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Duals of spaces of compact operators

by

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and

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Abstract. This is a study of dual spaces of various spaces of compact operators.
We derive representations for duals of (a) spaces $E(X, Y)$ of compact linear operators,
(b) injective tensor products $X\otimes Y$, and (c) spaces of vector-valued continuous
functions, in terms of the completed projective tensor product of the strong duals or
biduals of the factor spaces $X$ and $Y$. The results are specified for (i) $X$ and $Y$ Banach
spaces, (ii) $X$ and $F$ Fréchet spaces, and (iii) $X$ and $Y$ DF spaces. Applications are
given to problems of weak convergence in any of the above types of operator or function
spaces. The fundamental tool is a kind of localized Radon–Nikodym property
for locally convex spaces introduced by A. Grothendieck.

0.1. Introduction. The present work is a sequel to our previous article [7] on weak compactness in spaces of compact operators. In that
paper, our main object was to describe the weak topology and the notions
related to it, like weak compactness, weak convergence for sequences, and
reflexivity, in the general context of the operator space $L_1(X, Y)$ of weak*-weakly continuous linear operators from $X'$ into $Y$ which transform
equicontinuous subsets of $X'$ into relatively compact subsets of $Y$, endowed
with the topology of uniform convergence on the equicontinuous subsets of
$X'$, $X$ and $Y$ arbitrary locally convex spaces. The results were then applied
to the various spaces of analysis that can be represented as a suitable
operator space of the form $L_1(X, Y)$. Here our object is to describe the
dual space of $L_1(X, Y)$ in terms of the (presumably well known) duals $X'$
and $Y'$ of the factor spaces $X$ and $Y$, and, again, to specialize our results

to the concrete spaces representable as (a linear subspace of) an operator
space $L_1(X, Y)$: (a) spaces of compact operators, (b) injective tensor
products, and (c) spaces of vector-valued continuous, or holomorphic
functions.

* Research performed while the second named author was a visitor in the
Department of Mathematics at Louisiana State University, Baton Rouge.
For the convenience of the reader, we recall here three of the basic examples of such representations:

\[ (*) \quad K(X, Y) = L_0(X^*_a, Y) \quad \text{(isometrically),} \]
\[ h \mapsto h'' \quad \text{(X and Y Banach spaces),} \]
\[ (**) \quad X \otimes_y Y \subseteq L_0(X^*_a, Y) \quad \text{(topologically),} \]
\[ \sum_{i} (x_i, y_i) \mapsto \left[ x' \mapsto \sum_{i} x'_i y_i \right] \quad \text{(X and Y complete locally convex spaces),} \]
\[ (***) \quad O(X, X) = L_0(X^*_a, O(X)) \quad \text{(isometrically),} \]
\[ F \mapsto (x' \mapsto x' \circ F) \quad \text{(K compact Hausdorff, X a Banach space).} \]

For further details on the space \( L_0(X^*_a, Y) \), as well as for the basic terminology and notations, the reader is kindly asked to consult our previous paper [7].

We now describe the contents of this paper in more detail. It is known from the work of A. Grothendieck ([16], I. 4.4, Prop. 18, pp. 86, 96) that, for general \( X \) and \( Y \), the continuous linear functionals on \( L_0(X^*_a, Y) \) are given by certain integral linear forms, represented by Radon measures on the products \( U^* \times V^* \) of polars of zero neighbourhoods \( U \) and \( V \) in \( X \) and \( Y \), respectively. For more details on the space \( L_0(X^*_a, Y) \), as well as for the basic terminology and notations, the reader is kindly asked to consult our previous paper [7].

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Whenever \( X \) and \( Y \) are locally convex spaces such that \( X \) or \( Y \) is quasinormable and semi-reflexive, then for every \( T \in \mathcal{L}^0(X^*, Y) \), there exist zero neighbourhoods \( U \) and \( V \) in \( X \) and \( Y \) respectively, and an element \( \tilde{v} \in \sum \lambda_i x'_i \otimes y'_i \in X^{'*}_a \otimes Y_{o'} \) such that

\[ Th = \sum \lambda_i (h(x'_i) y'_i) \quad \text{for all} \quad h \in L_0(X^*_a, Y). \]

(Starting from the fact that reflexive Banach spaces do have the Radon–Nikodym property, the reader may realize that, in comparison to the Banach space result, for our general result the property of being normed had to be replaced by "quasinormable", and that of having the Radon–Nikodym property by "semi-reflexivity". The notion of quasi-normability is that of Grothendieck ([15], III. 1, Def. 4, p. 107), see the definition following Proposition 1.3 in Section 1.)

Section 1 contains the technical details this result is based on, together with a first application to our original problem: the dual of \( L_0(X^*_a, Y) \) (algebraically) is a quotient of \( X^{'*}_a \otimes Y \), whenever both \( X \) and \( Y \) are Fréchet spaces, one of which is reflexive and quasinormable, or both \( X \) and \( Y \) are generalized DF spaces (gDF), one of which is semi-reflexive. Moreover, we use our representation of the dual of \( L_0(X^*_a, Y) \) to derive various supplementary results to our paper [7] on weak compactness in operator spaces.

The subsequent sections are then devoted to a specialization of the general technical results of Section 1 to the particular cases of:

1. \( X \) and \( Y \) generalized DF spaces (Section 2),
2. \( X \) and \( Y \) Banach spaces (Section 3), and
3. \( X \) and \( Y \) Fréchet spaces (Section 4).

Our results comprise representations of duals of:

(a) spaces \( K(X, Y) \) of compact operators, \( X \) and \( Y \) Banach or, more generally, \( X \) gDF and \( Y \) Fréchet,
(b) completed injective tensor products \( X \otimes Y \), and of
(c) spaces of vector valued continuous functions, like \( C_b(S, X) \), \( C_b(S, Y) \), and \( C_b(S, X) \), \( S \) locally compact Hausdorff and \( X \) Banach, or gDF, or Fréchet.

To give a flavour of the results, we take up the three examples considered above:

- \( X^*_a \otimes Y^* \) (isometrically), \( X \) and \( Y \) Banach, \( X^*_a \) or \( Y^* \) RMP, \( X^* \) or \( Y^* \) a.p. (Section 3, Thm. 3.3).
C₀(S, X): continuous functions vanishing at infinity, with sup-norm
(S locally compact Hausdorff, X Banach).

C₀(S, X): bounded continuous functions, with the strict topology of
R. C. Buck ([(4), (5)]) (S locally compact Hausdorff, X quasi-complete
locally convex).

C₀(S, X): continuous functions with compact support, with the
usual inductive limit topology (S locally compact Hausdorff σ-compact,
X Banach).

(a) Spaces of compact operators: Given locally convex spaces X and Y,
we denote by K(X, Y) the space of weakly compact linear operators
from X into Y which transform bounded sets into relatively compact
sets, and, as usual, by K(X, Y) the space of compact linear operators
from X into Y. Throughout, these spaces will be assumed to be endowed
with the topology of uniform convergence on the bounded subsets of X
(= operator norm whenever X and Y are normed), as indicated by
K₀(X, Y) and K₀(X, Y). (Recall that a linear operator from X into Y is called
weakly compact if it transforms a certain zero neighbourhood in X
into a weakly relatively compact subset of Y.) Whenever X is normed,
or a generalized DF space (see the definition below), and Y a Fréchet
space, then K₀(X, Y) ∼ K(X, Y). For X and Y locally convex spaces
such that Y is quasi-complete (closed bounded sets are complete), we have
the following topological linear isomorphisms:

K₀(X, Y) ≃ K₀(X, Y) ≃ L₀(X₀, Y) ≃ (X₀) ′ ⊂ Y ,

k → k ′.

(b) Spaces of vector-valued continuous functions: Let T be a completely
regular Hausdorff space, X a quasi-complete locally convex space, and
V > 0 a Nachbin family of weights on T such that T is a V-space. Q₉(T, V)
denotes the associated space of X-valued continuous functions on T:

Q₉(T, X) = {F : T → X} continuous (F(t) = F(t) = F(t) vanishes
at infinity for all v ∈ V).

with the topology generated by the seminorms

b₉(F) = sup ||F(t)|| : t ∈ T ,

v ∈ V, and q a continuous semi-norm on X. For details consult ([7], Section
4). We have the following topological linear isomorphism:

Q₉(T, X) ≃ L₀(X₀, V₉(T)) ≃ (Q₉(T)) ′ ⊂ X,

F → {v → F(v) *}.

The following are the most common examples of weighted spaces.

Q₉(T, X₀): continuous functions, with compact open topology (T
completely regular Hausdorff k₀, X quasi-complete locally convex).

1. Grothendieck’s localized Radon–Nikodym property and the
dual of the operator space L₀(X₀, X). This section contains the fundamental
technical results on the representation of the dual of L₀(X₀, X) in
terms of the completed projective tensor product X₀ ⊗ X₀, together
with applications to weak convergence criteria in spaces of compact
operators.

Given locally convex spaces X and Y, we consider the natural linear
embedding of the algebraic tensor product X ⊗ Y into

\[ j : X \otimes Y \to (L₀(X₀, Y))' \]

\[ \sum_{i \in I} aᵢ \otimes yᵢ \mapsto \sum_{i \in I} (aᵢ yᵢ) \].
Our object is to specify conditions on $X$ and $Y$ under which
(a) $j_2$ is continuous from $X_1 \otimes Y_1'$ into $[L_2(X_1, Y)]_b'$, and has a continuous extension $j$ to $X_1 \otimes Y_1'$ which is still mapping into $[L_2(X_1, Y)]_b'$,
(b) $j$ is a surjection from $X_1 \otimes Y_1'$ onto $[L_2(X_1, Y)]_b'$, and
(c) $j$ is a one-one map from $X_1 \otimes Y_1'$ into $[L_2(X_1, Y)]_b'$.

As is to be expected, problem (c) is tied up with the approximation property (for $X_1$ or $Y_1$).

Problem (b) is the most important point of our investigation. The fundamental tool here is a kind of localized Radon–Nikodym property, introduced by A. Grothendieck [16].

Problem (a) has easy satisfactory solutions in the two special cases we are interested in:

1.1. Lemma. Let $X$ and $Y$ be both metrizable spaces or both gDF spaces. Then the map $j_2$ of (1.1) is continuous from $X_1 \otimes Y_1'$ into $[L_2(X_1, Y)]_b'$, and has a continuous linear extension $j$ to $X_1 \otimes Y_1'$ with range still in $[L_2(X_1, Y)]_b'$.

