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The ideal property of tensor products of Banach lattices with applications to the local structure of spaces of absolutely summing operators

by

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Abstract. Let E be a Banach space and X a Banach lattice with best convexity p_X and best concavity q_X. Under certain assumptions on p_X and q_X we give a necessary and sufficient condition for the pair (E, X) to have the property that for every bounded operator S on X, $I \otimes S$ is a bounded operator on the lattice tensor product $E \otimes_m X$.

This condition shows that for certain E and X we have for every subspace F of X that $\pi_1(E^*, F)$ is equal to the closure $E \tilde{\otimes}_m F$ of $E \otimes F$ in $E \otimes_m X$. This is then used to investigate when $\pi_1(E^*, F)$ has local unconditional structure and when it has the uniform approximation property.

Introduction. In this paper we study the question when a pair (E, X) of a Banach space E and a Banach lattice X has the property that there is a constant K so that for every bounded operator S on X, $I \otimes S$ is a bounded operator on the m-tensor product $E \otimes_m X$ with $\|I \otimes S\| \leqslant K\|S\|$, where I denotes the identity operator on E. We then apply these results to the study of the local structure of spaces of absolutely summing operators.

In Section 1 of the paper we give a necessary and sufficient condition for a pair (E, X) to have the property above (called the ideal property in this work) for a rather large class of Banach lattices X. We prove e.g. that if the best concavity q_X of X is attained and $p_X < q_X < 2$ (p_X the best convexity of X) then (E, X) has the ideal property if and only if every bounded operator from l_1 to E^* is q'_X-summing. A similar result holds by duality when $2 < p_X < q_X$. If X contains (l_p^n) and (l_q^n) uniformly on disjoint block for some $p, q, p \leqslant 2 \leqslant q$, then (E, X) has the ideal property if and only if E is isomorphic to a Hilbert space. Section 1 also contains several results on the local structure of $E \otimes_m X$ in case (E, X) has the ideal property.

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In Section 2 we investigate when a pair (E, F) of a Banach space E and a subspace F of a Banach lattice X has the ideal property relative to X, i.e. that there is a constant K so that for every bounded operator $S\colon F\to X$ $I\otimes S$ is a bounded operator from $E\overline{\otimes}_m F$ to $E\otimes_m X$ with $\|I\otimes S\| \leqslant K\|S\|$. Here $E\overline{\otimes}_m F$ denotes the closure of $E\otimes F$ in $E\otimes_m X$.

Since in most cases where (E, F) has the ideal property an operator T belongs to $E\overline{\otimes}_m F$ if and only if it is absolutely summing, the results of Sections 1 and 2 give some estimates of the absolutely summing norm of operators from E^* to F. These estimates are also presented in Section 2.

The results of Sections 1 and 2 reduce the investigation of the local structure of the space $\pi_1(E^*, F)$ of absolutely summing operators from E^* to F to the study of $E\overline{\otimes}_m F$ in case (E, F) has the ideal property relative to X and this is in general much easier. This investigation is done in Section 3. We present here results on when $\pi_1(E^*, F)$ has local unconditional structure and when it has an unconditional basis. Some of these theorems were already proved by Gordon and Lewis [4] using other methods.

The results of the Sections 1 and 2 are then combined with [7] to get theorems, which state that for certain E and F $\pi_1(E, F)$ has the uniform appoximation property. As a corollary we obtain e.g. that if $1\leqslant s\leqslant 2 \leqslant r\leqslant\infty$, $r\neq s'$ unless $s=1,2$ and E is an $\mathscr{L}_r$-space, F an $\mathscr{L}_s$-space complemented in F^{**} then $\pi_1(E, F)$ has the uniform approximation property.

Section 4 contains some further remarks on m-tensor products and some open problems.

0. Notation and preliminaries. In this paper we shall use the terminology and notation commonly used in Banach space theory as it appears in [16] and [17]. All vector spaces are assumed to be over the reals unless otherwise stated.

If E is a Banach space and $(x_j)\subseteq E$ is a finite or infinite sequence then we let $[x_j]$ denote the closed linear span in E of the sequence (x_j). The term "subspace of a Banach space" shall always refer to closed linear subspace.

If $1\leqslant p\leqslant\infty$ then we write $E \hookrightarrow L_p$, respectively $E \hookrightarrow QL_p$, if there is a measure μ so that E is isomorphic to a subspace of $L_p(\mu)$, respectively to a subspace of a quotient of $L_p(\mu)$.

In the paper we shall use several operator ideals and our general reference to this theory is [26]. If E and F are Banach spaces and $1\leqslant p\leqslant\infty$ then $N_p(E, F)$ denotes the space of all p-nuclear operators from E to F with the p-nuclear norm n_p, $I_p(E, F)$ denotes the space of all p-integral operators from E to F equipped with the p-integral norm i_p and $\Pi_p(E, F)$ denotes the space of all p-summing operators from E to F with the p-summing norm π_p. Finally, $\Gamma_\infty(E, F)$ shall denote the space of all operators $T\colon E\to F$, for which there is a factorization $T = BA$, $A\colon E\to C(S)$, $B\colon C(S)\to F$, where S is a compact Hausdorff space and A and B are bounded. We put $\gamma_\infty(T) = \inf\{\|A\|\,\|B\|\}$ where the inf is extended over all possible factorizations of the form above. γ_∞ is a norm on $\Gamma_\infty(E, F)$ turning it into a Banach space.

If $\mathscr{A}(E, F)$ is one of the operator ideals above we denote by $\mathscr{A}^f(E, F)$ the closure in $\mathscr{A}(E, F)$ of the space of all finite dimensional operators from E to F.

$B(E, F)$ is the space of all bounded operators from E to F equipped with the operator norm. We put $B(E) = B(E, E)$.

If X is a Banach lattice and f is a continuous 1-homogeneous real function on $\mathbf{R}^n$ then it follows from [17], Section 1 that for all finite set $(x_j)_{j=1}^n\subseteq X$ the expression $f(x_1, x_2, \ldots, x_n)$ can be given a unique meaning as an element of X. This calculus of 1-homogeneous expressions in Banach lattices was first developed by Krivine [10] and it is among other things used to define the concept of p-convexity and q-concavity of Banach lattices [10], [17].

Throughout the paper we put

$$p_X = \sup\{p \mid 1\leqslant p\leqslant\infty,\ X \text{ is } p\text{-convex}\},$$
$$q_X = \inf\{q \mid 1\leqslant q\leqslant\infty,\ X \text{ is } q\text{-concave}\}.$$

If E is a Banach space and X is a Banach lattice then we recall that an operator $T\in B(E, X)$ is called *order bounded* if T maps the unit ball of E into an order bounded subset of X. The space of all order bounded operators from E to X is denoted by $\mathscr{B}(E, X)$. If $T\in\mathscr{B}(E, X)$ then the order bounded norm $\|T\|_m$ of T is defined by

$$\|T\|_m = \inf\{\|z\| \mid z\in X,\ |Tx|\leqslant\|x\|z \text{ for all } x\in E\}.$$

$\mathscr{B}(E, X)$ is a Banach space under the norm $\|\ \|_m$ [23].

Further we let $E\otimes_m X$ denote the closure of $E\otimes X$ in $\mathscr{B}(E^*, X)$. The properties of this tensor product was investigated in [7]. If $e_1, e_2, \ldots, e_n\in E$ and we put $f(t_1, t_2, \ldots, t_n) = \|\sum_{j=1}^n t_je_j\|_E$ for all $(t_1, t_2, \ldots, t_n)\in\mathbf{R}^n$ then f is continuous and 1-homogeneous and therefore the expression $\|\sum_{j=1}^n x_je_j\|_E$ has a well-defined meaning as an element of X for all $x_1, x_2, \ldots, x_n\in X$. If $T = \sum_{j=1}^n e_j\otimes x_j\in E\otimes X$ then it follows from [7], Lemma 1.1 that $\|T\|_m = \|\,\|\sum_{j=1}^n x_je_j\|_E\|_X$. This fact will be used extensively in this paper.

We shall call X a *$\mathscr{P}$-lattice* if there is a positive bounded projection of X^{**} onto X. A fact which shall be very useful for us is:

0.1. LEMMA. *If X is an order continuous $\mathscr{P}$-lattice then X is weakly sequentially complete (hence there is a band projection of X^{**} onto X).*

Proof. From [17], Theorem 1.c.4, it follows that is is enough to show that every norm bounded increasing sequence in X is convergent. Hence let $(x_n) \subseteq X$ with $x_n \leqslant x_{n+1}$ for all n and $\sup_n \|x_n\| < \infty$. Define

$$x^{**}(x^*) = \lim_n x^*(x_n) \quad \text{for all } x^* \in X^*. \tag{1}$$

Clearly $x^{**} \in X^{**}$.

If P is a positive bounded projection of X^{**} onto X then $x_n \leqslant Px^{**}$ for all $n \in \boldsymbol{N}$. Since $Px^{**} \in X$ and X is order continuous, it follows that (x_n) is convergent. ■

If $(\Omega, \mathscr{S}, \mu)$ is a measure space and E is a Banach space then for $1 \leqslant p \leqslant \infty$ we let $L_p(\mu, E)$ denote the space of those measurable functions $f\colon \Omega \to E$ for which $\int \|f\|^p d\mu < \infty$ (ess sup $\|f\| < \infty$ if $p = \infty$). It is readily verified that for $1 \leqslant p \leqslant \infty$ we have $E \otimes_m L_p(\mu) = L_p(\mu, E)$ [7]. The abreviation RNP stands for the Radon–Nikodym property.

Finally, if $1 \leqslant p \leqslant \infty$ we let p' denote the dual number to p, i.e. $1/p + 1/p' = 1$.

1. Some general results on $E \otimes_m X$. Throughout this paper E and F will denote Banach spaces and X and Y Banach lattices. If $F \subseteq X$ is a subspace we put

$$E \overline{\otimes}_m F = \overline{E \otimes F}^{E \otimes_m X}.$$

1.1. DEFINITION. We shall say that the pair (E, X) *has the ideal property* if there is a constant K, so that for every $T \in E \otimes_m X$ and every $S \in B(X)$ $ST \in E \otimes_m X$ and $\|ST\|_m \leqslant K \|S\| \|T\|_m$, in other words if $I \otimes S$ is a bounded operator on $E \otimes_m X$ for every $S \in B(X)$, where I denotes the identity operator on E. If $F \subseteq X$ is a subspace of X then we say that (E, F) *has the ideal property relative to* X if the above holds for every $S \in B(F, X)$.

