STUDIA MATHEMATICA, T. LXXIV. (1982)

Some approximation properties in Orlicz–Sobolev spaces

by

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Abstract. We prove that weak derivatives in general Orlicz spaces are globally strong derivatives with respect to the modular convergence. Other approximation theorems involving the modular convergence are presented, which improve known density results of interest in the existence theory for strongly nonlinear boundary value problems.

Statement of results. Let $L_M(\Omega)$ be the Orlicz space on an open subset Ω of $\boldsymbol{R}^N$ corresponding to an N-function M and let $E_M(\Omega)$ be the norm closure in $L_M(\Omega)$ of the $L^\infty(\Omega)$ functions with compact support in $\bar{\Omega}$. The Sobolev space of functions u such that u and its distributional derivatives up to order m lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$) is denoted by $W^m L_M(\Omega)$ (resp. $W^m E_M(\Omega)$). Standard references about these spaces include [11], [1], [12].

A well-known theorem of Meyers–Serrin [13] states that for $1 \leqslant p < \infty$, $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is norm dense in $W^{m,p}(\Omega)$, i.e. weak derivatives in $L^p(\Omega)$ are globally strong derivatives (the local version goes back to Friedrichs and his mollifiers [7]). This was extended to the Orlicz spaces setting by Donaldson–Trudinger [4] who proved that $C^\infty(\Omega) \cap W^m E_M(\Omega)$ is norm dense in $W^m E_M(\Omega)$. The corresponding statement with E_M replaced by L_M is not true, even locally, simply because an $L_M(\Omega)$ function may not belong to $E_M(\Omega')$ for $\Omega' \Subset \Omega$ (take $\Omega =]-1, +1[$, $M(t) = e^{t^2} - 1$ and $u(x) = (\log |x|^{-1})^{1/2}$). Our first result concerns the density of $C^\infty(\Omega) \cap W^m L_M(\Omega)$ in $W^m L_M(\Omega)$ with respect to a weaker convergence, the so-called modular convergence [14].

THEOREM 1. *Let $u \in W^m L_M(\Omega)$. Then there exist $\lambda > 0$ and a sequence $u_k \in C^\infty(\Omega) \cap W^m L_M(\Omega)$ such that for $|\alpha| \leqslant m$,*

$$\int_\Omega M\big((D^\alpha u_k - D^\alpha u)/\lambda\big) \to 0 \quad \text{as} \quad k \to \infty.$$

As will be seen in the proof, it suffices to choose λ such that $16/\lambda\ D^\alpha u \in \mathscr{L}_M(\Omega)$ for $|\alpha| = m$, where $\mathscr{L}_M(\Omega)$ denotes the Orlicz class. Consequently, when $u \in W^m E_M(\Omega)$, λ can be taken arbitrary small, and we recover from Theorem 1 the result of [4] mentioned above.

The space $W^m L_M(\Omega)$ will be, as usual, identified with a subspace of the product $\prod_{|\alpha|\leqslant m} L_M(\Omega) \equiv \prod L_M$. Denoting by $\overline{M}$ the N-function conjugate to M, we consider the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$. Density results involving the later play an important role in the existence theory for strongly nonlinear boundary value problems (cf. [8], [3]). Comparing the modular convergence with $\sigma(\prod L_M, \prod L_{\overline{M}})$ (Lemma 6 below), we obtain the following

COROLLARY 2. *$C^\infty(\Omega)\cap W^mL_M(\Omega)$ is $\sigma(\prod L_M, \prod L_{\overline{M}})$ dense in $W^mL_M(\Omega)$.*

We now turn to the approximation by functions which are smooth up to the boundary, assuming some regularity on Ω. Recall that Ω is said to have the *segment property* if there exist an open covering $\{U_i\}$ of $\overline{\Omega}$ and corresponding vectors $\{y_i \in \boldsymbol{R}^N\}$ such that, for $x \in \overline{\Omega}\cap U_i$ and $0 < t < 1$, $x+ty_i \in \Omega$; Ω has the *cone property* if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a cone $C_x \subset \Omega$ congruent to C. It was proved in [8] that if Ω has the segment property, then $\mathscr{D}(\overline{\Omega})$ is $\sigma(\prod L_M, \prod L_{\overline{M}})$ dense in $W^mL_M(\Omega)$, where $\mathscr{D}(\overline{\Omega})$ denotes the restrictions to Ω of the functions in $\mathscr{D}(\boldsymbol{R}^N)$. This is improved in our next result under the assumption that Ω also has the cone property.

