Some approximation properties in Orlicz–Sobolev spaces

by

JEAN-PIERRE GOSSEZ (Bruxelles)

Abstract. We prove that weak derivatives in general Orlicz spaces are globally strong derivatives with respect to the modular convergence. Other approximation theorems involving the modular convergence are presented, which improve known density results of interest in the existence theory for strongly nonlinear boundary value problems.

Statement of results. Let \( L_M(\Omega) \) be the Orlicz space on an open subset \( \Omega \) of \( \mathbb{R}^n \) corresponding to an N-function \( M \) and let \( E_M(\Omega) \) be the norm closure in \( L_M(\Omega) \) of the \( L^m(\Omega) \) functions with compact support in \( \Omega \). The Sobolev space of functions \( u \) such that \( u \) and its distributional derivatives up to order \( m \) lie in \( L_M(\Omega) \) (resp. \( E_M(\Omega) \)) is denoted by \( W^mL_M(\Omega) \) (resp. \( W^mE_M(\Omega) \)). Standard references about these spaces include [11], [1], [12].

A well-known theorem of Meyers–Serrin [13] states that for \( 1 < p < \infty \), \( C^m(\Omega) \cap W^{m,p}(\Omega) \) is norm dense in \( W^{m,p}(\Omega) \), i.e. weak derivatives in \( L^p(\Omega) \) are globally strong derivatives (the local version goes back to Friedrichs and his mollifiers [7]). This was extended to the Orlicz spaces setting by Donaldson–Trudinger [4] who proved that \( C^m(\Omega) \cap W^mE_M(\Omega) \) is norm dense in \( W^mE_M(\Omega) \). The corresponding statement with \( E_M(\Omega) \) replaced by \( L_M(\Omega) \) is not true, even locally, simply because an \( L_M(\Omega) \) function may not belong to \( E_M(\Omega) \) for \( \Omega \subseteq \Omega \) (take \( \Omega = ]-1, -1[ \), \( M(t) = t^{\delta - 1} \) and \( u(x) = (\log |x|)^{-\Theta} \)). Our first result concerns the density of \( C^m(\Omega) \cap \cap W^mE_M(\Omega) \) in \( W^mL_M(\Omega) \) with respect to a weaker convergence, the so-called modular convergence [14].

**Theorem 1.** Let \( u \in W^mE_M(\Omega) \). Then there exist \( \lambda > 0 \) and a sequence \( u_k \in C^m(\Omega) \cap W^mL_M(\Omega) \) such that for \( |a| \leq m \),

\[
\int_{\Omega} M(|D^a u_k - D^a u|) \lambda \to 0 \quad \text{as} \quad k \to \infty .
\]

As will be seen in the proof, it suffices to choose \( \lambda \) such that \( 16/\lambda \) \( D^a u \in \mathcal{L}_M(\Omega) \) for \( |a| = m \), where \( \mathcal{L}_M(\Omega) \) denotes the Orlicz class. Consequently, when \( u \in W^mE_M(\Omega) \), \( \lambda \) can be taken arbitrary small, and we recover from Theorem 1 the result of [4] mentioned above.

The space \( W^mL_M(\Omega) \) will be, as usual, identified with a subspace of
the product \( \prod_{\mathcal{L}^m/\mathcal{L}} = \prod_{\mathcal{L}} \). Denoting by \( \mathcal{M} \) the \( \mathcal{N} \)-function conjugate to \( \mathcal{M} \), we consider the weak topologies \( \sigma(\prod \mathcal{L}^m, \prod \mathcal{L}) \) and \( \sigma(\prod \mathcal{L}^m, \prod \mathcal{L}) \). Density results involving the latter play an important role in the existence theory for strongly nonlinear boundary value problems (cf. [8], [3]). Comparing the normal convergence with \( \sigma(\prod \mathcal{L}^m, \prod \mathcal{L}) \) (Lemma 6 below), we obtain the following:

**Corollary 2.** \( C^0(\Omega) \cap W^{m,1}(\Omega) \) is \( \sigma(\prod \mathcal{L}^m, \prod \mathcal{L}) \) dense in \( W^{m,1}(\Omega) \).