Kwapień [13] proved that if μ is a measure and $1 \leqslant p < \infty$ then $(E, L_p(\mu))$ has the ideal property if and only if $E \hookrightarrow QL_p$.

Other examples are: $X = L_1(\mu)$, $X = C(S)$, S compact, $X = c_0(\Gamma)$ and E arbitrary; X arbitrary and E isomorphic to a Hilbert space. The case $X = L_1(\mu)$ follows from a result of Grothendieck [5] which states that $E \otimes_m L_1(\mu) = N_1(E^*, L_1(\mu))$ for all E (it also follows from Kwapień's result above), the case where $X = C(S)$ is immediate and the case $X = c_0(\Gamma)$ follows from [23], Chapter 4, since an operator with range in $c_0(\Gamma)$ is order bounded if and only if it is compact. If E is isomorphic to a Hilbert space and X is arbitrary then the result follows from Grothendieck's inequality but we shall treat it below for the sake of completeness. Later in this

section we shall show that for certain X (E, X) has the ideal property if and only if $B(l_1, E^*) = \Pi_p(l_1, E^*)$, where p depends on X.

It follows immediately that if (E, X) has the ideal property then so does (E_1, X) for every subspace E_1 of E and if (E_1, X) has the ideal property with the same constant for every separable subspace E_1 of E, then so does (E, X). By [30] $(E \otimes_m X)^* = \mathscr{B}(E, X^*)$, where the duality is given by

$$\langle S, T\rangle = \operatorname{trace}(S^* T) \quad \text{for} \quad T \in E \otimes_m X,\ S \in \mathscr{B}(E, X^*). \tag{1}$$

Using (1) it is readily verified that (E, X) has the ideal property if and only (E^*, X^*) has. Combining these results we get that if (E, X) has the ideal property so does (F, X) for every subspace F of a quotient of E.

The following theorem which can be found in [23], Proposition 4.9 will be very useful for us in the sequel.

1.2. THEOREM. *Let X be a $\mathscr{P}$-lattice. If $T \in B(E, F)$ with T^* absolutely summing then $ST \in \mathscr{B}(E, X)$ for every $S \in B(F, X)$ and $\|ST\|_m \leqslant \|S\| \pi_1(T^*)$.*

If F is finite dimensional then it is easily seen that Theorem 1.2 holds for every Banach lattice X.

1.3. PROPOSITION. *If $T \in l_2 \otimes_m X$ then*

$$K_G^{-1} \pi_1(T^*) \leqslant \|T\|_m \leqslant \pi_1(T^*), \tag{1}$$

where K_G is Grothendieck's constant. If in addition X is weakly sequentially complete every $T \in B(l_2, X)$ with $T^* \in \Pi_1(X^*, l_2)$ belongs to $l_2 \otimes_m X$.

Proof. Let $T \in l_2 \otimes_m X$. We can then find a compact Hausdorff space S and operators $T_1\colon l_2 \to C(S)$, $T_2\colon C(S) \to X$ so that $\|T_1\| \leqslant 1$, $\|T_2\| = \|T\|_m$, $T_2 \geqslant 0$ and $T = T_2 T_1$ [23]. By Grothendieck's inequality [15] $T_1^* \in \Pi_1(C(S)^*, l_2)$ and

$$\pi_1(T^*) \leqslant \|T_2\| \pi_1(T_1^*) \leqslant K_G \|T_1\| \|T_2\| \leqslant K_G \|T\|_m. \tag{i}$$

The other inequality in (i) follows from Theorem 1.2 for finite dimensional operators and hence by continuity for all $T \in l_2 \otimes_m X$.

If X is weakly sequentially complete and $T \in B(l_2, X)$ with $T^* \in \Pi_1(X^*, l_2)$ then $T \in \mathscr{B}(l_2, X)$ by Theorem 1.2, and since l_2 has the Radon–Nikodym property $\mathscr{B}(l_2, X) = l_2 \otimes_m X$ by [7], Theorem 2.6. ■

The following lemma, the roots of which go back to Grothendieck [5] and Kwapień [12], shall be very useful for us in the sequel.

1.4. LEMMA. (i) *Let E and F be Banach spaces so that $E \hookrightarrow L_1$. If $T \in \Pi_1(E^*, F)$ then $T^* \in \Pi_1(F^*, E^{**})$.*

(ii) *If $2 < p < q$ and $E \hookrightarrow QL_p$ then $B(l_1, E) = \Pi_q(l_1, E)$.*

Proof. (i): Without loss of generality we may assume that E is a subspace of $L_1(\mu)$ for some measure μ. If $T \in \Pi_1(E^*, F)$ then $T^* \in \mathscr{B}(F^*, L_1(\mu)^{**})$ by Theorem 1.2 and therefore $T^* \in \Pi_1(F^*, E^{**})$.

(ii): It is easily seen that it is no loss of generality to assume that E is separable and hence isomorphic to a subspace of a quotient of $L_p(0,1)$. Since the statement is hereditary it suffices to prove it when E is a quotient of $L_p(0,1)$, so let us assume that.

Since E is of cotype p it follows from [20] and [22] that $B(L_\infty(\nu), E) = \Pi_q(L_\infty(\nu), E)$ for every measure ν and hence by the Persson–Pietsch duality theory [26] $\Pi_1(E, G) = \Pi_{q'}(E, G)$ for every Banach space G. We shall use it for $G = l_1$.

Since E^* is a subspace of $L_{p'}(0,1)$ E^* is isomorphic to a subspace of $L_1(0,1)$. Hence if $T \in \Pi_1(E, l_1)$ then $T^* \in \Pi_1(l_\infty, E^*)$ by (i) and hence $T \in \mathscr{B}(E, l_1)$. Clearly $\mathscr{B}(E, l_1) = E^* \otimes_m l_1 = E^* \otimes_\pi l_1 = N_1(E, l_1)$. Combining this with the above we get $N_1(E, l_1) = \Pi_{q'}(E, l_1) = N_{q'}(E, l_1)$ and therefore by duality $B(l_1, E) = \Pi_q(l_1, E)$. ■

We can now show:

1.5. Theorem. 1° *If* $1 < q \leqslant 2$, X *is* q*-concave and* $B(l_1, E^*) = \Pi_{q'}(l_1, E^*)$, *then*

$$\text{(i)} \qquad T \in E \otimes_m X \Leftrightarrow T \in I_q(E^*, X) \Leftrightarrow T^* \in \Pi_1(X^*, E).$$

Dually,

2° *If* $2 \leqslant p < \infty$, X *is* p*-convex and* $B(l_1, E) = \Pi_p(l_1, E)$, *then*

$$\text{(ii)} \qquad T \in E \otimes_m X \Leftrightarrow T^* \in \Pi_p^f(X^*, E) \Leftrightarrow T \in \Gamma_\infty^f(E^*, X).$$

If, furthermore, X is weakly sequentially complete the superscripts "f" can be removed in (ii).

Consequently, if E and X satisfy either 1° *and* 2° (E, X) *has the ideal property.*

Proof. We shall only prove 1°. (ii) in 2° can either be obtained from 1° by using duality theory or proved directly using [7], Theorem 1.3 and it generalization [27], Theorem 1.2. The second statement in 2° follows from the above and [7], Theorem 2.6.

Note that the assumptions on E in 1° imply that $E \hookrightarrow QL_q$ so that E is reflexive.

Let $T \in E \otimes_m X$. Then there exists a compact Hausdorff space S and operators $T_1\colon E^* \to C(S)$, $T_2\colon C(S) \to X$ so that $T = T_2 T_1$, $\|T_1\| \leqslant 1$, $T_2 \geqslant 0$, $\|T_2\| = \|T\|_m$.

Since X is q-concave, T_2 is q-integral by [17], Theorem 1.d.10 and hence T is q-integral as well.

Assume next that $T \in I_q(E^*, X)$. Let μ be a measure so that there is a quotient map S of $L_1(\mu)$ onto E^*. Since S is q'-summing, by assumption it follows from [26] that TS and hence also $S^* T^*$ are 1-integral. S^* is an isometry and therefore $T^* \in \Pi_1(X^*, E)$.

If $T^* \in \Pi_1(X^*, E)$ then $T \in \mathscr{B}(E^*, X)$ by Theorem 1.2 (X is weakly sequentially complete since it is q-concave), but the reflexivity of E imply that $\mathscr{B}(E^*, X) = E \otimes_m X$ ([7], Theorem 2.6). ■

In case of $q = 1$, i.e. X an L_1-space we get Grothendieck's result.

1.6. Proposition. *Let* $(\Omega, \mathscr{S}, \mu)$ *be a measure space and* $T \in B(E^*, L_1(\mu))$. *Then*

$$1^\circ \qquad T \in \mathscr{B}(E^*, L_1(\mu)) \Leftrightarrow T \in I_1(E^*, L_1(\mu)) \Leftrightarrow T^* \in \Pi_1(L_\infty(\mu), E^{**}).$$

2° *If in addition E is isomorphic to a dual space and has the* RNP *then we have for every* $T \in B(E^*, L_1,(\mu))$ *with* $T^*(L_\infty(\mu)) \subseteq E$:

$$\text{(i)} \qquad T \in E \otimes_m L_1(\mu) \Leftrightarrow T \in I_1(E^*, L_1(\mu)) \Leftrightarrow T^* \in \Pi_1(L_\infty(\mu), E).$$

Proof. 1° can easily be proved using the methods of Theorem 1.5 and Theorem 1.2, so let us here only prove 2°.

