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Individual boundedness condition for positive definite sesquilinear form valued kernels

by

J. STOCHEL (Kraków)

Abstract. In the present paper we get some equivalent forms of the individual boundedness condition. We show that every positive definite kernel on a product of $*$-semigroups with dilatable sections is itself dilatable. The last part of the paper deals with the question when a positive definite operator function on a $*$-semigroup is simply a $*$-representation. Our result relates to that of [5].

1. In the sequel $\boldsymbol{F}$ stands for either the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$. Let X and Y be either vector spaces or topological vector spaces over $\boldsymbol{F}$. Denote by $L(X, Y)$ and $CL(X, Y)$, respectively, the space of all linear operators and the space of all continuous linear operators on X to Y. We write $CL(X) = CL(X, X)$. I_X stands for the identity operator on X. The space of all sesquiliear forms and the space of all jointly continuous sesquilinear forms on $X \times X$ are denoted by $L_2(X, \boldsymbol{F})$ and $CL_2(X, \boldsymbol{F})$, respectively. $\langle Bx, x' \rangle$ stands for the value of $B \in L_2(X, \boldsymbol{F})$ on $(x, x') \in X \times X$.

Let H be a Hilbert space over $\boldsymbol{F}$. Denote by $(h, h')_H$ the inner product of h and h'; $h, h' \in H$. We write $\|h\|_H = (h, h)_H^{1/2}$ for the norm of h, $h \in H$. The norm of $A \in CL(H)$ is denoted by $\|A\|$, the adjoint of A by A^*. If $(H_t)_{t \in T}$ is a family of subsets of H, then $\bigvee_{t \in T} H_t$ stands for the smallest closed linear subspace which includes the union $\bigcup_{t \in T} H_t$.

2. Let T be a set. A kernel $B\colon T \times T \to L_2(X, \boldsymbol{F})$ is said to be *positive definite* (PD) if the following conditions hold true: (1)

$$(1) \quad \sum_{i,j=1}^{n} \langle B(t_j, t_i) x_i, x_j \rangle \geqslant 0, \qquad t_1, \ldots, t_n \in T;\ x_1, \ldots, x_n \in X;\ n = 1, 2, \ldots,$$

$$(2) \qquad \langle B(t', t) x, x' \rangle = \overline{\langle B(t, t') x', x \rangle}, \qquad t, t' \in T;\ x, x' \in X.$$

(1) When $\boldsymbol{F} = \boldsymbol{C}$, (2) follows from (1) (see [2], [4]).

It is known (see [2], [7], [13]) that for every PD kernel $B\colon T\times T\to L_2(X,F)$ there exists a Hilbert space K over F and a function $D\colon T\to L(X,K)$ such that

$$(3)\qquad \langle B(t',t)x,x'\rangle = \big(D(t)x, D(t')x'\big)_K,\quad t,t'\in T;\ x,x'\in X,$$

$$(4)\qquad K = \bigvee_{t\in T} D(t)X.$$

(K,D) is called the *minimal factorization of* B. Notice that a minimal factorization of B is determined up to a unitary isomorphism ([4]).

Now let S be a $*$-semigroup of actions on T (we do not require S to have a unit). Write $s(t)$ for the action of $s\in S$ on $t\in T$. Suppose we are given a PD kernel $B\colon T\times T\to L_2(X,F)$ satisfying the following condition:

$$(5)\qquad B\big(t, s(t')\big) = B\big(s^*(t), t'\big),\quad t,t'\in T;\ s\in S.$$

If (K,D) is a minimal factorization of B, then (see [2], [3]) there exists a family $C(s)$, $s\in S$, of closed, densely defined linear operators on K such that

(6) the set $K_0 = \bigcup_{t\in T} D(t)X$ is included in the domain of each $C(s)$, $s\in S$,

$$(7)\qquad C(s)D(t) = D\big(s(t)\big),\quad t\in T;\ s\in S,$$

$$(8)\qquad C(s^*) = C(s)^* \quad \text{on}\quad K_0.$$

The family $\big(C(s)\big)_{s\in S}$ is called the *propagator of* $\big(D(t)\big)_{t\in T}$.