THEOREM 3. *Suppose that Ω satisfies both the segment and the cone property. Let $u \in W^mL_M(\Omega)$. Then there exist $\lambda > 0$ and a sequence $u_k \in \mathscr{D}(\overline{\Omega})$ such that for $|\alpha| \leqslant m$,*

$$\int_\Omega M\big((D^\alpha u_k - D^\alpha u)/\lambda\big)\to 0 \quad \text{as} \quad k\to\infty.$$

The proof will show that it suffices to choose λ such that $16(N+1)/\lambda\, D^\alpha u \in \mathscr{L}_M(\Omega)$ for $|\alpha| \leqslant m$. Moreover, the cone property is only used to guarantee that an element $v \in W^1L_M(\Omega)$ with compact support in $\overline{\Omega}$ lies in $E_M(\Omega)$ (the imbedding theorem of [4] implies $v \in C(\Omega)\cap L^\infty(\Omega)$ or $v \in L_{M^*}(\Omega)$ with M^* an N-function which, by Lemma 4.14 of [8], increases essentially more rapidly than M, and so, in any case, $v \in E_M(\Omega)$). This fact probably holds under a weaker assumption on Ω. Anyway, taking again λ arbitrary small, we recover from Theorem 3 the result of [4] that if Ω has the segment property, then $\mathscr{D}(\overline{\Omega})$ is norm dense in $W^mE_M(\Omega)$.

Finally we consider the analogue of the $W_0^{m,p}$ spaces. $W_0^mL_M(\Omega)$ is defined as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathscr{D}(\Omega)$ in $W^mL_M(\Omega)$ and $W_0^mE_M(\Omega)$ as the norm closure of $\mathscr{D}(\Omega)$ in $W^mL_M(\Omega)$ (or equivalently in $W^mE_M(\Omega)$). When $\partial\Omega$ is sufficiently regular, one can define the trace on $\partial\Omega$ of $D^\alpha u$ for $u \in W^mL_M(\Omega)$ and $|\alpha| \leqslant m-1$, and prove that the functions in $W_0^mL_M(\Omega)$ (resp. $W_0^mE_M(\Omega)$) are precisely those in $W^mL_M(\Omega)$ (resp. $W^mE_M(\Omega)$) whose trace and normal derivatives up to order $m-1$ on $\partial\Omega$ vanish (cf. [6], [9]).

THEOREM 4. *Suppose that Ω satisfies the segment property. Let $u \in W_0^mL_M(\Omega)$. Then there exist $\lambda > 0$ and a sequence $u_k \in \mathscr{D}(\Omega)$ such that for $|\alpha| \leqslant m$,*

$$\int_\Omega M\big((D^\alpha u_k - D^\alpha u)/\lambda\big)\to 0 \quad \text{as} \quad k\to\infty.$$

The same estimate on λ as in Theorem 3 holds here. Theorem 4 improves our result of [8] that if Ω has the segment property, then $\mathscr{D}(\Omega)$ is $\sigma(\prod L_M, \prod L_{\overline{M}})$ dense in $W_0^mL_M(\Omega)$.

The proofs of the $W^mE_M(\Omega)$ results of [4] referred to above as well as those of the $\sigma(\prod L_M, \prod L_{\overline{M}})$ versions of Theorems 3 and 4 given in [8] are rather simple modifications of the standard L^p proofs. To get Corollary 2 or Theorems 1, 3 and 4 requires more involved calculations, although the construction of the approximants is basically the same. An inequality which will be used repeatedly is

$$M\Big(\sum_i r_i\Big) \leqslant r^{-1}\sum_i M(rr_i)$$

where $r_i \in \boldsymbol{R}$ and r is the maximum number of nonzero r_i's. When the r_i's originate from a partition of unity, a control on r can be obtained from simple topological considerations (Lemma 7 below).