We now turn to the approximation by functions which are smooth up to the boundary, assuming some regularity on \( \Omega \). Recall that \( \Omega \) is said to have the *segment property* if there exists an open covering \( \{ U_i \} \) of \( \Omega \) and corresponding vectors \( y_i \in \mathbb{R}^n \) such that, for \( x \in \Omega \cap E_i \) and \( 0 < t < 1 \), \( x + ty_i \in \Omega \). \( \Omega \) has the cone property if there exists a finite cone \( \mathcal{C}_d \) such that each \( x \in \Omega \) is the vertex of a cone \( \mathcal{C}_d \) congruent to \( C \). It was proved in [8] that if \( \Omega \) has the segment property, then \( \sigma(\mathcal{D}) \) is \( \sigma(\prod \mathcal{L}^m, \prod \mathcal{L}) \) dense in \( W^{1,1}(\Omega) \), where \( \sigma(\mathcal{D}) \) denotes the restrictions to \( \Omega \) of the functions in \( \sigma(\mathcal{D}) \). This is improved in our next result under the assumption that \( \Omega \) also has the cone property.

**Theorem 3.** Suppose that \( \Omega \) satisfies both the segment and the cone property. Let \( u \in W^{1,1}(\Omega) \). Then there exist \( \lambda > 0 \) and a sequence \( u_n \in \sigma(\mathcal{D}) \) such that for \( |\alpha| \leq m \),

\[
\int_{\Omega} |\nabla^\alpha u - \nabla^\alpha u_n|/|\lambda| \to 0 \quad \text{as} \quad k \to \infty.
\]

The proof will show that it suffices to choose \( \lambda \) such that \( 16(N+1)/2 \lambda^2 \in L^1(\Omega) \) for \( |\alpha| \leq m \). Moreover, the cone property is only used to guarantee that an element \( v \in W^{1,1}(\Omega) \) with compact support in \( \mathcal{D} \) lies in \( L^1(\Omega) \) (the imbedding theorem of [4] implies \( v \in C(\Omega) \cap L^\infty(\Omega) \) or \( v \in L^1(\Omega) \) with \( \mathcal{M} \) an \( N \)-function which, by Lemma 4.14 of [3], increases essentially more rapidly than \( M_0 \) and \( s \), in any case, \( v \in L^1(\Omega) \)). This fact probably holds under a weaker assumption on \( \Omega \). Anyway, taking again \( \lambda \) arbitrary small, we recover from Theorem 3 the result of [4] that if \( \Omega \) has the segment property, then \( \sigma(\mathcal{D}) \) is norm dense in \( W^{1,1}(\Omega) \).

Finally we consider the analogue of the \( W^{1,p} \)-spaces. \( W^{1,p}(\Omega) \) is defined as the \( \sigma(\prod \mathcal{L}^m, \prod \mathcal{L}) \) closure of \( \mathcal{D}(\Omega) \) in \( W^{1,p}(\Omega) \) and \( W^{1,p}(\Omega) \) as the norm closure of \( \mathcal{D}(\Omega) \) in \( W^{1,p}(\Omega) \) (equivalently in \( W^{1,\infty}(\Omega) \)). When \( \mathcal{D} \) is sufficiently regular, one can define the trace on \( \mathcal{D} \) of \( s \) for \( u \in W^{1,p}(\Omega) \) and \( |\alpha| = m-1 \), and prove that the functions in \( W^{1,p}(\Omega) \) (resp. \( W^{1,p}(\Omega) \)) are precisely those in \( W^{1,p}(\Omega) \) (resp. \( W^{1,p}(\Omega) \)) whose trace and normal derivatives up to order \( m-1 \) on \( \partial \Omega \) vanish \( \text{cf. [6], [9]} \).

**Theorem 4.** Suppose that \( \Omega \) satisfies the segment property. Let \( u \in W^{1,p}(\Omega) \). Then there exist \( \lambda > 0 \) and a sequence \( u_n \in \sigma(\mathcal{D}) \) such that for \( |\alpha| \leq m \),

\[
\int_{\Omega} |\nabla^\alpha u - \nabla^\alpha u_n|/|\lambda| \to 0 \quad \text{as} \quad k \to \infty.
\]

The same estimate on \( \lambda \) as in Theorem 3 holds here. Theorem 4 improves our result of [8] if \( \Omega \) has the segment property, then \( \sigma(\mathcal{D}) \) is \( \sigma(\prod \mathcal{L}^m, \prod \mathcal{L}) \) dense in \( W^{1,1}(\Omega) \).

The proofs of the \( W^{1,p}(\Omega) \) results of [4] referred to above as well as those of the \( \sigma(\prod \mathcal{L}^m, \prod \mathcal{L}) \) versions of Theorems 3 and 4 given in [8] are rather simple modifications of the standard \( \nabla^\alpha \) proofs. To get Corollary 2 or Theorems 1, 3 and 4 requires more involved calculations, although the construction of the approximants is basically the same. An inequality which will be used repeatedly is

\[
M \left( \sum_{\gamma\in\Gamma} r_\gamma \right) \leq r^{-1} \sum_{\gamma\in\Gamma} M(r_\gamma)
\]

where \( r_\gamma \in \mathbb{R} \) and \( r \) is the maximum number of nonzero \( r_\gamma \)’s. When the \( r_\gamma \)’s originate from a partition of unity, \( \gamma \)’s control on \( r \) can be obtained from simple topological considerations (Lemma 7 below).