Denote by S_B the set of all $s\in S$ such that $C(s)$ is a bounded operator(²). It is easy to see that $s\in S_B$ if and only if there exists a real number $c_1\geqslant 0$ such that

$$(9)\qquad \sum_{i,j=1}^{n}\langle B\big(s(t_j), s(t_i)\big)x_i, x_j\rangle \leqslant c_1 \sum_{i,j=1}^{n}\langle B(t_j,t_i)x_i,x_j\rangle$$
$$\text{for each } t_1,\dots,t_n\in T;\ x_1,\dots,x_n\in X;\ n=1,2,\dots$$

Therefore the definition of S_B does not depend on the choice of a minimal factorization of B. In the case of $S=S_B$ we say that B satisfies the *boundedness condition* (BC). As is known, there exist many equivalent forms of the boundedness condition (see [2], [3], [9], [10]). There arises a natural question if some of them can be used to describe the set S_B. The following lemma is a necessary preliminary to answering the question.

LEMMA 1. *Let S be a $*$-semigroup of actions on T. If $B\colon T\times T\to L_2(X,F)$ is a PD kernel satisfying* (5), *then the set S'_B defined by*

(²) It is possible that $S_B=\emptyset$; however, if S has a unit e then $e\in S_B$.

$s\in S'_B$ if and only if $s\in S$ and there exists a real number $c_2\geqslant 0$ such that

$$(10)\qquad \langle B\big(s(t),s(t)\big)x,x\rangle \leqslant c_2\langle B(t,t)x,x\rangle,\quad t\in T,\ x\in X,$$

is a $*$-subsemigroup of S. Moreover, if $c_{2m}(s)$ stands for the minimal real number c_2 satisfying (10), *then $c_{2m}\colon S'_B\to \mathbf{R}_+$ is a submultiplicative function such that $c_{2m}(s^*) = c_{2m}(s)$, $s\in S'_B$.*

Proof. It is easy to see that S'_B is a subsemigroup of S and that $c_{2m}\colon S'_B\to\mathbf{R}_+$ is a submultiplicative function. We have only to prove that $s^*\in S'_B$ if $s\in S'_B$.

Suppose that $s\in S'_B$. Then, using (5), (10) and the Schwarz inequality (see [4], the inequality (4), p. 18; [9]), we obtain

$$\begin{aligned}\langle B\big(s^*(t), s^*(t)\big)x,x\rangle &= \Big\langle B\big(s\big(s^*(t)\big),t\big)x,x\Big\rangle\\ &\leqslant \Big\langle B\Big(s\big(s^*(t)\big)\Big), s\big(s^*(t)\big)x,x\Big\rangle^{1/2}\langle B(t,t)x,x\rangle^{1/2}\\ &\leqslant c_{2m}(s)^{1/2}\langle B(t,t)x,x\rangle^{1/2}\langle B\big(s^*(t),s^*(t)\big)x,x\rangle^{1/2},\quad t\in T,\ x\in X.\end{aligned}$$

This leads to

$$\langle B\big(s^*(t),s^*(t)\big)x,x\rangle \leqslant \big(c_{2m}(s)\langle B(t,t)x,x\rangle\big)^{1-2^{-n}}\langle B\big(s^*(t),s^*(t)\big)x,x\rangle^{2^{-n}},\quad n=1,2,\dots;$$

so by a limit passage we get

$$\langle B\big(s^*(t),s^*(t)\big)x,x\rangle \leqslant c_{2m}(s)\langle B(t,t)x,x\rangle,\quad t\in T,\ x\in X.$$

This means that $s^*\in S'_B$ and $c_{2m}(s^*)\leqslant c_{2m}(s)$, which completes the proof.

In the following theorem we will describe the set S_B with the aid of some inequalities appearing in dilation theory.

THEOREM 1. *Let S be a $*$-semigroup of actions on T. Suppose that $B\colon T\times T\to L_2(X,F)$ is a PD kernel satisfying* (5). *If $s\in S$, then the following conditions are equivalent*:

(i) *there exists a real number $c_1\geqslant 0$ such that* (9) *holds true*,

(ii) *there exists a real number $c_2\geqslant 0$ such that* (10) *holds true*,

(iii) *there exists a real number $c_3\geqslant 0$ such that*

$$(11)\qquad \liminf_{k\to\infty}\langle B\big((s^*s)^{2^k}(t),(s^*s)^{2^k}(t)\big)x,x\rangle^{2^{-k-1}}\leqslant c_3,\quad t\in T,\ x\in X,$$

(iv) *there exists a real number $c_4\geqslant 0$ such that*

$$(12)\qquad \liminf_{k\to\infty}\Big(\sum_{i,j=1}^{n}\langle B\big((s^*s)^{2^k}(t_j),(s^*s)^{2^k}(t_i)\big)x_i,x_j\rangle\Big)^{2^{-k-1}}\leqslant c_4,$$

for each $t_1,\dots,t_n\in T$; $x_1,\dots,x_n\in X$; $n=1,2,\dots$,

(v) $s\in S_B$.

