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Duals of spaces of compact operators

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Abstract. This is a study of dual spaces of various spaces of compact operators. We derive representations for duals of (a) spaces $K(X, Y)$ of compact linear operators, (b) injective tensor products $X\tilde{\otimes}_\varepsilon Y$, and (c) spaces of vector-valued continuous functions, in terms of the completed projective tensor product of the strong duals or biduals of the factor spaces X and Y. The results are specified for (i) X and Y Banach spaces, (ii) X and Y Fréchet spaces, and (iii) X and Y DF spaces. Applications are given to problems of weak convergence in any of the above types of operator or function spaces. The fundamental tool is a kind of localized Radon–Nikodym property for locally convex spaces introduced by A. Grothendieck.

0.1. Introduction. The present work is a sequel to our previous article [7] on weak compactness in spaces of compact operators. In that paper, our main object was to describe the weak topology and the notions related to it, like weak compactness, weak convergence for sequences, and reflexivity, in the general context of the operator space $L_e(X'_c, Y)$ of weak*-weakly continuous linear operators from X' into Y which transform equicontinuous subsets of X' into relatively compact subsets of Y, endowed with the topology of uniform convergence on the equicontinuous subsets of X', X and Y arbitrary locally convex spaces. The results were then applied to the various spaces of analysis that can be represented as a suitable operator space of the form $L_e(X'_c, Y)$. Here our object is to describe the dual space of $L_e(X'_c, Y)$ in terms of the (presumably well known) duals X' and Y' of the factor spaces X and Y, and, again, to specialize our results to the concrete spaces representable as (a linear subspace of) an operator space $L_e(X'_c, Y)$: (a) spaces of compact operators, (b) injective tensor products, and (c) spaces of vector-valued continuous, or holomorphic functions.

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For the convenience of the reader, we recall here three of the basic examples of such representations:

(*) $K(X, Y) = L_e(X''_c, Y)$ (isometrically),
$h \mapsto h''$ (X and Y Banach spaces),

(**) $X \tilde{\otimes}_\varepsilon Y \rightleftarrows L_e(X'_c, Y)$ (topologically),
$\sum_1^n x_i \otimes y_i \mapsto \{x' \mapsto \sum_1^n (x_i, x') y_i\}$ (X and Y complete locally convex spaces),

($_*{}^*{}_*$) $C(K, X) = L_e(X'_c, C(K))$ (isometrically),
$F \mapsto \{x' \mapsto x' \circ F\}$ (K compact Hausdorff, X a Banach space).

For further details on the space $L_e(X'_c, Y)$, as well as for the basic terminology and notations, the reader is kindly asked to consult our previous paper [7].

We now describe the contents of this paper in more detail.

It is known from the work of A. Grothendieck ([16], I. 4.1, Prop. 18, pp. 95, 96) that, for general X and Y, the continuous linear functionals on $L_e(X'_c, Y)$ are given by certain integral linear forms, represented by Radon measures on the products $U^\circ \times V^\circ$ of polars of zero neighbourhoods U and V in X and Y, respectively. Our object is to specify conditions on X and Y such that these linear forms are representable as elements of the completed projective tensor product $X'_b \tilde{\otimes}_\pi Y'_b$ of the strong duals of X and Y, and thus appear in a form much closer to the original factor spaces X and Y and their duals. Again, the prototype result in this direction is to be found in Grothendieck's work ([16], I. 4.2, Thm. 8, p. 122). In the modern terms of the Radon–Nikodym property for Banach spaces, it reads as follows:

($^{**}_{**}$) *Whenever X and Y are Banach spaces such that X' or Y' has the Radon–Nikodym property, and either of X' and Y' has the approximation property, then the dual of $X \tilde{\otimes}_\varepsilon Y$ is isometrically isomorphic to $X' \tilde{\otimes}_\pi Y' = N(X, Y')$ (the space of nuclear operators from X into Y').*

It is a kind of "localized" Radon–Nikodym property for general locally convex spaces, introduced by Grothendieck ([16], I. 4.1, Def. 6, p. 104) as "propriété Φ" ("Φ" for R. S. Phillips), which enables us to transfer this duality result to more general situations. (For a definition of the "Phillips property", see Section 1, Definition 1.5.) We arrive at the following extension of ($^{**}_{**}$):

($^{(*)}_{**}$) *Whenever X and Y are locally convex spaces such that X or Y is quasinormable and semi-reflexive, then, for every $T \in (L_e(X'_c, Y))'$, there exist zero neighbourhoods U and V in X and Y, respectively, and an element*

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i \in X'_{U^\circ} \tilde{\otimes}_\pi Y'_{V^\circ},$$

such that

$$Th = \sum_1^\infty \lambda_i (hx'_i, y'_i) \quad \textit{for all} \quad h \in L(X'_c, Y).$$

(Starting from the fact that reflexive Banach spaces do have the Radon–Nikodym property, the reader may realize that, in comparison to the Banach space result, for our general result the property of being normed had to be replaced by "quasinormable", and that of having the Radon–Nikodym property by "semi-reflexivity". The notion of quasinormability is that of Grothendieck ([15], III. 1, Def. 4, p. 106), see the definition following Proposition 1.3 in Section 1.)

Section 1 contains the technical details this result is based on, together with a first application to our original problem: the dual of $L_e(X'_c, Y)$ (algebraically) is a quotient of $X'_b \tilde{\otimes}_\pi Y'_b$ whenever both X and Y are Fréchet spaces, one of which is reflexive and quasinormable, or both X and Y are generalized DF spaces (gDF), one of which is semi-reflexive. Moreover, we use our representation of the dual of $L_e(X'_c, Y)$ to derive various supplementary results to our paper [7] on weak compactness in operator spaces.

The subsequent sections are then devoted to a specialization of the general technical results of Section 1 to the particular cases of

(1) X and Y generalized DF spaces (Section 2),

(2) X and Y Banach spaces (Section 3), and

(3) X and Y Fréchet spaces (Section 4).

Our results comprise representations of duals of

(a) spaces $K_b(X, Y)$ of compact operators, X and Y Banach or, more generally, X gDF and Y Fréchet,

(b) completed injective tensor products $X \tilde{\otimes}_\varepsilon Y$, and of

(c) spaces of vector valued continuous functions, like $C_0(S, X)$, $C_b(S, X)_\beta$, and $C_c(S, X)_t$, S locally compact Hausdorff and X Banach, or gDF, or Fréchet.

To give a flavour of the results, we take up the three examples considered above:

(*)′ $K'(X, Y) = X'' \tilde{\otimes}_\pi Y'$ (isometrically), X and Y Banach, X'' or Y' RNP, X'' or Y' a.p. (Section 3, Thm. 3.2),

(**)′ $(X\tilde{\otimes}_\varepsilon Y)' = X_b'\tilde{\otimes}_\pi Y_b'$ (algebraically), X and Y gDF, X or Y semi-reflexive, X_b' or Y_b' a.p. (Section 2, Thm. 2.1),

$({}^{*}_{**})'$ $C(K, X)' = M(K)\tilde{\otimes}_\pi X'$ (isometrically), K compact Hausdorff, X Banach, X' RNP (Section 3, Thm. 3.4).

0.2. Notation and terminology. Generally, we kindly ask the reader to consult the corresponding section in our previous paper [7]. For convenience, we recall some of the fundamental definitions.

(a) *Spaces of compact operators*: Given locally convex spaces X and Y, we denote by $K^b(X, Y)$ the space of weakly continuous linear operators from X into Y which transform bounded sets into relatively compact sets, and, as usual, by $K(X, Y)$ the space of compact linear operators from X into Y. Throughout, these spaces will be assumed to be endowed with the topology of uniform convergence on the bounded subsets of X (= operator norm whenever X and Y are normed), as indicated by $K_b^b(X, Y)$ and $K_b(X, Y)$. (Recall that a linear operator from X into Y is called [*weakly*] *compact* if it transforms a certain zero neighbourhood in X into a [weakly] relatively compact subset of Y.) Whenever X is normed, or a generalized DF space (see the definition below), and Y a Fréchet space, then $K^b(X, Y) = K(X, Y)$. For X and Y locally convex spaces such that Y is quasi-complete (closed bounded sets are complete), we have the following topological linear isomorphisms:

$$K_b(X, Y) \hookrightarrow K_b^b(X, Y) \cong L_e(X_c'', Y) \cong (X_b')\,\varepsilon\, Y,$$
$$h \mapsto h''.$$

(b) *Spaces of vector-valued continuous functions*: Let T be a completely regular Hausdorff space, X a quasi-complete locally convex space, and $V > 0$ a Nachbin family of weights on T such that T is a $V_{\mathbf{R}}$-space. $CV_0(T, X)$ denotes the associated space of X-valued continuous functions on T:

$$CV_0(T, X) = \{F\colon T\to X \text{ continuous} \mid vF\colon t\to v(t)F(t) \text{ vanishes at infinity for all } v\in V\},$$

with the topology generated by the seminorms

$$b_{v,q}(F) = \sup\{v(t)\,q(F(t)) \mid t\in T\},$$

$v\in V$, and q a continuous semi-norm on X. For details consult ([7], Section 4). We have the following topological linear isomorphism:

$$CV_0(T, X) \cong L_e(X_c', CV_0(T)) \cong (CV_0(T))\,\varepsilon\, X,$$
$$F \mapsto \{x' \mapsto x'\circ F\}.$$

The following are the most common examples of weighted spaces.

$C(T, X)_{co}$: continuous functions, with compact open topology (T completely regular Hausdorff $k_{\mathbf{R}}$, X quasi-complete locally convex).

$C_0(S, X)$: continuous functions vanishing at infinity, with sup-norm (S locally compact Hausdorff, X Banach).

$C_b(S, X)_\beta$: bounded continuous functions, with the strict topology of R. C. Buck ([4], [5]) (S locally compact Hausdorff, X quasi-complete locally convex).

$C_c(S, X)_i$: continuous functions with compact support, with the usual inductive limit topology (S locally compact Hausdorff σ-compact, X Banach).

(c) *Special classes of spaces*: A locally convex space X is called *semi-reflexive* if its bounded subsets are weakly relatively compact. X is called a *generalized* DF *space* (gDF) if (i) its strong dual X_b' is a Fréchet space, and (ii) linear operators into other locally convex spaces are continuous as soon as their restrictions to the bounded sets are. Besides their classical ancestors, and thus all normed spaces and strong duals of Fréchet spaces, this class includes Mackey duals and c-duals Z_c' of Fréchet spaces Z, as well as all function spaces with any of the extensions of R. C. Buck's ([4], [5]) strict topology β, c.f. ([26], [28]).

(d) *Additional notations*: Let X and Y be locally convex spaces, A a subset of X, and U a zero neighbourhood in X.

$B(X, Y)$ is the space of continuous bilinear forms on $X\times Y$, usually being endowed with the topology of uniform convergence on the products $B\times C$ of bounded sets B and C in X and Y, respectively, as indicated by $B_{bb}(X, Y)$.

$N(X, Y)$ is the space of nuclear operators from X into Y.

$A^{\circ\bullet}$ is the (absolute) bipolar of A in $X'' = (X_b')'$.

X'_{U° is the span of U° in X' endowed with the Banach space norm having U° as unit ball.

The term "Radon–Nikodym property" (for a Banach space X) is, as usual, abbreviated by "RNP".

1. A. Grothendieck's localized Radon–Nikodym property and the dual of the operator space $L_e(X_c', Y)$. This section contains the fundamental technical results on the representation of the dual of $L_e(X_c', Y)$ in terms of the completed projective tensor product $X_b'\tilde{\otimes}_\pi Y_b'$, together with applications to weak convergence criteria in spaces of compact operators.