Proof. For the continuity part of the assertion, let $H$ be a bounded subset of $L_2(X_1, Y)$. According to [29], Prop. 1.9, there exist bounded subsets $B$ and $C$ of $X$ and $Y$, respectively, such that $H(B') \subseteq C$. This implies that $j_2[B \otimes C'] \subseteq H$, and shows that $j_2$ is continuous.

For the other part of the assertion, consider first the case when $X$ and $Y$ are metrizable. Then $L_2(X_1, Y)$ is metrizable as well, so that $[L_2(X_1, Y)]_b'$ is a complete DDF space. Hence, the continuous linear extension $j$ of $j_2$ trivially maps into that space. In case $X$ and $Y$ are gDF, $L_2(X_1, Y)$ has a fundamental sequence of bounded sets [29], so that $[L_2(X_1, Y)]_b'$ is metrizable. However, the question whether this space is complete in this case, figures as "Probleme 10" in [16], and has been answered affirmatively only in special cases; see the discussion proceeding Theorem 2.2 in Section 2.

Nevertheless, we are able to show that $j$ still maps into $[L_2(X_1, Y)]_b'$ itself: Let $v \in X_1 \otimes Y_1'$. According to [16], Thm. 1, p. 51), there exist null sequences $(\lambda_n)_n$ and $(\nu_n)_n$ in $X_1$ and $Y_1$, respectively, and $(\lambda_n)_n$ in $v$ such that

$$
\hat{v} = \sum_n \lambda_n \nu_n' \otimes y_n'.
$$

Hence, $j(\hat{v})$ is the limit of $[\sum_n \lambda_n \nu_n(\mu_n')]_n$ in the completion of $[L_2(X_1, Y)]_b'$.

We now show that the map

$$
(*) \quad T : L_2(X_1, Y) \to K,
$$

$$
\lambda \mapsto \lim \sum_n \lambda_n (\mu_n, y_n')
$$
is an element of $(L_2(X_1, Y)]_b'$, and that the sequence $[\sum_n \lambda_n (\mu_n, y_n')]_n$ converges to $T$ in $[L_2(X_1, Y)]_b'$. This will complete the proof.

First, the sequences $(\lambda_n)_n$ and $(\nu_n)_n$ are contained in the polars $U_n$ and $V_n$ of zero neighbourhoods $U$ and $V$ in $X$ and $Y$, respectively, for strong nullsequences in the dual of a gDF space are equiconvergent ([29], Prop. 2.2). Hence, given any $\lambda \in L_2(X_1, Y)$, the sequence $(\lambda_n(\mu_n, y_n'))_n$ is a bounded sequence of scalars, and, therefore, the series $\sum \lambda_n (\mu_n, y_n')$ is convergent. In order to prove that $T$ is continuous, let $(\lambda_n)_n \subseteq L_2(X_1, Y)$ be a net that converges to zero in $L_2(X_1, Y)$. Then, given $\varepsilon > 0$ and $U$ and $V$ as above, there exists $\lambda_0 \in A$ such that

$$
h_2((\lambda_0, U, V)) = \varepsilon \left(\sum_n |\lambda_n|^{-1}V\right) \quad \text{for all} \quad \lambda \geq \lambda_0.
$$

Hence, we have:

$$
|T(\lambda)| = \left| \sum_n \lambda_n(\mu_n, y_n') \right| \leq \sum_n |\lambda_n| \left(\sum_n |\lambda_n|^{-1}V\right) = \varepsilon \quad \text{for all} \quad \lambda \geq \lambda_0.
$$

It remains to prove that $[\sum_n \lambda_n (\mu_n, y_n')]_n$ converges to $T$ in $[L_2(X_1, Y)]_b'$; as indicated before, given a bounded subset $H$ of $L_2(X_1, Y)$, there exist $B$ and $C$ bounded in $X$ and $Y$, respectively, such that $H(B') \subseteq C$.

For an arbitrary $\varepsilon > 0$, there exists $n_0 \in N$ such that $(\lambda_n)_n \subseteq C'$, and $\|\lambda_n\| < \varepsilon$. Hence, we have:

$$
|T(\lambda_n)| = \left| \sum_n \lambda_n(\mu_n, y_n') \right| \leq \sum_n |\lambda_n| < \varepsilon
$$

for all $n \geq n_0$ and all $\lambda \in H$.

This proves our assertion.

For a discussion of injectivity of the map $j$, consider the following linear map $p_2$ from $X' \otimes Y'$ into $B(X, X)$:

$$
p_2 : X' \otimes Y' \to B(X, X)
$$

$$
\sum_n \alpha_n' \otimes \beta_n' \mapsto \{(\alpha, y) \mapsto \sum_n (\alpha_n', \beta_n')(\alpha, y)\}
$$

1.2. Lemma. (a) The map $p_2$ is continuous from $X_1 \otimes Y_1'$ into $B_{ab}(X, X)$ for any locally convex spaces $X$ and $Y$.

(b) Whenever $X$ and $Y$ are both gDF spaces or both Fréchet spaces, then $B_{ab}(X, Y)$ is complete, and the continuous linear extension of $p_2$ maps $X_1 \otimes Y_1'$ into $B_{ab}(X, Y)$. The map $j$ is injective whenever $p$ is.
Connections between injectivity of \( p \) and the approximation property for \( X_1 \) or \( X_2 \) will be investigated separately in any of the particular cases to be discussed in the subsequent sections.

Proof of Lemma 1.2. The proof of part (b) being obvious, we immediately turn to a proof for part (b). In case \( X \) and \( Y \) are both \( gDF \), the solution of Grothendieck's "Problème des Topologies" for \( gDF \) spaces ([37], Thm. 1.9) reveals that \( B_{bg}(X, Y) \) is topologically isomorphic to the strong dual \( (X^\#)_{cm} \) of the \( gDF \) space \( X^\# \), and thus is a Fréchet space. In case \( X \) and \( Y \) both are Fréchet, we first note ([37], II. 34.2, Cor., p. 354) that \( B(X, Y) \) is equal to the space \( \mathcal{A}(X, Y) \) of \( \kappa \)-unilaterally continuous bilinear forms on \( X \times Y \), and then use ([16], Intr. III. 4, III.6) to conclude that \( \mathcal{A}_{bg}(X, Y) = L_1(X, Y) \) is complete.

Now assume that \( p \) is injective and let \( \tilde{v} \in X_1^\# \otimes_{\kappa} X_2 \) such that \( j(\tilde{v}) = 0 \). There exists a net
\[
(\tilde{v}_n)_{n\in\mathbb{N}} \subset X_1^\# \otimes Y', \quad v_n = \sum_{i=1}^{n_1} x_i^\# \otimes y_i^{d_i}
\]
converging to \( \tilde{v} \) in \( X_1^\# \otimes_{\kappa} X_2 \). By assumption on \( \tilde{v} \), we have \( 0 = j(\tilde{v})(h) = (b - \lim)j((\tilde{v}_n))(h) = \lim \sum_{i=1}^{n_1} (x_i^\#, \tilde{x}_i(y, y_i^{d_i})) \) for all \((x, y) \in X \times Y\).

Hence, we have:
\[
p(\tilde{v})(x, y) = (b - \lim)p_j(\tilde{v}')(x, y) = \lim \sum_{i=1}^{n_1} (x_i^\#, \tilde{x}_i(y, y_i^{d_i})) = 0
\]
for all \((x, y) \in X \times Y\), and thus by assumption on \( p, \tilde{v} = 0 \).

We now turn to the central problem of this paper, namely the question of when \( j \) is surjective. We shall derive the following result:

1.3. Proposition. (a) Let \( X \) and \( Y \) be locally convex spaces such that
(i) \( X \) or \( Y \) is a Banach space whose dual has RNP, or
(ii) \( X \) or \( Y \) is semi-reflexive and quasi-normable**.

Then for every \( T \in L_1(X_1^\#, Y') \) there exist zero neighbourhoods \( U \) and \( V \) in \( X \) and \( Y \), respectively, and
\[
\tilde{v} = \sum_{i=1}^{n_1} \lambda_i x_i^\# \otimes y_i^{d_i} \in X_1^\# \otimes_{\kappa} X_2 \quad \text{for all } h \in L(X_1^\#, Y),(h),
\]
such that \( Th = \sum_{i=1}^{n_1} \lambda_i(hx_i', y_i) \) for all \( h \in L(X_1^\#, Y) \).

** Actually, it is enough to assume that \( X \) or \( Y \) is a subspace of a product of Banach spaces whose duals have RNP. (Note that a semi-reflexive quasi-normable locally convex space is a subspace of a product of reflexive Banach spaces.)

(b) In particular, the map \( j: X_1^\# \otimes_{\kappa} X_2 \to L_1(X_1^\#, Y)' \) is surjective, whenever
(i) \( X \) and \( Y \) are Fréchet spaces one of which is reflexive and quasi-normable, or
(ii) \( X \) and \( Y \) are gDF spaces one of which is semi-reflexive or a Banach space whose dual has RNP.

We recall the definition of quasi-normability.

Definition ([15], III. 4.3, Def. 4, p. 106). A locally convex space \( X \) is called quasi-normable whenever for every equicontinuous subset \( H \) of \( X' \) there exists a zero neighbourhood \( U \) in \( X \) such that on \( H \) the strong dual topology and the topology of uniform convergence on \( U \) coincide.

Every DF and, more generally, every gDF space is quasi-normable [26]. Also recall that a Scheepe space is exactly a quasi-normable locally convex space whose bounded sets are precompact.

The fundamental tool for a proof of Proposition 1.3 is based on the notion of the "Phillips property" as introduced by A. Grothendieck.

First we need to fix some notations.

Notation. A convex circled subset of a linear space is called a disk. If \( B \) is a bounded disk in a locally convex space \( X \), then we denote by \( X_B \) the linear span of \( B \) in \( X \) endowed with the norm with unit ball \( B \). \( B \) is called complete if \( X_B \) is a Banach space.

1.4. Definition ([16], I. 4.1, Def. 6, p. 104). Let \( \mathcal{A} \) be a family of bounded complete disks in a locally convex space \( X \). A subset \( C_0 \) of \( X \) has the Phillips property with respect to \( \mathcal{A} \) if its closed convex circled hull \( C \) is weakly compact, and if the following condition is fulfilled:

(1.4.1) If \( X \) is any compact Hausdorff space \( E \), any Radon measure \( \mu \) on \( E \), and any continuous linear operator \( T \) from \( L^1(\mu) \) into \( X_0 \), there exist \( B \in \mathcal{A} \) and a bounded measurable function \( f: K \to X_B \) such that
\[
Tg = \int g \, df \quad \text{in } X_B \quad \text{for all } g \in L^1(\mu).
\]

The following facts on the Phillips property are crucial for our considerations.