If $c_{km}(s)$ stands for the minimal extended real number c_k satisfying $(8+k)$, $k=1,2,3,4$, *then* $c_{1m}(s)=c_{2m}(s)=c_{3m}(s)=c_{4m}(s)$ *for all* $s\in S$.

In the sequel $c_B(s)$ stands for one of the extended real numbers $c_{km}(s)$, $k=1,2,3,4$, $s\in S$.

Proof. The implications (iv) ⇒ (iii) and (i) ⇒ (ii) are obvious. It is plain that $c_{2m}(s)\leqslant c_{1m}(s)$ and $c_{3m}(s)\leqslant c_{4m}(s)$, $s\in S$. The Szafraniec inequality (see [4], Lemma 1, p. 28; [9]; [10]) shows that the implications (iii) ⇒ (ii) and (iv) ⇒ (i) hold true and that $c_{2m}(s)\leqslant c_{3m}(s)$, $c_{1m}(s)\leqslant c_{4m}(s)$, $s\in S$. Summing up, we have only to prove the implication (ii) ⇒ (iv) and the inequality $c_{4m}(s)\leqslant c_{2m}(s)$, $s\in S$.

Let (K, D) be a minimal factorization of B. Suppose that $s\in S'_B$. Then $(s^*s)^{2^k}\in S'_B$, $k=0,1,2,\ldots$ (by Lemma 1). It follows that

$$(13)\quad \big\|D\big((s^*s)^{2^k}(t)\big)x\big\|_K\leqslant c_{2m}\big((s^*s)^{2^k}\big)^{1/2}\|D(t)x\|_K \leqslant c_{2m}(s)^{2^k}\|D(t)x\|_K,\qquad t\in T,\ x\in X,\ k=0,1,2,\ldots$$

Using (13), we have

$$\Big(\sum_{i,j=1}^{n}\big\langle B\big((s^*s)^{2^k}(t_j),\,(s^*s)^{2^k}(t_i)\big)x_i,\,x_j\big\rangle\Big)^{2^{-k-1}} = \Big(\sum_{i,j=1}^{n}\big(D\big((s^*s)^{2^k}(t_i)\big)x_i,\,D\big((s^*s)^{2^k}(t_j)\big)x_j\big)_K\Big)^{2^{-k-1}}$$
$$\leqslant\Big(\sum_{i=1}^{n}\big\|D\big((s^*s)^{2^k}(t_i)\big)x_i\big\|_K\Big)^{2^{-k}}\leqslant c_{2m}(s)\Big(\sum_{i=1}^{n}\|D(t_i)x_i\|\Big)^{2^{-k}},$$

for each $t_1,\ldots,t_n\in T$, $x_1,\ldots,x_n\in X$, $n,k=1,2,\ldots$, so, by a limit passage, we get (iv) with $c_4=c_{2m}(s)$. This completes the proof.

COROLLARY 1. *Let S, T, X, B be as in Theorem 1. Suppose that B is a* PD *kernel satisfying* (5). *Then S_B is a $*$-subsemigroup of S with the following properties:*

(i) *if there exists a natural number k such that $(s^*s)^{2^k}\in S_B$ then $s\in S_B$,*

(ii) $c_B(s)=\big(c_B\big((s^*s)^{2^k}\big)\big)^{2^{-k-1}}$, $s\in S$, $k=0,1,2,\ldots$

The proof of (i) and (ii) follows from the Szafraniec inequality (see [4], Lemma 1, p. 28; [9]; [10]).

COROLLARY 2. *Let S be a $*$-algebra. Suppose that $B\colon S\times S\to L_2(X, C)$ is a* PD *kernel satisfying* (5). *If, for every $s\in S$, the mapping $B(s,\cdot)\colon S\to L_2(X, C)$ is linear, then S_B is a $*$-subalgebra of S. Moreover, $c_B(\cdot)^{1/2}$ is a seminorm.*

Notice that the Paschke dilation theorem ([1], Th. 1, p. 413) for completely positive linear maps on U^*-algebras can directly be obtained from Corollary 2 and the general dilation theorem by Sz.-Nagy ([12], Principal Theorem).