It is clearly no restriction to assume that $E = F^*$ for some Banach space F. If $T \in B(F^{**}, L_1(\mu))$ with $T^* \in \Pi_1(L_\infty(\mu), F^*)$ then by 1° $T \in \mathscr{B}(F^{**}, L_1(\mu))$, but since $T^*(L_\infty(\mu)) \subseteq F^*$ we may identify T with $T|F \in \mathscr{B}(F, L_1(\mu))$ in a canonical manner. However since F^* has RNP, Theorem 2.6 of [7] shows that $F^* \otimes_m L_1(\mu) = \mathscr{B}(F, L_1(\mu))$. ■

In case $E \hookrightarrow L_p$ for some $p \leqslant 2$ or E is a quotient of an L_p-space for $p > 2$ we get from Theorem 1.5:

1.7. Theorem. (i) *Let* $1 < q < p < 2$ *or* $1 \leqslant q < \infty$ *and* $p = 2$ *or* $q = p = 1$. *If X is q-concave and* $E \hookrightarrow L_p$ *(and E is isomorphic to a dual space with the* RNP *if* $p = 1$) *then we have for every operator* $T \in B(E^*, X)$ ($T^* X^* \subseteq E$ *if* $p = 1$)

$$T \in E \otimes_m X \Leftrightarrow T \in \Pi_q(E^*, X) \ (\Leftrightarrow T \in \Pi_1(E^*, X) \ \textit{if } q \leqslant 2).$$

Dually,

(ii) *Let* $2 < p < q < \infty$ *or* $p = 2$ *and* $1 \leqslant q < \infty$. *If X is q-convex and E is isomorphic to a quotient of an L_p-space then for every operator* $T \in B(E^*, X)$

$$T \in E \otimes_m X \Leftrightarrow T^* \in I_q^f(X^*, E) \ (\Leftrightarrow T^* \in \Gamma_\infty^f(X^*, E) \ \textit{if } q \geqslant 2).$$

Proof. It is clearly enough to prove (i). Since our assumptions there imply that $B(l_1, E^*) = \Pi_{q'}(l_1, E^*)$, it follows from Theorems 1.3, 1.5 and 1.6 that it is enough to show that

(1) $T \in \Pi_q(E^*, X) \Rightarrow T \in \Pi_1(E^*, X) \Rightarrow T^* \in \Pi_1(X^*, E)$ if $q \leqslant 2$, and

(2) $T \in \Pi_q(E^*, X) \Rightarrow T^* \in \Pi_1(X^*, E)$ if E is isomorphic to a Hilbert space and $2 \leqslant q < \infty$.

Since $q < p$ the first implication of (1) follows from [22], (this result was also used in the proof of Lemma 1.3). Since $E \hookrightarrow L_1$, the second implication in (1) follows directly from [12], Theorem 4. ■

We can now prove the following partial result on the ideal property.

1.8. THEOREM. (i) *If $p_X < q_X < 2$ and X is q_X concave then (E, X) has the ideal property if and only if $B(l_1, E^*) = \Pi_{q'_X}(l_1, E^*)$.*

Dually if $2 < p_X < q_X \leq \infty$ and X is p_X-convex then (E, X) has the ideal property if and only if $B(l_1, E) = \Pi_{p_X}(l_1, E)$.

(ii) *If (E, X) has the ideal property and X is p_X-convex with $p_X < q_X < 2$ (resp. X is q_X-concave $2 < p_X < q_X \leq \infty$) then $B(l_1, E^*) = \Pi_{q'_X}(l_1, E^*)$ (resp. $B(l_1, E) = \Pi_{p_X}(l_1, E)$).*

(iii) *If $p_X \leq 2 \leq q_X$ and either p_X or q_X is attained or X contains (l_2^n) uniformly complemented on disjoint blocks then (E, X) has the ideal property if and only if E is isomorphic to a Hilbert space.*

Proof. The "if" part of (i) and (iii) follows from Theorem 1.2 and Theorem 1.5.

The "only if" parts of (i)–(iii) are based on the following argument:

Assume that (E, X) has the ideal property and that X is q_X-concave or $q_X = \infty$. It follows from [7], Proposition 1.6 that every $T \in E\otimes_m X$ has p_X-summing adjoint. Since q_X is attained we get from [11] and [29] that for every n there is a sublattice F_n of X spanned by n mutually disjoint positive vectors, 2-equivalent to the unit vector basis of $l^n_{q_X}$ and so that the F_n's are uniformly complemented in X. Together with the above this shows that there is a constant K_1 so that for every n and every $T \in E\otimes_m l^n_{q_X}$ we have

$$\pi_{p_X}(T^*) \leq K_1\|T\|_m. \tag{1}$$

An approximation argument yields that (1) holds for every $T \in E\otimes_m l_{q_X}$.

Proof of (i). If $p_X < q_X < 2$ then it follows from [22] that there is a constant K_2 so that $\pi_1(S) \leq K_2\pi_{p_X}(S)$ for all $S \in \Pi_{p_X}(l_{q'_X}, E)$. Hence combining this with (1) and Theorem 1.2 we conclude that $T \in E\otimes_m l_{q_X}$ if and only if $T^* \in \Pi_1(l_{q'_X}, E)$. In particular, (E, l_{q_X}) has the ideal property and therefore $E \hookrightarrow QL_{q_X}$ by [13] so that E is reflexive. By duality we get that $T \in E^*\otimes_m l_{q'_X}$ if and only if $T \in \Gamma_\infty(E, l_{q'_X})$. Let K_3 be a constant so that $\|T\|_m \leq K_3\gamma_\infty(T)$ for all $T \in \Gamma_\infty(E, l_{q'_X})$.

Now let $S \in B(l_1, E^*)$ and let $V \in B(l_\infty, l_{q'_X})$. Then $VS^* \in \Gamma_\infty(E, l_{q'_X})$ and hence

$$\|VS^*\|_m \leq K_3\gamma_\infty(VS^*) \leq K_3\|V\|\|S\|. \tag{2}$$

Since (2) holds for every $V \in B(l_\infty, l_{q'_X})$ it follows from [23], Corollary 4.6, that S is q'_X-summing with $\pi_{q'_X}(S) \leq K_3\|S\|$. Hence we have proved the first part of (i). The second part follows either by duality or using similar arguments as above.

Proof of (ii). We assume that $2 < p_X < q_X$ and that X is q_X-concave or $q_X = \infty$. In the case $q_X = \infty$ we get directly from (1) that every compact operator from l_1 to E is p_X-summing and by using the Persson–Pietsch duality theory twice we obtain $B(l_1, E) = \Pi_{p_X}(l_1, E)$. Assume next that $q_X < \infty$ and let $S \in B(l_1, E)$. Since $p_X < q_X$ we get from [22] that there is a constant K_4 so that

$$i_{q_X}(V) \leq K_4\|V\| \quad \text{for every } V \in B(l_\infty, l_{p_X}). \tag{3}$$

Hence if $V \in B(l_\infty, l_{p_X})$ we can for every $\varepsilon > 0$ find a measure μ and operators $A\colon l_\infty \to L_{q_X}(\mu)$, $B\colon L_{q_X}(\mu) \to l_{p_X}$ so that A is order bounded with $\|A\|_m \leq i_{q_X}(V) + \varepsilon$, $\|B\| \leq 1$ and $V = BA$. As before we can conclude that E is reflexive and therefore $AS^* \in \mathscr{B}(E^*, L_{q_X}(\mu)) = E\otimes_m L_{q_X}(\mu)$. From (1) we get therefore that $BAS^* = VS^* \in E\otimes_m l_{p_X}$ with:

$$\begin{aligned}\|VS^*\|_m &\leq \pi_{p_X}(S^{**}A^*B^*) \leq \pi_{p_X}(S^{**}A^*) \leq K_1\|AS^*\|_m \leq K_1\|S\|\|A\|_m \\ &\leq K_1K_4(\|V\| + \varepsilon)\|S\|.\end{aligned} \tag{4}$$

And again, we conclude from [23] that S is p_X-summing.

Proof of (iii). It is clearly enough to prove the only if part of (iii) in case either q_X is attained or X contains (l_2^n) uniformly and uniformly complemented on disjoint blocks. Put $r = q_X$ under the first assumption and $r = 2$ under the second. The argument leading to (1) together with an approximation procedure give that if (E, X) has the ideal property then there is a constant K_5 so that

$$\pi_2(T^*) \leq \pi_{p_X}(T^*) \leq K_5\|T\|_m \quad \text{for all } T \in E\otimes_m L_r(0, 1). \tag{5}$$

If $r = \infty$ then we can argue as in (ii) to get that $B(l_1, E) = \Pi_2(l_1, E)$. Hence if S is a quotient map from an L_1-space onto E it factors through a subspace of an L_2-space and therefore E is isomorphic to a subspace of a quotient of a Hilbert space.

If $r < \infty$ then (5) implies that $E \hookrightarrow QL_r$ so that E is of type 2. Let (r_n) be the sequence of Rademacher functions on $[0, 1]$ (e_n) the unit vector basis of l_2. There is a constant K_6 and an operator $S \in B(L_r(0, 1), l_2)$ so that $\|S\| \leq K_6$, $Sr_n = e_n$. If $(x_j)_{j=1}^n \subseteq E$ and $T = \sum_{j=1}^n x_j\otimes r_j$ then by (5),

$$\begin{aligned}\Big(\sum_{j=1}^n \|x_j\|^2\Big)^{1/2} &= \|ST\|_m \leq \|S\|\Pi_2(T^*) \leq K_5K_6\|T\|_m \\ &= K_5K_6\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(t)x_j\Big\|^r dt\Big)^{1/r}.\end{aligned} \tag{6}$$

(6) shows that E is also of cotype 2 and hence E is isomorphic to a Hilbert space by [14]. ■

Remark. Note that the condition that q_X (or p_X) is attained only was used in the above proof to conclude that operators defined on the spaces F_n could be extended to the whole of X preserving the norm up to a constant. We conjecture that (i)–(iii) hold also if p_X or q_X are not attained but we were unable to find a proof.

Another question is what happens if $p_X = q_X$. In that case we conjecture that (i)–(ii) still hold unless X is lattice isomorphic to an L_{p_X}-space and that (iii) still holds. We shall comment further on it in Section 2.

The condition $B(l_1, E^*) = \Pi_{q'_X}(l_1, E^*)$ is not completely satisfactory in view of the foregoing results and we can pose

1.9. Problem. Let $1 < p < 2$ and assume that $B(l_1, E^*) = \Pi_{p'}(l_1, E^*)$. Does $E \hookrightarrow QL_r$ for some r, $p < r \leqslant 2$?

If the answer to this question is affirmative then it follows from Theorem 1.5 that we can write "if and only if" in condition (ii) of Theorem 1.8.