Proofs. Given a function $u(x)$, we denote by u_y its translate $u_y(x) = u(x-y)$, by u_δ its regularization $u_\delta = u*\varrho_\delta$ where $\varrho_\delta \in \mathscr{D}(\boldsymbol{R}^N)$, $\varrho_\delta(x) = 0$ for $|x| \geqslant \delta$, $\varrho_\delta \geqslant 0$ and $\int_{\boldsymbol{R}^N} \varrho_\delta = 1$, and by u_r the function $u_r = \varphi_r u$ where $\varphi_r(x) = \varphi(x/r)$ with $\varphi \in \mathscr{D}(\boldsymbol{R}^N)$, $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ for $|x| \leqslant 1$ and $\varphi(x) = 0$ for $|x| \geqslant 2$. The following lemma will be needed.

LEMMA 5. *If $u \in L_M(\boldsymbol{R}^N)$ with $2u \in \mathscr{L}_M(\boldsymbol{R}^N)$, then $\int_{\boldsymbol{R}^N} M(u_y - u)\to 0$ as $|y|\to 0$ and $\int_{\boldsymbol{R}^N} M(u_\delta - u)\to 0$ as $\delta\to 0$. If $u \in \mathscr{L}_M(\boldsymbol{R}^N)$, then $\int_{\boldsymbol{R}^N} M(u_r - u)\to 0$ as $r\to\infty$.*

Proof. We consider u_δ; the case of u_y is treated similarly and that of u_r is immediate. Since $u_\delta\to u$ in $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$, one has, for a subsequence, $u_\delta\to u$ a.e. and so $M(u_\delta - u)\to 0$ a.e. Moreover,

$$M(u_\delta - u) \leqslant \tfrac{1}{2}M(2u_\delta) + \tfrac{1}{2}M(2u) \leqslant \tfrac{1}{2}\big(M(2u)\big)_\delta + \tfrac{1}{2}M(2u)$$

where we have used Jensen's integral inequality. Since $\big(M(2u)\big)_\delta\to M(2u)$ in $L^1(\boldsymbol{R}^N)$, we conclude by Vitali's theorem that $M(u_\delta - u)\to 0$ in $L^1(\boldsymbol{R}^N)$. ∎

Thus we see that for any $u \in L_M(\boldsymbol{R}^N)$, $u_\delta\to u$ with respect to the modular convergence (i.e. there exists $\lambda > 0$ such that $\int_{\boldsymbol{R}^N} M\big((u_\delta - u)/\lambda\big)\to 0$). In general, u_δ does not converge to u in the mean (i.e. the above with $\lambda = 1$) and a fortiori in norm. If $u \in E_M(\boldsymbol{R}^N)$, then $2u/\lambda \in \mathscr{L}_M(\boldsymbol{R}^N)$ for all $\lambda > 0$ and by taking λ arbitrary small, we eventually derive from Lemma 5 that u_δ converges to u in norm, a result originally given in [2]. Similar remarks apply to u_y.

Proof of Theorem 1. Let $u \in W^m L_M(\Omega)$ and choose $\lambda > 0$ such that $16/\lambda D^\alpha u \in \mathscr{L}_M(\Omega)$ for $|\alpha| = m$. Let $\varepsilon > 0$, $\varepsilon \leqslant 1$. We will prove that there exists $v \in C^\infty(\Omega) \cap W^m L_M(\Omega)$ such that

$$\int_\Omega M\big((D^\alpha v - D^\alpha u)/\lambda\big) \leqslant \varepsilon \tag{1}$$

for $|\alpha| \leqslant m$. Define for $i = 1, 2, \ldots,$

$$\Omega_i = \{x \in \Omega;\ |x| < i \text{ and } \operatorname{dist}(x, \partial\Omega) > 1/i\},$$