Given locally convex spaces X and Y, we consider the natural linear embedding of the algebraic tensor product $X'\otimes Y'$ into $(L_e(X_c', Y))'$ ([7], Section 0):

$$j_0\colon X'\otimes Y' \to (L_e(X_c', Y))',$$
(1.1)
$$\sum_1^n x_i'\otimes y_i' \mapsto \Big\{h \mapsto \sum_1^n (hx_i', y_i')\Big\}.$$

Our object is to specify conditions on X and Y under which

(a) j_0 is continuous from $X'_b \otimes_\pi Y'_b$ into $(L_e(X'_c, Y))'_b$, and has a continuous extension j to $X'_b \tilde{\otimes}_\pi Y'_b$ which is still mapping into $(L_e(X'_c, Y))'$,

(b) j is a surjection from $X'_b \tilde{\otimes}_\pi Y'_b$ onto $(L_e(X'_c, Y))'$, and

(c) j is a one-one map from $X'_b \tilde{\otimes}_\pi Y'_b$ into $(L_e(X'_c, Y))'$.

As is to be expected, problem (c) is tied up with the approximation property (for X'_b or Y'_b).

Problem (b) is the most important point of our investigation. The fundamental tool here is a kind of localized Radon–Nikodym property, introduced by A. Grothendieck [16].

Problem (a) has easy satisfactory solutions in the two special cases we are interested in:

1.1. LEMMA. *Let X and Y be both metrizable spaces or both* gDF *spaces. Then the map j_0 of* (1.1) *is continuous from $X'_b \otimes_\pi Y'_b$ into $(L_e(X'_c, Y))'_b$, and has a continuous linear extension j to $X'_b \tilde{\otimes}_\pi Y'_b$ with range still in $(L_e(X'_c, Y))'$.*

Proof. For the continuity part of the assertion, let H be a bounded subset of $L_e(X'_c, Y)$. According to ([29], Prop. 1.9), there exist bounded subsets B and C of X and Y, respectively, such that $H(B^\circ) \subset C$. This implies that $j_0(\operatorname{ac}(B^\circ \otimes C^\circ)) \subset H^\circ$, and shows that j_0 is continuous.

For the second part of the assertion, consider first the case when X and Y are metrizable. Then $L_e(X'_c, Y)$ is metrizable as well, so that $(L_e(X'_c, Y))'_b$ is a complete DF space. Hence, the continuous linear extension j of j_0 trivially maps into that space. In case X and Y are gDF, $L_e(X'_c, Y)$ has a fundamental sequence of bounded sets [29], so that $(L_e(X'_c, Y))'_b$ is metrizable. However, the question whether this space is complete in this case, figures as "Problème 10" in [16], and has been answered affirmatively only in special cases; see the discussion preceeding Theorem 2.2 in Section 2.

Nevertheless we are able to show that j still maps into $(L_e(X'_c, Y))'$ itself: Let $\tilde{v} \in X'_b \tilde{\otimes}_\pi Y'_b$. According to ([16], Thm. 1, p. 51), there exist nullsequences $(x'_i)_{i\in N}$ and $(y'_i)_{i\in N}$ in X'_b and Y'_b, respectively, and $(\lambda_i)_{i\in N} \in l^1$ such that

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i.$$

Hence, $j(\tilde{v})$ is the limit of $\left(j_0\left(\sum_1^n \lambda_i x'_i \otimes y'_i\right)\right)_{n\in N}$ in the completion of $(L_e(X'_c, Y))'_b$.

We now show that the map

$$(*) \qquad T\colon L(X'_c, Y) \to \boldsymbol{K}, \qquad h \mapsto \lim \sum_1^n \lambda_i (hx'_i, y'_i)$$

is an element of $(L_e(X'_c, Y))'$, and that the sequence $\left(j_0\left(\sum_1^n \lambda_i x'_i \otimes y'_i\right)\right)_{n\in N}$ converges to T in $(L_e(X'_c, Y))'_b$. This will complete the proof.

First, the sequences $(x'_i)_{i\in N}$ and $(y'_i)_{i\in N}$ are contained in the polars U° and V° of zero neighbourhoods U and V in X and Y, respectively, for strong nullsequences in the dual of a gDF space are equicontinuous ([26], Prop. 2.2). Hence, given any $h \in L_e(X'_c, Y)$, the sequence $((hx'_i, y'_i))_{i\in N}$ is a bounded sequence of scalars, and, therefore, the series $\sum_1^\infty \lambda_i (hx'_i, y'_i)$ is convergent. In order to prove that T is continuous, let $(h_\lambda)_{\lambda\in\Lambda} \subset L(X'_c, Y)$ be a net that converges to zero in $L_e(X'_c, Y)$. Then, given $\varepsilon > 0$ and U and V as chosen above, there exists $\lambda_0 \in \Lambda$ such that

$$h_\lambda(U^\circ) \subset \varepsilon \cdot \left(\sum_1^\infty |\lambda_i|\right)^{-1} V \quad \text{for all} \quad \lambda \geqslant \lambda_0.$$

Hence, we have:

$$|T(h_\lambda)| = \left|\sum_1^\infty \lambda_i (h_\lambda x'_i, y'_i)\right| \leqslant \sum_1^\infty |\lambda_i| \left(\varepsilon \cdot \left(\sum_1^\infty |\lambda_i|\right)^{-1}\right) = \varepsilon \quad \text{for all} \quad \lambda \geqslant \lambda_0.$$

It remains to prove that $\left(j_0\left(\sum_1^n \lambda_i x'_i \otimes y'_i\right)\right)_{n\in N}$ converges to T in $(L_e(X'_c, Y))'_b$: as indicated before, given a bounded subset H of $L_e(X'_c, Y)$, there exist B and C bounded in X and Y, respectively, such that $H(B^\circ) \subset C$.

For an arbitrary $\varepsilon > 0$, there exists $n_0 \in N$ such that $(x'_i)_{i \geqslant n_0} \subset C^\circ$, and $\sum_{n_0}^\infty |\lambda_i| < \varepsilon$. Hence, we have:

$$\left|Th - \sum_1^n \lambda_i (hx'_i, y'_i)\right| = \left|\sum_{n+1}^\infty \lambda_i (hx'_i, y'_i)\right| \leqslant \sum_{n+1}^\infty |\lambda_i| < \varepsilon$$

for all $n \geqslant n_0$ and all $h \in H$.

This proves our assertion.

For a discussion of injectivity of the map j, consider the following linear map p_0 from $X' \otimes Y'$ into $B(X, Y)$:

$$(1.2) \qquad p_0\colon X' \otimes Y' \to B(X, Y), \qquad \sum_1^n x'_i \otimes y'_i \mapsto \left\{(x, y) \mapsto \sum_1^n (x, x'_i)(y, y'_i)\right\}$$

1.2. LEMMA. (a) *The map p_0 is continuous from $X'_b \otimes_\pi Y'_b$ into $B_{bb}(X, Y)$ for any locally convex spaces X and Y.*

(b) *Whenever X and Y are both* gDF *spaces or both Fréchet spaces, then $B_{bb}(X, Y)$ is complete, and the continuous linear extension p of p_0 maps $X'_b \tilde{\otimes}_\pi Y'_b$ into $B_{bb}(X, Y)$. The map j is injective whenever p is.*

Connections between injectivity of p and the approximation property for X'_b or Y'_b will be investigated separately in any of the particular cases to be discussed in the subsequent sections.

Proof of Lemma 1.2. The proof of part (a) being obvious, we immediately turn to a proof for part (b). In case X and Y are both gDF, the solution of Grothendieck's "Problème des Topologies" for gDF spaces ([28], Thm. 1.9) reveals that $B_{bb}(X, Y)$ is topologically isomorphic to the strong dual $(X\tilde{\otimes}_\pi Y)'_b$ of the gDF space $X\tilde{\otimes}_\pi Y$, and thus is a Fréchet space. In case X and Y both are Fréchet, we first note ([33], II. 34.2, Cor., p. 354) that $B(X, Y)$ is equal to the space $\mathscr{B}(X, Y)$ of separately continuous bilinear forms on $X\times Y$, and then use ([16], Intr. III. 4, III.6) to conclude that $\mathscr{B}_{bb}(X, Y) = L_b(X, Y'_b)$ is complete.

Now assume that p is injective and let $\tilde{v}\in X'_b\tilde{\otimes}_\pi Y'_b$ such that $j(\tilde{v}) = 0$. There exists a net

$$(v_\lambda)_{\lambda\in\Lambda}\subset X'\otimes Y', \qquad v_\lambda = \sum_1^{n_\lambda} x'_{i\lambda}\otimes y'_{i\lambda}$$

converging to $\tilde{v}$ in $X'_b\tilde{\otimes}_\pi Y'_b$. By assumption on $\tilde{v}$, we have $0 = j(\tilde{v})(h) = \big((b-\lim) j(v_\lambda)\big)(h)$ for all $h\in L(X'_c, Y)$, in particular

$$0 = j(\tilde{v})(x\otimes y) = \lim \sum_1^{n_\lambda} (x, x'_{i\lambda})(y, y'_{i\lambda}) \quad \text{for all} \quad (x, y)\in X\times Y.$$

Hence, we have:

$$p(\tilde{v})(x, y) = \big((bb-\lim)p_0(v_\lambda)\big)(x, y) = \lim \sum_1^{n_\lambda} (x, x'_{i\lambda})(y, y'_{i\lambda}) = 0$$

for all $(x, y)\in X\times Y$, and thus, by assumption on p, $\tilde{v} = 0$.

We now turn to the central problem of this paper, namely the question of when j is surjective. We shall derive the following result:

1.3. Proposition. (a) *Let X and Y be locally convex spaces such that*

(i) *X or Y is a Banach space whose dual has* RNP, *or*

(ii) *X or Y is semi-reflexive and quasinormable***.

Then for every $T\in\big(L_e(X'_c, Y)\big)'$ there exist zero neighbourhoods U and V in X and Y, respectively, and

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i\otimes y'_i \in X'_{U^\circ}\tilde{\otimes}_\pi Y'_{V^\circ},$$

such that $Th = \sum_1^\infty \lambda_i(hx'_i, y'_i)$ *for all* $h\in L(X'_c, Y)$.

** Actually, it is enough to assume that X or Y is a subspace of a product of Banach spaces whose duals have RNP. (Note that a semi-reflexive quasinormable locally convex space is a subspace of a product of *reflexive* Banach spaces.)

(b) *In particular, the map* $j\colon X'_b\tilde{\otimes}_\pi Y'_b\to\big(L_e(X'_c, Y)\big)'$ *is surjective, whenever*

(i) *X and Y are Fréchet spaces one of which is reflexive and quasinormable, or*

(ii) *X and Y are* gDF *spaces one of which is semi-reflexive or a Banach space whose dual has* RNP.

We recall the definition of quasinormability.

Definition ([15], III. 1, Déf. 4, p. 106). A locally convex space X is called *quasinormable* whenever for every equicontinuous subset H of X' there exists a zero neighbourhood U in X such that on H the strong dual topology and the topology of uniform convergence on U coincide.

Every DF and, more generally, every gDF space is quasinormable [26]. Also recall that a *Schwartz space* is exactly a quasinormable locally convex space whose bounded sets are precompact.

The fundamental tool for a proof of Proposition 1.3 is based on the notion of the "Phillips property" as introduced by A. Grothendieck.

First we need to fix some notations.

Notation. A convex circled subset of a linear space is called a *disk*. If B is a bounded disk in a locally convex space X, then we denote by X_B the linear span of B in X endowed with the norm with unit ball B. B is called *completing* if X_B is a Banach space.