1.5. Facts.

(1.5.1) If \( X \) is a Banach space whose dual has RNP, then the dual unit ball \( X_1^\# \) has the Phillips property with respect to itself (take \( X' \) with the weak*-topology).

(1.5.2) ([16], I. 4.1, Thm. 4, p. 104) Every weakly compact subset of a Fréchet space \( X \) has the Phillips property with respect to the family of all bounded complete disks in \( X \).

(1.5.3) ([16], I.4.1, Thm. 6, p. 108) Let \( E \) be a locally convex space, \( \mathcal{A} \) a family of bounded complete disks in \( E \), and \( C \) a weakly compact...
defines a continuous linear functional on \( L_c(X^*, Y) \) by formula (1.6). Furthermore, the corresponding linear map
\[
f_{U,W}: X_U^* \otimes_\beta Y^*_W \to [L_c(X^*, Y)]_0^*
\]
is continuous.

Proof. For a proof of part (a), recall first the topological linear isomorphism ([7], Section 0, (0.6.1))
\[
L_c(X^*, Y) \cong \mathcal{K}(E, X^*),
\]
\[
h \mapsto E^2: (x', y') \mapsto (hx', y'),
\]
and recall that each \( E \) has continuous restrictions to
\[(U^*, \text{weak}^*) \times (W^*, \text{weak}^*),\]
\(U\) and \(W\) zero neighbourhoods in \( X\) and \( Y\), respectively, by ([16], Intr. IV, Lemma D, p. 27). Hence, letting
\[
E = X^*,
\]
\(\mathcal{B} = \{V\} \mid V\ a\ zero\ neighbourhood\ in\ X\), \(U = U^*\), \(F = Y^*,\)
\(\mathcal{G} = \{W\} \mid W\ a\ zero\ neighbourhood\ in\ Y\), and \(H = \mathcal{K}(E, X^*_U),\)
part (a) comes out to be nothing but a special case of fact (1.5.3) above.

The proof of part (b) is similar to the proof of Lemma 1.1. The reasoning there first shows that for every
\[
\tilde{v} = \sum_{i} \lambda_i \otimes y'_i \in X_U^* \otimes_\beta Y^*_W^*,
\]
the map \( T_{\tilde{v}}: L_c(X^*, Y) \to K \)
\[
T_{\tilde{v}}(h) = \sum_{i} \lambda_i (h x'_i, y'_i)
\]
is well-defined, continuous and linear. In order to show that the continuous linear extension of
\[
f_{U,W}: X_U^* \otimes_\beta Y^*_W \to [L_c(X^*, Y)]_0^*,
\]
\[
j_{U,W} \left( \sum_{i} \lambda_i \otimes y'_i \right)(h) = \sum_{i} \lambda_i (h x'_i, y'_i),
\]
to the completion \( X_U^* \otimes_\beta Y^*_W^* \) still maps into \( [L_c(X^*, Y)]_0^* \), we also follow the reasoning of the proof of Lemma 1.1, this time using the general characterization ([29], Prop. 1.9) of bounded subsets in \( L_c(X^*, Y) \); a subset \( H \) of \( L_c(X^*, Y) \) is bounded if and only if \( H(U^*) \) is bounded in \( Y \) for all zero neighborhoods \( U \) in \( X \), or, equivalently, \( H \) is an equicontinuous subset of \( L_c(X^*_U, Y) \). We omit the details.
We now turn to the proof of Proposition 1.3: Condition (i) of part (a) being clear, we immediately consider the second condition, and assume that \( X \) is quasinormable and semi-reflexive. According to Proposition 1.6, we have to show that for every \( U \) neighbourhood in \( X \), there exists another such, \( V \) say, such that \( U^* \) has the Phillips property with respect to \( V \). By the quasinormability of \( X \), there exists a zero neighbourhood \( V \) in \( X \), \( V \subset U \), such that on \( U^* \) the strong topology coincides with the topology induced by the Banach space \( X_{r_0} \). But the strong topology on \( X \) is equal to the Mackey topology, for \( X \) is semi-reflexive. Hence, \( U^* \) is a weakly compact disk in \( X' \), and thus weakly compact in the Banach space \( X_{r_0} \) as well. An appeal to Phillips' result (1.5.2) now completes the proof.

We close this circle of ideas by deriving (from Proposition 1.3) our fundamental general results on the representation of continuous linear functionals on compact operator spaces and on spaces of vector-valued continuous functions.

1.7. Theorem. Let \( X \) and \( Y \) be locally convex spaces.

(a) If \( X \) and \( Y \) are complete, and either of \( X \) and \( Y \) is semi-reflexive and quasinormable, or a Banach space whose dual has BNP, then for every \( F \in (X \otimes Y)' \) there exist zero neighbourhoods \( U \) and \( V \) in \( X \) and \( Y \), respectively, and

\[
\tilde{v} = \sum_{i=1}^{m} \lambda_i x_i \otimes y_i \in X_{r_0} \otimes Y_{r_0},
\]

such that

\[
T \tilde{v} = \sum_{i=1}^{m} \lambda_i (h x_i', y_i') \quad \text{for all } h \in X_{r'_0} Y_{r_0}.
\]

(b) If \( Y \) is quasi-complete, and either of \( X \), and \( Y \) is semi-reflexive and quasinormable, or a Banach space whose dual has BNP, then for every \( F \in (X \otimes Y)' \) there exist \( B \) bounded in \( X \) and \( V \) a zero neighbourhood in \( Y \), and

\[
\tilde{v} = \sum_{i=1}^{m} \lambda_i x_i' \otimes y_i' \in X_{r_0} \otimes Y_{r_0},
\]

such that

\[
T \tilde{v} = \sum_{i=1}^{m} \lambda_i (h x_i', y_i') \quad \text{for all } h \in X_{r'_0} Y_{r_0}.
\]

(c) If \( X \) is semi-reflexive and quasinormable, or a Banach space whose dual has BNP, \( T \) a completely regular Hausdorff space, and \( V > 0 \) a Nachbin family of weights on \( T \) such that \( T \) is a \( V_{m} \)-space, then for every

\[
S \in (OV_e(T, X))', \text{ there exist zero neighbourhoods } U \text{ and } V \text{ in } OV_e(T) \text{ and } X, \text{ respectively, and}
\]

\[
\tilde{V} = \sum_{i=1}^{m} \lambda_i x_i' \otimes y_i' \in (OV_e(T))' \otimes Y_{r_0} X_{r_0},
\]

such that

\[
S \tilde{V} = \sum_{i=1}^{m} \lambda_i (h x_i', y_i') \quad \text{for all } F \in OV_e(T, X).
\]

For the definition of the operator space \( E(X, Y) \) and of weighted spaces of vector-valued continuous functions and the most common examples, consult Section 0.3, (a) and (b). (Any of the examples \( C(T, X)_{0,0}, C_0(X, X), C^*_0(S, X), \) and \( C^*_0(S, X) \)) will be investigated separately in the sequel.)

The object of the subsequent sections is to refine the representations of Theorem 1.7 to algebraical or even topological isomorphisms between \( (L_e(X'_e, Y))' \), \( X_{r_0} \otimes Y_{r_0} \), \( E(X, Y)' \) for various special classes of locally convex spaces.

The present section will be closed with applications of the results given so far to questions of weak convergence in the operator space \( L_e(X'_e, Y) \).

1.8. Theorem. Let \( X \) and \( Y \) be locally convex spaces, and assume that either of \( X \) and \( Y \) is semi-reflexive and quasinormable, or a Banach space whose dual has BNP.

Then the algebraic tensor product \( X \otimes Y \) is sequentially dense in \( (L_e(X'_e, Y))' \) with respect to the strong dual topology on \( [L_e(X'_e, Y)]' \).

In particular, on bounded subsets of \( L_e(X'_e, Y) \), the weak topology of \( L_e(X'_e, Y) \) and the \( X \otimes Y \)-weak operator topology coincide: a bounded net \( (h_s)_{s \in J} \) in \( L_e(X'_e, Y) \) converges weakly to \( h \in L_e(X'_e, Y) \) if and only if \( (h_s w'_s)_{s \in J} \) converges weakly in \( Y \) to \( h w' \) for all \( w' \in X' \).

This is a direct consequence of Proposition 1.3.

Remarks. For general locally convex spaces \( X \) and \( Y \), we only know that

(a) \( X \otimes Y \) is weak*-dense in \( (L_e(X'_e, Y))' \); and that

(b) the families of weakly compact and of \( X \otimes Y \)-wet compact subsets of \( L_e(X'_e, Y) \) coincide ([7], Prop. 1.2). Thus, Theorem 1.8 adds the information that under the given assumptions on \( X \) and \( Y \), the two topologies even coincide on every bounded subset of \( L_e(X'_e, Y) \).

1.9. Corollary. Let \( X \) and \( Y \) be complete locally convex spaces, such that either of \( X \) and \( Y \) is semi-reflexive and quasinormable, or a Banach space whose dual has BNP. Then on bounded subsets of \( X \otimes Y, \) the weak topology (of \( X \otimes Y \)) and the \( X \otimes Y \)-weak operator topology coincide.
1.10. **Theorem.** Let $X$ and $Y$ be locally convex spaces, $X$ quasi-complete, and assume that $X'$ or $Y$ is semi-reflexive and quasi-normable, or a Banach space whose dual has RNP.

Then the algebraic tensor product $X''\otimes Y'$ is strongly sequentially dense in $(K(X, Y))'$, and a bounded net $(h_{n})_{n\in\mathbb{N}}$ in $K(X, Y)$ converges weakly to $h \in K(X', Y)$ if and only if $(h_{n}'' \omega')_{n\in\mathbb{N}}$ converges weakly in $Y'$ to $h'' \omega'$ for all $\omega' \in X''$. In particular, these assertions hold for the space $K_{0}(X, Y)$ of compact linear operators from $X$ to $Y$.

This is a consequence of Theorem 1.8 and the topological linear isomorphisms $K_{0}(X, Y) \sim K(X, Y)$, Section 0, (12), Example 0.2 (b)). For the special case of Banach spaces $X$ and $Y$, this has previously been proved by Feder/Saphar ([11], Cor. 1.2).