and also, for convenience, $\Omega_{-2} = \Omega_{-1} = \Omega_0 = \emptyset$. Let $\{\psi_i;\ i = 1, 2, \ldots\}$ be a C^∞ partition of unity on Ω such that $\operatorname{supp}\psi_i \subset \Omega_{i+1} \setminus \Omega_{i-1}$. For each $i = 1, 2, \ldots$, let $\varrho_{\delta_i} \equiv \varrho_i$ be a mollifier satisfying

$$\delta_i < 1/(i+1)(i+2), \tag{2}$$

$$\|(D^\beta \psi_i D^\gamma u) * \varrho_i - D^\beta \psi_i D^\gamma u\|_{M,\Omega} \leqslant \varepsilon\lambda/a2^{i+1} \tag{3}$$

for all $|\beta + \gamma| \leqslant m$ with $|\gamma| < m$, and

$$\int_\Omega M\big(8((\psi_i D^\alpha u) * \varrho_i - \psi_i D^\alpha u)/\lambda\big) \leqslant \varepsilon/2^i \tag{4}$$

for $|\alpha| = m$. Here $\|\ \|_{M,\Omega}$ denotes the Luxemburg norm in $L_M(\Omega)$:

$$\|w\|_{M,\Omega} = \inf\Big\{h > 0;\ \int_\Omega M(w/h) \leqslant 1\Big\}$$

and the number a is defined in terms of the coefficients which appear in Leibniz formula:

$$a = \max\Big\{\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta};\ |\alpha| \leqslant m\Big\}. \tag{5}$$

Condition (3) can be fulfilled because on a smooth bounded domain, $W^m L_M$ is (compactly) imbedded into $W^{m-1}E_M$ (see the introduction) and so $D^\gamma u$, $|\gamma| < m$, lies in $E_M^{\mathrm{loc}}(\Omega)$. Condition (4) can be fulfilled by applying Lemma 5. It follows from (2) that $(\psi_i u) * \varrho_i$ has support in $\Omega_{i+2} \setminus \Omega_{i-2}$. Thus the series

$$v = \sum_{i=1}^\infty (\psi_i u) * \varrho_i$$

is trivially convergent and $v \in C^\infty(\Omega)$. The fact that $v \in W^m L_M(\Omega)$ will follow once (1) is proved.

To verify (1), take $j = 1, 2, \ldots$ and write

$$\begin{aligned}\int_{\Omega_j} M\big((D^\alpha v - D^\alpha u)/\lambda\big) &= \int_{\Omega_j} M\Big(\sum_{i=1}^{j+1} \big(D^\alpha(\psi_i u) * \varrho_i - D^\alpha(\psi_i u)\big)/\lambda\Big) \\ &\leqslant 2^{-1} \int_{\Omega_j} M\Big(2\sum_{i=1}^{j+1} \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<m}} \binom{\alpha}{\beta} \big((D^\beta \psi_i D^\gamma u) * \varrho_i - D^\beta \psi_i D^\gamma u\big)/\lambda\Big) + \\ &\quad + 2^{-1} \int_{\Omega_j} M\Big(2\sum_{i=1}^{j+1} \big((\psi_i D^\alpha u) * \varrho_i - \psi_i D^\alpha u\big)/\lambda\Big) = I_1 + I_2\end{aligned}$$

where the term I_2 does not appear when $|\alpha| < m$. We have

$$I_1 \leqslant \varepsilon 2^{-1} a^{-1} \sum_{i=1}^{j+1} 2^{-i} \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<m}} \binom{\alpha}{\beta} \int_{\Omega_j} M\big(2^{i+1} a((D^\beta \psi_i D^\gamma u) * \varrho_i - D^\beta \psi_i D^\gamma u)/\varepsilon\lambda\big) \leqslant \varepsilon/2$$

by (3) and the definition of the Luxemburg norm. To study I_2, we observe that for a.e. $x \in \Omega_j$,

$$M\Big(2\sum_{i=1}^{j+1} \big((\psi_i D^\alpha u) * \varrho_i - \psi_i D^\alpha u\big)/\lambda\Big) \leqslant \tfrac{1}{4} \sum_{i=1}^{j+1} M\big(8((\psi_i D^\alpha u) * \varrho_i - \psi_i D^\alpha u)/\lambda\big)$$