1.4. Definition ([16], I. 4.1, Déf. 6, p. 104). Let $\mathscr{B}$ be a family of bounded completing disks in a locally convex space X. A subset C_0 of X has the *Phillips property with respect to* $\mathscr{B}$ if its closed convex circled hull C is weakly compact, and if the following condition is fulfilled:

Given any compact Hausdorff space K, any Radon measure μ on K, and any continuous linear operator T from $L^1(\mu)$ into X_C, there exist $B\in\mathscr{B}$ and a bounded measurable function $f\colon K\to X_B$ such that

$$Tg = \int gf\,d\mu \quad (\text{in } X_B) \quad \text{for all } g\in L^1(\mu).$$

The following facts on the Phillips property are crucial for our considerations.

1.5. Facts.

(1.5.1) If X is a Banach space whose dual has RNP, then the dual unit ball $B_{X'}$ has the Phillips property with respect to itself (take X' with the weak*-topology).

(1.5.2) (Phillips, [16], I.4.1, Thm. 4, p. 104) Every weakly compact subset of a Fréchet space X has the Phillips property with respect to the family of all bounded completing disks in X.

(1.5.3) ([16], I.4.1, Thm. 6, p. 108) Let E be a locally convex space, $\mathscr{B}$ a family of bounded completing disks in E, and C a weakly compact disk in E which is contained in any $B \in \mathscr{B}$ and has the Phillips property with respect to $\mathscr{B}$. Furthermore, let F be any other locally convex space, $\mathscr{D}$ an upwards directed family of weakly compact disks in F, and H a linear space of separately continuous bilinear forms on $E \times F$ whose restrictions to $(C, \text{weak}) \times (D, \text{weak})$ are continuous for all $D \in \mathscr{D}$. Consider H to be endowed with the topology of uniform convergence on the products $C \times D$, $D \in \mathscr{D}$. Then any continuous linear functional in the polar of a zero neighbourhood of the form

$$W(C, D) = \{h \in H |\ |h(x, y)| \leqslant 1 \text{ for all } x \in C, y \in D\}, \qquad D \in \mathscr{D},$$

"originates" from the unit ball of the space $E_B \tilde{\otimes}_\pi F_D$ for a certain $B \in \mathscr{B}$. More precisely, this means that there exists $B \in \mathscr{B}$ with the property that for any $T \in W(C, D)^\circ$ there exist nullsequences $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$ in E_B and F_D, respectively, and an l^1-sequence $(\lambda_n)_{n\in N}$ such that

$$Th = \sum_1^\infty \lambda_i h(x_i, y_i) \quad \text{for all } h \in H.$$

We now apply this very last result to the space $\mathscr{H}_{ee}(X'_c, Y'_c) = L_e(X'_c, Y)$ ([7], Section 0, (0.6.1)) and thus obtain our fundamental technical result on the representation of the dual of the operator space $L_e(X'_c, Y)$.

1.6. Proposition. *Let X and Y be locally convex spaces, U and V zero neighbourhoods in X, and W a zero neighbourhood in Y.*

(a) *Consider the zero neighbourhood*

$$N(U^\circ, W) = \{h \in L(X'_c, Y) |\ h(U^\circ) \subset W\}$$

in $L_e(X'_c, Y)$. Assume that U° (considered as a weakly compact disk in $(X', \text{weak*})$) has the Phillips property with respect to V°.

Then for every $T \in (N(U^\circ, W))^\circ$ there exists

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i$$

in the unit ball of $X'_{V^\circ} \tilde{\otimes}_\pi Y'_{W^\circ}$, $(\lambda_i)_{i\in N} \in l^1$, $(x'_i)_{i\in N}$ and $(y'_i)_{i\in N}$ nullsequences in X'_{U° and Y'_{W°, such that

$$Th = \sum_1^\infty \lambda_i (hx'_i, y'_i) \quad \text{for all} \quad h \in L_e(X'_c, Y). \tag{1.6}$$

(b) *Conversely, any*

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i \in X'_{U^\circ} \tilde{\otimes}_\pi Y'_{W^\circ}$$

defines a continuous linear functional on $L_e(X'_c, Y)$ by formula (1.6). Furthermore, the corresponding linear map

$$j_{U,W}: X'_{U^\circ} \tilde{\otimes}_\pi Y'_{W^\circ} \to (L_e(X'_c, Y))'_b$$

is continuous.

Proof. For a proof of part (a), recall first the topological linear isomorphism ([7], Section 0, (0.6.1))

$$L_e(X'_c, Y) \cong \mathscr{H}_{ee}(X'_c, Y'_c),$$
$$h \mapsto B_h: \{(x', y') \mapsto (hx', y')\},$$

and recall that each B_h has continuous restrictions to

$$(U^\circ, \text{weak*}) \times (W^\circ, \text{weak*}),$$

U and W zero neighbourhoods in X and Y, respectively, by ([16], Intr. IV, Lemme D, p. 27). Hence, letting

$E = X'_c$,

$\mathscr{B} = \{V^\circ |\ V \text{ a zero neighbourhood in } X\}$, $C = U^\circ$, $F = Y'_c$,

$\mathscr{D} = \{W^\circ |\ W \text{ a zero neighbourhood in } Y\}$, and $H = \mathscr{H}_{ee}(X'_c, Y'_c)$,

part (a) comes out to be nothing but a special case of fact (1.5.3) above.

The proof of part (b) is similar to the proof of Lemma 1.1. The reasoning there first shows that for every

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i \in X'_{U^\circ} \tilde{\otimes}_\pi Y'_{W^\circ},$$

the map $T_{\tilde{v}}: L_e(X'_c, Y) \to K$,

$$T_{\tilde{v}}(h) = \sum_1^\infty \lambda_i (hx'_i, y'_i)$$

is well-defined, continuous and linear. In order to show that the continuous linear extension of

$$j_{U,W}: X'_{U^\circ} \otimes_\pi Y'_{W^\circ} \to (L_e(X'_c, Y))'_b,$$
$$j_{U,W}\Big(\sum_1^n x'_i \otimes y'_i\Big)(h) = \sum_1^n (hx'_i, y'_i),$$

to the completion $X'_{U^\circ} \tilde{\otimes}_\pi Y'_{W^\circ}$ still maps into $(L_e(X'_c, Y))'$, we also follow the reasoning of the proof of Lemma 1.1, this time using the general characterization ([29], Prop. 1.9) of bounded subsets in $L_e(X'_c, Y)$: a subset H of $L_e(X'_c, Y)$ is bounded if and only if $H(U^\circ)$ is bounded in Y for all zero neighbourhoods U in X, or, equivalently, H is an equicontinuous subset of $L(X'_b, Y)$. We omit the details.

We now turn to the proof of Proposition 1.3: Condition (i) of part (a) being clear, we immediately consider the second condition, and assume that X is quasinormable and semi-reflexive. According to Proposition 1.6, we have to show that for every zero neighbourhood U in X, there exists another such, V say, such that U° has the Phillips property with respect to V. By the quasinormability of X, there exists a zero neighbourhood V in X, $V \subset U$, such that on U° the strong topology coincides with the topology induced by the Banach space X'_{V°. But the strong topology on X is equal to the Mackey topology, for X is semi-reflexive. Hence, U° is a weakly compact disk in X'_b, and thus weakly compact in the Banach space X'_{V° as well. An appeal to Phillips' result (1.5.2) now completes the proof.

We close this circle of ideas by deriving (from Proposition 1.3) our fundamental general results on the representation of continuous linear functionals on compact operator spaces and on spaces of vector-valued continuous functions.

1.7. Theorem. *Let X and Y be locally convex spaces.*

(a) *If X and Y are complete, and either of X and Y is semi-reflexive and quasinormable, or a Banach space whose dual has* RNP, *then for every $T \in (X \tilde{\otimes}_\varepsilon Y)'$ there exist zero neighbourhoods U and V in X and Y, respectively, and*

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i \in X'_{U^\circ} \tilde{\otimes}_\pi Y'_{V^\circ}$$

such that

$$Th = \sum_1^\infty \lambda_i (hx'_i, y'_i) \quad \text{for all } h \in X \tilde{\otimes}_\varepsilon Y.$$

(b) *If Y is quasi-complete, and either of X'_b and Y is semi-reflexive and quasinormable, or a Banach space whose dual has* RNP, *then for every $T \in (K^b_b(X, Y))'$ there exist B bounded in X and V a zero neighbourhood in Y, and*

$$\tilde{v} = \sum_1^\infty \lambda_i x''_i \otimes y'_i \in X''_{B^{\circ\circ}} \tilde{\otimes}_\pi Y'_{V^\circ},$$

such that

$$Th = \sum_1^\infty \lambda_i (h'' x''_i, y'_i) \quad \text{for all } h \in K^b_b(X, Y).$$

(c) *If X is semi-reflexive and quasinormable, or a Banach space whose dual has* RNP, *T a completely regular Hausdorff space, and $V > 0$ a Nachbin family of weights on T such that T is a $V_{\mathbf{R}}$-space, then for every $S \in (CV_0(T, X))'$ there exist zero neighbourhoods U and V in $CV_0(T)$ and X, respectively, and*

$$\tilde{v} = \sum_1^\infty \lambda_i s_i \otimes x'_i \in (CV_0(T))'_{U^\circ} \tilde{\otimes}_\pi X'_{V^\circ},$$

such that

$$SF = \sum_1^\infty \lambda_i (x'_i \circ F, s_i) \quad \text{for all } F \in CV_0(T, X).$$

For the definition of the operator space $K^b(X, Y)$ and of weighted spaces of vector-valued continuous functions and the most common examples, consult Section 0.2, (a) and (b). (Any of the examples $C(T, X)_{co}$, $C_0(S, X)$, $C_b(S, X)_\beta$ and $C_c(S, X)_i$ will be investigated separately in the sequel.)

The object of the subsequent sections is to refine the representations of Theorem 1.7 to algebraical or even topological isomorphisms between $(L_e(X'_c, Y))'_b$ and $X'_b \tilde{\otimes}_\pi Y'_b$ for various special classes of locally convex spaces.

The present section will be closed with applications of the results given so far to questions of weak convergence in the operator space $L_e(X'_c, Y)$.

1.8. Theorem. *Let X and Y be locally convex spaces, and assume that either of X and Y is semi-reflexive and quasinormable, or a Banach space whose dual has* RNP.

Then the algebraic tensor product $X' \otimes Y'$ is sequentially dense in $(L_e(X'_c, Y))'$ with respect to the strong dual topology on $(L_e(X'_c, Y))'$.

In particular, on bounded subsets of $L_e(X'_c, Y)$, the weak topology of $L_e(X'_c, Y)$ and the $X' \otimes Y'$-weak operator topology coincide: a bounded net $(h_\lambda)_{\lambda \in \Lambda}$ in $L_e(X'_c, Y)$ converges weakly to $h \in L(X'_c, Y)$ if and only if $(h_\lambda x')_{\lambda \in \Lambda}$ converges weakly in Y to hx' for all $x' \in X'$.

This is a direct consequence of Proposition 1.3.

Remarks. For general locally convex spaces X and Y, we only know that

(a) $X' \otimes Y'$ is weak*-dense in $(L_e(X'_c, Y))'$, and that

(b) the families of weakly compact and of $X' \otimes Y'$-wot compact subsets of $L_e(X'_c, Y)$ coincide ([7], Prop. 1.2). Thus, Theorem 1.8 adds the information that under the given assumptions on X and Y, the two topologies even coincide on every bounded subset of $L_e(X'_c, Y)$.

