us to avoid the cone property for Ω. ■

We remark that when Ω is bounded, the above arguments can be carried through without using Lemma 7. The coefficient λ is then chosen so that $8b/\lambda\, D^\alpha u \in \mathscr{L}_M(\Omega)$ for $|\alpha| = m$, where b is the number of pieces of the covering $\{U_i\}$ needed to cover $\bar{\Omega}$.

As for Theorem 1, one can show by a simple modification of the proofs that the u_k's in the statement of Theorems 3 and 4 can be taken so that $D^\alpha u_k \to D^\alpha u$ in norm for $|\alpha| < m$.

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A simple proof of the atomic decomposition for $H^p(\boldsymbol{R}^n)$, $0 < p \leqslant 1$

by

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Abstract. A simple proof of the atomic decomposition for distributions in $H^p(\boldsymbol{R}^n)$, $0 < p < 1$, is given. The proof uses Green's Theorem and a result of Fefferman and Stein. It does not use their "grand" maximal function.

We define $H^p(\boldsymbol{R}^n)$ to be the space of functions $u_0(x, y)$ harmonic in $\boldsymbol{R}_+^{n+1} = \{(x, y) = x \in \boldsymbol{R}^n, y > 0\}$ for which the maximal function

$$u_0^*(x) = \sup_{|x-t|<100y\sqrt{n}} |u(t, y)|$$

is in $L^p(\boldsymbol{R}^n)$. We can define an H^p "norm" by

$$\|u_0\|_{H^p} = \|u_0^*\|_p .$$

Fefferman and Stein showed that for $u_0 \in H^p$, $\lim_{y\to 0} u_0(\circ, y) = f$ exists in the sense of tempered distributions and that f uniquely determines u_0. We may thus define H^p as a space of distributions with "norm" $\|f\|_{H^p} = \|u_0\|_{H^p}$. See [3].

We call a function $b(x)$ a *p-atom* if:

(A) b is supported on a cube $Q \subset \boldsymbol{R}^n$.

(B) $\|b\|_\infty \leqslant |Q|^{-1/p}$ ($|Q|$ is the volume of Q).

(C) $\int_Q b(x) x^\alpha dx = 0$ for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant [n(p^{-1} - 1)]$, the integer part of $n(p^{-1} - 1)$.

We prove the following:

THEOREM. *A tempered distribution f is in $H^p(\boldsymbol{R}^n)$, $0 < p \leqslant 1$, if and only if there exist p-atoms a_i and positive numbers λ_i such that*

$$f = \sum_{i=1}^\infty \lambda_i a_i$$

in the sense of distributions. The λ_i satisfy:

$$C_1 \|f\|_{H^p}^p \leqslant \sum \lambda_i^p \leqslant C_2 \|f\|_{H^p}^p \tag{1}$$

C_1 and C_2 only depend on p and n. ■

This theorem was proved for $n = 1$ by Coifman [1] and for general n by Latter [4]. Their proofs used the "grand" maximal function of Fefferman and Stein [2]. Ours does not. It uses these (somewhat) simpler facts:

(I) For $u_0 \in H^p$ we may define the *conjugate harmonic functions* $u_1, \ldots, u_n$ such that

$$F(x, y) = (u_0, u_1, \ldots, u_n) = \left(\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}\right)$$

for some H harmonic in $\boldsymbol{R}_+^{n+1}$.

(II) Define $|F| = (\sum_{j=0}^{n} |u_j|^2)^{1/2}$ and set

$$F^*(x) = \sup_{|x-t|<100y\sqrt{n}} |F(t, y)|.$$

Then $\|F^*\|_p \leqslant C\|u_0\|_{H^p}$. See [3].

Before going on, we wish to thank John Garnett for his encouragement and prodding.

The "if" half of the theorem and, with it, the left inequality in (1) are easy. See Latter [5]. We prove the other parts. We shall first do the case $p = 1$ and then show how to adapt the proof to $p < 1$.

Let $f \in H^1 \subset L^1$ and define for $k = 0, \pm 1, \pm 2, \ldots$

$$E_k = \{x \in \boldsymbol{R}^n,\ F^*(x) > 2^k\}.$$

Clearly E_k is open. Let $\{\Omega_j^k\}_{j=1}^{\infty}$ be a decomposition of E_k into Whitney cubes. These are dyadic cubes which satisfy:

(α) $E_k = \bigcup_{j=1}^{\infty} \Omega_j^k$,

(β) $j \neq l \Rightarrow (\Omega_j^k)^\circ \cap (\Omega_l^k)^\circ = \emptyset$,

(γ) $\operatorname{diameter}(\Omega_j^k) \leqslant \operatorname{distance}(\Omega_j^k, E_k^c) \leqslant 4\operatorname{diameter}(\Omega_j^k)$. (See [5].)