since at most 4 terms of the sum are nonzero at x. Consequently, using (4), we get $I_2 \leqslant \varepsilon/8$. So

$$\int_{\Omega_j} M\big((D^\alpha v - D^\alpha u)/\lambda\big) \leqslant \varepsilon$$

and letting $j \to \infty$, we obtain (1). ∎

By a simple modification of this proof one can show that the u_k's in the statement of Theorem 1 can be taken so that $D^\alpha u_k \to D^\alpha u$ in norm for $|\alpha| < m$.

Corollary 2 is a direct consequence of Theorem 1 and the following

LEMMA 6. *Let $u_k, u \in L_M(\Omega)$. If $u_k \to u$ with respect to the modular convergence, then $u_k \to u$ for $\sigma(L_M, L_{\overline{M}})$.*

Proof. Let $\lambda > 0$ be such that $\int_\Omega M((u_k - u)/\lambda) \to 0$. Thus, for a subsequence, $u_k \to u$ a.e. in Ω. Take $v \in L_{\overline{M}}(\Omega)$. Multiplying v by a suitable constant, we can assume $\lambda v \in \mathscr{L}_{\overline{M}}(\Omega)$. By Young's inequality,

$$|(u_k - u)v| \leqslant M\big((u_k - u)/\lambda\big) + \overline{M}(\lambda v),$$

which implies, by Vitali's theorem, that $\int_\Omega |(u_k - u)v| \to 0$. ∎

The proofs of Theorems 3 and 4 uses the following lemma from general topology.

LEMMA 7. *Let A be a closed subset of $\mathbf{R}^N$ and let $\{U_i\}$ be an open covering of A. Then $\{U_i\}$ can be refined into a locally finite countable open covering*

$\{V_j\}$ of A with the property that at most $N+1$ distinct V_j's have a nonempty intersection.

Lemma 7 follows from the fact that the covering dimension of $\boldsymbol{R}^N$ is N (cf. [5]). All we need actually for our purposes is an estimate on the maximum number of intersecting V_j's which is independent of A and of the covering $\{U_i\}$. Such an estimate can be obtained directly by elementary means.

Proof of Theorem 3. Let $u \in W^m L_M(\Omega)$ and choose $\lambda > 0$ such that $16(N+1)/\lambda\, D^\alpha u \in \mathscr{L}_M(\Omega)$ for $|\alpha| \leqslant m$. We will show that for $|\alpha| \leqslant m$,

$$\int_\Omega M\big(2(D^\alpha u - D^\alpha u_r)/\lambda\big) \to 0 \tag{6}$$

as $r \to \infty$ where $u_r = \varphi_r u$ is the function involved in Lemma 5. We will also show that for each r and for each $\eta > 0$, $\eta \leqslant 1$, there exists $v \in \mathscr{D}(\overline{\Omega})$ such that for $|\alpha| \leqslant m$.

$$\int_\Omega M\big(2(D^\alpha u_r - D^\alpha v)/\lambda\big) \leqslant \eta. \tag{7}$$

The conclusion of Theorem 3 then follows easily.

To verify (6), we write

$$\int_\Omega M\big(2(D^\alpha u - D^\alpha u_r)/\lambda\big) \leqslant 2^{-1} \int_\Omega M\big(4(D^\alpha u - \varphi_r D^\alpha u)/\lambda\big) + \\ + 2^{-1} a^{-1} \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|>0}} \binom{\alpha}{\beta} \int_\Omega M\big(4ar^{-|\beta|}(D^\beta \varphi)(x/r) D^\gamma u/\lambda\big).$$

Here a is the number defined by (5). Each term on the right hand side goes to zero as $r \to \infty$, the first one by Lemma 5 and the second one by direct examination.