The condition $B(l_1, E^*) = \Pi_{p'}(l_1, E^*)$ for some p, $1 < p < 2$ implies that $E \hookrightarrow QL_p$ and that E is of type p-stable (and hence of type r for some $r > p$). Indeed, we can find a measure μ and a quotient map S of $L_1(\mu)$ onto E, which is p'-summing. The first statement follows immediately from this and if $T \in E \otimes_m l_p$ then $TS \in I_1(E^*, l_p)$ and therefore $T^* \in \Pi_1(l_{p'}, E)$. Now let (f_n) be a sequence of stochastically independent p-stables on $[0, 1]$ and let $r < p$. (f_n) spans l_p in $L_r(0, 1)$ and hence if (e_n) denotes the unit vector basis of l_p we can find $V \in B(l_p, L_r(0, 1))$ so that $Ve_n = f_n$ for all n. The above gives that there is a constant K (independent of V) so that

$$\|VT\|_m \leqslant \pi_1(T^*V^*) \leqslant K\|V\|\|T\|_m \quad \text{for all } T \in E \otimes_m l_p.$$

If $(x_j)_{j=1}^n \subseteq X$ and $T = \sum_{j=1}^n x_j \otimes e_j$, this gives:

$$\Big(\int_0^1 \Big\|\sum_{j=1}^n f_j(t)x_j\Big\|^r dt\Big)^{1/r} = \|ST\|_m \leqslant K\|S\|\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p}$$

which shows that E is of type p-stable. Hence we can also pose

1.10. Problem. Let $1 < p < 2$ and let $E \hookrightarrow QL_p$ be p-stable. Does there exist an $r > p$ so that $E \hookrightarrow QL_r$?

It follows from Rosenthal [28] that the answer to Problem 1.10 is affirmative if actually $E \hookrightarrow L_p$. The same is true if E is isomorphic to a quotient of an L_p-space. Indeed, then $E^* \hookrightarrow L_{p'}$ and is of better cotype than p' and therefore E^* is isomorphic to a Hilbert space by [9].

In Section 2 we shall give a necessary and sufficient condition on a Banach space E and a Banach lattice X in order that (E, F) has the ideal property relative to X for all subspace $F \subseteq X$ and we shall give some applications of the foregoing results.

The rest of the theorems in this section are results needed in Sections 2 and 3.

We start with the following proposition which follows from [7], Theorem 1.3 and Proposition 1.6 together with [27], Theorem 1.2.

1.11. Proposition. *Let X be p_X-convex. If $F \subseteq X$ is a subspace so that (E, F) has the ideal property then $T \in E\overline{\otimes}_m F \Leftrightarrow T^* \in \Pi^f_{p_X}(F^*, E)$.*

We can now prove

1.12. Theorem. *Let $(f_n) \subseteq E$, $(x_k) \subseteq X$ be unconditional basic sequences so that $(E, [x_k])$ has the ideal property relative to X. If both E and X are of finite cotype then $\{f_n \otimes x_k \mid n, k \in N\}$ is an unconditional basic sequence in $E \otimes_m X$.*

Proof. Since $[f_n] \otimes_m X$ embeds isometrically into $E \otimes_m X$ we can without loss of generality assume that $[f_n] = E$. Put $F = [x_k]$ and let K_1, respectively K_2, be the unconditional basis constant for (f_n), respectively (x_k) and let $(x_k^*) \subseteq F^*$ be the biorthogonal system to (x_k). If $(t_{n,k})$ is a finite sequence of scalars and (α_n) and (β_k) are sequences of signs, then (compare with Lemma 2.3 of [7])

$$(1)\qquad \Big\|\sum_{n,k}\alpha_n\beta_k t_{n,k} f_n \otimes x_k\Big\|_m = \Big\|\Big\|\sum_n \alpha_n\Big(\sum_k \beta_k t_{n,k} x_k\Big) f_n\Big\|_E\Big\|_X \leqslant K_1 \Big\|\Big\|\sum_n\Big|\sum_k \beta_k t_{n,k} x_k\Big| f_n\Big\|_E\Big\|_X = K_1\Big\|\sum_{n,k}\beta_k t_{n,k} f_n \otimes x_k\Big\|_m.$$

We define $S \in B(F, X)$ by

$$(2)\qquad Sx = \sum_{k=1}^\infty \beta_k x_k^*(x) x_k \quad \text{for all } x \in F.$$

By assumption there is a constant K_3 so that

$$(3)\qquad \|ST\|_m \leqslant K_3\|S\|\|T\|_m \leqslant K_2K_3\|T\|_m \quad \text{for all } T \in E\overline{\otimes}_m F.$$

Comparing (1) and (3) we get

$$(4)\qquad \Big\|\sum_{n,k}\alpha_n\beta_k t_{n,k} f_n \otimes x_k\Big\|_m \leqslant K_1K_2K_3\Big\|\sum_{n,k} t_{n,k} f_n \otimes x_k\Big\|_m.$$

Since both E and X are assumed to be of finite cotype it follows from Theorem 2.9 of [7] that the Banach lattice $E \otimes_m X$ is of finite cotype and therefore we can apply the two-dimensional version of the Maurey–Khintchine inequality ([17], Theorem 1.d.6 and Proposition 2.d.6).

Hence there is a constant K_4 so that if (r_n) denotes the Rademacher system on $[0,1]$ then

$$(5)\qquad K_4^{-1}\Big\|\Big(\sum_{n,k}|t_{n,k}|^2|f_n\otimes x_k|^2\Big)^{1/2}\Big\|_m \leqslant \int_0^1\int_0^1\Big\|\sum_{n,k}r_n(t)r_k(s)t_{n,k}f_n\otimes x_k\Big\|\,dt\,ds \leqslant K_4\Big\|\Big(\sum_{n,k}|t_{n,k}|^2|f_n\otimes x_k|^2\Big)^{1/2}\Big\|_m.$$

This together with (4) shows that for $K = K_1K_2K_3K_4$,

$$(6)\qquad K^{-1}\Big\|\Big(\sum_{n,k}|t_{n,k}|^2|f_n\otimes x_k|^2\Big)^{1/2}\Big\|_m \leqslant \Big\|\sum_{n,k}t_{n,k}f_n\otimes x_k\Big\|_m \leqslant K^{-1}\Big\|\Big(\sum_{n,k}|t_{n,k}|^2|f_n\otimes x_k|^2\Big)^{1/2}\Big\|_m.$$

(6) shows immediately that $(f_n\otimes x_k)$ is an unconditional basic sequence. ∎

Recall that if $K\geqslant 1$ then a Banach space E is said to have the *K-local unconditional structure* (K-l.u.st.) if for every finite dimensional subspace $F\subseteq E$ there is a finite dimensional subspace $F\subseteq F_1\subseteq E$, having an unconditional basis with constant less than K. We shall say that E has l.u.st. if it has K-l.u.st. for some constant K.

The concept of l.u.st. was defined originally in [2]. Gordon and Lewis [4] define l.u.st. differently (called GL-l.u.st. in this paper).

1.13. Corollary. *Let E and X be of finite cotype and $F\subseteq X$ a subspace so that for every finite dimensional subspace $F_1\subseteq F$, (E, F_1) has the ideal property relative to X with a uniform constant K. If E and F have* l.u.st. *then $E\overline{\otimes}_m F$ has* l.u.st.

Proof. Assume that E has the K_1-l.u.st. and F has the K_2-l.u.st. and let $G\subseteq E\overline{\otimes}_m F$ be a finite dimensional subspace. By choosing an Auerbach basis for G and approximating its elements with elements from $E\otimes F$ we can for a given $\varepsilon>0$ find finite dimensional subspaces $E_1\subseteq E$ and $F_1\subseteq F$ and an isomorphism S of G into $E_1\otimes_m F_1$ so that $\|Sx-x\|\leqslant\varepsilon\|x\|$ for all $x\in G$. A standard pertubation argument [8], Lemma 2.4 then shows that we can assume without loss of generality that $G\subseteq E_1\otimes_m F_1$.

By assumption we can find finite dimensional subspaces $E_1\subseteq E_2\subseteq E$, $F_1\subseteq F_2\subseteq F$ so that E_2, respectively F_2, have a basis with unconditional constant K_1, respectively K_2. Proposition 1.12 shows that $E_2\otimes_m F_2$ has an unconditional basis with a constant KK_1K_2, where K only depends on E and F. ∎

Remark. A similar result hold for GL-l.u.st.

2. Further results on the ideal property and some estimates of ideal norms. In this section we shall first give some immediate consequences of the results in Section 1 especially concerning factorization theorems and

estimates of norms of absolutely summing operators. Later we shall also for certain X give a necessary and sufficient condition on E in order that (E, F) has the ideal property relative to X for all subspaces $F\subseteq X$. This will among other things show that if $1\leqslant q<p\leqslant 2$, $E\hookrightarrow L_p$, X is q-concave and $F\subseteq X$ is a subspace then $\pi_1(E^*, F) = E\overline{\otimes}_m F$. This fact shall be important for us when we investigate the local structure of these spaces in Section 3.

An immediate consequence of the results of Section 1 is that if E and X both have unconditional bases and $E\hookrightarrow QL_p$ $1<p\leqslant 2$ and X is q-concave for some q, $1\leqslant q<p$ then $E\otimes_m X$ has an unconditional basis. The same is true if $2\leqslant p<q<\infty$ and $E\hookrightarrow QL_p$ and X is q-convex.

Together with an interpolation argument this can e.g. be used to prove the well known result that if X is a rearrangement invariant function space on $[0,1]$ of type strictly larger than one and $E\hookrightarrow QL_p$ for some p, $1<p<\infty$ has an unconditional basis then $E\otimes_m X = X(E)$ has an unconditional basis.

The next three results are special cases of Theorem 1.5 and Theorem 1.7. The first should be compared with [25].

2.1. Proposition. *Let $1\leqslant q<p<2$ or $1\leqslant q<\infty$ and $p=2$. If $E\hookrightarrow QL_p$ and X is q-concave and either E or X have the bounded approximation property then $T\in E\otimes_m X$ if and only if T is q-nuclear.*

Proof. Under the assumptions above we have that $N_q(E^*, X) = I_q^f(E^*, X)$. Now apply Theorem 1.5. ∎

2.2. Proposition. *Let $1\leqslant q<p<2$ or $p=2$ and $1\leqslant q<\infty$ and let $(\Omega, \mathscr{S}, \mu)$ be a finite measure space. If $E\subseteq L_1(\mu)$ is of type p and X is q-concave then there is a measurable function $\varphi\colon \Omega\to[0,\infty]$, $\int\varphi\,d\mu\leqslant 1$ and a constant K so that if $T=\sum_{j=1}^n f_j\otimes x_j\in E\otimes X$ then*

$$\text{(i)}\qquad K^{-1}\Big(\int\Big\|\sum_{j=1}^n f_j(t)x_j\Big\|^q\varphi^{1-q}(t)\,d\mu(t)\Big)^{1/q}\leqslant\|T\|_m = \Big\|\int\Big|\sum_{j=1}^n f_j(t)x_j\Big|\varphi\mu(t)\Big\|\leqslant\int\Big\|\sum_{j=1}^n f_j(t)x_j\Big\|\,d\mu(t).$$

Proof. By a result of Rosenthal [28] there is a $\varphi\in L_1(\mu)$ $\varphi>0$ a.e. $\int\varphi\,d\mu\leqslant 1$ and a constant K_1 so that

$$(1)\qquad \Big(\int|f|^q\varphi^{1-q}\,d\mu\Big)^{1/q}\leqslant K_1\int|f|\,d\mu\quad\text{for all } f\in E.$$

By Theorem 1.5 there is a constant K_2 so that

$$(2)\qquad \pi_q(T)\leqslant K_2\|T\|_m\quad\text{for all } T\in E\otimes_m X.$$

Put $\nu=\varphi\,d\mu$ and let $S\colon E\to L_q(\nu)$ be defined by $Sf=f\varphi^{-1}$ for all $f\in E$.