Notes. The inequality (10) belongs to Masani (see [2], [3]). The inequality (12) is due to Szafraniec (see [9], [10]). The boundedness condition (BC) has been introduced by Sz.-Nagy ([12]).

3. Now we consider positive definite kernels on generalized direct products of *$*$-semigroups* and *$*$-algebras*. We show that if sections of such a kernel are dilatable (for this terminology see [4] and [7]) then the kernel is itself dilatable.

To begin with we introduce the following definition. We say that a *$*$-semigroup* (resp. a *$*$-algebra*) S is a *generalized direct* (g.d.) *product* of $*$-semigroups (resp. $*$-algebras) $(S_i)_{i\in I}$ if the following conditions hold true:

(14) S_i is a $*$-subsemigroup (resp. a $*$-subalgebra) of S, $i\in I$,

$$(15)\qquad s_is_j=s_js_i,\qquad s_i\in S_i,\ s_j\in S_j;\ i,j\in I,\ i\neq j,$$

(16) S is a $*$-semigroup (resp. a $*$-algebra) generated by $\bigcup_{i\in I}S_i$.

A few examples are now in order.

EXAMPLE 1. Suppose that S is a direct product of $*$-semigroups S_i, $i\in I$, with units e_i, $i\in I$, respectively, i.e.

$$S=\{(s_i)_{i\in I}\colon \text{there exists a finite set } I_0\subset I \text{ such that } s_i=e_i,\ i\in I\setminus I_0\}.$$

Denote by $\hat S_i$ the $*$-semigroup $\{s\in S\colon s_j=e_j,\ j\in I\setminus\{i\}\}$, $i\in I$. Then S is a g.d. product of $*$-semigroups $\hat S_i$, $i\in I$.

EXAMPLE 2. Let S be a tensor product of $*$-algebras S_i, $i=1,2,\ldots,n$, with units e_i, $i=1,2,\ldots,n$, respectively. Denote by $\hat S_i$ the $*$-algebra

$$e_1\otimes\ldots\otimes e_{i-1}\otimes S_i\otimes e_{i+1}\otimes\ldots\otimes e_n,\qquad i=1,2,\ldots,n.$$

Then S is a g.d. product of $*$-algebras $\hat S_i$, $i=1,2,\ldots,n$.

Now we are able to prove the following

THEOREM 2. *Let S be a* g.d. *product of $*$-semigroups S_i, $i\in I$. Suppose that $B\colon S\times S\to L_2(X, F)$ is a* PD *kernel satisfying* (5). *If for every $i\in I$ the kernel $B_i=B|_{S_i\times S_i}$ satisfies* (BC), *then B itself satisfies* (BC).

Proof. Let us fix $i\in I$, $s_i\in S_i$ and $t\in S$. The conditions (14), (15) and (16) imply that there exists a finite non-empty set $I_0\subset I$ such that $t=\prod_{j\in I_0}t_j$, where $t_j\in S_j$, $j\in I_0$. Examine three cases:

Case 1. $i\notin I_0$. Then by the Schwarz inequality (see [4], the inequality (4), p. 18; [9]) we have

$$d_k(t,x)\overset{\mathrm{df}}{=}\big\langle B\big((s_i^*s_i)^{2^k}t,\,(s_i^*s_i)^{2^k}t\big)x,\,x\big\rangle^{2^{-k-1}}$$
$$\leqslant\langle B(t^*t,\,t^*t)x,\,x\rangle^{2^{-k-2}}\big\langle B_i\big((s_i^*s_i)^{2^{k+1}-1}s_i^*s_i,\,(s_i^*s_i)^{2^{k+1}-1}s_i^*s_i\big)x,\,x\big\rangle^{2^{-k-2}}$$
$$\leqslant\big(\langle B(t^*t,\,t^*t)x,\,x\rangle\langle B_i(s_i^*s_i,\,s_i^*s_i)x,\,x\rangle\big)^{2^{-k-2}}\big(c_{B_i}\big((s_i^*s_i)^{2^{k+1}-1}\big)\big)^{2^{-k-2}}$$
$$\leqslant\big(\langle B(t^*t,\,t^*t)x,\,x\rangle\langle B_i(s_i^*s_i,\,s_i^*s_i)x,\,x\rangle\big)^{2^{-k-2}}c_{B_i}(s_i)^{1-2^{-k-1}}$$