Let $(\Omega_j^k)^*$ be the cube in $\boldsymbol{R}_+^{n+1}$ whose bottom face is Ω_j^k. Define:

$$\partial_f(\Omega_j^k)^* = \{(t, y) \in \partial(\Omega_j^k)^*:\ y > 0 \text{ and } (t, y) \notin \partial(\Omega_l^k)^* \text{ for any } l \neq j\}.$$

We call this the *free boundary* of $(\Omega_j^k)^*$. Recall that $F = \nabla H$ and set

$$h_k(x) = \begin{cases} \dfrac{1}{|\Omega_j^k|} \displaystyle\int_{\partial_f(\Omega_j^k)^*} \nabla H \cdot d\vec{n}, & x \in \Omega_j^k, \\ f(x), & x \notin E_k \end{cases}$$

where $d\vec{n}$ denotes the outer normal with respect to $(\Omega_j^k)^*$.

We claim that $|h_k| \leqslant C2^k$. This is clear for $x \notin E_k$, since $u_0 =$ the Poisson integral of f. For $x \in E_k$ it will follow from

(2) $$|\nabla H| \leqslant C2^k \quad \text{on} \quad \partial_f(\Omega_j^k)^*$$

since the surface area of $\partial_f(\Omega_j^k)^*$ is $< (2n+2)|\Omega_j^k|$.

Let $(t, y) \in \partial_f(\Omega_j^k)^*$. If Ω_l^k touches Ω_j^k, and we set $l_j^k =$ side length (Ω_j^k), then:

$$l_l^k \geqslant \tfrac{1}{4} l_j^k.$$

Therefore $y \geqslant \frac{1}{4} l_j^k$ (see Figure 1).

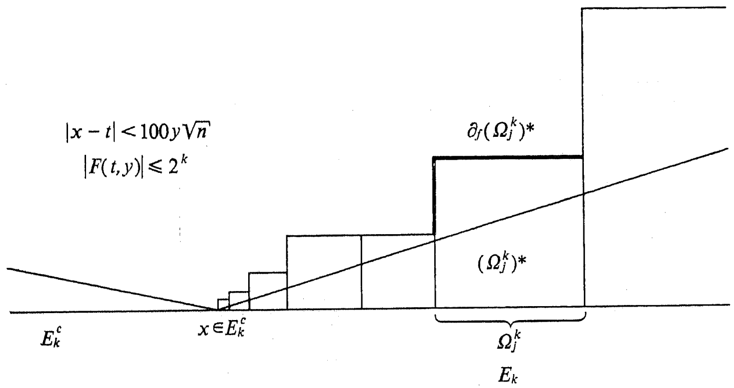

Fig. 1

By (γ), there is an $x \in E_k^c$ such that:

$$|x - t| \leqslant 5\operatorname{diameter}(\Omega_j^k) \leqslant 5\sqrt{n}\, l_j^k \leqslant 20\ y\sqrt{n}.$$

Thus $|\nabla H(t, y)| = |F(t, y)| \leqslant F^*(x) \leqslant 2^k$. Inequality (2) and the fact that $F^* < \infty$ a.e. imply

$$f = \sum_{k=-\infty}^{+\infty} h_{k+1} - h_k \quad \text{a.e.}$$

Now consider the integral

$$\int_{\Omega_j^k} (h_{k+1} - h_k)\, dx.$$

It equals the integral

(3) $$\int_{\Sigma(\Omega_j^k)} \nabla H \cdot d\vec{n}$$

where we define:

$$\sum(\Omega_j^k) = \{(t, y) \in \partial(\Omega_j^k)^*:\ y > 0,\ (t, y) \notin \partial_f(\Omega_j^k)^*, \text{ and either}$$

$$(t, y) \notin \bigcup_{\Omega_l^{k+1} \subset \Omega_j^k} \partial(\Omega_l^{k+1})^* \text{ or } (t, y) \in \bigcup_{\Omega_l^{k+1} \subset \Omega_j^k} \partial_f(\Omega_l^{k+1})^*\}.$$

Here is why. The integral $\int_{\Omega_j^k} h_{k+1}\,dx$ equals the integral of H's normal derivative (in the direction indicated in Figure 2) on the surface

$$\Gamma = (E_{k+1}^c \cap \Omega_j^k) \cap \Big(\bigcup_{\Omega_l^{k+1} \subset \Omega_j^k} \partial_f(\Omega_l^{k+1})^* \Big)$$

(on $\boldsymbol{R}^n \cap \Gamma$ we may treat $f(x)$ as though it were H's upward normal derivative by doing everything at $y = 2^{-m}$ and letting $m \to \infty$).