1.11. **Corollary.** Let $T$ be a completely regular Hausdorff space, and $Y > 0$ a Nachbin family on $T$ such that $T$ is a $V_{K}$-space. Furthermore, let $X$ be a quasi-normable semi-reflexive locally convex space, or a Banach space whose dual has RNP.

Then $(GV_{0}(T))' \otimes Y$ is sequentially dense in $(GV_{0}(T, X))''$, and a bounded net $(G_{n})_{n\in\mathbb{N}}$ in $GV_{0}(T, X)$ converges weakly to $F \in GV_{0}(T, X)$ if and only if $(G_{n} \omega')_{n\in\mathbb{N}}$ converges weakly in $(GV_{0}(T))'$ to $G \omega'$ for all $\omega' \in X'$.

A further quite convenient weak convergence criterion can be derived from Proposition 1.3 by analyzing the following composition of continuous maps

$$X'_{0} \times Y'_{0} \xrightarrow{\psi} X_{0} \otimes_{\Delta} Y_{0} \xrightarrow{j} (I_{0}(X', Y'))'.$$

1.12. **Proposition.** Let $X$ and $Y$ be both $D$-spaces one of which is semi-reflexive, or a Banach space whose dual has RNP, or let both be Fréchet spaces one of which is reflexive and quasi-normable. Moreover, let $M$ and $N$ be subspaces of $X'$ and $Y'$ whose linear spans are strongly dense in $X'$ and $Y'$, respectively.

Then, on bounded subsets of $L_{0}(X', Y)$, the weak topology and the $M \otimes N$-weak operator topology coincide.

We note two particularly interesting cases.

**Notation.** Given a Banach space $Z$, we denote by $\text{sexp } B_Z$ the set of strongly exposed points of its unit ball $B_Z$.

1.13. **Theorem.** Let $X$ and $Y$ be Banach spaces.

(a) If $X'$ and $Y'$ have RNP, then a bounded net $(h_{n})_{n\in\mathbb{N}}$ in $X \otimes_{\Delta} Y$ converges weakly to $h \in X \otimes_{\Delta} Y$ if and only if $(h_{n}, \omega')_{n\in\mathbb{N}}$ converges to $(h, \omega')$ for all $\omega' \in X' \times Y'$ and all $\omega' \in X' \times Y'$.

(b) If $X''$ and $Y'$ have RNP, then a bounded net $(h_{n})_{n\in\mathbb{N}}$ in $K_{0}(X, Y)$ converges weakly to $h \in K(X, Y)$ if and only if $(h_{n}'' \omega')_{n\in\mathbb{N}}$ converges to $(h'' \omega', y')$ for all $\omega' \in X'' \times Y'$ and all $\omega' \in X'' \times Y'$.

1.14. **Corollary.** Let $(\Omega, \Sigma, \mu)$ be a finite measure space, $X$ a Banach space whose dual has RNP, and denote by $K(\mu, X)$ the space of all $\mu$-continuous vector measures $F : \Sigma \to X$ whose range is relatively compact, equipped with the seminorm of norm.

Then $K(\mu, X)$ is isometrically isomorphic to $L \mu \otimes_{\Delta} X$, and a bounded net $(F_{n})_{n\in\mathbb{N}}$ in $K(\mu, X)$ converges weakly to $F \in K(\mu, X)$ if and only if $(x' \circ F_{n}(\omega))_{n\in\mathbb{N}}$ converges weakly in $x' \circ F(\omega)$ for all $x' \in X'$ and all $\omega \in \Sigma$.

For a proof of the isometry $K(\mu, X) = L(\mu) \otimes_{\Delta} X$, the reader is referred to ([10], VIII. 1, Thm. 5, p. 224).

2. **Spaces of compact operators on $DF$ spaces.** In this section, our general results are based on the representation of the dual of the operator $\text{sexp } B_{X'}$. Structures for the case of $D$-spaces $X$ and $Y$ are specified in Theorem 1.3.

The following results are the fundamental results.

2.1. **Theorem.** Let $X$ and $Y$ be $D$-spaces such that $X$ or $Y$ is semi-reflexive, or a Banach space whose dual has RNP.

(a) The linear map

$$j : X'_{0} \otimes_{\Delta} Y'_{0} \to (L_{0}(X'_{0}, Y'))'$$

is surjective: for every $T \in L_{0}(X'_{0}, Y')$ there exists

$$\tilde{T} = \sum_{i=1}^{n} \lambda_{i} x_{i} \otimes_{\Delta} y_{i} \in X'_{0} \otimes_{\Delta} Y'_{0}$$

such that

$$T h = \sum_{i=1}^{n} \lambda_{i} (h x_{i}, y_{i}) \quad \text{for all } h \in L(X'_{0}, Y).$$

(b) If the map

$$P : X'_{0} \otimes_{\Delta} Y'_{0} \to B(X, Y)$$

is injective, then $j$ is surjective:

$$L_{0}(X'_{0}, Y')' = X'_{0} \otimes_{\Delta} Y'_{0} = N(X, Y)_{0} = \text{nuclear operators } X \to Y'_{0}.$$

The map $P$ is injective whenever $X'_{0}$ or $X'_{0}$ or $X'_{0}$ has the approximation property.
(c) If $X$ and $Y$ are complete, and

$p : X \hat{\otimes}_\iota Y \rightarrow B(X,Y)$

is injective, then we have the following isomorphisms:

(01) $I_p(X, Y) \cong X \hat{\otimes}_\iota Y$ (topologically).

(02) $(X \hat{\otimes}_\iota Y, \tau') = X \hat{\otimes}_\iota Y$ (algebraically).

The question of when the isomorphism $(I_p(X, Y)) \cong X \hat{\otimes}_\iota Y$ of Theorem 2.1 (b) is topological for the strong topology on $(L(X, Y))'$, is connected with one of the problems in Grothendieck’s work, problem 10 in (118), 11, questions non résolus, p. 137: under which conditions on two DF spaces $X$ and $Y$ is the completed injective tensor product $X \hat{\otimes}_\iota Y$ again DF, or, at least, its strong dual Fréchet? It follows from a result of H. Buchwalter ([3], Prop. (2.7)) that $(X \hat{\otimes}_\iota Y)'$ is Fréchet whenever $X$ and $Y$ are semi-Montel gDF spaces. We are able to extend this result to the situation of two semi-reflexive gDF spaces only one of which needs to be semi-Montel.

2.2. Theorem. Let $X$ and $Y$ be semi-reflexive gDF spaces such that $X \hat{\otimes}_\iota Y$ is semi-Montel. Then we have:

(a) $L_p(X, Y)$ is semi-reflexive.

(b) $L_p(X, Y)' \cong X \hat{\otimes}_\iota Y \cong X \hat{\otimes}_\iota Y_0$ (topologically).

(c) $(X \hat{\otimes}_\iota Y)' \cong (X \hat{\otimes}_\iota Y)' \cong X \hat{\otimes}_\iota Y_0$ (topologically),

hence $(X \hat{\otimes}_\iota Y)'$ is a Fréchet space.

In conclusion, let’s recall that the class of semi-reflexive gDF spaces includes the Mackey duals of Fréchet spaces, and that the class of semi-Montel gDF spaces is exactly the class of $\epsilon$-duals of Fréchet spaces.

We now turn to the proofs of Theorems 2.1 and 2.2. Clearly, proposition (a) and the first part of proposition (b) of Theorem 2.1 are direct consequences of the general results established in Section 1 (Lemma 1.2 and Proposition 1.3). Thus, what essentially remains to be proven is the indicated connection between the injectivity of the map $p$ and the approximation property for $X \hat{\otimes}_\iota Y_0$ and $X \hat{\otimes}_\iota Y'_0$. At the beginning of this proof, we would like to apologize for the annoying fact (both for the reader and for us) that, although everybody will take this connection for granted, it takes us lengthy technical arguments to really establish it.

First we need the following result:

2.3. Lemma. Let $X$ and $Y$ be locally convex spaces such that

(a) $X$ is $\epsilon$-bounded, and

(b) every hypocontinuous bilinear form on $X \times Y$ is continuous.

Then $(X \hat{\otimes}_\iota Y)$ is algebraically isomorphic to $L(X, Y)$. Condition (b) is fulfilled whenever $X$ and $Y$ are both Fréchet or both gDF spaces.

Proof. It is known that $(X \hat{\otimes}_\iota Y)$ is equal to $B(X,Y)$ and that $B(X,Y)$ is a linear subspace of $L(X,Y)$ for any locally convex spaces $X$ and $Y$. Let $B(X,Y) \rightarrow L(X,Y)$,

$$B \rightarrow (\sigma \mapsto (\sigma, x)).$$

It remains to prove that under the given assumptions, every $\sigma \in L(X,Y)$ defines a continuous bilinear form on $X \times Y$. Let $u \in L(X,Y)$ and consider $B_u : X \times Y \rightarrow K$,

$$(\sigma, y) \mapsto (y, u)\sigma.$$  

It is sufficient to show that $B_u$ is hypocontinuous: if $A$ is a bounded subset of $X$, then $\sigma(B)$ is bounded in $Y$, and thus, by the usability of $Y$, equicontinuous. There exists a zero neighborhood $V$ in $Y$ such that $u(B) \subset V$. Hence, we have:

$$|B_u(B, V)| = |V, u(B)| \leq 1.$$  

If $C$ is a bounded subset of $Y$, then there exists a zero neighborhood $U$ in $X$ such that $u(U) \subset C$. Hence, we have:

$$|B_u(U, C)| = |C, u(U)| \leq 1.$$  

Finally, hypocontinuous bilinear forms on $X \times Y$ are continuous whenever $X$ and $Y$ are Fréchet, cf. ([33], II, 13, Prop. p. 534), or whenever $X$ and $Y$ are gDF ([28], Thm. 1.4).

Next, we need the following result of our map $p$ given arbitrary locally convex spaces $X$ and $Y$, consider the linear map

$$p : X \hat{\otimes}_\iota Y \rightarrow B_m((X \hat{\otimes}_\iota Y)^0),$$

$$(x_0, y_0) \mapsto \sum_{i=1}^{\infty} x_i \otimes y_i \mapsto (y', y) \mapsto \sum_{i=1}^{\infty} (x_i, x_i')(y_i', y_i').$$

2.1. $p$ is continuous.