**Proofs.** Given a function \( u(x) \), we denote by \( u_\gamma(x) \) its translate \( u(x-y) \), by \( u_\gamma \) its regularization \( u_\gamma = u + \varepsilon \eta \), where \( \varepsilon \in C(\mathbb{R}^n) \) \( \eta_\gamma \)(\( \gamma \)) \( \left( \varepsilon \in C(\mathbb{R}^n) \right) \) \( \left( \varepsilon \in C(\mathbb{R}^n) \right) \). However, this fact probably holds under a weaker assumption on \( \Omega \). Anyway, taking again \( \lambda \) arbitrary small, we recover from Theorem 3 the result of [4] that if \( \Omega \) has the segment property, then \( \sigma(\mathcal{D}) \) is norm dense in \( W^{1,1}(\Omega) \).

Finally we consider the analogue of the \( W^{1,p} \)-spaces. \( W^{1,p}(\Omega) \) is defined as the \( \sigma(\prod \mathcal{L}^m, \prod \mathcal{L}) \) closure of \( \mathcal{D}(\Omega) \) in \( W^{1,p}(\Omega) \) and \( W^{1,p}(\Omega) \) as the norm closure of \( \mathcal{D}(\Omega) \) in \( W^{1,p}(\Omega) \) (equivalently in \( W^{1,\infty}(\Omega) \)). When \( \mathcal{D} \) is sufficiently regular, one can define the trace on \( \partial \Omega \) of \( u \) for \( u \in W^{1,p}(\Omega) \) and \( |\alpha| = m-1 \), and prove that the functions in \( W^{1,p}(\Omega) \) (resp. \( W^{1,p}(\Omega) \)) are precisely those in \( W^{1,p}(\Omega) \) (resp. \( W^{1,p}(\Omega) \)) whose trace and normal derivatives up to order \( m-1 \) on \( \partial \Omega \) vanish \( \text{cf. [6], [9]} \).
all $\lambda > 0$ and by taking $\lambda$ arbitrary small, we eventually derive from Lemma 5 that $u_0$ converges to $u$ in norm, a result originally given in [2]. Similar remarks apply to $u_0$.

Proof of Theorem 1. Let $u \in W^{m}_L(M)$ and choose $\lambda > 0$ such that $16\lambda D^u \in L^m(M)$ for $|a| = m$. Let $\varepsilon > 0$, $\varepsilon \leq 1$. We will prove that there exists $v \in C^m(M \cap W^{m}_L(M))$ such that

$$
\int_{\partial} M(D^v \cdot D^u(\partial))/|a| \leq \varepsilon
$$

for $|a| \leq m$. Define for $i = 1, 2, \ldots$,

$$
\Omega_i = \{x \in \Omega; |x| < i \text{ and dist}(x, \partial \Omega) > \frac{1}{a_i}\},
$$

and also, for convenience, $\Omega_{-1} = \Omega_{-2} = \Omega = \emptyset$. Let $\{\psi_i; i = 1, 2, \ldots\}$ be a $C^m$ partition of unity on $\Omega$ such that supp$\psi_i \subset \Omega_{i+1} \setminus \Omega_{i-1}$. For each $i = 1, 2, \ldots$, let $\theta_i = \theta_0$ be a mollifier satisfying

$$
\delta_i < \lambda/(i+1)(i+2),
$$

and all $|\beta + \gamma| \leq m$ with $|\gamma| < m$, and

$$
\int_{\partial} M(\delta(|(\psi_i D^v \cdot u_{\theta_i} - D^v \cdot u_{\theta_i})/|a|) \leq \varepsilon/2^i
$$

for $|a| = m$. Here $||u||_{L^m(M)}$ denotes the Luxemburg norm in $L^m(M)$:

$$
||u||_{L^m(M)} = \inf\{h > 0; \int_{\partial} M(u, |a|) \leq h\}
$$

and the number $a$ is defined in terms of the coefficients which appear in Leibniz formula:

$$
a = \max\{\sum_{i \in \mathbb{N}} |a_i|; |a| \leq m\}.
$$

Condition (3) can be fulfilled because on a smooth bounded domain, $W^m_2(M)$ is (compactly) imbedded into $W^{m-1}E_2(M)$ (see the introduction) and so $D^v u_0$, $|\gamma| < m$, lies in $E_2(M)$. Condition (4) can be fulfilled by applying Lemma 3. It follows from (2) that $v(u_0 \cdot \theta_0$ has support in $\Omega_{i+1} \setminus \Omega_{i-2}$. Thus the series

$$
v = \sum_{i \in \mathbb{N}} (\psi_i u_0) \cdot \theta_i
$$

is trivially convergent and $v \in C^m(M)$. The fact that $v \in W^m_2(M)$ will follow once (1) is proved.