To verify (7) we use the covering $\{U_i\}$ of $\overline{\Omega}$ given by the segment property. Refining $\{U_i\}$ if necessary, we can assume that it satisfies the properties stated in Lemma 7. In particular, $i = 1, 2, \ldots$ Let ψ_i be a C^∞ partition of unity on $\overline{\Omega}$ subordinate to $\{U_i\}$. Clearly $\operatorname{supp}\psi_i \subset U_i'$ for some open set U_i' with compact closure $\overline{U}_i' \subset U_i$. Write $\Gamma_i = \overline{U}_i' \cap \partial\Omega$ and $\Gamma_{i,t} = \Gamma_i - ty_i$ where y_i is the vector associated to U_i by the segment property. Extend u_r outside Ω by zero and note that $\psi_i u_r$ vanishes identically for $i \geqslant$ some i_r. Note also that the translate $(\psi_i u_r)_t(x) = (\psi_i u_r)(x + ty_i)$, $0 < t < 1$, belongs to $W^m L_M(\boldsymbol{R}^N \setminus \Gamma_{i,t})$ and that, by the segment property, $\operatorname{dist}(\Gamma_{i,t}, \Omega) > 0$. For each $i = 1, 2, \ldots, i_r$, choose $0 < t_i < 1$ and $\varrho_{\delta_i} \equiv \varrho_i$, $\delta_i < \operatorname{dist}(\Gamma_{i,t_i}, \Omega)$, such that

$$\|(D^\beta \psi_i D^\gamma u_r)_{t_i} * \varrho_i - D^\beta \psi_i D^\gamma u_r\|_{M,\Omega} \leqslant \lambda\eta/2^{i+2} a \tag{8}$$

for all $|\beta+\gamma| \leqslant m$ with $|\gamma| < m$, and

$$\int_\Omega M\big(4(N+1)\big((\psi_i D^\alpha u_r)_{t_i} * \varrho_i - \psi_i D^\alpha u_r\big)/\lambda\big) \leqslant \eta/2^i \tag{9}$$

for $|\alpha| = m$. Condition (8) can be fulfilled because $D^\gamma u_r \in E_M(\Omega)$ for $|\gamma| < m$ (the cone property is used here, see the introduction). Condition (9) can be fulfilled by two consecutive applications of Lemma 5. Taking t_i and δ_i smaller if necessary, we can assume that $\operatorname{supp}(\psi_i u_r)_{t_i} * \varrho_i \subset U_i$. Define

$$v = \sum_{i=1}^{i_r} (\psi_i u_r)_{t_i} * \varrho_i \in \mathscr{D}(\overline{\Omega}) \tag{10}$$

and observe that by the property of Lemma 7, at each $x \in \Omega$, the sum above contains at most $(N+1)$ nonzero terms. We have

$$\int_\Omega M\big(2(D^\alpha v - D^\alpha u_r)/\lambda\big) = \int_\Omega M\Big(2\sum_{i=1}^{i_r}\big((D(\psi_i u_r))_{t_i} * \varrho_i - D^\alpha(\psi_i u_r)\big)/\lambda\Big) \\ \leqslant 2^{-1} \int_\Omega M\Big(4\sum_{i=1}^{i_r} \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<m}} \binom{\alpha}{\beta} \big((D^\beta \psi_i D^\gamma u_r)_{t_i} * \varrho_i - D^\beta \psi_i D^\gamma u_r\big)/\lambda\Big) \\ + 2^{-1} \int_\Omega M\Big(4\sum_{i=1}^{i_r}\big((\psi_i D^\alpha u_r)_{t_i} * \varrho_i - \psi_i D^\alpha u_r\big)/\lambda\Big) = I_1 + I_2$$

where the term I_2 does not appear when $|\alpha| < m$. Now

$$I_1 \leqslant \eta 2^{-1} a^{-1} \sum_{i=1}^{i_r} 2^{-i} \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<m}} \int_\Omega M\big(2^{i+2} a\big((D^\beta \psi_i D^\gamma u_r)_{t_i} * \varrho_i - D^\beta \psi_i D^\gamma u_r\big)\lambda\eta\big) \\ \leqslant \eta/2$$