1.9. Corollary. *Let X and Y be complete locally convex spaces, such that either of X and Y is semi-reflexive and quasinormable, or a Banach space whose dual has* RNP. *Then on bounded subsets of $X \tilde{\otimes}_\varepsilon Y$, the weak topology (of $X \tilde{\otimes}_\varepsilon Y$) and the $X' \otimes Y'$-weak operator topology coincide.*

1.10. THEOREM. *Let X and Y be locally convex spaces, Y quasi-complete, and assume that X'_b or Y is semi-reflexive and quasinormable, or a Banach space whose dual has* RNP.

Then the algebraic tensor product $X''\otimes Y'$ is strongly sequentially dense in $(K_b^b(X, Y))'$, and a bounded net $(h_\lambda)_{\lambda\in\Lambda}$ in $K_b^b(X, Y)$ converges weakly to $h\in K^b(X, Y)$ if and only if $(h''_\lambda x'')_{\lambda\in\Lambda}$ converges weakly in Y to $h''x''$ for all $x''\in X''$.

In particular, these assertions hold for the space $K_b(X, Y)$ of compact linear operators from X into Y.

This is a consequence of Theorem 1.8 and the topological linear isomorphisms $K_b(X, Y)\hookrightarrow K_b^b(X, Y)\cong L_e(X''_c, Y)$ ([7], Section 0, example 0.2 (b)). For the special case of Banach spaces X and Y, this has previously been proved by Feder/Saphar ([11], Cor. 1.2).

1.11. COROLLARY. *Let T be a completely regular Hausdorff space, and $V>0$ a Nachbin family on T such that T is a V_R-space, Furthermore, let X be a quasinormable semi-reflexive locally convex space, or a Banach space whose dual has* RNP.

Then $(CV_0(T))'\otimes X'$ is sequentially dense in $(CV_0(T, X))'_b$, and a bounded net $(F_\lambda)_{\lambda\in\Lambda}$ in $CV_0(T, X)$ converges weakly to $F\in CV_0(T, X)$ if and only if $(x'\circ F_\lambda)_{\lambda\in\Lambda}$ converges weakly (in $CV_0(T)$) to $x'\circ F$ for all $x'\in X'$.

A further quite convenient weak convergence criterion can be derived from Proposition 1.3 by analyzing the following composition of continuous maps

$$X'_b\times Y'_b\xrightarrow{\otimes}X'_b\tilde{\otimes}_\pi Y'_b-j\rightarrow(L_e(X'_c, Y))'_b.$$

1.12. PROPOSITION. *Let X and Y be both* gDF *spaces one of which is semi-reflexive, or a Banach space whose dual has* RNP, *or let both be Fréchet spaces one of which is reflexive and quasinormable. Moreover, let M and N be subsets of X' and Y' whose linear spans are strongly dense in X' and Y', respectively.*

Then, on bounded subsets of $L_e(X'_c, Y)$, the weak topology and the $M\otimes N$-weak operator topology coincide.

We note two particularly interesting cases.

NOTATION. Given a Banach space Z, we denote by $\operatorname{sexp} B_Z$ the set of strongly exposed points of its unit ball B_Z.

1.13. THEOREM. *Let X and Y be Banach spaces.*

(a) *If X' and Y' have* RNP, *then a bounded net $(h_\lambda)_{\lambda\in\Lambda}$* in $X\tilde{\otimes}_\varepsilon Y$ *converges weakly to $h\in X\tilde{\otimes}_\varepsilon Y$ if and only if $((h_\lambda x', y'))_{\lambda\in\Lambda}$ converges to (hx', y') for all $x'\in\operatorname{sexp} B_{X'}$ and all $y'\in\operatorname{sexp} B_{Y'}$.*

(b) *If X'' and Y' have* RNP, *then a bounded net $(h_\lambda)_{\lambda\in\Lambda}$ in $K_b(X, Y)$ converges weakly to $h\in K(X, Y)$ if and only if $(h''_\lambda x'', y')$ converges to $(h''x'', y')$ for all $x''\in\operatorname{sexp} B_{X''}$ and all $y'\in\operatorname{sexp} B_{Y'}$.*

For a proof we need only recall that a Banach space Z has RNP if and only if each nonempty closed bounded convex subset of Z is the (norm) closed convex hull of its strongly exposed points, cf. ([10], VIII. 3, Cor. 4, p. 203).

Further consequences of Proposition 1.12 can be based on the fact ([17], Thm. 3.3) that a Banach space Z contains no isomorph of l^1 if and only if every weak*-compact convex subset of Z' is the norm closed convex hull of its extreme points.

1.14. COROLLARY. *Let (Ω, Σ, μ) be a finite measure space, X a Banach space whose dual has* RNP, *and denote by $K(\mu, X)$ the space of all μ-continuous vector measures $F\colon \Sigma\to X$ whose range is relatively compact, equipped with the semivariation norm.*

Then $K(\mu, X)$ is isometrically isomorphic to $L^1(\mu)\tilde{\otimes}_\varepsilon X$, and a bounded net $(F_\lambda)_{\lambda\in\Lambda}$ in $K(\mu, X)$ converges weakly to $F\in K(\mu, X)$ if and only if $(x'\circ F_\lambda(E))_{\lambda\in\Lambda}$ converges to $x'\circ F(E)$ for all $x'\in\operatorname{sexp} B_{X'}$, and all $E\in\Sigma$.

For a proof of the isometry $K(\mu, X)=L^1(\mu)\tilde{\otimes}_\varepsilon X$, the reader is referred to ([10], VIII. 1, Thm. 5, p. 224).

2. Spaces of compact operators on DF **spaces.** In this section, our general results on the representation of the dual of the operator space $L_e(X'_c, Y)$ are specified for the case of gDF spaces X and Y.

The following are the fundamental results.

2.1. THEOREM. *Let X and Y be* gDF *spaces such that X or Y is semi-reflexive, or a Banach space whose dual has* RNP.

(a) *The linear map*

$$j\colon X'_b\tilde{\otimes}_\pi Y'_b\to(L_e(X'_c, Y))'$$

is surjective: for every $T\in(L_e(X'_c, Y))'$ there exists

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i\otimes y'_i\in X'_b\tilde{\otimes}_\pi Y'_b$$

such that

$$Th = \sum_1^\infty \lambda_i(hx'_i, y'_i)\quad \textit{for all } h\in L(X'_c, Y).$$

(b) *If the map*

$$p\colon X'_b\tilde{\otimes}_\pi Y'_b\to B(X, Y)$$

is injective, then j sets up the following algebraical isomorphism:

$$(L_e(X'_c, Y))' = X'_b\tilde{\otimes}_\pi Y'_b = N(X, Y'_b) = \text{nuclear operators } X\to Y'_b.$$

The map p is injective whenever X'_b or Y'_b or Y''_b has the approximation property.

(c) *If X and Y are complete, and*

$$p\colon X_b'\tilde{\otimes}_\pi Y_b' \to B(X, Y)$$

is injective, then we have the following isomorphisms:

(c1) $L_e(X_c', Y) \cong X\tilde{\otimes}_\varepsilon Y$ (*topologically*).

(c2) $(X\tilde{\otimes}_\varepsilon Y)' = X_b'\tilde{\otimes}_\pi Y_b' = N(X, Y_b')$ (*algebraically*).

The question of when the isomorphism $(L_e(X_c', Y))' = X_b'\tilde{\otimes}_\pi Y_b'$ of Theorem 2.1 (b) is topological for the strong topology on $(L_e(X_c', Y))'$, is connected with one of the problems in Grothendieck's work, problem 10 in ([16], II, questions non résolues, p. 137): under which conditions on two DF spaces X and Y is the completed injective tensor product $X\tilde{\otimes}_\varepsilon Y$ again DF, or, at least, its strong dual Fréchet? It follows from a result of H. Buchwalter ([3], Prop. (2.7)) that $(X\tilde{\otimes}_\varepsilon Y)_b'$ is Fréchet whenever X and Y are semi-Montel gDF spaces. We are able to extend this result to the situation of two semi-reflexive gDF spaces only one of which needs to be semi-Montel.

2.2. THEOREM. *Let X and Y be semi-reflexive* gDF *spaces such that X or Y is semi-Montel. Then we have*:

(a) $L_e(X_c', Y)$ *is semi-reflexive*.

(b) $(L_e(X_c', Y))_b' \cong X_b'\tilde{\otimes}_\pi Y_b' \cong N(X, Y_b')$ (*topologically*).

(c) $(X\tilde{\otimes}_\varepsilon Y)_b' \cong (X_b'\tilde{\otimes}_\pi Y_b')/(X\tilde{\otimes}_\varepsilon Y)^\perp$ (*topologically*),

hence $(X\tilde{\otimes}_\varepsilon Y)_b'$ is a Fréchet space.

In connection with this result, recall that the class of semi-reflexive gDF spaces includes the Mackey duals of Fréchet spaces, and that the class of semi-Montel gDF spaces is exactly the class of c-duals of Fréchet spaces.

We now turn to the proofs of Theorems 2.1 and 2.2. Clearly, proposition (a) and the first part of proposition (b) of Theorem 2.1 are direct consequences of the general results established in Section 1 (Lemma 1.2 and Proposition 1.3). Thus, what essentially remains to be proven is the indicated connection between the injectivity of the map p and the approximation property for X_b', Y_b', or Y_b''. At the beginning of this proof, we would like to apologize for the annoying fact (both for the reader and for us) that, although everybody will take this connection for granted, it takes us lengthy technical arguments to really establish it.

First we need the following result:

2.3. LEMMA. *Let X and Y be locally convex spaces such that*

(α) *Y is evaluable, and*

(β) *every hypocontinuous bilinear form on $X\times Y$ is continuous.*

Then $(X\tilde{\otimes}_\pi Y)'$ is algebraically isomorphic to $L(X, Y_b')$. Condition (β) *is fulfilled whenever X and Y are both Fréchet or both* gDF *spaces.*

Proof. It is known that $(X\tilde{\otimes}_\pi Y)'$ is equal to $B(X, Y)$ and that $B(X, Y)$ is a linear subspace of $L(X, Y_b')$ for any locally convex spaces X and Y:

$$B(X, Y)\to L(X, Y_b'),$$
$$B\mapsto \{x\mapsto B(x,.)\}.$$

It remains to prove that under the given assumptions, every $u\in L(X, Y_b')$ defines a continuous bilinear form on $X\times Y$. Let $u\in L(X, Y_b')$ and consider

$$B_u\colon X\times Y\to \boldsymbol{K},$$
$$(x, y)\mapsto (y, ux).$$

It suffices to show that B_u is hypocontinuous: if B is a bounded subset of X, then $u(B)$ is bounded in Y_b', and thus, by the evaluability of Y, equicontinuous. There exists a zero neighbourhood V in Y such that $u(B)\subset V^\circ$. Hence, we have:

$$|B_u(B, V)| = |\langle V, u(B)\rangle| \leqslant 1.$$

If C is a bounded subset of Y, then there exists a zero neighbourhood U in X such that $u(U)\subset C^\circ$. Hence, we have:

$$|B_u(U, C)| = |\langle C, u(U)\rangle| \leqslant 1.$$

Finally, hypocontinuous bilinear forms on $X\times Y$ are continuous whenever X and Y are Fréchet, cf. ([33], II. 34-5, Cor. on p. 354), or whenever X and Y are gDF ([28], Thm. 1.4).

Next, we need the following relative of our map p: given arbitrary locally convex spaces X and Y, consider the linear map

$$p_1\colon X_b'\otimes_\pi Y_b'\to B_{ee}\big((X_b')_c', (Y_b')_c'\big),\tag{2.1}$$
$$\sum_1^n x_i'\otimes y_i'\mapsto \Big\{(x'', y'')\mapsto \sum_1^n (x_i', x'')(y_i', y'')\Big\}.$$

(2.1.1) p_1 is continuous.