2.1.2) Whenever $X$ and $Y$ are both gDF or both Fréchet spaces, then the map $p$ has a continuous linear extension

$$p : X \hat{\otimes}_\iota Y \rightarrow B_m((X \hat{\otimes}_\iota Y)^0).$$

2.1.3) Let $X$ and $Y$ be both gDF or both Fréchet spaces, and let $v \in X \hat{\otimes}_\iota Y$. Then $p(v) = p(v) \hat{\otimes}_\iota Y \rightarrow B_m((X \hat{\otimes}_\iota Y)^0)$, and $p(v) = 0$ whenever $p(v) = 0$.  

2. — Studia Mathematica 37
Proof. Let \( p(\tilde{\epsilon}) = 0 \), and let \((x'',y'') \in \mathcal{X}'' \times \mathcal{Y}''\). There exist closed bounded disks \( B' \) and \( C' \) in \( \mathcal{X}'' \) and \( \mathcal{Y}'' \), respectively, such that

\[
(x'',y'') \in B' \times C' = B \times C,
\]

\( (c\text{-closures of } B \text{ and } C \text{ in } (\mathcal{X}''_p, \mathcal{Y}''_p), \text{ respectively}. \)\ Since \( \tilde{p}_i(\tilde{\epsilon})B \times C = 0 \) and \( \tilde{p}_i(\tilde{\epsilon})B' \times C' = 0 \), \( \tilde{p}_i(\tilde{\epsilon}) \) is \( c \times c \)-continuous, cf. ([16]), I Intr. VI., Lemme D, p. 27, we conclude that

\[
\tilde{p}_i(\tilde{\epsilon}) (x'', y'') = 0.
\]

Proof of Theorem 2.1 (b). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be gDF spaces,

\[
\tilde{\epsilon} \in \mathcal{X} \otimes \mathcal{Y} \Rightarrow \tilde{\epsilon} = \sum_1^\infty \lambda_i x'_i \otimes y'_i,
\]

\((x'_i, y'_i)\) e nulsequences in \( \mathcal{X}'' \) and \( \mathcal{Y}'' \), respectively, and assume that \( p(\tilde{\epsilon}) = 0 \). We have to show that \( \tilde{\epsilon} = 0 \) whenever \( X'' \) or \( Y'' \) has the approximation property (a.p.).

Step 1. Assume that \( X'' \) or \( Y'' \) has a.p. According to (2.3.3) we have

\[
0 = \tilde{p}_i(\tilde{\epsilon}) (x''', y''') = \sum_1^\infty \lambda_i x'_i (y''') \quad \text{for all } (x''', y''') \in \mathcal{X}'' \times \mathcal{Y}''.
\]

Since, by Lemma 2.3, \( (\mathcal{X}''_p \otimes \mathcal{Y}''_p)' = L(\mathcal{X}'', \mathcal{Y}'') \), we can interpret (1) in the following way:

\[
k(\tilde{\epsilon}) = 0 \quad \text{for all } k \in \mathcal{X}'' \otimes \mathcal{Y}''.
\]

Furthermore, if \( X'' \) or \( Y'' \) has a.p., then \( \mathcal{X}'' \otimes \mathcal{Y}'' \) is dense in \( L(\mathcal{X}'', \mathcal{Y}'') \), cf. ([15]), I 5.1, Prop. 35, pp. 164, 165.

Now, let \( u \in L(\mathcal{X}'', \mathcal{Y}'') \) and set \( u = \sum_1^\infty \lambda_i x'_i \otimes y'_i \), and \( \epsilon > 0 \). Then there exists \( k \in \mathcal{X}'' \otimes \mathcal{Y}'' \) such that \( k - u \in \mathcal{X}'' \otimes \mathcal{Y}'' \) is dense in \( L(\mathcal{X}'', \mathcal{Y}'') \), where \( \epsilon < \|k\| \) is chosen such that \((x'_i, y'_i)\) e nulsequences in \( \mathcal{X}'' \) and \( \mathcal{Y}'' \), respectively, \( \sum_1^\infty \lambda_i x'_i \otimes y'_i \).

Hence, \( u(\tilde{\epsilon}) = 0 \) for all \( u \in (\mathcal{X}'' \otimes \mathcal{Y}'')' \), i.e., \( \tilde{\epsilon} = 0 \). This completes the proof.

Step 2. Assume that \( X'' \) has a.p.

(2.3.4) Let \( \mathcal{X} \) and \( \mathcal{Y} \) be locally convex spaces such that \( X'' \) and \( Y'' \) are evaluative. Then the map

\[
S: L(\mathcal{X}'', \mathcal{Y}'') \to L(\mathcal{X}'', \mathcal{Y}''),
\]

\[
u \mapsto \tilde{\nu}: \nu \mapsto \sum_1^\infty \lambda_i x'_i \otimes y'_i \]

is a linear bijection, and \( \tilde{\nu} = s \).

This is a consequence of the fact that \( \tilde{\nu} = s \) is a p.daf isomorphic to the space \( \mathcal{X}'' \otimes \mathcal{Y}'' \) of gDF spaces isomorphic to \( \mathcal{X}'' \otimes \mathcal{Y}'' \) of nuclear operators from \( \mathcal{X} \) into \( \mathcal{Y} \), whenever \( p \) is injective. To this end, consider the map

\[
\gamma : \mathcal{X} \otimes \mathcal{Y} \to L_0(\mathcal{X}, \mathcal{Y}),
\]

\[
\sum_1^n x'_i \otimes y'_i \mapsto \sum_1^n \gamma(x'_i, y'_i).
\]

Since \( L_0(\mathcal{X}, \mathcal{Y}) \) is Fréchet (\( \mathcal{X} \) and \( \mathcal{Y} \) are gDF), \( \gamma \) has a continuous linear extension : \( \mathcal{X} \otimes \mathcal{Y} \to L(\mathcal{X}, \mathcal{Y}) \). According to the definition of nuclear operators, the space \( N(\mathcal{X}, \mathcal{Y}) \) is even topologically isomorphic to \( \mathcal{X} \otimes \mathcal{Y} \) ker. It is easy to see that ker is 0 whenever \( p \) is injective. This completes the proof of part (b) of Theorem 2.1.

For a proof of part (c), let \( p \) be injective, and let

\[
T = f(\tilde{\epsilon}) \in L(\mathcal{X}'', \mathcal{Y}''),
\]

\[
\tilde{\epsilon} \in \mathcal{X} \otimes \mathcal{Y}, \text{ be such that } T \mathcal{X} \otimes \mathcal{Y} = 0. \text{ This implies that } p(\tilde{\epsilon}) = 0, \text{ and thus } \tilde{\epsilon} = 0, \text{ and } T = 0. \text{ Under the given assumptions this means that } \mathcal{X} \otimes \mathcal{Y} \text{ is semi-Montel}. \text{ Then we note that}
\]

\[
(\mathcal{X}'' \otimes \mathcal{Y}'')' = L(\mathcal{X}'', \mathcal{Y}'') = L(\mathcal{X}''', \mathcal{Y}''),
\]

(Lemma 2.3 and semi-reflexivity of \( \mathcal{Y} \), and that \( L(\mathcal{X}'', \mathcal{Y}'') = L(\mathcal{X}''', \mathcal{Y}''), \mathcal{Y} \) and \( \mathcal{Y} \) have the same bounded sets. Hence, every \( u \in L(\mathcal{X}'', \mathcal{Y}'') \) transforms bounded subsets of \( \mathcal{X}'' \) into bounded subsets of \( \mathcal{Y}'' \). But \( \mathcal{X}'' = \mathcal{X}'' \) is Fréchet, so that \( u = L(\mathcal{X}'', \mathcal{Y}'') \). Thus we have shown that \( \mathcal{X}'' \otimes \mathcal{Y}'' \) and \( \{L(\mathcal{X}''', \mathcal{Y}'')\} \) have the same dual (according to part (a), \( L(\mathcal{X}''', \mathcal{Y}'') \) is semi-reflexive).

Hence, if \( j(\tilde{\epsilon}) = 0 \),

\[
\tilde{\epsilon} = \sum_1^\infty \lambda_i x'_i \otimes y'_i \in \mathcal{X} \otimes \mathcal{Y},
\]

is a linear bijection, and \( \tilde{\nu} = s \).
then we have:

\[ 0 = j(\tilde{e})(u) = \sum_{i=1}^{\infty} \lambda_i(u_i, y_i) = u(\tilde{e}) \quad \text{for all } u \in L(X', Y) = (X' \otimes \mathcal{A})', \]

and thus \( \tilde{e} = 0 \). Altogether, we have shown that \( X' \otimes \mathcal{A} \) and \( L_0(X', Y)' \) are algebraically isomorphic and have the same dual. Since \( X' \otimes \mathcal{A} \) is Fréchet and \( L_0(X', Y)' \) is metrizable [29], we conclude that both spaces are topologically isomorphic.

Part (c): According to part (b), \( L_0(X', Y)' \) is Fréchet. \( X \otimes \mathcal{A} \) is a closed linear subspace of the (semi-reflexive) space \( L_0(X', Y) \). Hence, the assumptions of Proposition 2.7 (i), of [26] are fulfilled, and we can conclude that

\[ \beta((X \otimes \mathcal{A})', (X \otimes \mathcal{A})) = \beta((L_0(X', Y)')', L_0(X', Y)) \]

We now turn to applications and special cases of Theorems 3.1 and 2.2.

Recall that, given locally convex spaces \( X \) and \( Y \), we denote by \( \mathcal{R}^X(X, Y) \) the space of weakly continuous linear operators from \( X \) into \( Y \) which transform bounded sets into relatively compact sets, endowed with the topology of uniform convergence on bounded subsets of \( X \). Whenever \( Y \) is quasi-complete, then \( \mathcal{R}^X(X, Y) \approx L_0(X', Y) \), by (17), Example 2.0 (c). Note that \( X_0(X, Y) \) is a topological linear subspace of \( \mathcal{R}^X(X, Y) \). We first deduce a representation for the continuous linear functionals on \( \mathcal{R}^X(X, Y) \).