for each $x \in X$, $k = 0, 1, 2, \ldots$; so

$$\liminf_{k\to\infty} d_k(t, x) \leqslant c_{B_i}(s_i). \tag{17}$$

Case 2. $i \in I_0$ and the set $I_0 \setminus \{i\}$ is non-empty. Then $t = t_i t'$, where $t' = \prod_{j \in I_0 \setminus \{i\}} t_j$. Using (15) and the Schwarz inequality, we obtain

$$\begin{aligned} d_k(t, x) &\overset{\mathrm{df}}{=} \langle B\big((s_i^* s_i)^{2^k} t, (s_i^* s_i)^{2^k} t\big) x, x\rangle^{2^{-k-1}} \\ &= \langle B\big(t'^* t, (s_i^* s_i)^{2^{k+1}} t_i\big) x, x\rangle^{2^{-k-1}} \\ &\leqslant \langle B(t'^* t, t'^* t) x, x\rangle^{2^{-k-2}} \langle B_i\big((s_i^* s_i)^{2^{k+1}} t_i, (s_i^* s_i)^{2^{k+1}} t_i\big) x, x\rangle^{2^{-k-2}} \end{aligned}$$

for each $x \in X$, $k = 0, 1, 2, \ldots$; so, by Theorem 1, (17) holds true.

Case 3. $I_0 = \{i\}$. The assumptions of Theorem 2 immediately imply the condition (17).

Summing up, we have $\bigcup_{i \in I} S_i \subset S_B$, by Theorem 1. This completes the proof.

The following theorem is a consequence of Theorem 2 and Corollary 2.

THEOREM 2'. *Let S be a g.d. product of $*$-algebras $(S_i)_{i \in I}$. Suppose that B: $S \times S \to L_2(X, C)$ is a PD kernel satisfying* (5). *If for every $s \in S$ the mapping $B(s, \cdot)$: $S \to L_2(X, C)$ is linear and for every $i \in I$ the kernel $B_i = B|_{S_i \times S_i}$ satisfies* (BC), *then B itself satisfies* (BC).

4. This part of the paper deals with the question when a positive definite operator function on a $*$-semigroup is simply a $*$-representation.

The following theorem is discussed under stronger assumptions by Mlak ([4], Prop. 2, p. 12). The first formulation in the context of U^*-algebras is due to Paschke (see [1], Th. 2, p. 414; [8]).

THEOREM 3. *Let S be a $*$-semigroup (resp. a $*$-algebra) and let H be a complex Hilbert space. Suppose that B: $S \to CL(H)$ is an operator function (resp. a linear operator function) satisfying the following conditions*:

(18) *there exists a positive real number $q \leqslant 1$ such that*

$$\Big\| \sum_{i=1}^n B(s_i) h_i \Big\|_H^2 \leqslant q \sum_{i,j=1}^n \big(B(s_j^* s_i) h_i, h_j\big)_H,$$

for each $s_1, \ldots, s_n \in S$, $h_1, \ldots, h_n \in H$, $n = 1, 2, \ldots$,

$$B(s^*) = B(s)^*, \quad s \in S. \tag{19}$$

Then the sets

$$S_0 = \{s \in S\colon B(s^* s) = B(s)^* B(s)\}$$

and

$$S_0^* = \{s \in S\colon s^* \in S_0\}\,(^3)$$

are subsemigroups (resp. subalgebras) of S and

$$B(s s_0) = B(s) B(s_0), \quad s \in S,\ s_0 \in S_0, \tag{20}$$

$$B(s_0 s) = B(s_0) B(s), \quad s \in S,\ s_0 \in S_0^*. \tag{21}$$

Proof. Let S_1 be a unitization of S. Denote by 1 the adjoined unit (if S has a unit e then we require $1 \neq e$). Then by (18) and (19) B can be extended to a PD function(4) B_1: $S_1 \to CL(H)$ such that $B_1(1) = I_H$ (see [11], Prop. 1). Therefore without loss of generality we may assume that S has a unit e and B: $S \to CL(H)$ is a PD function such that $B(e) = I_H$.