(2.1.2) Whenever X and Y are both gDF or both Fréchet spaces, then the space $\mathscr{H}_{ee}\big((X_b')_c', (Y_b')_c'\big)\cong X_b'\,\varepsilon\, Y_b'$ is complete, cf. ([7], Section 0, (0.6.2)), and, according to the inclusion $B_{ee}\big((X_b')_c', (Y_b')_c'\big)\subset \mathscr{H}_{ee}\big((X_b')_c', (Y_b')_c'\big)$, the map p_1 has a continuous linear extension

$$\tilde{p}_1\colon X_b'\tilde{\otimes}_\pi Y_b'\to \mathscr{H}_{ee}\big((X_b')_c', (Y_b')_c'\big).$$

(2.1.3) Let X and Y be both gDF or both Fréchet spaces, and let $\tilde{v}\in X_b'\tilde{\otimes}_\pi Y_b'$. Then $p(\tilde{v}) = \tilde{p}_1(\tilde{v})|X\times Y$, and $\tilde{p}_1(\tilde{v}) = 0$ whenever $p(\tilde{v}) = 0$.

Proof. Let $p(\tilde{v}) = 0$, and let $(x'', y'') \in X'' \times Y''$. There exist closed bounded disks B and C in X and Y, respectively, such that

$$(x'', y'') \in B^{\circ\bullet} \times C^{\circ\bullet} = \bar{B}^c \times \bar{C}^c$$

(c-closures of B and C in $(X'_b)'_c$ and $(Y'_b)'_c$, respectively). Since $\tilde{p}_1(\tilde{v})|B \times C = 0$ and $\tilde{p}_1(\tilde{v})|B^{\circ\bullet} \times C^{\circ\bullet}$ is $c \times c$-continuous, cf. ([16], I, Intr. VI, Lemme D, p. 27), we conclude that

$$\tilde{p}_1(\tilde{v})(x'', y'') = 0.$$

Proof of Theorem 2.1 (b). Let X and Y be gDF spaces,

$$\tilde{v} \in X'_b \tilde{\otimes}_\pi Y'_b, \qquad \tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i,$$

$(\lambda_i)_{i\in N} \in l^1$, $(x'_i)_{i\in N}$ and $(y'_i)_{i\in N}$ nullsequences in X'_b and Y'_b, respectively, and assume that $p(\tilde{v}) = 0$. We have to show that $\tilde{v} = 0$ whenever X'_b, or Y'_b, or Y''_b has the approximation property (a.p.).

Step 1. Assume that X'_b or Y''_b has a.p. According to (2.1.3) we have

(1)
$$0 = \tilde{p}_1(\tilde{v})(x'', y'') = \sum_1^\infty \lambda_i (x'_i, x'')(y'_i, y'') \quad \text{for all } (x'', y'') \in X'' \times Y''.$$

Since, by Lemma 2.3, $(X'_b \tilde{\otimes}_\pi Y'_b)' = L(X'_b, Y''_b)$, we can interpret (1) in the following way:

(2)
$$k(\tilde{v}) = 0 \quad \text{for all } k \in X'' \otimes Y''.$$

Furthermore, if X'_b or Y''_b has a.p., then $X'' \otimes Y''$ is dense in $L_c(X'_b, Y''_b)$, cf. ([15], I. 5.1, Prop. 35, pp. 164, 165).

Now, let $u \in L(X'_b, Y''_b) = (X'_b \tilde{\otimes}_\pi Y'_b)'$, and $\varepsilon > 0$. Then there exists $k \in X'' \otimes Y''$ such that $(k-u)\big((x'_i)_{i\in N}\big) \subset \varepsilon \big(\sum_1^\infty |\lambda_i|\big)^{-1} V^{\circ\bullet}$, where $V \in \mathcal{U}_Y$ is chosen such that $(y'_i)_{i\in N} \subset V^\circ$. We conclude that

$$|u\tilde{v}| \leqslant |(u-k)(\tilde{v})| = \Big|\sum_1^\infty \lambda_i \big((u-k)x'_i, y'_i\big)\Big| \leqslant \sum_1^\infty |\lambda_i| \big|\big((u-k)x'_i, y'_i\big)\big| \leqslant \varepsilon.$$

Hence, $u(\tilde{v}) = 0$ for all $u \in (X'_b \tilde{\otimes}_\pi Y'_b)'$, i.e. $\tilde{v} = 0$. This completes the proof.

Step 2. Assume that Y'_b has a.p.

(2.1.4) Let X and Y be locally convex spaces such that X'_b and Y'_b are evaluable. Then the map

$$S: L(X'_b, Y''_b) \to L(Y'_b, X''_b),$$

$$u \mapsto \overset{*}{u}: \{y' \mapsto (u(.), y')\}$$

is a linear bijection, and $(\overset{*}{u})^* = u$.

This is a consequence of the fact that $\overset{*}{u} = u'|Y'_b$, and that evaluable locally convex spaces Z are topological linear subspaces of their strong bidual $Z''_b = (Z'_b)'_b$.

Given gDF spaces X and Y such that Y'_b has a.p., we can now use the symmetry $L(X'_b, Y''_b) = L(Y'_b, X''_b)$ of (2.1.4) together with the denseness of $Y'' \otimes X''$ in $L_c(Y'_b, X''_b)$ to follow the reasoning of step 1 above in order to establish injectivity of p also in this case. We omit the details.

To complete the proof of Theorem 2.1 (b), it remains to show that $X'_b \tilde{\otimes} Y'_b$ is isomorphic to the space $N(X, Y'_b)$ of nuclear operators from X into Y'_b, whenever p is injective. To this end, consider the map

$$\iota_0: X'_b \otimes_\pi Y'_b \to L_b(X, Y'_b),$$

$$\sum_1^n x'_i \otimes y'_i \mapsto \Big\{x \mapsto \sum_1^n (x, x'_i) y'_i\Big\}.$$

Since $L_b(X, Y'_b)$ is Fréchet (X and Y are gDF!), ι_0 has a continuous linear extension $\iota: X'_b \tilde{\otimes}_\pi Y'_b \to L_b(X, Y'_b)$. According to the definition of nuclear operators, the space $N(X, Y'_b)$ is even *topologically* isomorphic to $X'_b \tilde{\otimes}_\pi Y'_b / \ker \iota$. But it is easy to see that $\ker \iota = \{0\}$ whenever p is injective. This completes the proof of part (b) of Theorem 2.1.

For a proof of part (c), let p be injective, and let

$$T = j(\tilde{v}) \in \big(L_e(X'_c, Y)\big)',$$

$\tilde{v} \in X'_b \tilde{\otimes}_\pi Y'_b$, be such that $T|X \tilde{\otimes}_\varepsilon Y = 0$. This implies that $p(\tilde{v}) = 0$, and thus $\tilde{v} = 0$, and $T = 0$. Under the given assumptions this means that $X \tilde{\otimes}_\varepsilon Y = L_e(X'_c, Y)$.

The proof of Theorem 2.1 is now complete.

Proof of Theorem 2.2: Part (a) is a special case of a result in our previous paper ([7], Section 2, Thm. 2.13).

Part (b): We may assume that X is semi-Montel. Then we note that

$$(X'_b \tilde{\otimes}_\pi Y'_b)' = L(X'_b, Y''_b) = L(X'_c, Y_b)$$

(Lemma 2.3 and semi-reflexivity of Y), and that $L(X'_c, Y_b) = L(X'_c, Y)$: Y and Y_b have the same bounded sets. Hence, every $u \in L(X'_c, Y)$ transforms bounded subsets of X'_c into bounded subsets of Y_b. But $X'_c = X'_b$ is Fréchet, so that $u \in L(X'_c, Y_b)$. We thus have shown that $X'_b \tilde{\otimes}_\pi Y'_b$ and $\big(L_e(X'_c, Y)\big)'_b$ have the same dual (according to part (a), $L_e(X'_c, Y)$ is semi-reflexive!). Hence, if $j(\tilde{v}) = 0$,

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i \in X'_b \tilde{\otimes}_\pi Y'_b,$$

then we have:

$$0 = j(\tilde{v})(u) = \sum_1^\infty \lambda_i(ux_i', y_i') = u(\tilde{v}) \quad \text{for all } u \in L(X_c', Y) = (X_b' \tilde{\otimes}_\pi Y_b')',$$

and thus $\tilde{v} = 0$. Altogether, we have shown that $X_b' \tilde{\otimes}_\pi Y_b'$ and $(L_e(X_c', Y))_b'$ are algebraically isomorphic and have the same dual. Since $X_b' \tilde{\otimes}_\pi Y_b'$ is Fréchet and $(L_e(X_c', Y))_b'$ is metrizable [29], we conclude that both spaces are topologically isomorphic.

Part (c): According to part (b), $(L_e(X_c', Y))_b'$ is Fréchet. $X \tilde{\otimes}_\varepsilon Y$ is a closed linear subspace of (the semi-reflexive) space $L_e(X_c', Y)$. Hence, the assumptions of Proposition 2.7 (i), of [26] are fulfilled, and we can conclude that

$$\beta((X \tilde{\otimes}_\varepsilon Y)', X \tilde{\otimes}_\varepsilon Y) = \beta((L_e)', L_e)/(X \tilde{\otimes}_\varepsilon Y)^\perp.$$

We now turn to applications and special cases of Theorems 2.1 and 2.2.

Recall that, given locally convex spaces X and Y, we denote by $K_b^b(X, Y)$ the space of weakly continuous linear operators from X into Y which transform bounded sets into relatively compact sets, endowed with the topology of uniform convergence on bounded subsets of X. Whenever Y is quasi-complete, then $K_b^b(X, Y) \cong L_e(X_c'', Y)$, by ([7], Example 0.2 (c)). Note that $K_b(X, Y)$ is a topological linear subspace of $K_b^b(X, Y)$. We first deduce a representation for the continuous linear functionals on $K_b^b(X, Y)$.

2.4. Theorem. *Let X be a metrizable space and Y a complete* gDF *space such that X_b' or Y is semi-reflexive, or a Banach space whose dual has* RNP. *Then we have:*

(a) *The map*

$$j\colon X_b'' \tilde{\otimes}_\pi Y_b' \to (K_b^b(X, Y))'$$

is surjective: for every $T \in (K_b^b(X, Y))'$ there exist $(\lambda_i)_{i\in N} \in l^1$ and nullsequences $(x_i'')_{i\in N}$ and $(y_i')_{i\in N}$ in X_b'' and Y_b', respectively, such that

$$Th = \sum_1^\infty \lambda_i(h''x_i'', y_i') \quad \text{for all } h \in K^b(X, Y).$$

(b) *If, in addition to the assumptions, X_b'', or Y_b', or Y_b'' has the approximation property, then the map j of* (a) *is a linear isomorphism:*

$$(K_b^b(X, Y))' = X_b'' \tilde{\otimes}_\pi Y_b' = N(X_b', Y_b') \quad (\textit{algebraically}).$$

(c) *If, in addition to the assumptions, X_b' and Y are semi-reflexive and one of them is semi-Montel, then we have:*

(c1) *$K_b^b(X, Y)$ is semi-reflexive.*

(c2) $(K_b^b(X, Y))_b' \cong X_b'' \tilde{\otimes}_\pi Y_b' \cong N(X_b', Y_b')$ (*topologically*).