**Theorem.** Let \( X \) be a metrizable space and \( Y \) a complete gDF space such that \( X_0 \) or \( Y_0 \) is semi-reflexive, or a Banach space whose dual has HNP. Then we have:

(a) The map

\[ j: X'_0 \otimes \mathcal{A} \to (\mathcal{R}^X(X, Y)')' \]

is surjective: for every \( T \in (\mathcal{R}^X(X, Y)')' \), there exist \( (\lambda_i)_{i\in \mathbb{N}} \in \ell^1 \) and null-sequences \( (a_i')_{i\in \mathbb{N}} \) in \( X'_0 \) and \( (y_i')_{i\in \mathbb{N}} \) in \( X_0 \), respectively, such that

\[ Th = \sum_{i=1}^{\infty} \lambda_i (a_i', y_i') \quad \text{for all } h \in \mathcal{R}^X(X, Y). \]

(b) If, in addition to the assumptions, \( X'_0 \) or \( X_0 \) or \( Y'_0 \) or \( Y_0 \) has the approximation property, then the map \( j \) of (a) is a linear isomorphism:

\[ (\mathcal{R}^X(X, Y)')' = X'_0 \otimes \mathcal{A} \otimes Y_0 \approx (X_0)' \approx (N(X_0, Y_0)' \text{ (algebraically).}) \]

(c) If, in addition to the assumptions, \( X'_0 \) and \( X_0 \) are semi-reflexive and one of them is semi-Montel, then we have:

(a) \( \mathcal{R}^X(X, Y) \) is semi-reflexive.

(b) \( (\mathcal{R}^X(X, Y)')' \approx X'_0 \otimes \mathcal{A} \otimes Y_0 \approx (X_0)' \approx (N(X_0, Y_0)' \text{ (topologically).}) \]

2.5. Corollary. Let \( X \) be a Banach space and \( Y \) a complete gDF space such that \( X'' \) has HNP or that \( Y \) is semi-reflexive. Then we have:

(a) The dual of \( K_0(X, Y) \) is algebraically isomorphic to a quotient of \( X'_0 \otimes \mathcal{A} \).

(b) If, in addition to the assumptions, \( X \) is a reflexive Banach space and \( Y \) is semi-Montel, then the space \( K_0(X, Y) \) is semi-reflexive, and

\[ (K_0(X, Y))': \approx X'_0 \otimes \mathcal{A} \otimes Y_0 \approx (N(X_0, Y_0)' \text{ (topologically).}) \]

2.6. Corollary. Let \( X \) be a Banach space and \( Y \) the c-dual of a Banach space with the approximation property. Then we have:

(a) \( (L_0(X, Y))' = X'_0 \otimes \mathcal{A} \otimes Y_0 \approx (N(X_0, Y_0)' \text{ (algebraically).}) \)

(b) If, in addition to the assumptions, \( X \) is reflexive, then \( L_0(X, Y) \) is semi-reflexive, and we have:

\[ (L_0(X, Y))' \approx X'_0 \otimes \mathcal{A} \otimes Y_0 \approx (N(X_0, Y_0)' \text{ (topologically).}) \]

The spaces \( L''(\mathcal{G}) \) and \( H''(\mathcal{G}) \), the strict topology of R. C. Buck ([41], [51]), \( \mathcal{G} \) a simply connected region in the plane, are (non-trivial) concrete examples for c-duals of Banach spaces with a p. (for \( H''(\mathcal{G}) \), consult [17], Satz 9).

2.7. Theorem. Let \( X \) be a Fréchet space and \( Y \) a gDF space.

(a) For every continuous linear functional \( T \) on \( L_0(X, Y) \) there exist \( (\lambda_i)_{i\in \mathbb{N}} \in \ell^1 \) and null-sequences \( (a_i')_{i\in \mathbb{N}} \) in \( X_0 \) and \( (y_i')_{i\in \mathbb{N}} \) in \( X_0 \), respectively, such that

\[ Th = \sum_{i=1}^{\infty} \lambda_i (a_i', y_i') \quad \text{for all } h \in L_0(X, Y). \]

(b) If, in addition to the assumptions, \( X \) or \( Y_0 \) or \( Y''_0 \) has the approximation property, then \( (L_0(X, Y))' = X'_0 \otimes \mathcal{A} \otimes Y_0 \approx (N(X'_0, Y'_0)' \text{ (algebraically).}) \)

(c) If, in addition to the assumptions, \( X \) is semi-reflexive, then \( L_0(X, Y) \) is semi-reflexive and

\[ (L_0(X, Y))' \approx X'_0 \otimes \mathcal{A} \otimes Y_0 \approx (N(X_0, Y'_0)' \text{ (topologically).}) \]

This is a consequence of Theorems 2.1 and 2.2 and the topological isomorphism \( L_0(X, Y) = L_0((X_0)'_0, Y) \) of Example 0.2 [7]. For a DF space \( X \), proposition (a) of Theorem 2.5 is a result of Grothendieck [16], L.4.2, Prop. 22, p. 114.

2.8. Corollary. Let \( X \) be a Fréchet–Montel space and \( Y \) a gDF space.

(a) If \( X \) or \( Y'_0 \) or \( Y''_0 \) has the approximation property, then

\[ (L_0(X, Y))' = X'_0 \otimes \mathcal{A} \otimes Y_0 \approx (N(X'_0, Y'_0)' \text{ (algebraically).}) \]

(b) If \( Y \) is semi-reflexive, then \( L_0(X, Y) \) is semi-reflexive and

\[ (L_0(X, Y))' \approx X'_0 \otimes \mathcal{A} \otimes Y_0 \approx (N(X'_0, Y'_0)' \text{ (topologically).}) \]
In the context of vector-valued continuous functions, Singer [32] and Bogdanovics [22, Thm. 4] determined the dual of \( C(K, X) \) (\( K \) compact Hausdorff, \( X \) Banach), and Wells [34, Thm. 1] and Fontenot [13, Thm. 3.13] the dual of \( C_0(S, X)_p \) (\( S \) locally compact or completely regular Hausdorff, \( X \) a locally convex space, \( \beta \) the strict topology of H. C. Buck [4], [5]). In any case, the dual turned out to be a space of certain \( X' \)-valued measures. It appears to be one of the nicest applications of our general duality results, that we are able to derive a much more specific result for the special case that the range space \( X \) is a semi-reflexive gDF space or a Banach space whose dual has RNP.

2.9. Theorem. Let \( S \) be a locally compact Hausdorff space, \( X \) a semi-reflexive gDF space, or a Banach space whose dual has RNP, and denote by \( C_0(S, X)_p \) the space of bounded continuous \( X \)-valued functions on \( S \), endowed with the strict topology \( \beta \) of H. C. Buck [4], [5]. Furthermore, let \( M_0(S) \) be the space of bounded regular Borel measures on \( S \), endowed with the total variation norm. Then we have:

\[
(C_0(S, X)_p)' = M_0(S) \otimes X' \quad \text{(algebraically)}.
\]

For every \( T \in (C_0(S, X)_p)' \) there exist \( (\lambda_i)_i \in F_1 \), and nullsequences \( (\nu_i)_i \in M_0(S) \otimes X_r \), respectively, such that

\[
TF = \sum_i \lambda_i \int (a_i \circ T) \, d\nu_i \quad \text{for all } T \in C_0(S, X).
\]

This result is just a special case of Theorem 2.1 (c): \( C_0(S, X)_p \) and \( M_0(S) \) have the approximation property (for \( C_0(S, X) \), see [6]) and, according to the results of Section 4 of our previous paper [7], we have

\[
C_0(S, X)_p \simeq L_1((C_0(S, X)_p)'(\mathbb{R})) \quad \text{(topologically)}.
\]

2.10. Theorem. Let \( S \) be a locally compact \( \sigma \)-compact Hausdorff space, \( X \) a Banach space whose dual has RNP, and denote by \( C_0(S, X) \) the space of continuous \( X \)-valued functions on \( S \) with compact support, endowed with the usual inductive limit topology. Furthermore, denote by \( M(S) \) the space of Eberlein measures on \( S \), endowed with the strong dual topology of \( (C_0(S))' \).

Then, for every \( T \in (C_0(S, X))' \), there exists

\[
\tilde{e} = \sum_i \lambda_i \rho_i \otimes a_i \in M(S) \otimes X'
\]

such that

\[
TF = \sum_i \lambda_i \int (a_i \circ T) \, d\nu_i \quad \text{for all } T \in C_0(S, X).
\]

Again, this is a special case of Theorem 2.1, for we have the topological isomorphism \( C_0(S, X)_p \simeq L_1((C_0(S, X)_p)) \), cl. ([7], Section 4).

We close this section with an application of Theorems 2.1 and 2.2 to spaces of holomorphic vector-valued functions with the strict topology \( \beta \).

2.11. Theorem. Let \( G \) be a simply connected region in the complex plane, and \( X \) a complete gDF space, and denote by

\[
M_0(G) = M_0(G)/\|H^\infty(G)\|^{-1}
\]

the (strong) dual of \( H^\infty(G) \). Then we have:

(a) \( H^\infty(G, X)_p \) is semi-reflexive, \( H^\infty(G, X)_p \) is semi-reflexive, and we have:

\[
(C_0(G, X)_p)' \simeq M_0(G) \otimes X'_r \quad \text{(topologically)}.
\]

(b) If, in addition to the assumptions, \( X \) is semi-reflexive, then \( H^\infty(G, X)_p \) is semi-reflexive, and we have:

\[
(C_0(G, X)_p)' \simeq M_0(G) \otimes X'_r \quad \text{is semi-reflexive, and we have:}
\]

This result is a special case of Theorems 2.1 and 2.2: \( H^\infty(G) \) is a semi-Montel gDF space [28], and, for \( G \) simply connected, has the approximation property ([11], Satz 9). According to ([21], § 43.4.9, (9)), \( M_0(G) \) has the a.p. as well.

3. Spaces of compact operators on Banach spaces. In the context of Banach spaces, the general duality results of the foregoing sections nicely specialize to isometrical representations of duals of spaces of compact operators. The following is the fundamental result. (The dual of a normed space \( E \) endowed with the dual norm, is denoted by \( E' \).)

3.1. Theorem. Let \( X \) and \( Y \) be Banach spaces such that \( X' \) or \( Y' \) has RNP. Then we have:

(a) The map

\[
j : X' \otimes Y' \to (L_1(X', Y))'
\]

is surjective, and \( (L_1(X', Y))' \) is isometrically isomorphic to \( (X' \otimes Y')/\ker j \).

For every \( T \in (L_1(X', Y))' \) there exists

\[
\tilde{e} = \sum_i \lambda_i \rho_i \otimes y_i \otimes X' \otimes Y' \quad \| \tilde{e} \| = \| T \|
\]

such that

\[
\tilde{T} = \sum_i \lambda_i \int (h \circ T) \, dy_i \quad \text{for all } h \in L_1(X', Y).
\]
(b) If, in addition to the assumptions, the map
\[ p : X' \otimes_{\alpha} Y \to B(X, Y) \]
is injective, then we have the following isometrical isomorphisms:

(b1) \( X \otimes_{\alpha} Y = L_c(X', Y) \) \( \cong \) the space of compact weak*–weakly continuous linear operators from \( X' \) into \( Y \).