Let (K, D) be a minimal factorization of B, and let $C(s)$, $s \in S$, be a propagator of $D(s)$, $s \in S$. Since $B(s) \in CL(H) = CL_2(H, C)$, $s \in S$, we have $D(s) \in CL(H, K)$, $s \in S$; so

$$B(s) h = B(e^* s) h = D(e)^* D(s) h = D(e)^* C(s) D(e) h, \quad h \in H,\ s \in S.$$

In particular, $I_H = B(e) = D(e)^* D(e)$. This means that $V \overset{\mathrm{df}}{=} D(e)$ is an isometry. If $s_0 \in S_0$ and $h \in H$, then by (8) we have

$$\begin{aligned} \|VB(s_0)h - C(s_0)Vh\|_K^2 &= \|VB(s_0)h\|_K^2 + \|C(s_0)Vh\|_K^2 - \\ &\qquad - 2\mathrm{re}\big(VB(s_0)h, C(s_0)Vh\big)_K \\ &= \|B(s_0)h\|_H^2 + \big(B(s_0^* s_0)h, h\big)_H - 2\mathrm{re}\big(B(s_0)h, B(s_0)h\big)_H = 0. \end{aligned}$$

Suppose that $s_0 \in S_0$, $s \in S$ and $h \in H$. Then, since $VB(s_0)h = C(s_0)Vh$, we have

$$\begin{aligned} B(s s_0) h &= V^* C(s s_0) Vh = V^* C(s) C(s_0) Vh = V^* C(s) V B(s_0) h \\ &= B(s) B(s_0) h. \end{aligned}$$

To prove that S_0 is a subsemigroup we take $s, s' \in S_0$ and then, by the previous property, we successively obtain

$$\begin{aligned} B\big((ss')^* ss'\big) &= B\Big(\big((ss')^* s\big) s'\Big) = B\big((ss')^* s\big) B(s') \\ &= B\big((ss')^*\big) B(s) B(s') = B(ss')^* B(ss'). \end{aligned}$$

The equality (21) is a simple consequence of (20). This completes the proof.

Remark 1. Notice that under the assumptions of Theorem 3 the set $S_0 \cap S_0^*$ is a $*$-subsemigroup (resp. a $*$-subalgebra) of S.

THEOREM 3'. *Let S be a topological $*$-semigroup*(5) *(resp. a topological $*$-algebra*(6)) *and let H be a complex Hilbert space. Suppose that B: $S \to CL(H)$ is a strongly continuous (resp. strongly continuous and linear) operator function satisfying* (18) *and* (19). *Then the sets S_0 and S_0^* defined as in Theorem 3 are closed subsemigroups (resp. closed subalgebras) of S.*

(3) It is possible that $S_0 = \emptyset$. If S has a unit e then $e \in S_0$ if and only if $B(e)$ is an orthogonal projection.

(4) I.e. a kernel $B_1(s^* s')$, $(s, s') \in S_1 \times S_1$ is PD.

(5) I.e. S is a topological semigroup with a continuous involution$*$.

(6) I.e. S is a topological algebra with a continuous involution$*$.

COROLLARY 3. *Let S be a $*$-semigroup (resp. a $*$-algebra) with a unit e. Suppose that $B\colon S\to CL(H)$ is a PD operator function (resp. a linear PD operator function) such that $\|B(e)\|\leqslant 1$. Then the sets S_0 and S_0^* defined as in Theorem 3 are subsemigroups (resp. subalgebras) of S.*

Proof. Let (K, D) be a minimal factorization of B. Then

$$\Big\|\sum_{i=1}^{n} B(s_i)h_i\Big\|_H^2 = \Big\|D(e)^*\sum_{i=1}^{n} D(s_i)h_i\Big\|_H^2 \leqslant \|D(e)^*D(e)\|_H\sum_{i,j=1}^{n}\big(B(s_j^*s_i)h_i, h_j\big)_H$$
$$= \|B(e)\|\sum_{i,j=1}^{n}\big(B(s_j^*s_i)h_i, h_j\big)_H,$$
$$h_1, \ldots, h_n\in H,\ s_1, \ldots, s_n\in S,\ n = 1, 2, \ldots;$$

so the condition (18) holds true. This completes the proof.

COROLLARY 4. *Let S be a tpological $*$-semigroup (resp. a topological $*$-algebra) and let H be a complex Hilbert space. Suppose that $B\colon S\to CL(H)$ is a strongly continuous operator function (resp. a strongly continuous linear operator function) satisfying* (18) *and* (19). *If there exists a subset $\tilde{S}$ of the set S with the following properties:*

(i) *S is the smallest closed subsemigroup (resp. the smallest closed subalgebra) of S which includes $\tilde{S}$,*

(ii) $B(s^*s) = B(s)^*B(s)$, $s\in\tilde{S}$,

then B is a $*$-representation, i.e. an involution preserving semigroup homomorphism (resp. an involution preserving algebra homomorphism).