2.5. Corollary. *Let X be a Banach space and Y a complete* gDF *space such that X'' has* RNP *or that Y is semi-reflexive. Then we have:*

(a) *The dual of $K_b(X, Y)$ is algebraically isomorphic to a quotient of $X_b'' \tilde{\otimes}_\pi Y_b'$.*

(b) *If, in addition to the assumptions, X is a reflexive Banach space and Y is semi-Montel, then the space $K_b(X, Y)$ is semi-reflexive, and*

$$(K_b(X, Y))_b' \cong X \tilde{\otimes}_\pi Y_b' \cong N(X_b', Y_b') \quad (\textit{topologically}).$$

2.6. Corollary. *Let X be a Banach space and Y the c-dual of a Banach space with the approximation property. Then we have:*

(a) $(L_b(X, Y))' = X_b'' \tilde{\otimes}_\pi Y_b' = N(X_b', Y_b')$ (*algebraically*).

(b) *If, in addition to the assumptions, X is reflexive, then $L_b(X, Y)$ is semi-reflexive, and we have:*

$$(L_b(X, Y))_b' \cong X \tilde{\otimes}_\pi Y_b' \cong N(X_b', Y_b') \quad (\textit{topologically}).$$

The spaces l_β^∞ and $H^\infty(G)_\beta$, β the strict topology of R. C. Buck ([4], [5]), G a simply connected region in the plane, are (non-trivial) concrete examples for c-duals of Banach spaces with a.p. (for $H^\infty(G)_\beta$, consult ([1], Satz 9)).

2.7. Theorem. *Let X be a Fréchet space and Y a* gDF *space.*

(a) *For every continuous linear functional T on $L_c(X, Y)$ there exist $(\lambda_i)_{i\in N} \subset l^1$ and nullsequences $(x_i)_{i\in N}$ and $(y_i')_{i\in N}$ in X and Y_b', respectively, such that*

$$Th = \sum_1^\infty \lambda_i(hx_i, y_i') \quad \text{for all } h \in L(X, Y).$$

(b) *If, in addition to the assumptions, X or Y_b' or Y_{b1}'' has the approximation property, then $(L_c(X, Y))' = X \tilde{\otimes}_\pi Y_b'$ (algebraically).*

(c) *If, in addition to the assumptions, Y is semi-reflexive, then $L_c(X, Y)$ is semi-reflexive and*

$$(L_c(X, Y))_b' \cong X \tilde{\otimes}_\pi Y_b' \cong N(X_c', Y_b') \quad (\textit{topologically}).$$

This is a consequence of Theorems 2.1 and 2.2 and the topological isomorphism $L_c(X, Y) = L_c((X_c')_c', Y)$ of Example 0.3 in [7]. For a DF space Y, proposition (a) of Theorem 2.5 is a result of Grothendieck [16], I.4.2, Prop. 22, p. 114).

2.8. Corollary. *Let X be a Fréchet–Montel space and Y a* gDF *space.*

(a) *If X or Y_b' or Y_b'' has the approximation property, then*

$$(L_b(X, Y))' = X \tilde{\otimes}_\pi Y_b' = N(X_b', Y_b') \quad (\textit{algebraically}).$$

(b) *If Y is semi-reflexive, then $L_b(X, Y)$ is semi-reflexive and*

$$(L_b(X, Y))_b' \cong X \tilde{\otimes}_\pi Y_b' \cong N(X_b', Y_b') \quad (\textit{topologically}).$$

In the context of vector-valued continuous functions, Singer [32] and Bogdanowicz ([2], Thm. 4) determined the dual of $C(K, X)$ (K compact Hausdorff, X Banach), and Wells ([34], Thm. 1) and Fontenot ([13], Thm. 3.13) the dual of $C_b(S, X)_\beta$ (S locally compact or completely regular Hausdorff, X a locally convex space, β the strict topology of R. C. Buck ([4], [5])). In any case, the dual turned out to be a space of certain X'-valued measures. It appears to be one of the nicest applications of our general duality results, that we are able to derive a much more specific result for the special case that the range space X is a semi-reflexive gDF space or a Banach space whose dual has RNP.

2.9. THEOREM. *Let S be a locally compact Hausdorff space, X a semi-reflexive gDF space, or a Banach space whose dual has RNP, and denote by $C_b(S, X)_\beta$ the space of bounded continuous X-valued functions on S, endowed with the strict topology β of R. C. Buck* ([4], [5]). *Furthermore, let $M_b(S)$ be the space of bounded regular Borel measures on S, endowed with the total variation norm.*

Then we have:

$$\big(C_b(S, X)_\beta\big)' = M_b(S)\tilde{\otimes}_\pi X'_b \quad (\textit{algebraically}).$$

For every $T \in \big(C_b(S, X)_\beta\big)'$ there exist $(\lambda_i)_{i\in N} \in l^1$, and nullsequences $(\mu_i)_{i\in N}$ and $(x'_i)_{i\in N}$ in $M_b(S)$ and X'_b, respectively, such that

$$TF = \sum_1^\infty \lambda_i \int (x'_i \circ F)\, d\mu_i \quad \textit{for all } F \in C_b(S, X).$$

This result is just a special case of Theorem 2.1 (c): $C_b(S)_\beta$ and $M_b(S)$ have the approximation property (for $C_b(S)_\beta$, see [6]), and, according to the results of Section 4 of our previous paper [7], we have

$$C_b(S, X)_\beta \cong L_e\big(((C_b(S))_\beta)'_c, X\big) \quad (\cong C_b(S)_\beta \tilde{\otimes}_\varepsilon X).$$

2.10. THEOREM. *Let S be a locally compact σ-compact Hausdorff space, X a Banach space whose dual has RNP, and denote by $C_c(S, X)_i$ the space of continuous X-valued functions on S with compact support, endowed with the usual inductive limit topology. Furthermore, denote by $M(S)$ the space of Radon measures on S, endowed with the strong dual topology of $\big(C_c(S)_i\big)'$.*

Then, for every $T \in \big(C_c(S, X)_i\big)'$, there exists

$$\tilde{v} = \sum_1^\infty \lambda_i \mu_i \otimes x'_i \in M(S)\tilde{\otimes}_\pi X'$$

such that

$$TF = \sum_1^\infty \lambda_i \int (x'_i \circ F)\, d\mu_i \quad \textit{for all } F \in C_c(S, X).$$

Again, this is an obvious special case of Theorem 2.1, for we have the topological isomorphism $C_c(S, X)_i \cong L_e\big(X'_c, C_c(S)_i\big)$, cf. ([7], Section 4).

We close this section with an application of Theorems 2.1 and 2.2 to spaces of holomorphic vector-valued functions with the strict topology β.

2.11. THEOREM. *Let G be a simply connected region in the complex plane, and X a complete gDF space, and denote by*

$$M_0(G) = M_b(G)/\big(H^\infty(G)\big)^\perp$$

the (strong) dual of $H^\infty(G)_\beta$. Then we have:

(a) $\big(H^\infty(G, X)_\beta\big)' = M_0(G)\tilde{\otimes}_\pi X'_b$ *(algebraically).*

For every $T \in \big(H^\infty(G, X)_\beta\big)'$ there exist $(\lambda_i)_{i\in N} \in l^1$, and nullsequences $(\mu_i)_{i\in N}$ and $(x'_i)_{i\in N}$ in $M_0(G)$ and X'_b, respectively, such that

$$TF = \sum_1^\infty \lambda_i \int (x'_i \circ F)\, d\mu_i \quad \textit{for all } F \in H^\infty(G, X).$$

(b) *If, in addition to the assumptions, X is semi-reflexive, then $H^\infty(G, X)_\beta$ is semi-reflexive, and we have:*

$$\big(H^\infty(G, X)_\beta\big)'_b \cong M_0(G)\tilde{\otimes}_\pi X'_b \quad (\textit{topologically}).$$

This result is a special case of Theorems 2.1 and 2.2: $H^\infty(G)_\beta$ is a semi-Montel gDF space [26], and, for G simply connected, has the approximation property ([1], Satz 9). According to ([21], § 43.4, (9)], $M_0(G)$ has the a.p. as well.

3. Spaces of compact operators on Banach spaces.

In the context of Banach spaces, the general duality results of the foregoing sections nicely specialize to isometrical representations of duals of spaces of compact operators. The following is the fundamental result. (The dual of a normed space Z, endowed with the dual norm, is denoted by Z'.)

3.1. THEOREM. *Let X and Y be Banach spaces such that X' or Y' has RNP. Then we have:*

(a) *The map*

$$j\colon X'\tilde{\otimes}_\pi Y' \to \big(L_e(X'_c, Y)\big)'$$

is surjective, and $\big(L_e(X'_c, Y)\big)'$ is isometrically isomorphic to $(X'\tilde{\otimes}_\pi Y')/\ker j$. For every $T \in \big(L_e(X'_c, Y)\big)'$ there exists

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i \in X'\tilde{\otimes}_\pi Y', \quad \|\tilde{v}\|_\pi = \|T\|,$$

such that

$$Th = \sum_1^\infty \lambda_i (h x'_i, y'_i) \quad \textit{for all } h \in L_e(X'_c, Y).$$

(b) *If, in addition to the assumptions, the map*

$$p\colon X'\tilde{\otimes}_\pi Y'\to B(X, Y)$$

is injective, then we have the following isometrical isomorphisms:

(b1) $X\tilde{\otimes}_\varepsilon Y = L_e(X'_c, Y)$ = *the space of compact weak*-weakly continuous linear operators from* X' *into* Y.

(b2) $(X\tilde{\otimes}_\varepsilon Y)' = N(X, Y')$ = *the space of nuclear operators from* X *into* Y'.

(b3) $(X\tilde{\otimes}_\varepsilon Y)'' = L(X', Y'')$ = *the space of bounded linear operators from* X' *into* Y''.

The map p *is injective whenever* X' *or* Y' *has the approximation property.*

In view of the applications of Theorem 3.1 to quite recent results in Banach space theory, it seems particularly remarkable to note that, essentially, this theorem can be traced back to Grothendieck's work ([16], I.4.2, Thm. 8, p. 122), compare ([14], Thms. 5.2 and 5.3).

Here, Theorem 3.1 appears just as a special case of Theorem 2.1.

We now discuss consequences and special cases of Theorem 3.1.

3.2. THEOREM. *Let* X *be a normed space, or, more generally, a* gDF *space whose strong dual* X'_b *is a Banach space,* Y *a Banach space, and assume that* X'' *or* Y' *has* RNP. *Furthermore, consider the linear map*

$$j\colon X''\tilde{\otimes}_\pi Y'\to (K(X, Y))',$$

$$\sum_1^n x''_i\otimes y'_i \mapsto \Big\{h\mapsto \sum_1^n (h''x''_i, y'_i)\Big\}.$$

Then we have:

(a) $(K(X, Y))' = X''\tilde{\otimes}_\pi Y'/\ker j$ (*isometrically*).

For every $T\in (K(X, Y))'$, *there exists*

$$\tilde{v} = \sum_1^\infty \lambda_i x''_i\otimes y'_i \in X''\tilde{\otimes}_\pi Y', \qquad \|T\| = \|\tilde{v}\|_\pi,$$

such that

$$Th = \sum_1^\infty \lambda_i (h''x''_i, y'_i) \quad \textit{for all } h\in K(X, Y).$$

(b) *The map* j *is injective, whenever the map*

$$p\colon X''\tilde{\otimes}_\pi Y'\to B(X', Y)$$

is injective, in particular, whenever X'' *or* Y' *has the approximation property. In any of these cases, we have the following isometrical isomorphisms:*

(b1) $(K(X, Y))' = N(X', Y')$.

(b2) $(K(X, Y))'' = L(X'', Y'')$.

This follows from Theorem 3.1 and the isometrical isomorphism $K(X, Y) = L_e(X''_c, Y)$ of ([7], Example 0.2 (a)). For the case of Banach spaces X and Y, Theorem 3.2 quite recently has been (re)proven (see the notes preceeding Theorem 3.2) by Feder and Saphar ([11], Thm. 1) in a different way, using further deep results on the Radon–Nikodym property for Banach spaces.