By combining (1) and (2) with [23], Corollary 4.8 we get for $T = \sum_{j=1}^{n} f_j \otimes x_j \in E \otimes X$

$$(3)\qquad \Big(\int \Big\| \sum_{j=1}^{n} f_j(t) x_j \Big\|^q \varphi^{1-q}(t)\, d\mu(t)\Big)^{1/q} = \|ST^*\|_m \leqslant \|S\|\, \pi_q(T) \leqslant K_1 K_2 \|T\|_m$$
$$= K_1 K_2 \Big\| \int \Big| \sum_{j=1}^{n} f_j(t) x_j \Big|\, d\mu(t) \leqslant K_1 K_2 \int \Big\| \sum_{j=1}^{n} f_j(t) x_j \Big\| d\mu(t)$$

which proves the proposition with $K = K_1 K_2$. ■

2.3. PROPOSITION. *Let $1 \leqslant q < p < 2$ or $1 \leqslant q < \infty$ and $p = 2$ and let (e_n) denote the unit vector basis of l_p, (e_n^*) the unit vector basis of $l_{p'}$. If X is q-concave and $q < p$, then $T \in \Pi_1(l_{p'}, X)$ if and only if $T^* \in \Pi_1(X^*, l_p)$ (if $p = 2$ $T \in \Pi_q(l_2, X)$ if and only if $T^* \in \Pi_1(X^*, l_2)$) and there is a constant K, so that if $T = \sum_{j=1}^{n} e_j \otimes Te_{j*}$, then*

$$\text{(i)}\qquad K^{-1}\pi_1(T) \leqslant \Big\| \Big(\sum_{j=1}^{n} |Te_{j*}|^p\Big)^{1/p} \Big\| \leqslant K\pi_q(T) \leqslant K\pi_1(T) \quad \text{if} \quad q < p$$

and

$$\text{(ii)}\qquad K^{-1}\pi_q(T) \leqslant \Big\| \Big(\sum_{j=1}^{n} |Te_{j*}|^2\Big)^{1/2} \Big\| \leqslant K\pi_q(T) \quad \text{if} \quad p = 2 \text{ and } 1 \leqslant q < \infty.$$

Proof. The first part follows immediately from Theorem 1.7. The second part also follows from that theorem by noting that if T has the form above then $\|T\|_m = \|(\sum_{j=1}^{n} |Te_{j*}|^p)^{1/p}\|$. ■

The following theorem, which shall be very useful to us in this section, shows that to some extent Theorem 1.7 can be generalized to the case where X is substituted with a subspace F of X:

2.4. THEOREM. *Let $1 \leqslant q < p < 2$ or $p = 2$ and $1 \leqslant q < \infty$, $E \hookrightarrow L_p$, X q-concave and let $F \subseteq X$ be a subspace.*

1° *For every $T \in B(E^*, F)$ we have*

$$\text{(i)}\qquad T \in E\overline{\otimes}_m F \Leftrightarrow T^* \in \Pi_1^f(F^*, E) \Leftrightarrow T \in \Pi_q^f(E^*, F)$$
$$(\Leftrightarrow T \in \Pi_1^f(E^*, F) \text{ if } 1 < q < p \leqslant 2).$$

2° *If either E or F and X have the bounded approximation property, or if F is complemented in X then we have*

$$\text{(ii)}\qquad E\overline{\otimes}_m F = \Pi_q(E^*, F) \;(= \Pi_1(E^*, F) \text{ if } q < p < 2)$$

and

$$\text{(iii)}\qquad T \in E\overline{\otimes}_m F \Leftrightarrow T^* \in \Pi_1(F^*, E).$$

Proof. From Lemma 1.4 and the proof of Theorem 1.7 it follows that the implications

$$(1)\qquad T \in \Pi_1(E^*, F) \Rightarrow T^* \in \Pi_1(F^*, E); \qquad T \in \Pi_q(E^*, F) \Rightarrow T \in \Pi_1(E^*, F)$$
$$\text{if} \quad q < p \leqslant 2; \quad \text{and} \quad T \in \Pi_q(l_2, F) \Rightarrow T^* \in \Pi_1(F^*, l_2)$$

do not depend on F. The proof of Theorem 1.7 now shows that (i) holds. Assume next that the assumptions in 2° are satisfied. In view of the above remarks it is enough to show that if $T \in E \otimes_m X = \Pi_q(E^*, X) = \Pi_q^f(E^*, X)$ and $T(E^*) \subseteq F$, then $T \in E\overline{\otimes}_m F$.

If E or F and X have the bounded approximation property then a result of Persson and Pietsch [26] show that $\Pi_q^f(E^*, X)$, respectively $\Pi_q^f(E^*, F)$ is equal to the space of all quasi-q-nuclear operators from E^* to X, respectively to F.

Hence T is quasi-q-nuclear considered as a map into X and therefore quasi-q-nuclear into F as well, and hence $T \in \Pi_q^f(E^*, F) = E\overline{\otimes}_m F$.

Assume that there is a projection P of X onto F. By Theorem 1.7 there is a constant K so that $\|PS\|_m \leqslant K\|P\|\,\|S\|_m$ for all $S \in E\otimes_m X$.

If now $(T_n) \subseteq E\otimes X$ converges to T in the m-norm, we have that $(PT_n) \subseteq E\otimes F$ and the inequality above shows that it is a Cauchy sequence in the m-norm. Clearly its limit has to be T. ■

As a corollary we get:

2.5. COROLLARY. *Let E and X satisfy the assumptions in the beginning of Theorem 2.4. There is a constant K so that for all subspaces $F \subseteq X$ (E, F) has the ideal property with constant K relative to X.*

For the sake of completeness we state the result on L_p-spaces corresponding to Theorem 2.4.

2.6. THEOREM. *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and $1 \leqslant p < \infty$. If $E \hookrightarrow L_p$ and F is isomorphic to a subspace of $L_p(\mu)$ then we have for every $T \in B(E^*, F)$ (with $T^*F^* \subseteq E$ for $p = 1$)*

$$T \in E\overline{\otimes}_m F \Leftrightarrow T \in \Pi_p^f(E^*, F) \Leftrightarrow T^* \in \Pi_p^f(F^*, E).$$

Further, if either E or F have the bounded approximation property and for $p = 1$, $E = G^*$ for some Banach space G and has the RNP, then

$$E\overline{\otimes}_m F = \Pi_p(E^*, F) \quad \text{for} \quad 1 < p < \infty,$$
$$E\overline{\otimes}_m F = \Pi_p(G, F) \quad \text{for} \quad p = 1$$

and $T \in E\overline{\otimes}_m F \Leftrightarrow T^* \in \Pi_p(F^*, E)$.

A direct application of Theorems 1.2, 1.5, 1.7, and 2.4 give

2.7. COROLLARY. (i) *Let $1 \leqslant q < p < 2$ or $p = 2$ and $1 \leqslant q < \infty$, $E \hookrightarrow L_p$, F a subspace of a q-concave Banach lattice X and Y a weakly sequentially complete Banach lattice. If $S \in B(F, Y)$ then $I\otimes S$ is a bounded operator from $E\overline{\otimes}_m F$ to $E\otimes_m Y$. This remains valid if $E \hookrightarrow QL_p$ and $F = X$.*

(ii) *If $2 < p < q$, $E \to QL_p$, X is q-convex and Y is weakly sequentially complete then for every $S \in B(X, Y)$ $I \otimes S$ is a bounded operator from $E \otimes_m X$ to $E \otimes_m Y$.*

Since $L_p(0,1)$ embeds into $L_1(0,1)$ for all $1 \leqslant q \leqslant 2$ Corollary 2.7 shows e.g. that if $E \subseteq L_p(0,l)$ is a subspace, $1 < p \leqslant 2$ and $1 \leqslant q \leqslant p$, then $L_q([0,1], E)$ embeds into $L_1([0,1], E)$ which in turn embeds into $L_1([0,1]^2)$ (which is isometric to $L_1(0,1)$).

Theorem 1.7 and Corollary 2.7 also show that if $1 \leqslant q < p \leqslant 2$ and X is q-concave then there exists a constant K so that for every subspace $F \subseteq X$, every Banach lattice Y and every $S \in B(F, Y)$

$$(*) \qquad \Big\|\Big(\sum_{j=1}^{n} |Sx_j|^p\Big)^{1/p}\Big\| \leqslant K\|S\| \Big\|\Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p}\Big\|$$

for all finite sets $(x_j)_{j=1}^n \subseteq F$. (The remark just after Theorem 1.2 shows that we can take Y arbitrary since $(*)$ only involves finite expressions.)

We now wish to find a necessary and sufficient condition on E and X in order that (E, F) has the ideal property relative to X for all subspaces $F \subseteq X$.

We need the following definition.

2.8. Definition. Let $1 < p < 2$. X is said to *contain* (l_p^n) *uniformly p-stable-like* if there is a $\lambda \geqslant 1$ so that for every n there exists $(x_{jn})_{j=1}^n \subseteq X$, λ-equivalent to the unit vector basis of l_p^n with the following property:

$(*)$ For all m there is an n and scalars $a_1, a_2, \ldots, a_n$ so that

$$\Big\|\Big(\sum_{j=1}^{n} |a_j|^p |x_{jn}|^p\Big)^{1/p}\Big\| \geqslant m \Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p}.$$

2.9. Example. If $1 \leqslant r < p$ and $(f_n) \subseteq L_r(0,1)$ is a stochastically independent sequence of p-stables then (f_n) is equivalent to the unit vector basis of l_p and from [32] it follows that $(*)$ of Definition 2.8 is satisfied.