(b2) \( (X \otimes_{\alpha} Y)' = X(X, Y) \) \( \cong \) the space of nuclear operators from \( X \) into \( Y' \).

(b3) \( (X \otimes_{\alpha} Y)' = L(X', Y') \) \( \cong \) the space of bounded linear operators from \( X' \) into \( Y' \).

The map \( p \) is injective whenever \( X' \) or \( X \) has the approximation property.

In view of the applications of Theorem 3.1 to quite recent results in Banach space theory, it seems particularly remarkable to note that, essentially, this theorem can be traced back to Grothendieck's work ([16], I.4.2, Thm. 8, p. 123), compare ([14], Thm. 5.2 and 5.3).

Here, Theorem 3.1 appears just as a special case of Theorem 2.1. We now discuss consequences and special cases of Theorem 3.1.

3.2. THEOREM. Let \( X \) be a normed space, or, more generally, a qDF space whose strong dual \( X' \) is a Banach space, \( Y \) a Banach space, and assume that \( X' \) or \( Y \) has RNP. Furthermore, consider the linear map
\[ j : X'' \otimes_{\alpha} Y' \to \mathcal{K}(X, Y), \]
\[ \sum_{i=1}^{n} a_i \otimes y_i' \mapsto \{ h \mapsto \sum_{i=1}^{n} \langle h(a_i), y_i' \rangle \} . \]

Then we have:

(a) \( \mathcal{K}(X, Y) \cong X'' \otimes_{\alpha} Y' \) (isometrically).

For every \( Y \in \mathcal{K}(X, Y) \), there exists \( \tilde{v} = \sum_{i=1}^{n} \lambda_i a_i \otimes y_i' \in X'', \mathcal{K}(X, Y) \) such that \( \| Y \| = \| \tilde{v} \| \).

(b) The map \( j \) is injective, whenever the map
\[ p : X' \otimes_{\alpha} Y' \to B(X', Y) \]
is injective, in particular, whenever \( X' \) or \( X \) has the approximation property.

In one of these cases, we have the following isometrical isomorphisms:

(b1) \( (X \otimes_{\alpha} Y)' \cong \mathcal{K}(X', Y) \).

(b2) \( (X \otimes_{\alpha} Y)' \cong \mathcal{K}(X'', Y) \).

This follows from Theorem 3.1 and the isometrical isomorphism \( \mathcal{K}(X, Y) = L_c(X', Y) \) of ([17], Example 0.2 (a)). For the case of Banach spaces \( X \) and \( Y \), Theorem 3.1 quite recently has been (re)proven (see the notes preceding Theorem 3.3) by Pecar and Sapar (20, Thm. 1) in a different way, using further deep results on the Radon–Nikodým property for Banach spaces.

Note. In a series of papers, Ruckle [24], Holub [19], Kalton [20] and Heinrich [18] dealt with the conjecture that, for reflexive Banach spaces \( X \) and \( Y \), the space \( L(X, Y) \) is reflexive if and only if every bounded linear operator from \( X \) into \( Y \) is compact. Roughly, this connection holds whenever, in addition, \( X \) or \( Y \) has a.p. For a detailed discussion of the problem we refer to Section 3.2, Theorem 2.8, of our previous paper ([7]). Theorem 3.2 allows to add the following more specific information:

3.3. COROLLARY. Let \( X \) and \( Y \) be reflexive Banach spaces.

(a) If \( L(X, Y) = K(X, Y) \), then \( L(X, Y) \) is reflexive.

(b) Conversely, if \( K(X, Y) \) is weakly sequentially complete and the map \( j : X \otimes_{\alpha} Y \to \mathcal{K}(X, Y) \) of Theorem 3.2 above is injective, then we have \( L(X, Y) = K(X, Y) \).

In particular, \( L(X, Y) = K(X, Y) \) if and only if \( K(X, Y) \) is reflexive and \( j \) is injective.

This result contains the corresponding ones of the authors cited above as special cases. It is a consequence of Theorem 2.3 and of ([7], Thm. 2.8). Furthermore, it reduces the question of whether the conjecture \( L(X, Y) \) is reflexive if and only if the space \( L(X, Y) \) is reflexive Banach spaces is true to the problem of injectivity of \( j \).

Problem. Given reflexive Banach spaces \( X \) and \( Y \), is the map \( j : X \otimes_{\alpha} Y \to \mathcal{K}(X, Y) \) of Theorem 3.2 injective? More specifically, given reflexive Banach spaces \( X \) and \( Y \), and
\[ \tilde{v} = \sum_{i=1}^{n} \lambda_i a_i \otimes y_i' \in X \otimes_{\alpha} Y', \]
\[ (\lambda_i)_{i=1}^{n} \in F', (a_i)_{i=1}^{n} \in X' \text{ and } (y_i')_{i=1}^{n} \text{ nullsequences in } X \text{ and } Y, \]
such that \( \tilde{v}(k) = \sum_{i=1}^{n} \lambda_i \langle k(a_i), y_i' \rangle = 0 \) for all \( h \in K(X, Y) \), is it true that then \( \tilde{v}(\omega) = \sum_{i=1}^{n} \lambda_i \langle u(w), y_i' \rangle = 0 \) for all \( u \in L(X, Y) \) as well?

Trivially, this is true whenever \( X \) or \( Y \) has a.p. That it very well can happen without \( X \) or \( Y \) having a.p. can be seen by combining a result of Pitt's with the counterexamples to the a.p.: it is a consequence of ([25], Thm. 1) that \( L(P, P) = K(P, P) \) for \( 1 \leq p < q < \infty \). Furthermore, according to results of A. M. Davie [9] and T. Figiel [12], there exist
closed linear subspaces of $P$ without a.p. for any $p$ with $0 < p < \infty$. Thus, we arrive at the desired example if we choose subspaces $M$ and $N$ of $P$ and $P'$, respectively, without a.p. for $p$ and $q$ such that $2 < p < q < \infty$. For then we still have $L(N, M) = K(N, M)$, as can be seen by employing techniques of Rosenthal's [235], proof of Theorem A5, pp. 206, 207. (We are grateful to our colleague Lutz Weis for working out this example.)

We close this section with applications to spaces of vector-valued continuous functions and to spaces of compact operators on function spaces with the strict topology.

3.4. THEOREM. Let $S$ be a locally compact Hausdorff space, $X$ a Banach space whose dual has RNP, and denote by $C_b(S, X)$ the space of continuous $X$-valued functions on $S$ vanishing at infinity, endowed with the sup-norm topology. Furthermore, denote by $M_b(S)$ the space of bounded Radon measures on $S$. Then we have

$$(C_b(S, X))' = M_b(S) \hat{\otimes}_a X'$

(isometrically).

For every $T \in (C_b(S, X))'$, there exists

$$\hat{\varphi} = \sum \lambda_i \mu_i \otimes \varphi_i \in M_b(S) \hat{\otimes}_a X'$$

such that

$$T \varphi = \sum \lambda_i \int (\varphi_i \otimes F) d\mu_i$$

for all $T \in (C_b(S, X))'$.

This follows from Theorem 3.1 (b) and the known facts that $C_b(S, X)$ and $M_b(S)$ have a.p.

TERMINOLOGY AND NOTATION. (a) For a completely regular Hausdorff space $T$, $\beta_T$, $\beta$, and $\beta_T$ denote the substrict, strict, and superstrict topology of $T$, respectively, on the space $C_b(T)$ of bounded continuous scalar-valued functions on $T$. (Note that, whenever $T = S$ is locally compact, $\beta_\omega = \beta$ = Buck's original strict topology). By $M_b(T)$, $\xi \in [\beta_T, \beta, \beta_\omega]$, we denote the respective duals of $C_b(T)$, the spaces of tight, $\tau$-additive, and $\sigma$-additive measures on $T$, respectively.

(b) Given a region $G$ in the complex plane, we denote by $H^\infty(G)$ the space of bounded holomorphic functions on $G$ endowed with the strict topology $\beta$ (resp. the sup-norm topology).

We are now ready to state the following case of Theorem 3.2:

3.5. THEOREM. Let $T$ be a completely regular Hausdorff space, $G$ a plane region, and $X$ a Banach space whose dual has RNP and the approximation property. Then we have the following isomorphic isomorphisms:

(a) $X(C_b(T), X) = M_b(T) \hat{\otimes}_a X$,
(b) $[X(C_b(T), X)]' = X(M_b(T), X')$,
(c) $[X(C_b(T), X)]'' = X(M_b(T)', X')$ ($\xi \in [\beta_T, \beta, \beta_\omega]$).

4. Spaces of compact operators on Fréchet spaces. This section is devoted to the study of spaces of compact operators acting between Fréchet spaces $X$ and $Y$. Again, we start with the fundamental duality results for the space $L_1(X', Y)$.

4.1. THEOREM. Let $X$ and $Y$ be Fréchet spaces one of which is reflexive and quasinormable, or a Banach space whose dual has RNP.

(a) The $c$-dual of $L_1(X', Y)$ is isometrically isomorphic to $X'_b \hat{\otimes}_a Y'_b$:

$$L_1(X', Y)' = X'_b \hat{\otimes}_a Y'_b.$$

(b) The linear map

$$j : X'_1 \hat{\otimes}_a Y'_1 \to (L_1(X', Y))'$$

is surjective. More precisely, we have: for every $T \in (L_1(X', Y))'$, there exist $U \in X'_2, V \in Y'_2$, and

$$\hat{\varphi} = \sum \lambda_i \varphi_i \hat{\otimes} y_i \in X'_2 \hat{\otimes}_a Y'_2$$

such that

$$Th = \sum \lambda_i \varphi_i (h' , y_i')$$

for all $h \in L_1(X', Y)$.

(c) If the map

$$p : X'_1 \hat{\otimes}_a Y'_1 \to B(X, Y)$$

is injective, then we have:

$$(c1) X \hat{\otimes}_a Y = L_1(X', Y)$$

(topologically),

$$(c2) (X \hat{\otimes}_a Y)' = X'_2 \hat{\otimes}_a Y'_2$$

(algebraically).

If $p$ is injective if any of the following additional conditions on $X$ and $Y$ are fulfilled:

Case 1. $X$ is Fréchet–Montel, $Y'_b$ is barrelled, and $X$ or $Y'_b$ has a.p.