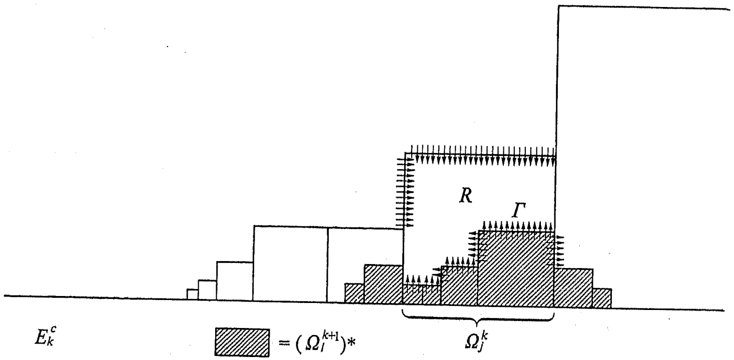

Fig. 2

The integral $-\int_{\Omega_j^k} h_k\,dx$ equals the integral of H's inward normal derivative on $\partial_f(\Omega_j^k)^*$. Now look at Figure 2 and see where the arrows, indicating normal derivatives, are pointing. Some of them point into the region R. The rest point out of $(\Omega_j^k)^*$. The integral over the first set of arrows and an application of Green's Theorem to R, plus the contribution of the second set, yield (3). See Figure 3.

We claim that for $j \neq l$:

$$(4) \qquad \sum(\Omega_j^k) \cap \partial(\Omega_l^k)^* = \sum(\Omega_l^k) \cap \partial(\Omega_j^k)^*$$

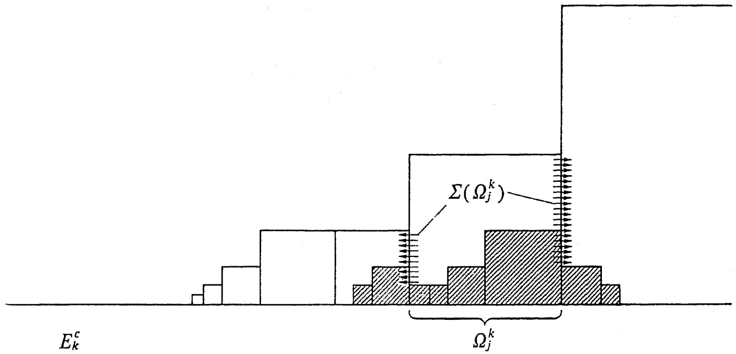

Fig. 3

almost everywhere relative to surface measure (in $\boldsymbol{R}_+^{n+1}$). To prove (4), let (t, y) belong to the left-hand side. Then $(t, y) \in \partial(\Omega_l^k)^*$. If $(t, y) \notin \sum(\Omega_l^k)$ then there is an $\Omega_r^{k+1} \subset \Omega_l^k$ such that $(t, y) \in \partial(\Omega_r^{k+1})^* \setminus \partial_f(\Omega_r^{k+1})^*$ (since $(t, y) \in \partial(\Omega_j^k)^*$ implies $(t, y) \notin \partial_f(\Omega_l^k)^*$). Thus (t, y) lies on an edge of $(\Omega_r^{k+1})^*$ or also lies on some $(\Omega_s^{k+1})^*$, with $\Omega_s^{k+1} \subset \Omega_j^k$. The set of edges has measure zero. The second case contradicts $(t, y) \in \sum(\Omega_l^k)$. Therefore the left-hand side is contained in the right, and the other conclusion comes from symmetry.

Define:

$$\varphi_{jl}^k(x) = \frac{\chi_{\Omega_j^k}}{|\Omega_j^k|} \int_{\Sigma(\Omega_j^k) \cap \partial(\Omega_l^k)^*} \nabla H \cdot d\vec{n}$$

($d\vec{n}$ is outward relative to $(\Omega_j^k)^*$).