Note. In a series of papers, Ruckle [24], Holub [19], Kalton [20] and Heinrich [18] dealt with the conjecture that, for reflexive Banach spaces X and Y, the space $L(X, Y)$ is reflexive if and only if every bounded linear operator from X into Y is compact. Roughly, this connection holds whenever, in addition, X or Y has a.p. For a detailed discussion of the problem we refer to Section 2, Theorem 2.8, of our previous paper [7]. Theorem 3.2 allows to add the following more specific information:

3.3. COROLLARY. *Let* X *and* Y *be reflexive Banach spaces.*

(a) *If* $L(X, Y) = K(X, Y)$, *then* $L(X, Y)$ *is reflexive.*

(b) *Conversely, if* $K(X, Y)$ *is weakly sequentially complete and the map* $j\colon X\tilde{\otimes}_\pi Y'\to (K(X, Y))'$ *of Theorem* 3.2 *above is injective, then we have* $L(X, Y) = K(X, Y)$.

In particular, $L(X, Y) = K(X, Y)$ *if and only if* $K(X, Y)$ *is reflexive and* j *is injective.*

This result contains the corresponding ones of the authors cited above as special cases. It is a consequence of Theorem 3.2 and of ([7], Thm. 2.8). Furthermore, it reduces the question of whether the conjecture "$L(X, Y)$ is reflexive if and only of $L(X, Y) = K(X, Y)$", (X and Y reflexive Banach spaces) is true to the problem of injectivity of j.

PROBLEM. Given reflexive Banach spaces X and Y, is the map $j\colon X\tilde{\otimes}_\pi Y'\to (K(X, Y))'$ of Theorem 3.2 injective? More specifically: given reflexive Banach spaces X and Y, and

$$\tilde{v} = \sum_1^\infty \lambda_i x_i\otimes y'_i \in X\tilde{\otimes}_\pi Y',$$

$(\lambda_i)_{i\in N}\in l^1$, $(x_i)_{i\in N}$ and $(y'_i)_{i\in N}$ nullsequences in X and Y', such that $\tilde{v}(k) = \sum_1^\infty \lambda_i(kx_i, y'_i) = 0$ for all $k\in K(X, Y)$, is it true that then $\tilde{v}(u) = \sum_1^\infty \lambda_i(ux_i, y'_i) = 0$ for all $u\in L(X, Y)$ as well?

Trivially, this is true whenever X or Y has a.p. That it very well can happen without X or Y having a.p. can be seen by combining a result of Pitt's with the counterexamples to the a.p.: it is a consequence of ([22], Thm. 1) that $L(l^q, l^p) = K(l^q, l^p)$ for $1\leqslant p < q < \infty$. Furthermore, according to results of A. M. Davie [9] and T. Figiel [12], there exist closed linear subspaces of l^p without a.p. for any p with $2 < p \leqslant \infty$. Thus, we arrive at the desired example if we choose subspaces M and N of l^p and l^q, respectively, without a.p., for p and q such that $2 < p < q < \infty$. For then we still have $L(N, M) = K(N, M)$, as can be seen by employing techniques of Rosenthal's ([23], proof of Theorem A2, pp. 206, 207). (We are grateful to our colleague Lutz Weis for working out this example.)

We close this section with applications to spaces of vector-valued continuous functions and to spaces of compact operators on function spaces with the strict topology.

3.4. THEOREM. *Let S be a locally compact Hausdorff space, X a Banach space whose dual has* RNP, *and denote by $C_0(S, X)_\infty$ the space of continuous X-valued functions on S vanishing at infinity, endowed with the sup-norm topology. Furthermore, denote by $M_b(S)$ the space of bounded Radon measures on S. Then we have*

$$\big(C_0(S, X)_\infty\big)' = M_b(S)\tilde{\otimes}_\pi X' \quad (\textit{isometrically}).$$

For every $T \in \big(C_0(S, X)_\infty\big)'$, there exists

$$\tilde{v} = \sum_1^\infty \lambda_i \mu_i \otimes x_i' \in M_b(S)\tilde{\otimes}_\pi X', \qquad \|\tilde{v}\|_\pi = \|T\|,$$

such that

$$TF = \sum_1^\infty \lambda_i \int (x_i' \circ F)\, d\mu_i \quad \textit{for all } F \in C_0(S, X).$$

This follows from Theorem 3.1 (b) and the known facts that $C_0(S, X)_\infty = C_0(S)\tilde{\otimes}_\varepsilon X$, and that $M_b(S)$ has a.p.

TERMINOLOGY AND NOTATION. (a) For a completely regular Hausdorff space T, β_0, β, and β_1 denote the substrict, strict, and superstrict topology of [31], respectively, on the space $C_b(T)$ of bounded continuous scalar-valued functions on T. (Note that, whenever $T = S$ = locally compact, $\beta_0 = \beta$ = Buck's original strict topology). By $M_\zeta(T)$, $\zeta \in \{\beta_0, \beta, \beta_1\}$, we denote the respective duals of $C_b(T)_\zeta$, the spaces of tight, τ-additive, and σ-additive measures on T, respectively.

(b) Given a region G in the complex plane, we denote by $H^\infty(G)_\beta$ (resp. $H^\infty(G)_\infty$) the space of bounded holomorphic functions on G endowed with the strict topology β (resp. the sup-norm topology).

We are now ready to state the following special case of Theorem 3.2:

3.5. THEOREM. *Let T be a completely regular Hausdorff space, G a plane region, and X a Banach space whose dual has* RNP *and the approximation property. Then we have the following isometrical isomorphisms:*

(a) (i) $K\big(C_b(T)_\zeta, X\big) = M_\zeta(T)\tilde{\otimes}_\varepsilon X$,
(ii) $\big(K(C_b(T)_\zeta, X)\big)' = N\big(M_\zeta(T), X'\big)$,
(iii) $\big(K(C_b(T)_\zeta, X)\big)'' = L\big((M_\zeta(T))', X''\big)$ $(\zeta \in \{\beta_0, \beta, \beta_1\})$.

(b) (i) $L\big(H^\infty(G)_\beta, X\big) = K\big(H^\infty(G)_\beta, X\big) = M_0(G)\tilde{\otimes}_\varepsilon X$,
(ii) $\big(L(H^\infty(G)_\beta, X)\big)' = N\big(M_0(G), X'\big)$,
(iii) $\big(L(H^\infty(G)_\beta, X)\big)'' = L\big(H^\infty(G)_\infty, X''\big)$.

4. Spaces of compact operators on Fréchet spaces. This section is devoted to a study of spaces of compact operators acting between Fréchet spaces X and Y. Again, we start with the fundamental duality result for the space $L_e(X_c', Y)$.

4.1. THEOREM. *Let X and Y be Fréchet spaces one of which is reflexive and quasinormable, or a Banach space whose dual has* RNP.

(a) *The c-dual of $L_e(X_c', Y)$ is topologically isomorphic to $X_c'\tilde{\otimes}_\pi Y_c'$:*

$$\big(L_e(X_c', Y)\big)_c' = X_c'\tilde{\otimes}_\pi Y_c'.$$

(b) *The linear map*

$$j\colon X_b'\tilde{\otimes}_\pi Y_b' \to \big(L_e(X_c', Y)\big)'$$

is surjective. More precisely, we have: for every $T \in \big(L_e(X_c', Y)\big)'$ there exist $U \in \mathscr{U}_X$, $V \in \mathscr{U}_Y$, and

$$\tilde{v} = \sum_1^\infty \lambda_i x_i' \otimes y_i' \in X_{U^\circ}'\tilde{\otimes}_\pi Y_{V^\circ}'$$

such that

$$Th = \sum_1^\infty \lambda_i (h x_i', y_i') \quad \textit{for all } h \in L_e(X_c', Y).$$

(c) *If the map*

$$p\colon X_b'\tilde{\otimes}_\pi Y_b' \to B(X, Y)$$

is injective, then we have:

(c1) $X\tilde{\otimes}_\varepsilon Y = L_e(X_c', Y)$ *(topologically).*
(c2) $(X\tilde{\otimes}_\varepsilon Y)' = X_b'\tilde{\otimes}_\pi Y_b'$ *(algebraically).*

p is injective if any of the following additional conditions on X and Y are fulfilled:

Case 1. *X is Fréchet–Montel, Y_b' is barrelled, and X or Y_b'' has* a.p.

Case 2. *Y is Fréchet–Montel, X_b' is barrelled, and Y or X_b'' has* a.p.

Remarks. 1. We are not able to establish a connection between injectivity of p and the approximation property as nice and general as in the case of gDF spaces X and Y (Theorem 2.1!), but need the additional assumption that at least one of the spaces X or Y is Montel. As will become clear from the proof of part (c) of Theorem 4.1 below, the reason for this lies in the fact, that, in general, for the elements $\tilde{v} \in X_b'\tilde{\otimes}_\pi Y_b'$, we do not have a series representation in the form of an (infinite) absolutely convex combination of the tensor product of two nullsequences in X_b' and

Y_b', respectively. Note that, conversely, we arrive at such a representation whenever the map $j\colon X_b'\tilde{\otimes}_\pi Y_b' \to (L_e(X_c', Y))'$ is injective (Lemma 4.3 below).

2. Again, in general ,we are also not able to extend the algebraic isomorphism of part (c2) to a topological isomorphism between $X_b'\tilde{\otimes}_\pi Y_b'$ and $(L_e(X_c', Y))_b'$. But we do get the desired result in a special situation, analogous to that of the corresponding result for gDF spaces X and Y in Section 2 (Theorem 2.2).

4.2. THEOREM. *Let X and Y be reflexive Fréchet spaces such that X or Y is quasinormable and X or Y is Montel. Then we have:*

(a) *$L_e(X_c', Y)$ and $X\tilde{\otimes}_\varepsilon Y$ are reflexive Fréchet spaces.*

(b) *$(L_e(X_c', Y))_b' \cong X_b'\tilde{\otimes}_\pi Y_b'$ (topologically).*

Note. The assumptions of Theorem 4.2 are fulfilled whenever X and Y are reflexive Fréchet spaces one of which is even a Fréchet–Schwartz space.

Proof of Theorem 4.1. Part (a): We have the following topological isomorphisms:

$$L_e(X_c', Y) \cong B_{bb}(X_c', Y_c') \cong (X_c'\tilde{\otimes}_\pi Y_c')_b',$$

the second one being a special case of Theorem 1.9 of [28]. According to that same result, $X_c'\tilde{\otimes}_\pi Y_c'$ is semi-Montel gDF, hence we have $(L_e(X_c', Y))_c' \cong X_c'\tilde{\otimes}_\pi Y_c'$ topologically, for semi-Montel gDF spaces are exactly the c-duals of their strong duals.

Part (b) is a consequence of Theorem 1.3, Proposition 1.7 and Lemma 1.8.

Part (c): In case $p\colon X_b'\tilde{\otimes}_\pi Y_b' \to B(X, Y)$ is injective, the map

$$j\colon X_b'\tilde{\otimes}_\pi Y_b' \to (L_e(X_c', Y))'$$

is injective too, and, as in the proof of the corresponding fact in Theorem 2.1, it is easy to see that

$$X\tilde{\otimes}_\varepsilon Y \cong L_e(X_c', Y).$$

We now turn to the connection between injectivity of p and the approximation property. We give a proof for case 1. The other case follows by symmetry.