Case 2. $Y$ is Fréchet–Montel, $X'_b$ is barrelled, and $Y$ or $X'_b$ has a.p.

Remarks. 1. We are not able to establish a connection between invertibility of $p$ and the approximation property as nice and general as in the case of $\alpha$DF spaces $X$ and $Y$ (Theorem 2.1!), but need the additional assumption that at least one of the spaces $X$ or $Y$ is Montel. As will become clear from the proof of part (c) of Theorem 4.1 below, the reason for this lies in the fact that, in general, for the elements $\hat{\varphi} \in X'_1 \hat{\otimes}_a Y'_1$, we do not have a series representation in the form of an (finite) absolutely convex combination of the tensor product of two nullsequences in $X'_1$ and $Y'_1$. However, this in turn follows from the duality results of Theorem 3.2 and the reflexive and quasinormable assumptions on $X$ and $Y$.
the space $X'' \otimes Y''$ is dense in $L_0(X_1', Y_1')$. At this point, if we would know that $\tilde{\varphi}$ has a series representation

$$\tilde{\varphi} = \sum_{i=1}^{\infty} \lambda_i \varphi_i' \otimes y_i', \quad \text{with} \quad (\lambda_i)_{i=1}^\infty \in l^1,$$

and $(\varphi_i')_{i=1}^\infty$ and $(y_i')_{i=1}^\infty$ nullsequences in $X_1'$ and $Y_1'$, respectively, then we would be able to follow the arguments of the proof of the corresponding part of Theorem 2.1 to conclude that $\tilde{\varphi} = 0$. Since such a series representation is not guaranteed ($X_1'$ and $Y_1'$ are DF spaces), we use here the additional assumption that $X$ is Montel, instead. We thus can start from the situation, that $\tilde{\varphi}$ is an element of the completion of the tensor product $X_1' \overset{\sigma}{\otimes} Y_1'$, where $X_1'$ is a Montel DF space and $Y_1'$ a DF space.

According to Theorem 1.9 of [28], we conclude that there exist a closed bounded (and, hence, compact) disk $B$ in $X_1'$ and a bounded disk $\mathcal{C}$ in $Y_1'$ such that $\tilde{\varphi} \in \text{ac}(B \otimes \mathcal{C})$; there exists a net

$$v_1 = \sum_{i=1}^{n_1} a_i \varphi_i' \otimes y_i', \quad \text{such that} \quad (v_i)_{i=1}^{\infty} \text{converges to} \tilde{v} \in X_1' \overset{\sigma}{\otimes} Y_1'.$$

Now let $u \in L(X_1', Y_1') = (X_1' \overset{\sigma}{\otimes} Y_1')'$ (Lemma 2.3), and $\varepsilon > 0$. There exists $k \in X'' \otimes Y''$ such that

$$(u-k)(B) \subseteq \varepsilon \mathcal{O}. \tag{2}$$

According to (1), we conclude that

$$|u(\tilde{\varphi})| \leq |(u-k)(\tilde{\varphi})| \leq \limsup_{i \to \infty} |(u-k)(v_i)| \leq \lim_{i \to \infty} \sum_{i}(|a_i| |(u-k)x_i'|, y_i')| \leq \varepsilon.$$

Hence $\tilde{\varphi} = 0$, and the proof of Theorem 4.1 is complete.

Proof of Theorem 4.2. Part (a) is a special case of a result of our previous paper ([27], Thm. 2.13).

Part (b). Step 1: $j : X_1' \overset{\sigma}{\otimes} Y_1' \to (L_0(X_1', Y_1'))'$ is injective: let $\tilde{\varphi} \in X_1' \overset{\sigma}{\otimes} Y_1'$ such that $j(\tilde{\varphi}) = 0$. There exists a (bounded) net

$$v_1 = \sum_{i=1}^{n_1} a_i \varphi_i' \otimes y_i' \in X_1' \overset{\sigma}{\otimes} Y_1'$$

such that $(v_i)_{i=1}^{\infty}$ converges to $\tilde{v} \in X_1' \overset{\sigma}{\otimes} Y_1'$. According to the continuity of $j$ into $L_0(X_1', Y_1')$ (Lemma 1.1), we conclude that

$$(1) \quad 0 = j(\tilde{\varphi})(k) = \lim_{i \to \infty} j(v_i)(k) = \lim_{i \to \infty} \sum_{i} (kx_i', y_i') \quad \text{for all} \quad k \in L(X_1', Y_1').$$

Furthermore, we have

$$(2) \quad (X_1' \overset{\sigma}{\otimes} Y_1')' = L_0(X_1', Y_1').$$
4.4. Theorem. Let $X$ be a $\mathcal{DF}$ space and $Y$ a Fréchet space such that $X_s$ or $Y$ is reflexive and quasinormable or a Banach space whose dual has EXP.

(a) The linear map $j: X'_s \bar{\otimes}_o Y'_s \to (E(X,Y))'$ is surjective. More precisely, we have: for every $T \in (E(X,Y))'$, there exist $B$ bounded in $X$, $V \in \mathcal{U}_Y$, and

$$\hat{v} = \sum_{j=1}^n \lambda_j a_j' \otimes y_j' \in (X'_s \bar{\otimes}_o Y'_s)$$

such that

$$T = \sum_{j=1}^n \lambda_j (a_j', y_j') \quad \text{for all} \; h \in E(X,Y).$$

(b) $E'$ is the bipolar of $E$ in $X'' = (X'_s)'$.

(c) If, in addition to the assumptions, the map

$$p: X'_s \bar{\otimes}_o Y'_s \to B(X'_s, Y)$$

is injective, then we have:

(b1) $E(X,Y) \cong X'_s \bar{\otimes}_o Y'_s$ (topologically).

(b2) $E(X,Y) \cong X'_s \bar{\otimes}_o Y'_s$ (algebraically).

Conditions ensuring injectivity of $p$ can be read from part (c) of Theorem 4.1.

(c) If, in addition to the assumptions, $X_s$ and $Y$ are reflexive, and $X'_s$ or $Y$ is Montel, and $X'_s$ or $Y$ is quasinormable, then we have:

(c1) $E(X,Y)$ is a reflexive Fréchet space.

(c2) $E(X,Y)$ is a reflexive Fréchet space.

This is a consequence of Theorems 4.1 and 4.2, and the topological isomorphism $E(X,Y) \cong L_q(X'_s, Y)$ of ([7], Example 0.3 (b)).

4.5. Examples. Theorem 4.4 can be used to determine the form of continuous linear functionals on the following spaces of compact operators:

4.5.1) $E(X,Y)$, $X$ a Banach space whose bidual has RNP and $Y$ a Fréchet space.

4.5.2) (i) $E_0(C(T, Y), X)$ completely regular Hausdorff, $\xi \in (\beta_0, \beta, \beta)$.

(ii) $E_0(C_0(S), X)$, $S$ locally compact Hausdorff.

(iii) $E_0(C_0(S), X)$, $S$ locally compact Hausdorff and $\sigma$-compact.

And, in all three cases, $X$ a reflexive quasinormable Fréchet space, in particular, a Fréchet-Schwartz space.

We close this section (and this paper) with a representation theorem for the dual of $C(S, X_{\sigma})$. 

4.3. Lemma. Let $X$ and $Y$ be Fréchet spaces one of which is reflexive and quasinormable, and assume that

$$j: X'_s \bar{\otimes}_o Y'_s \to (L_q(X,Y))'$$

is injective.

Then, for every $\hat{v} \in X'_s \bar{\otimes}_o Y'_s$, there exist $U \in \mathcal{U}_X$, $V \in \mathcal{U}_Y$, $(\lambda_j)_{j=1}^n \in \mathbb{F}$, and null sequences $(a_j')_{j=1}^n$ and $(y_j')_{j=1}^n$, in $X'_s$ and $Y'_s$, respectively, such that

$$\hat{v} = \sum_{j=1}^n \lambda_j a_j' \otimes y_j'$$

in the topology of $X'_s \bar{\otimes}_o Y'_s$.

Remark. Assumptions on $X$ and $Y$ under which $j$ is injective, are to be found in Theorems 4.1 and 4.2.

Proof of Lemma 4.3. Let $\hat{v} \in X'_s \bar{\otimes}_o Y'_s$. Then $T = j(\hat{v}) \in (L_q(X,Y))' \setminus \{0\}$ (Lemma 1.1). According to Proposition 1.7 and Lemma 1.8, there exist $U \in \mathcal{U}_X$, $V \in \mathcal{U}_Y$, and

$$\hat{w} = \sum_{j=1}^n \lambda_j a_j' \otimes y_j' \in X'_s \bar{\otimes}_o Y'_s$$

such that $T = j(U, V)(\hat{w})$. Clearly, the sequence $\{\sum_{j=1}^n \lambda_j a_j' \otimes y_j'\}_{n=1}^\infty$ is Cauchy in $X'_s \bar{\otimes}_o Y'_s$, hence it converges to an element $\hat{w}_1 \in X'_s \bar{\otimes}_o Y'_s$. The continuity of the maps $j(U, V)$ and $j$ into $(L_q(X,Y))'$ allows us to conclude that $j(\hat{w}_1) = T = j(\hat{v})$. Hence, $\hat{w}_1 = \hat{v}$, for $j$ is supposed to be injective.

We now turn to applications and special cases of Theorems 4.1 and 4.2.
4.6. Theorem. Let $S$ be a locally compact $\sigma$-compact Hausdorff space, $X$ a reflexive quasi-invariant Fréchet space, or a Banach space whose dual has RNP, and denote by $O(S, X)_0$ the space of continuous $X$-valued functions on $S$, endowed with the compact-open topology. Furthermore, denote by $M_c(S)$ the space of Radon measures on $S$ with compact support. Then we have:

(a) $O(S, X)_0$ is a quotient of $M_c(S) \hat{\otimes} X'$. More precisely, we have:

$$\epsilon = \sum_{\mu_1} \lambda_1 \mu_1 \otimes \alpha_1 \in M_c(S) \hat{\otimes} X'$$

such that

$$TF = \sum_{\mu_1} \lambda_1 \int (a_0 \circ F) \, d \mu_1$$

for all $F \in O(S, X)$.  

(b) If, in addition to the assumptions, $X$ is a Fréchet–Schwartz space with the approximation property, then we have:

$$O(S, X)_0 \cong M_c(S) \hat{\otimes} X'$$

(algebraically).

This is a consequence of Theorem 4.1 and the topological isomorphism $O(S, X)_0 \cong O(S)_0 \hat{\otimes} X$.

References


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