2.10. Proposition. *Let X be a Banach lattice so that $p_X < q_X < 2$; then X contains $(l_{q_X}^n)$ uniformly q_X-stable-like.*

Proof. Let $(f_j) \subseteq L_{p_X}(0,1)$ be a sequence of stochastically independent q_X-stables and let n be given. We can then find an m and an isomorphism S: $[f_j]_{j=1}^n \to l_{p_X}^m$ so that $\|S\| \leqslant 2$, $\|S^{-1}\| \leqslant 1$.

By Corollary 2.7 there is a constant K (independent of n, m, and S) so that

$$(1) \qquad \frac{1}{K}\Big\|\Big(\sum_{j=1}^{n} |a_j|^{q_X} |f_j|^{q_X}\Big)^{1/q_X}\Big\|_{p_X} \leqslant \Big\|\Big(\sum_{j=1}^{n} |a_j|^{q_X} |Sf_j|^{q_X}\Big)^{1/q_X}\Big\|_{p_X} \leqslant 2K \Big\|\Big(\sum_{j=1}^{n} |a_j|^{q_X} |f_j|^{q_X}\Big)^{1/q_X}\Big\|_{p_X}$$

for all scalars $a_1, a_2, \ldots, a_n$.

From Krivine [11] and Rosenthal [29] it follows that there exist mutually disjoint positive elements $x_1, x_2, \ldots, x_m \in X$, so that $(x_j)_{j=1}^m$ is 2-equivalent to the unit vector basis $(e_j)_{j=1}^m$ of $l_{p_X}^m$. Let $V \in B(l_{p_X}^m, X)$ so that $Ve_j = x_j$, $1 \leqslant j \leqslant m$ and put $y_j = VSf_j$, $1 \leqslant j \leqslant n$. Clearly, $(y_j)_{j=1}^n$ is equivalent to the unit vector basis of $l_{q_X}^n$ with a constant independent of n. From (1), Example 2.9 and the fact that V is a lattice isomorphism of $l_{p_X}^m$ into X it follows that the conditions of Definition 2.8 are satisfied. ∎

We now need the following lemma:

2.11. Lemma. *Let $1 \leqslant p < 2$ and let $K \geqslant 1$ be a constant. Assume further that X contains a finite basic sequence $(x_j)_{j=1}^n$ consisting of normalized mutually disjoint elements, which is K-equivalent to the unit vector basis of l_p^n.*

If $(l_p, [x_j])$ has the ideal property relative to X with constant M, then for all basic sequences $(y_j)_{j=1}^n$ which are N-equivalent to the unit vector basis of l_p^n for some constant N we have

$$\text{(i)} \qquad \Big\|\Big(\sum_{j=1}^{n} |a_j|^p |y_j|^p\Big)^{1/p}\Big\| \leqslant K^2MN \Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p}$$

for all n-tuples $(a_j)_{j=1}^n$ of scalars.

Proof. Assume that $(y_j)_{j=1}^n$ is a basic sequence in X, N-equivalent to the unit vector basis of l_p^n. Then there is an operator S: $[x_j] \to [y_j]$ so that $Sx_j = y_j$ for $j \leqslant n$ and $\|S\| \leqslant KN$. Since $(l_p, [x_j])$ has the ideal property (relative to X) with constant M, we conclude that for all scalars $a_1, a_2, \ldots, a_n$

$$(1) \qquad \Big\|\Big(\sum_{j=1}^{n} |a_j|^p |y_j|^p\Big)^{1/p}\Big\| \leqslant M\|S\| \Big\|\Big(\sum_{j=1}^{n} |a_j|^p |x_j|^p\Big)^{1/p}\Big\| \leqslant KNM \Big\|\sum_{j=1}^{n} a_j x_j\Big\| \leqslant K^2MN \Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p}.$$

We are now able to prove

2.12. Theorem. 1° *If $1 < q_X < 2$ and X contains $(l_{q_X}^n)$ uniformly q_X-stable-like (which happens e.g. when $p_X < q_X$) then the following statements are equivalent:*

(i) *$\exists\, r$, $q_X < r \leqslant 2$, $q_X < r \leqslant 2$ so that $E \hookrightarrow L_r$.*

(ii) *There is a constant K, so that (E, F) has the ideal property relative to X with constant K for all subspaces F of X.*

2° *If $2 \leqslant q_X < \infty$ and $p_X < q_X$ unless $q_X = 2$ then* (ii) *above holds if and only if E is isomorphic to a Hilbert space.*

Proof of 1°. It follows from Corollary 2.5 that (i) ⇒ (ii).

Next assume (ii). Put $q = q_X$ and let (e_n) denote the unit vector basis of l_q. From [11] we get that for every n there exist mutually disjoint positive normalized vectors $x_1^n, x_2^n, \ldots, x_n^n \in X$ and an isomorphism $V_n\colon l_q^n \to [x_j^n]_{j=1}^n = F_n$ so that $V_n e_j = x_j^n$ for $j \leqslant n$ and $\|V_n\|\|V_n^{-1}\| \leqslant 2$.

Our assumptions together with the argument in part (i) of the proof of Theorem 1.8 show that there exists a constant K so that if $T \in E\otimes_m l_q$ then $T^* \in \Pi_q(l_{q'}, E)$ and

$$(1)\qquad \pi_q(T^*) \leqslant K\|T\|_m \quad \text{for all } T \in E\otimes_m l_q.$$

Therefore (E, l_q) has the ideal property so that $E \hookrightarrow QL_q$ ([13]) and hence E is reflexive.

We wish to show that $E \hookrightarrow L_q$ and according to [13], Corollary 6 and Remark 1, we have to show that

$$(2)\qquad T \in \Pi_q(E^*, l_q) \Rightarrow T^* \in \Pi_q(l_{q'}, E).$$

Hence let $T \in \Pi_q(E^*, l_q)$. (1) gives that if we show that $T \in E\otimes_m l_q$ then we are done. Let $(x_j) \subseteq E$ so that

$$(3)\qquad Tx^* = \sum_{j=1}^{\infty} x^*(x_j)e_j \quad \text{for all } x^* \in E^*$$

and put for every n

$$(4)\qquad T_n x^* = \sum_{j=1}^{n} x^*(x_j)e_j \quad \text{for all } x^* \in E^*.$$

Let $\varepsilon > 0$ be arbitrary. From the Pietsch factorization theorem [26] we conclude that for every n there is a k, a subspace $G_n \subseteq l_q^k$ and operators $A_n \in B(E^*, G_n)$ $B_n \in B(G_n, l_q^n)$ so that $T_n = B_n A_n$, $\|B_n\| \leqslant 1$ and A_n is order bounded considered as a map of E^* into l_q^k with $\|A_n\|_m \leqslant \pi_q(T) + \varepsilon$. Hence $V_n T_n = (V_n B_n V_k^{-1}) V_k A_n \in E\otimes_m X$ with

$$(5)\qquad \begin{aligned} \|V_n T_n\|_m &\leqslant K\|V_n\|\|B_n\|\|V_k^{-1}\|\|V_k A_n\|_m \\ &\leqslant K\|V_n\|\|B_n\|\|V_k^{-1}\|\|V_k\|\|A_n\|_m \leqslant 2K\|V_n\|\big(\pi_q(T_n)+\varepsilon\big) \\ &\leqslant 2K\|V_n\|\big(\pi_q(T)+\varepsilon\big), \end{aligned}$$

where we used that V_k is a lattice isomorphism into X. By [23], Proposition 2.7, $V_n T_n$ is order bounded into the sublattice F_n as well with the same norm and hence since V_n^{-1} is a lattice isomorphism we get for every n

$$(6)\qquad \Big\|\Big(\sum_{j=1}^{n} |x_j|^q\Big)^{1/q}\Big\| = \|T_n\|_m \leqslant \|V_n^{-1}\|\|V_n T_n\|_m \leqslant 4K\big(\pi_q(T)+\varepsilon\big).$$

Since the right hand side does not depend on n (6) shows that $T \in E\otimes_m l_q$. This shows (2).

Lemma 2.11 and our assumptions show that E cannot contain a subspace isomorphic to l_q and hence by [28], Theorem 8, there exists an r, $q < r \leqslant 2$ so that $E \hookrightarrow L_r$. Hence (ii) $\Rightarrow$ (i).

Proof of 2°. The "if"-part follows from Corollary 2.5. Assume that $p_X < q_X < \infty$ or $p_X = q_X = 2$ and that (ii) holds. An argument similar to the one in (i) shows that $E \hookrightarrow L_{p_X}$.

The assumption (ii) implies the arguments of the proof of Theorem 1.8 can be used to show that $B(l_1, E) = \Pi_p(l_1, E)$, where $p = \max(p_X, 2)$. If $p = 2$ we conclude directly that E is isomorphic to a Hilbert space and if $p = p_X > 2$ then E does not contain a subspace isomorphic to l_{p_X} (c.f. the remarks after Problem 1.9) and therefore E is isomorphic to a Hilbert space. ∎

Remark. Note that if $p_X < q_X = \infty$ then no Banach space E will satisfy condition (ii) of Theorem 2.12.

The next result should be compared with Problems 1.9 and 1.10.

2.13. Proposition. *Let $1 \leqslant q_X < 2$ and assume that X contains $(l_{q_X}^n)$ uniformly q_X-stable-like and that X is q_X-concave. If (E, X) has the ideal property then $E \hookrightarrow QL_{q_X}$ and E is of type r for some $r > q_X$.*

Proof. We can argue like in Theorem 1.8 to obtain that $E \hookrightarrow QL_{q_X}$. Put $p_0 = \sup\{p \mid E \text{ is of type } p\}$. Clearly $p_0 \geqslant q_X$, but by [22], E contains $(l_{p_0}^n)$ uniformly so that (l_{p_0}, X) has the ideal property, and hence by our assumptions and Lemma 2.11 $p_0 \neq q_X$. ∎

2.14. Example. Let X be a rearrangement invariant function space on $[0,1]$ so that $p_X = q_X = p < 2$ and so that X contains an element f with a p-stable distribution. Then the following statements are equivalent for an r, $1 \leqslant r \leqslant 2$.

(i) $\exists K$ so that for every subspace $F \subseteq X$ and every $S \in B(F, X)$

$$\Big\|\Big(\sum_{j=1}^{n} |Sx_j|^r\Big)^{1/r}\Big\| \leqslant K\|S\|\Big\|\Big(\sum_{j=1}^{n} |x_j|^r\Big)^{1/r}\Big\|$$

for all finite sets $(x_j)_{j=1}^n \subseteq X$.

(ii) $r > p$.

The equivalence still holds if we in (i) put $F = X$ and assume that q_X is attained.