Case 1: X Montel, X or Y_b'' has a.p. Let $\tilde{v} \in X_b'\tilde{\otimes}_\pi Y_b'$ such that $p(\tilde{v}) = 0$. Then, according to Proposition (2.1.3) in Section 2, we have:

$$(1)\qquad k(\tilde{v}) = 0 \quad \text{for all } k \in X''\otimes Y''.$$

Furthermore, since X has a.p. and is Fréchet–Montel, $X_b' = X_c'$ has a.p., too, cf. ([21], § 43.4, (10), p. 248). Hence, under the assumptions on X and Y, the space $X''\otimes Y''$ is dense in $L_c(X_b', Y_b'')$. At this point, if we would know that $\tilde{v}$ has a series representation

$$\tilde{v} = \sum_1^\infty \lambda_i x_i'\otimes y_i' \quad \text{with} \quad (\lambda_i)_{i\in N} \in l^1,$$

and $(x_i')_{i\in N}$ and $(y_i')_{i\in N}$ nullsequences in X_b' and Y_b', respectively, then we would be able to follow the arguments of the proof of the corresponding part of Theorem 2.1 to conclude that $\tilde{v} = 0$. Since such a series representation is not guaranteed (X_b' and Y_b' are DF spaces!), we use here the additional assumption that X is Montel, instead. We thus can start from the situation, that $\tilde{v}$ is an element of the completion of the tensor product $X_b'\otimes_\pi Y_b'$, where X_b' is a Montel DF space and Y_b' a DF space. According to Theorem 1.9 of [28], we conclude that there exist a closed bounded (and, hence, compact) disk B in X_b' and a bounded disk C in Y_b' such that $\tilde{v} \in \overline{\mathrm{ac}(B\otimes C)}$; there exists a net

$$v_\lambda = \sum_1^{n_\lambda} a_{i\lambda} x_{i\lambda}'\otimes y_{i\lambda}' \in \mathrm{ac}(B\otimes C)$$

such that $(v_\lambda)_{\lambda\in\Lambda}$ converges to $\tilde{v}$ in $X_b'\tilde{\otimes}_\pi Y_b'$. Now let $u \in L(X_b', Y_b'') = (X_b'\tilde{\otimes}_\pi Y_b')'$ (Lemma 2.3), and $\varepsilon > 0$. There exists $k \in X''\otimes Y''$ such that

$$(2)\qquad (u-k)(B) \subset \varepsilon C^\circ.$$

According to (1), we conclude that

$$|u(\tilde{v})| \leqslant |(u-k)(\tilde{v})| = \lim |(u-k)(v_\lambda)| \leqslant \overline{\lim} \sum_1^{n_\lambda} |a_{i\lambda}| \left|\left((u-k)x_{i\lambda}', y_{i\lambda}'\right)\right| \leqslant \varepsilon.$$

Hence $\tilde{v} = 0$, and the proof of Theorem 4.1 is complete.

Proof of Theorem 4.2. Part (a) is a special case of a result of our previous paper ([7], Thm. 2.13).

Part (b). *Step* 1: $j\colon X_b'\tilde{\otimes}_\pi Y_b' \to (L_e(X_c', Y))'$ is injective: let $\tilde{v} \in X_b'\tilde{\otimes}_\pi Y_b'$ such that $j(\tilde{v}) = 0$. There exists a (bounded) net

$$v_\lambda = \sum_1^{n_\lambda} x_{i\lambda}'\otimes y_{i\lambda}' \in X_b'\otimes_\pi Y_b'$$

such that $(v_\lambda)_{\lambda\in\Lambda}$ converges to $\tilde{v}$ in $X_b'\tilde{\otimes}_\pi Y_b'$. According to the continuity of j into $(L_e(X_c', Y))_b'$ (Lemma 1.1), we conclude that

$$(1)\qquad 0 = j(\tilde{v})(h) = \lim j(v_\lambda)(h) = \lim \sum_1^{n_\lambda} (hx_{i\lambda}', y_{i\lambda}') \quad \text{for all } h \in L(X_c', Y).$$

Furthermore, we have

$$(2)\qquad (X_b'\tilde{\otimes}_\pi Y_b')' = L(X_c', Y):$$

first, according to Lemma 2.3

$$(X'_b \tilde{\otimes}_\pi Y'_b)'' = L(X'_b, Y)$$

(X and Y are both reflexive Fréchet spaces). But, since X or Y is Montel, we also have the identity $L(X'_b, Y) = L(X'_c, Y)$. In case X is Montel, this is clear. In case Y is Montel, we use Theorem 3.1 (4) of [28]. (1) and (2) together imply that $\tilde{v} = 0$.

Step 2. According to step 1, j is a continuous bijection from $X'_b \tilde{\otimes}_\pi Y'_b$ onto $\big(L_e(X'_c, Y)\big)'_b$. But both these spaces are complete reflexive (hence also barrelled) DF spaces: for $\big(L_e(X'_c, Y)\big)'_b$, this is a consequence of part (a), for $X'_b \tilde{\otimes}_\pi Y'_b$, consult ([16], I.1.3, Cor. 1, p. 44, and Cor. 2, p. 45]). Hence, the closed graph theorem 3.8 (2) of [25] allows us to conclude that j is a topological isomorphism. This completes the proof of Theorem 4.2.

We close this circle of ideas with a result on the representation of the elements of $X'_b \tilde{\otimes}_\pi Y'_b$ for X and Y being Fréchet spaces.

4.3. Lemma. *Let X and Y be Fréchet spaces one of which is reflexive and quasinormable, and assume that*

$$j\colon X'_b \tilde{\otimes}_\pi Y'_b \to \big(L_e(X'_c, Y)\big)'$$

is injective.

Then, for every $\tilde{v} \in X'_b \tilde{\otimes}_\pi Y'_b$, there exist $U \in \mathscr{U}_X$, $V \in \mathscr{U}_Y$, $(\lambda_i)_{i\in N} \in l^1$, and nullsequences $(x'_i)_{i\in N}$ and $(y'_i)_{i\in N}$ in X'_{U° and Y'_{V°, respectively, such that

$$\tilde{v} = \sum_1^\infty \lambda_i x'_i \otimes y'_i$$

(in the topology of $X'_b \tilde{\otimes}_\pi Y'_b$).

Remark. Assumptions on X and Y under which j is injective, are to be found in Theorems 4.1 and 4.2.

Proof of Lemma 4.3. Let $\tilde{v} \in X'_b \tilde{\otimes}_\pi Y'_b$. Then $T = j(\tilde{v}) \in \big(L_e(X'_c, Y)\big)'$ (Lemma 1.1). According to Proposition 1.7 and Lemma 1.8, there exist $U \in \mathscr{U}_X$, $V \in \mathscr{U}_Y$, and

$$\tilde{w} = \sum_1^\infty \lambda_i x'_i \otimes y'_i \in X'_{U^\circ} \tilde{\otimes}_\pi Y'_{V^\circ}$$

such that $T = j_{U,V}(\tilde{w})$. Clearly, the sequence $\big(\sum_1^n \lambda_i x'_i \otimes y'_i\big)_{n\in N}$ is Cauchy in $X'_b \tilde{\otimes}_\pi Y'_b$, hence it converges to an element $\tilde{w}_1 \in X'_b \tilde{\otimes} Y'_b$. The continuity of the maps $j_{U,V}$ and j into $\big(L_e(X'_c, Y)\big)'_b$ allows us to conclude that $j(\tilde{w}_1) = T = j(\tilde{v})$. Hence, $\tilde{w}_1 = \tilde{v}$, for j is supposed to be injective.

We now turn to applications and special cases of Theorems 4.1 and 4.2.

4.4. Theorem. *Let X be a gDF space and Y a Fréchet space such that X'_b or Y is reflexive and quasinormable or a Banach space whose dual has RNP.*

(a) *The linear map $j\colon X''_b \tilde{\otimes}_\pi Y'_b \to \big(K_b(X, Y)\big)'$ is surjective. More precisely, we have: for every $T \in \big(K_b(X, Y)\big)'$, there exist B bounded in X, $V \in \mathscr{U}_Y$, and*

$$\tilde{v} = \sum_1^\infty \lambda_i x''_i \otimes y'_i \in (X''_{B^{\circ\circ}}) \tilde{\otimes}_\pi Y'_{V^\circ}$$

such that

$$Th = \sum_1^\infty \lambda_i (h'' x''_i, y'_i) \quad \text{for all } h \in K(X, Y).$$

($B^{\circ\circ}$ is the bipolar of B in $X'' = (X'_b)'$.)

(b) *If, in addition to the assumptions, the map*

$$p\colon X''_b \tilde{\otimes}_\pi Y'_b \to B(X'_b, Y)$$

is injective, then we have:

(b1) *$K_b(X, Y) \cong X'_b \tilde{\otimes}_\varepsilon Y$ (topologically).*

(b2) *$\big(K_b(X, Y)\big)' = X''_b \tilde{\otimes}_\pi Y'_b$ (algebraically).*

Conditions ensuring injectivity of p can be read from part (c) *of Theorem* 4.1.

(c) *If, in addition to the assumptions, X'_b and Y are reflexive, and X'_b or Y is Montel, and X'_b or Y is quasinormable, then we have:*

(c1) *$K_b(X, Y)$ is a reflexive Fréchet space.*

(c2) *$\big(K_b(X, Y)\big)'_b \cong X''_b \tilde{\otimes}_\pi Y'_b$ (topologically).*

This is a consequence of Theorems 4.1 and 4.2, and the topological isomorphism $K_b(X, Y) \cong L_e(X''_c, Y)$ of ([7], Example 0.2 (b)).

4.5. Examples. Theorem 4.4 can be used to determine the form of continuous linear functionals on the following spaces of compact operators:

(4.5.1) $K_b(X, Y)$, X a Banach space whose bidual has RNP, and Y a Fréchet space.

(4.5.2) (i) $K_b\big(C_b(T)_\zeta, X\big)$, T completely regular Hausdorff, $\zeta \in \{\beta_0, \beta, \beta_1\}$.
(ii) $K_b\big(C_0(S)_\infty, X\big)$, S locally compact Hausdorff
(iii) $K_b\big(C_c(S)_i, X\big)$, S locally compact Hausdorff and σ-compact.
And, in all three cases, X a reflexive quasinormable Fréchet space, in particular, a Fréchet–Schwartz space.

We close this section (and this paper) with a representation theorem for the dual of $C(S, X)_{\infty\infty}$.

4.6. THEOREM. *Let S be a locally compact σ-compact Hausdorff space, X a reflexive quasinormable Fréchet space, or a Banach space whose dual has RNP, and denote by $C(S, X)_{co}$ the space of continuous X-valued functions on S, endowed with the compact-open topology. Furthermore, denote by $M_c(S)$ the space of Radon measures on S with compact support.*

Then we have:

(a) *$(C(S, X)_{co})'$ is a quotient of $M_c(S)\tilde{\otimes}_\pi X'_b$. More precisely, we have: for every $T \in (C(S, X)_{co})'$, there exist zero neighbourhoods U in $C(S)_{co}$ and V in X, and*

$$\tilde{v} = \sum_1^\infty \lambda_i \mu_i \otimes x'_i \in M_c(S)_{U^\circ} \tilde{\otimes}_\pi X'_{V^\circ}$$

such that

$$TF = \sum_1^\infty \lambda_i \int (x'_i \circ F)\, d\mu_i \quad \text{for all } F \in C(S, X).$$

(b) *If, in addition to the assumptions, X is a Fréchet–Schwartz space with the approximation property, then we have:*

$$(C(S, X)_{co})' = M_c(S) \tilde{\otimes}_\pi X'_b \quad (\textit{algebraically}).$$

This is a consequence of Theorem 4.1 and the topological isomorphism $C(S, X)_{co} \cong C(S)_{co} \tilde{\otimes}_\varepsilon X$.

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