by (8) and the definition of the Luxemburg norm. Since at a.e. $x \in \Omega$,

$$M\Big(4\sum_{i=1}^{i_r}\big((\psi_i D^\alpha u_r)_{t_i} * \varrho_i - \psi_i D^\alpha u_r\big)/\lambda\Big) \\ \leqslant (N+1)^{-1} \sum_{i=1}^{i_r} M\big(4(N+1)\big((\psi_i D^\alpha u_r)_{t_i} * \varrho_i - \psi_i D^\alpha u_r\big)/\lambda\big),$$

we obtain from (9) that $I_2 \leqslant \eta/2$, and so (7) is proved. ∎

Proof of Theorem 4. It is essentially the same as that of Theorem 3 except that one replaces in (10) t_i by $-t_i$ and chooses δ_i with

$$\delta_i < \operatorname{dist}\big((\operatorname{supp}\psi_i \cap \overline{\Omega}) + t_i y_i,\ \boldsymbol{R}^N \setminus \Omega\big).$$

One also uses the fact that by extending a $W_0^m L_M(\Omega)$ function by zero outside Ω, one gets a $W^m L_M(\mathbf{R}^N)$ function. This allows us to avoid the cone property for Ω. ∎

We remark that when Ω is bounded, the above arguments can be carried through without using Lemma 7. The coefficient λ is then chosen so that $8b/\lambda\, D^\alpha u \in \mathscr{L}_M(\Omega)$ for $|\alpha| = m$, where b is the number of pieces of the covering $\{U_i\}$ needed to cover $\bar{\Omega}$.

As for Theorem 1, one can show by a simple modification of the proofs that the u_k's in the statement of Theorems 3 and 4 can be taken so that $D^\alpha u_k \to D^\alpha u$ in norm for $|\alpha| < m$.

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Received September 22, 1980 (1639)

STUDIA MATHEMATICA, LXXIV. (1982)

A simple proof of the atomic decomposition for $H^p(\mathbf{R}^n)$, $0 < p \leqslant 1$

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Abstract. A simple proof of the atomic decomposition for distributions in $H^p(\mathbf{R}^n)$, $0 < p < 1$, is given. The proof uses Green's Theorem and a result of Fefferman and Stein. It does not use their "grand" maximal function.

We define $H^p(\mathbf{R}^n)$ to be the space of functions $u_0(x, y)$ harmonic in $\mathbf{R}_+^{n+1} = \{(x, y) = x \in \mathbf{R}^n, y > 0\}$ for which the maximal function

$$u_0^*(x) = \sup_{|x-t|<100y\sqrt{n}} |u(t, y)|$$

is in $L^p(\mathbf{R}^n)$. We can define an H^p "norm" by

$$\|u_0\|_{H^p} = \|u_0^*\|_p.$$

Fefferman and Stein showed that for $u_0 \in H^p$, $\lim\limits_{y\to 0} u_0(\circ, y) = f$ exists in the sense of tempered distributions and that f uniquely determines u_0. We may thus define H^p as a space of distributions with "norm" $\|f\|_{H^p} = \|u_0\|_{H^p}$. See [3].

We call a function $b(x)$ a *p-atom* if:

(A) b is supported on a cube $Q \subset \mathbf{R}^n$.

(B) $\|b\|_\infty \leqslant |Q|^{-1/p}$ ($|Q|$ is the volume of Q).

(C) $\int_Q b(x) x^\alpha dx = 0$ for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant [n(p^{-1}-1)]$, the integer part of $n(p^{-1}-1)$.

We prove the following:

THEOREM. *A tempered distribution f is in $H^p(\mathbf{R}^n)$, $0 < p \leqslant 1$, if and only if there exist p-atoms a_i and positive numbers λ_i such that*

$$f = \sum_{i=1}^{\infty} \lambda_i a_i$$

in the sense of distributions. The λ_i satisfy:

$$C_1 \|f\|_{H^p}^p \leqslant \sum \lambda_i^p \leqslant C_2 \|f\|_{H^p}^p \tag{1}$$

C_1 and C_2 only depend on p and n. ∎