As examples of X's satisfying these conditions we can consider the Orlicz function $M(x) = x^p|\log x|^{-a}$, $x \in [0, \infty]$, where $a > 1$ and put $X = L_M(0,1)$. It is readily verified that X contains p-stables and that $q_X = p$ is attained.

Let us end this section with an application to convexity and concavity of Banach lattices.

2.15. Theorem. *Let $1 \leqslant r \leqslant q < \infty$ and let X be q-concave.*

(i) *If X is r-convex and $E \hookrightarrow QL_p$ with either $1 \leqslant q < p \leqslant 2$ or $p = 2 \leqslant q$ then $\Pi_1(X^*, E) = \Pi_r(X^*, E)$.*

(ii) *If $1 \leqslant r \leqslant p \leqslant 2$ and $\Pi_1(X^*, l_p) = \Pi_r(X^*, l_p)$ then X is of type r.*

Proof. (i): If $T \in \Pi_r(X^*, E)$ then T^* is order bounded by [7], Theorem 1.3 and [27], Theorem 1.2 and therefore $T^* \in E \otimes_m X^{**}$ as well. Theorem 1.5 now gives that $T \in \Pi_1(X^*, E)$.

(ii): Under the assumptions of (ii) we can find a constant $K \geqslant 1$ so that

$$\pi_1(T) \leqslant K\pi_r(T) \quad \text{for all } T \in \Pi_r(X^*, l_p). \tag{1}$$

Let (e_n) be the unit vector basis of l_p and let (r_n) be the sequence of Rademacher functions on $[0, 1]$.

If $(x_j)_{j=1}^n \subseteq X$ is arbitrary and $T = \sum_{j=1}^n x_j \otimes e_j$ then it follows from (1), Maurey–Khintchine's inequality [17] and the assumption $1 \leqslant r \leqslant p \leqslant 2$ that there is a constant K_1 so that

$$\begin{aligned} \int \Big\| \sum_{j=1}^n r_j(t) x_j \Big\| dt &\leqslant K_1 \Big\| \Big(\sum_{j=1}^n |x_j|^2 \Big)^{1/2} \Big\| \\ &\leqslant K_1 \Big\| \Big(\sum |x_j|^p \Big)^{1/p} \Big\| = K_1 \|T^*\|_m \leqslant K_1 \pi_1(T) \\ &\leqslant K K_1 \pi_r(T) \leqslant K K_1 \Big(\sum_{j=1}^n \|x_j\|^r \Big)^{1/r}. \end{aligned} \tag{2}$$

(2) shows that X is of type r. ■

3. Applications to local structures in spaces of absolutely summing operators. The results of the two previous sections not only give us conditions on E and X for (E, F) to have the ideal property for all subspaces $F \subseteq X$ but allow us also to use the results to describe the structure of spaces $\Pi_1(E^*, F)$ and other classical operator ideals. In this section we wish to study this in further detail.

We start with:

3.1. Theorem. *Let $1 \leqslant q < p \leqslant 2$ $E \hookrightarrow L_p$, X q-concave and F a subspace of X. Then the following statements hold.*

(i) *If E is a Banach lattice then $\Pi_1(E^*, X)$ $(= \Pi_s(E^*, X) \approx \Pi_1(X^*, E)$, $1 \leqslant s \leqslant q)$ is a Banach lattice under the usual ordering of operators and under a suitable renorming.*

(ii) *If E and F both have* l.u.st. *then $\Pi_1^f(E^*, F)$ and $\Pi_1^f(F^*, E)$ both have* l.u.st.

(iii) *If E and F both have unconditional bases then $\Pi_1(E^*, F)$ and $\Pi_1(F^*, E)$ have unconditional bases as well.*

Proof. (i) is clear, since $\Pi_1(E^*, X) = E \otimes_m X$ and the latter is a Banach lattice in the case considered.

(ii) and (iii) follows from the Theorems 1.7 and 2.4 together with Theorem 1.12 and Corollary 1.13. ■

We note that Theorem 1.5 ensures that if $E \hookrightarrow QL_p$, $1 < p < 2$, X is r-convex for some r, $p' < r \leqslant \infty$ and F is isomorphic to a complemented subspace of X then the conclusions of Theorem 3.1 hold for $\Pi_1(F, E)$.

From Theorem 1.5, (ii), we obtain a dual version of the above; it reads:

3.2. Theorem. *Let $2 < p < q < \infty$, let $E \hookrightarrow QL_p$ and let F be isomorphic to a complemented subspace of a q-convex weakly sequentially complete Banach lattice X. The conclusions of Theorem* 3.1, (i) *hold for $\Gamma_\infty(E^*, X)$,* (ii) *holds for $\Gamma_\infty^f(E^*, F)$ and* (iii) *holds for $\Gamma_\infty(E^*, F)$.*

Proof. A direct application of Theorems 1.5, 1.2, and Corollary 1.13. ■

The result corresponding to Theorem 3.1 in case $E = l_2$ looks like this

3.3. Theorem. *Let F be a Banach space so that F^* does not contain (l_∞^n) uniformly.*

If F has l.u.st. *then $\Pi_1(F, l_2)$ has* l.u.st. *If further F^* has an unconditional basis then $\Pi_1(F, l_2)$ has it as well.*

Proof. By [3], Corollary 2.2 and Proposition 2.6, F^* is isomorphic to a complemented subspace of a Banach lattice X not containing (l_∞^n) uniformly. Hence X is q-concave for some q, $1 \leqslant q < \infty$ and it follows that $\Pi_1(F, l_2)$ is naturally isomorphic to $l_2 \overline{\otimes}_m F^*$. Now Theorem 1.12 and Corollary 1.13 can be applied. ■

Remark. The condition that F^* does not contain (l_∞^n) uniformly cannot be removed. Indeed, $\pi_1(l_1, l_2) = B(l_1, l_2)$ does not have l.u.st.

From Theorem 3.1 we obtain e.g. that if M and N are Orlicz functions so that $1 < \alpha_M^\infty \leqslant \beta_M^\infty < 2 < \alpha_N^\infty \leqslant \beta_N^\infty < \infty$ then $\Pi_1(L_N(0,1), L_M(0,1))$ has an unconditional basis. Indeed both $L_{N^*}(0,1)$ and $L_M(0,1)$ have an unconditional basis (the Haar system) and by a result of [1], $L_M(0,1)$ embeds isomorphically into $L_1(0,1)$ and $L_{N^*}(0,1)$ embeds isomorphically into $L_p(0,1)$ for every p with $1 \leqslant p < \alpha_{N^*}^\infty = \beta_N^{\infty'}$. Hence we can apply Theorem 3.1.

Another special case of Theorem 3.1 is:

3.4. Corollary. *Let $1 \leqslant s \leqslant 2 \leqslant r \leqslant \infty$ and let E be an $\mathscr{L}_r$-space, F an $\mathscr{L}_s$-space. Then $\Pi_1(E, F)$ has* l.u.st. *and it has an unconditional basis if both E^* and F have unconditional bases. The same is true for $\pi_1(E, l_2)$ if E is an $\mathscr{L}_r$-space, $1 < r \leqslant \infty$.*

Proof. The local structure of $\mathscr{L}_p$-spaces, [15], show that in order to prove the statement about l.u.st. it is enough to show that for all n and m $\Pi_1(l_r^n, l_s^m)$ has an unconditional basis with a constant independent of n and m. In case $r < \infty$ this follows from Theorem 3.1 with $p = r'$ and using that l_s is isomorphic to a subspace of $L_1(0,1)$, and if $r = \infty$ then it follows from Theorem 2.6 and Theorem 1.12. Assume next that E^* and F

have unconditional bases. Again we conclude from Theorem 3.1 that $\Pi_1(E, F)$ has an unconditional basis when $r < \infty$. If $r = \infty$ then E^* is an $\mathscr{L}_1$-space with an unconditional basis and by [15] that basis is equivalent to the unit vector basis of l_1. The conclusion follows now from the second part of Theorem 2.6 together with Theorem 1.12.

The statements on $\Pi_1(E, l_2)$ follow from Theorem 3.3. ■

It was originally proved by Gordon and Lewis that $\Pi_1(E, F)$ has l.u.st. under the assumptions of Corollary 3.4. Schütt [31] proved that if $\max(r, 2) < s$ then $\Pi_1(E, F)$ does not have GL-l.u.st. (and hence not l.u.st.). It is not known what happens in the remaining cases.

We now turn our attention to the question of the uniform approximation property (the u.a.p.) for spaces $E \otimes_m X$ where E and X satisfy the conditions of Theorem 1.7. Let us recall that if $\lambda \geq 1$ and φ: $N \to N$ is a function then a Banach space E is said to have the (λ, φ)-u.a.p. if for every n-dimensional subspace $F \subseteq E$ there is an operator T on E, which is the identity on F and so that $\|T\| \leqslant \lambda$, $rk(T) \leqslant \varphi(n)$. E is said to have the λ-u.a.p. if it has the (λ, φ)-u.a.p. for some function φ. The u.a.p. was first introduced by Pełczyński and Rosenthal [24] and has since been studied by several authors [6], [18]. In [7] the u.a.p. of m-tensor products was investigated and we shall here apply the results from there to the situation of this paper.

Our first result is an improvement (at least formally) of Theorem 3.7 in [7] in the special case considered in this paper.

3.5. THEOREM. *Let $1 < q < p < 2$ and $E \hookrightarrow L_p$, X q-concave, or let E be isomorphic to a Hilbert space and X arbitrary or let E be an arbitrary Banach space and X an L_1-space. There is a constant K so that if E has the α-u.a.p. and X has the β-u.a.p. then $E \otimes_m X$ has the $(K\alpha\beta + \varepsilon)$-u.a.p. for all $\varepsilon > 0$.*

Proof. If X is an L_1-space it has the positive u.a.p. and the conclusion follows from Theorem 3.7 in [7], and if E is isomorphic to a Hilbert space the conclusion follows from Theorem 3.8 of [7].

For the remaining case we may assume that actually $E \subseteq L_p(\mu)$ for some measure μ.

If E is finite dimensional then we argue like in [7], Theorem 3.4 (i) ⇒ (ii), but assuming only that X has the β-u.a.p. For the operator T constructed there we get that $I \otimes T$ is a bounded operator on $E \otimes_m X$ with $\|I \otimes T\| \leqslant K\|T\| \leqslant K\beta$, where K is the constant for the ideal property of (E, X). The rest of the argument of the paper above then show that $E \otimes_m X$ has the $K\beta$-u.a.p.