An argument like that for (3) implies $|\varphi_{jl}^k| \leqslant C2^k$. For fixed j, $\varphi_{jl}^k \not\equiv 0$ for at most $(12)^n l$ (see [5]). If $\varphi_{jl}^k \not\equiv 0$, then $\varphi_{jl}^k + \varphi_{lj}^k$ is supported on a cube Ω_{jl}^k such that $|\Omega_{jl}^k| \leqslant C\min(|\Omega_j^k|, |\Omega_l^k|)$, and:

$$\int_{\Omega_{jl}^k} (\varphi_{jl}^k + \varphi_{lj}^k)\,dx = 0$$

since what is outward for one cube is inward for the other. Obviously:

$$\int_{\Omega_j^k} \Big(h_{k+1} - h_k - \sum_l \varphi_{jl}^k \Big)\,dx = 0.$$

Now we let $A > 0$ be large and define atoms:

$$a_j^k = \frac{\chi_{\Omega_j^k}[h_{k+1} - h_k - \sum_l \varphi_{jl}^k]}{A2^k|\Omega_j^k|}$$

$$b_{jl}^k = \frac{\varphi_{jl}^k + \varphi_{lj}^k}{A2^k|\Omega_{jl}^k|} \qquad (j < l)$$

and constants:

$$\lambda_j^k = A2^k|\Omega_j^k|, \qquad \gamma_{jl}^k = A2^k|\Omega_{jl}^k| \qquad (j < l).$$

Now:

$$\text{(5)} \qquad \sum_{k,j} \lambda_j^k + \sum_{\substack{k \\ j<l}} \gamma_{jl}^k \leqslant \sum_{k=-\infty}^{\infty} A2^k \Big\{ \sum_j |\Omega_j^k| + \sum_{j<l} |\Omega_{jl}^k| \Big\}$$

$$\leqslant A' \sum_{k=-\infty}^{\infty} 2^k |E_k| \leqslant C\|F^*\|_1 \leqslant C\|f\|_{H^1}.$$

And:

$$\sum_{k,j} \lambda_j^k a_j^k + \sum_{\substack{k \\ j<l}} \gamma_{jl}^k b_{jl}^k = \sum_{k=-\infty}^{\infty} \Big(h_{k+1} - h_k - \sum_{l,j} \varphi_{lj}^k\Big) + \sum_{\substack{k=-\infty \\ j<l}}^{\infty} (\varphi_{jl}^k + \varphi_{lj}^k)$$

$$= \sum_{k=-\infty}^{\infty} (h_{k+1} - h_k) = f \text{ a.e.}$$

Inequality (5) implies the convergence as distributions and justifies the change in the order of summation. This proves the case for $p = 1$.

To complete the proof in the spirit of the first part we define, for any monomial x^a in $x_1, \ldots, x_n$:

$$v_0^a(x, y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} y^{2k} \Delta^k(x^a),$$

$$v_i^a(x, y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} y^{2k+1} \frac{\partial}{\partial x_i} \Delta^k(x^a) \qquad (1 \leqslant i \leqslant n)$$

where Δ^k denotes the k-fold application of the Laplacean operator, with $\Delta^0 =$ the identity. Observe that each $v_i^a(x, y)$ is a polynomial in $x_1, \ldots, x_n, y$ (since $\Delta^k(x^a) \equiv 0$ for large k). Also, if we set $x_0 = y$ then $v_0, v_1, \ldots, v_n$ is a Riesz system, i.e.:

$$\text{(i)} \qquad \frac{\partial v_j^a}{\partial x_i} = \frac{\partial v_i^a}{\partial x_j}, \qquad 0 \leqslant i, j \leqslant n,$$

$$\text{(ii)} \qquad \sum_{i=0}^{n} \frac{\partial v_i^a}{\partial x_i} = 0,$$

as is easily verified. Note also that:

$$v_i^a(x, 0) = \begin{cases} x^a, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

Define $V^a(x, y) = \big(v_0^a(x, y), \ldots, v_n^a(x, y)\big)$. For P a polynomial in $x_1, \ldots, x_n$, $P = \sum_a a_a x^a$, let

$$\Phi(P)(x, y) = \sum_a a_a V^a(x, y) = \big((P_0(x, y), \ldots, P_n(x, y)\big).$$

Clearly $P_0, \ldots, P_n$ is a Riesz system.

Note that if $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are two Riesz systems in $\boldsymbol{R}_+^{n+1}$ then the vector field

$$\langle \vec{a}, \vec{b} \rangle = \Big(a_0 b_0 - \sum_{i=1}^{n} a_i b_i,\ a_0 b_1 + a_1 b_0, \ldots, a_0 b_n + a_n b_0\Big)$$

has zero divergence. See Koosis [4].