To verify (1), take $j = 1, 2, \ldots$ and write

$$
\int_{\partial} M(\delta(\lambda(D^v \cdot D^u(\partial))/|a|) = \int_{\partial} M(\sum_{i \in \mathbb{N}} (\delta(D^v(\psi_i u_0) \cdot \theta_0 - D^v(\psi_i u_0))/|a|) \leq 2^{-1} \int_{\partial} M(2 \sum_{j \in \mathbb{N} \setminus \mathbb{N}} \sum_{|\beta| + |\gamma| \leq m} (\delta(D^v(\psi_j u_0) \cdot \theta_0 - D^v(\psi_j u_0))/|a|) \leq 2^{-1} \int_{\partial} M(2 \sum_{i \in \mathbb{N}} (\delta(|(\psi_i u_0 \cdot \theta_i - D^v(\psi_i u_0))/|a|) \leq I_1 + I_2
$$

where the term $I_2$ does not appear when $|a| < m$. We have

$$
I_1 \leq \varepsilon/2 \delta_{i-1} \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{|\beta| + |\gamma| \leq m} (\delta(|(\psi_j u_0 \cdot \theta_j - D^v(\psi_j u_0))/|a|) \leq \varepsilon/2
$$

by (3) and the definition of the Luxemburg norm. To study $I_1$, we observe that for a.e. $x \in \Omega$,

$$
M(2 \sum_{i \in \mathbb{N}} (\delta(|(\psi_i u_0 \cdot \theta_i - D^v(\psi_i u_0))/|a|) \leq \varepsilon/2 \delta_{i-1} \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{|\beta| + |\gamma| \leq m} (\delta(|(\psi_j u_0 \cdot \theta_j - D^v(\psi_j u_0))/|a|) \leq \varepsilon/2
$$

since at most 4 terms of the sum are nonzero at $x$. Consequently, using (4), we get $I_2 \leq \varepsilon/8$. So

$$
\int_{\partial} M(D^v \cdot D^u(\partial))/|a| \leq \varepsilon
$$

and letting $j \to \infty$, we obtain (1).  

By a simple modification of this proof one can show that the $u_0$'s in the statement of Theorem 1 can be taken so that $D^v u_0 \to D^v u$ in norm for $|a| < m$.

Corollary 2 is a direct consequence of Theorem 1 and the following

**Lemma 6.** Let $u_0, u \in E^m(M)$. If $u_{0} \to u$ with respect to the modular convergence, then $u_{0} \to u$ for $\|u_{0}, u\|_{L^m(M)}$.

**Proof.** Let $\lambda > 0$ be such that $\int_{\partial} M(|u_{0} - u|)/|a| \leq \lambda$. Thus, for a subsequence, $u_{0} \to u$ a.e. in $\Omega$. Take $v \in E^m(M)$, $|a| \leq m$. Then $u_{0} \to u$ for $\|u_{0}, u\|_{L^m(M)}$.

The proofs of Theorems 3 and 4 uses the following lemma from general topology.

**Lemma 7.** Let $A$ be a closed subset of $\mathbb{R}^n$ and let $\{U_i\}$ be an open covering of $A$. Then $\{U_i\}$ can be refined into a locally finite countable open covering of $A$. 


for all $|\beta + \gamma| \leq m$ with $|\gamma| < m$, and

\[
\delta \left( \sum_{i=1}^{t_r} \Phi_i u_i \right) \leq \frac{\eta}{\delta'}
\]

for $|a| = m$. Condition (8) can be fulfilled because $D^m u \in E_{J_l}(D)$ for $|\gamma| < m$ (the cone property is used here, see the introduction). Condition (9) can be fulfilled by two consecutive applications of Lemma 5. Taking $t_i$ and $\delta_i$ smaller if necessary, we can assume that $\text{supp}(\Phi_i u_i) \cap \Omega \subset U_i$.