If E is infinite dimensional we let $\varepsilon > 0$ and let for every n, A_n denote the closure of the set of all n-dimensional subspaces of E in the set of all n-dimensional Banach spaces equipped with the Banach–Mazur distance. A_n is then compact and we choose an ε-net in it.

The space Y in the proof of Theorem 3.7 in [7] can then be constructed using only the ε-nets chosen above and a general result on embedding Banach spaces into L_p-spaces [15] then gives that Y embeds isometrically into $L_p(\mu)$. Hence $Y \otimes_m X$ has the $(K\beta + \varepsilon)$-u.a.p. from the first part of this proof, and the proof of Theorem 3.7 in [7] then shows that $E \otimes_m X$ has the $(K\alpha\beta + \varepsilon)$-u.a.p. for all $\varepsilon > 0$.

Remark. We note that it follows [7], Theorem 3.6 that if X is r-convex for some $1 < r < \infty$ and q-concave for some $q < \infty$ then X has u.a.p. if and only if it has the order u.a.p.

As a corollary we get:

3.6. COROLLARY. 1° *Let $1 < q < p < 2$ or $p = 2$ and $1 < q \leqslant 2$ and let $E \hookrightarrow L_p$, X q-concave. If E and X have the u.a.p. and F is isomorphic to a complemented subspace of X then $\Pi_1(E^*, F) = \Pi_s(E^*, F)$, $1 \leqslant s \leqslant q$ and $\Pi_1(F^*, E)$ have the u.a.p.*

2° *If $E \hookrightarrow L_1$ has the u.a.p., X is an L_1-space and F is isomorphic to a complemented subspace of X then $\Pi_1(E^*, F)$ has the u.a.p.*

3° *If F^* is isomorphic to a complemented subspace of a weakly sequentially complete Banach lattice X with the u.a.p. then $\Pi_1(F, l_2)$ has the u.a.p.*

Proof. 1°: By Theorems 1.7 and 2.4 it is enough to show that $E \overline{\otimes}_m F$ has the u.a.p. Theorem 3.5 gives that $E \otimes_m X$ has the u.a.p. and from Corollary 2.7 it readily follows that $E \overline{\otimes}_m F$ is complemented in $E \otimes_m X$ and therefore has the u.a.p. as well.

2°: Assume that $X = L_1(\mu)$ for some measure μ. By Theorem 2.6 we get that $\Pi_1(E^*, L_1(\mu)) = \mathscr{B}(E^*, L_1(\mu))$. Since $L_1(\mu)$ is a band in $L_1(\mu)^{**}$ it readily follows that $\mathscr{B}(E^*, L_1(\mu))$ is a complemented subspace of $\mathscr{B}(E^*, L_1(\mu)^{**}) = (E^* \otimes_m L_1(\mu)^*)^*$. The latter space has the u.a.p. since $E^* \otimes_m L_1(\mu)^*$ has it ([7]) and the u.a.p. is a self-dual property. Hence $\Pi_1(E^*, L_1(\mu))$ has the u.a.p. This finishes the proof since clearly $\Pi_1(E^*, F)$ is complemented in $\Pi_1(E^*, L_1(\mu))$.

3°: This can be proved as 1° by using Proposition 1.3. ■

As a special case of Corollary 3.6 we get:

3.7. COROLLARY. 1° *Let M and N be Orlicz functions so that $1 < \alpha_M^\infty \leqslant \beta_M^\infty < \alpha_N^\infty \leqslant \beta_N^\infty < 2$. If E is isomorphic to a complemented subspace of $L_M(0, 1)$ and F is isomorphic to a complemented subspace of $L_N(0, 1)$ then $\Pi_1(E^*, F)$ and $\Pi_1(F^*, E)$ have the u.a.p.*

2° *Let $1 \leqslant s \leqslant 2 \leqslant r \leqslant \infty$ and $r \neq s'$ unless $s = 1, 2$. If E is an $\mathscr{L}_r$-space and F is an $\mathscr{L}_s$-space and F is complimented in F^{**} then $\Pi_1(E, F)$ has the u.a.p.*

Proof. 1°: It follows from a result of Lindenstrauss and Tzafriri [18] that both E and F have the u.a.p. under the assumptions of 1°. Further if p, q are chosen so that $\beta_M^\infty < q < p < \alpha_N^\infty$ then it is readily verified that

$L_M(0,1)$ is q-concave and since $\beta_N^\infty < 2$ it follows from [1] that $L_N(0,1)$ embeds into $L_p(0,1)$. Now Corollary 3.6 applies

2°: It follows from [24] that every $\mathscr{L}_p$-space, $1 < p \leqslant \infty$, has the u.a.p. and hence we just have to verify that it is possible to use Corollary 3.6.

Assume first that $1 \leqslant r' < s$ $(1 \leqslant r' \leqslant 2$ if $s = 2)$. E^* is then isomorphic to a complemented subspace of an $L_{r'}$-space and $F \hookrightarrow L_s$ [15]. Hence Corollary 3.6 shows that $\pi_1(E, F)$ has the u.a.p.

If $1 \leqslant s < r'$ then again by [15] and our assumptions F is isomorphic to a complemented subspace of an L_s-space and $E^* \hookrightarrow L_{r'}$. Hence again Corollary 3.6 gives that $\pi_1(E, F)$ has the u.a.p. ∎

With the methods of this paper we have not been able to determine whether $\Pi_1(\mathscr{L}_r, \mathscr{L}_s)$ has the u.a.p. for the remaining values of r and s and we pose the question as an open problem. Of special interest is of course the case $\Pi_1(l_{p'}, l_p)$, $1 < p < 2$.

It follows easily from Theorem 2.6 that if $1 < p < \infty$ and E is an $\mathscr{L}_{p'}$-space, F an $\mathscr{L}_p$-space then $\Pi_p(E, F)$ has the u.a.p. Indeed it follows that $\Pi_p(E, F)$ is isomorphic to a complemented subspace of an L_p-space and hence has the u.a.p. by [24].

4. Some concluding remarks. There are several problems on composition of order bounded operators which we have not touched in this paper. One of them is:

4.1. Problem. Let E and X be given. Characterize those $S \in B(X)$ for which $I \otimes S$ acts as a bounded operator on $E \otimes_m X$.

This problem is related to the question whether $L_p([0,1], E)$ has an unconditional basis when $1 < p < \infty$ and E is super-reflexive with an unconditional basis. Indeed the whole problem is whether the bounded operator S on $L_p(0,1)$, which changes signs of coordinates in the expansions of the Haar basis, has the property that $I \otimes S$ is bounded on $E \otimes_m L_p(0,1) = L_p([0,1], E)$.

Another question which is often applicable in Banach space theory is the following:

4.2. Problem. Given E, X and $T_1, T_2 \in \mathscr{B}(E, X)$ so that there is an operator $S \in B(X)$ with $T_2 = ST_1$. When does there exist a constant K so that $\|T_2\|_m \leqslant K\|S\|\|T_1\|_m$?

There are several examples of operators which have the property in Problem 4.2. Let us here only mention the following:

Let E be a Banach space and let $g, h\colon [0,1] \to E$ be two Gaussian variables so that $\int_0^1 |x^*g(t)|^2\,dt \leqslant \int_0^1 |x^*h(t)|^2\,dt$ for all $x^* \in E^*$. It then follows from [19] that $\int_0^1 \|g(t)\|^2\,dt \leqslant \int \|h(t)\|^2\,dt$. If we define $(T_g x^*)(t) = x^*g(t)$, $(T_h x^*)(t) = x^*h(t)$ for all $x^* \in E^*$ and all $t \in [0,1]$ then $T_g, T_h \in E \otimes_m L_2(0,1)$ and the first inequality above gives that $T_g = ST_h$ for some operator S on $L_2(0,1)$ with $\|S\| \leqslant 1$, while the second means that $\|T_g\|_m \leqslant \|T_h\|_m$.

This fact about vector valued Gaussians has many applications and some of the material used in this paper can be traced back to it. Another application can be found in [21]. It can also be used to show the following:

Let E be a Banach space and (g_n) a sequence of independent Gaussian variables on $[0,1]$. E is then of type 2 if and only if there is a constant K so that if $T = \sum_{j=1}^n x_j \otimes g_j \in E \otimes_m L_2(0,1)$ then $\|T\|_m \leqslant K\pi_2(T)$.

This result can be considered as a Banach space version of Theorem 2.15 in case the p there is equal to 2.

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On weak restricted estimates and endpoints problems for convolutions with oscillating kernels (II)

by

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Abstract. Throughout we consider $K(t) = e^{i|t|^a}/|t|^b$, $a > 0$, $a \neq 1$, $b < 1$ and $t \in \mathbf{R}$. Here we consider for fixed λ, μ the function

$$M(\lambda, \mu;\, K) = M(\lambda, \mu) = \sup_{\chi_\mu} \left|\{x\colon\ |K*\chi_\mu(x)| > \lambda\}\right|$$

over all "characteristic" functions χ_μ with complex signs (i.e. χ_μ is a measurable function for which $|\chi_\mu| = 1$ on E, $|\chi_\mu| = 0$ off E and $|E| \leqslant \mu$ $(\mu > 0)$). We first note that

$$|K*\chi_\mu| \leqslant \int_{-\mu}^{\mu} \frac{dt}{|t|^b} = c\mu^{1-b},$$

and so if $c\mu^{1-b} < \lambda$ then $M(\lambda, \mu) = 0$. And so we assume throughout that $\lambda \leqslant c_1\mu^{1-b}$ for some constant c_1, independent of λ and μ (but may depend on K). Under these conditions ($\lambda \leqslant c_1\mu^{1-b}$) we estimate $M(\lambda, \mu)$ within constant factors from above and below.

This paper is a continuation of part I [3] where we estimated the function $B(\lambda, \mu)$ within constant factors from above and below. For fixed $\lambda, \mu > 0$ we set

$$B(\lambda, \mu) = \sup_{\chi_\lambda, \chi_\mu} \int \chi_\lambda(x) K*\chi_\mu(x)\, dx,$$

where the sup is taken over all "characteristic" functions χ_λ, χ_μ with complex signs.

§ 1. An interpolation theorem with respect to the kernel. Here we prove the analogue for M of the corresponding theorem for B given in part I. Again we consider a decomposition of the kernel

$$K = K_1 + K_2$$

and make use of the decreasing rearrangement K^* of K (if it exists), so that

$$\sup_{\chi_\mu} \left|\int K(x)\chi_\mu(x)\, dx\right| = \int_0^{\mu} K^*(t)\, dt \qquad (x \in \mathbf{R}^n,\ t \in \mathbf{R}).$$

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