Proof. By Theorem 3′, $S_0 = S$. This means that

$$B(s^*s) = B(s)^*B(s), \quad s\in S;$$

so Corollary 4 is a consequence of Theorem 3 of [6].

COROLLARY 5. *Let S be an abelian topological $*$-semigroup (resp. an abelian topological $*$-algebra) and let H be a complex Hilbert space. Suppose that $B\colon S\to CL(H)$ is a strongly continuous (resp. strongly continuous, linear) operator function satisfying* (18) *and* (19). *If there exists a subset $\tilde{S}$ of the set S with the following properties:*

(i) *S is the smallest closed $*$-subsemigroup (resp. the smallest closed $*$-subalgebra) of S, which includes $\tilde{S}$,*

(ii) $B(s^*s) = B(s)^*B(s)$, $s\in\tilde{S}$,

(iii) *$B(s)$ is a normal operator, $s\in\tilde{S}$,*

then B is a $*$-representation.

Proof. The conditions (i), (ii) and (iii) imply that $\tilde{S}\subset S_0\cap S_0^*$; so by Theorem 3′, $S_0\cap S_0^* = S$, which completes the proof ([6], Th. 3).

Remark 2. Theorem 3 remains true after removing the condition (19). The proof will be published in another paper (we will use a new general unbounded dilation theorem).

5. The last theorem relates to Theorem B of [5]. We show that the boundedness condition implicitly contained in its assumptions can be omitted.

THEOREM 4. *Let S be a* g.d. *product of $*$-semigroups (resp. a* g.d. *product of $*$-algebras) S_i, $i\in I$, and let H be a complex Hilbert space. Suppose that $B\colon S\to CL(H)$ is an operator function (resp. a linear operator function) satisfying* (18) *and* (19). *If, for every $i\in I$, the function $B_i = B|_{S_i}$ satisfies the following condition:*

(i) $B_i(s^*s) = B_i(s)^*B_i(s)$, $s\in S_i$,

then B is a $*$-representation.

Remark 3. Notice that the condition (ii) in Corollary 4 (resp. the condition (i) in Theorem 4) can be replaced by the following one:

$$B(ss^*) = B(s)B(s)^*, \quad s\in\tilde{S}$$

(resp. $B_i(ss^*) = B_i(s)B_i(s)^*$, $s\in S_i$).

Remark 4. Theorem 4 can also be formulated for topological tensor products of topological $*$-algebras.

Remark to Corollary 1. To prove that S_B is a $*$-subsemigroup of S, we do not need Theorem 1. Indeed, if $s\in S_B$ then $C(s)\in CL(K)$. Since $C(s)^*\in CL(K)$, we have, by (4), $C(s)^*D(t) = D\big(s^*(t)\big)$, for all $t\in T$. This means that $C(s^*)\in CL(K)$ and $s^*\in S_B$.

Remark to Theorem 2. If X is a Banach space over $\boldsymbol{F}$, S is a g.d. product of $*$-semigroups S_i, $i = 1, 2$, S, S_1 and S_2 have a common unit e and $B(s, s') = B'(s^*s')$, where $B'\colon S\to CL_2(X, \boldsymbol{F})$, the proof of Theorem 2 can be simplified. Indeed, let (K, D) be a minimal factorization of B and let $s = s_1s_2$, where $s_i\in S_i$, $i = 1, 2$, be a number of S. Then, by Proposition 1 of [10], we have

$$\|B'(s)\| = \big\|B'\big((s_1^*)^*s_2\big)\big\| = \|D(s_1^*)^*D(s_2)\| \leqslant \|B_1'(s_1s_1^*)\|^{1/2}\|B_2'(s_2^*s_2)\|^{1/2}$$
$$\leqslant A_1^{1/2}A_2^{1/2}\big(c_{B_1}(s_1)c_{B_2}(s_2)\big)^{1/2}$$

where A_1, A_2 are positive real numbers. Since $c_{B_1}\otimes c_{B_2}$ is submultiplicative, the kernel B satisfies (BC) (see [10], Prop. 1).

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
Oddział w Krakowie, Solskiego 30
31-127 Kraków, Poland

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