Define

\[
v = \sum_{i=1}^{t_r} \Phi_i u_i \epsilon \in \mathcal{D}(\Omega)
\]

and observe that by the property of Lemma 7, at each $x \in \Omega$, the sum above contains at most $(N+1)$ nonzero terms. We have

\[
\delta \left( \sum_{i=1}^{t_r} \Phi_i D^m u_i \right) \leq \delta \left( \sum_{i=1}^{t_r} \Phi_i D^m u_i \right) \leq \sum_{i=1}^{t_r} \delta \left( \Phi_i D^m u_i \right) 
\]

where the term $I_2$ does not appear when $|a| < m$. Now

\[
I_2 \leq \frac{\eta}{\delta} \delta \left( \sum_{i=1}^{t_r} \Phi_i D^m u_i \right)
\]

by (8) and the definition of the Luxemburg norm. Since at a.e. $x \in \Omega$,

\[
\delta \left( \sum_{i=1}^{t_r} \Phi_i D^m u_i \right) \leq \delta \left( \sum_{i=1}^{t_r} \Phi_i D^m u_i \right) 
\]

we obtain from (9) that $I_2 \leq \frac{\eta}{\delta}$, and so (7) is proved. $\blacksquare$

Proof of Theorem 4. It is essentially the same as that of Theorem 3 except that one replaces in (10) $\delta_i$ by $-\epsilon_i$ and chooses $\delta_i$ with

\[
\delta_i < \text{dist}(\text{supp} \Phi_i \cap \Omega) + \epsilon_i y_i, R^N \setminus D.
\]
One also uses the fact that by extending a $W^m_0 L^p(\Omega)$ function by zero outside $\Omega$, one gets a $W^m L^p(\mathbb{R}^n)$ function. This allows us to avoid the cone property for $\Omega$. □

We remark that when $\Omega$ is bounded, the above arguments can be carried through without using Lemma 7. The coefficient $\lambda$ is then chosen so that $8b/\lambda \bar{D}^p u \in L^p(\Omega)$ for $|\alpha| = m$, where $b$ is the number of pieces of the covering $\{U_j\}$ needed to cover $\Omega$.

As for Theorem 1, one can show by a simple modification of the proofs that the $u_{k,l}$'s in the statement of Theorems 3 and 4 can be taken so that $\bar{D}^\nu u_{k,l} \to \bar{D}^\nu u$ in norm for $|\nu| < m$.

References


DÉPARTEMENT DE MATHÉMATIQUES
C.P. 214, UNIVERSITÉ Libre DE BRUXELLES
1050 Bruxelles, Belgium

Received September 22, 1980 (1639)

STUDIA MATHEMATICA, LXXIV. (1982)

A simple proof of the atomic decomposition for $H^p(\mathbb{R}^n)$, $0 < p \leq 1$

by

J. MICHAEL WILSON (Los Angeles, Calif.)

Abstract. A simple proof of the atomic decomposition for distributions in $H^p(\mathbb{R}^n)$, $0 < p \leq 1$, is given. The proof uses Gossesz's theorem and is a result of Fefferman and Stein. It does not use their "grand" maximal function.

We define $H^p(\mathbb{R}^n)$ to be the space of functions $u_0(x, y)$ harmonic in $\mathbb{R}^n_{x+y}$ for which the maximal function $M_\alpha u_0(x) = \sup \frac{|u_0(x,y)|}{|\text{dist}(x,y)|}$ is in $L^p(\mathbb{R}^n)$. We can define an $H^p$ "norm" by

$$||u_0||_{H^p} = ||M_{\alpha}u_0||_{L^p}.$$  

Fefferman and Stein showed that for $u_0 \in H^p$, $\lim_{|\alpha| \to \infty} u_0(x, y) = f$ exists in the sense of tempered distributions and that $f$ uniquely determines $u_0$. We may thus define $H^p$ as a space of distributions with $''\text{norm}''$

$$||f||_{H^p} = ||u_0||_{H^p}.$$  

We call a function $b(x)$ a $p$-atom if:

(A) $b$ is supported on a cube $Q \subset \mathbb{R}^n$.

(B) $||b||_{L^p} \leq |Q|^{1/p} (|Q|)$ is the volume of $Q$.

(C) $\int b(x)z^\alpha dx = 0$ for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq [n(p^{-1})]$, the integer part of $n(p^{-1})$.

We prove the following:

Theorem. A tempered distribution $f$ is in $H^p(\mathbb{R}^n)$, $0 < p \leq 1$, if and only if there exist $p$-atoms $\alpha_i$ and positive numbers $\lambda_i$ such that

$$f = \sum_{i=1}^m \lambda_i \alpha_i$$

in the sense of distributions. The $\lambda_i$ satisfy:

$$C_1 ||f||_{H^p} \leq \sum_{i=1}^m \lambda_i^p \leq C_2 ||f||_{H^p}$$

$C_1$ and $C_2$ only depend on $p$ and $n$. □