Let Q_0 be the cube in $\boldsymbol{R}^n$ of side length one centered at the origin. Let $\Pi^1, \ldots, \Pi^L$ be polynomials, orthonormal in $L^2(Q_0, dx)$, that span the space of polynomials of degree $\leqslant [n(p^{-1} - 1)]$.

Let $f \in H^p \cap L^1$. Define, for $k = 0, \pm 1, \ldots$

$$E_k = \{x \in \boldsymbol{R}^n\colon F^*(x) > 2^{k/p}\}$$

and let $\{\Omega_j^k\}$ be a Whitney decomposition as before. Define

$$x_j^k = \text{the center of } \Omega_j^k$$

and let

$$h_k(x) = \begin{cases} \displaystyle\sum_{m=1}^{L} c_m^{kj} \frac{\Pi^m\big((x - x_j^k)/l_j^k\big)}{\sqrt{|\Omega_j^k|}}, & x \in \Omega_j^k, \\ f(x), & x \notin E^k \end{cases}$$

where we set:

$$c_m^{kj} = \frac{1}{\sqrt{|\Omega_j^k|}} \int_{\partial_f(\Omega_j^k)^*} \Big\langle F, \Phi(\Pi^m)\big((x - x_j^k)/l_j^k, y/l_j^k\big) \Big\rangle \cdot d\vec{n}.$$

By the same argument as for $p = 1$, we have $|h_k| \leqslant C\, 2^{k/p}$ and that

$$f = \sum_{k=-\infty} h_{k+1} - h_k \text{ a.e.}$$

Since $u_0, u_1, \ldots, u_n$ is a Riesz system, Green's Theorem and elementary Hilbert space theory imply, as in the proof of (3), that:

$$\int_{\Omega_j^k} (h_{k+1} - h_k) x^a\, dx = \int_{\Sigma(\Omega_j^k)} \langle F, V^a \rangle \cdot d\vec{n}$$

for all $|a| \leqslant [n(p^{-1} - 1)]$.

For $x \in \Omega_j^k$ define

$$\varphi_{jl}^k = \sum_{m=1}^{L} \dot{d}_m^{kjl} \frac{\Pi^m((x-x_j^k)/l_j^k)}{\sqrt{|\Omega_j^k|}}$$

where

$$d_m^{kjl} = \frac{1}{\sqrt{|\Omega_j^k|}} \int\limits_{\Sigma(\Omega_j^k)\cap\partial(\Omega_l^k)^*} \langle F, \Phi(\Pi^m)((x-x_j^k)/l_j^k, y/l_j^k)\rangle \cdot d\vec{n}\rangle .$$

Clearly $|\varphi_{jl}^k| \leqslant C\, 2^{k/p}$ and $|h_{k+1}-h_k-\sum_l \varphi_{jl}^k| \leqslant C\, 2^{k/p}$ (this is exactly as for $p=1$). Much as before, and for the same reason:

$$\int\limits_{\Omega_j^k} \Big(h_{k+1}-h_k-\sum_l \varphi_{jl}^k\Big) x^a\,dx = 0, \qquad \int\limits_{\Omega_{jl}^k} (\varphi_{jl}^k+\varphi_{lj}^k)\, x^a\,dx = 0$$

for all $|\alpha| \leqslant [n(p^{-1}-1)]$; Ω_{jl}^k is as before.

Define p-atoms:

$$a_j^k = \frac{\chi_{\Omega_j^k}\big(h_{k+1}-h_k-\sum_l \varphi_{jl}^k\big)}{A\, 2^{k/p}|\Omega_j^k|^{1/p}},$$

$$b_{jl}^k = \frac{\varphi_{jl}^k+\varphi_{lj}^k}{A\, 2^{k/p}|\Omega_{jl}^k|^{1/p}} \qquad (j<l)$$

and constants:

$$\lambda_j^k = A\, 2^{k/p}|\Omega_j^k|^{1/p}, \qquad \gamma_{jl}^k = A\, 2^{k/p}|\Omega_{jl}^k|^{1/p} \quad (j<l)$$

and proceed as in the first case.

The passage to general $f \in H^p$ uses the density (in the $\|\cdot\|_{H^p}^p$ topology) of $H^p \cap L^1$ in H^p (see [3]) and an easy limiting argument, given in Latter [5].

After writing this paper I was told of another construction by A.P. Calderón which resembles mine in some ways. See [1].

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