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Random ergodic theorems for sub-Markovian operators

by

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Abstract. We prove a random ergodic theorem for positive weak*-continuous contractions on L^∞. This theorem generalizes a random ergodic theorem of Ryll-Nardzewski [15] and some results of Nawrotzki [11].

In the classical random ergodic theory the average behavior of measure-preserving transformations chosen at random from a set Φ is studied. One of the problems is the structure of the limit function. This problem was considered by Kakutani [8], Ryll-Nardzewski [15] and Gładysz [6], [7]; the operator-theoretical generalizations were given by Cairoli [2]. The case of a finite phase space was considered by Nawrotzki [11].

In [2] Cairoli obtained an extension of Ryll-Nardzewski random ergodic theorem by considering a measurable family of positive contractions on L_1 having a strictly positive invariant function. The aim of the present paper is to give a natural extension of Ryll-Nardzewski theorem [15] and of the main results of Nawrotzki [11] by considering a measurable family $\mathscr{P} = \{P_s\colon s \in S\}$ of sub-Markovian operators, i.e. positive weak*-continuous contractions on L_∞.

Section 1 is preliminary. Having a measurable family $\mathscr{P} = \{P_s\colon s \in S\}$ of sub-Markovian operators we define a sub-Markovian operator U (which corresponds to the classical skew product transformation). Section 2 is devoted to the study of invariant functions and measures for the operator U. In Theorem 2.4 it is proved that the U-invariant functions do not depend essentially on random parameters on the conservative part of U. The analogous result for the U-invariant measures is given in Theorem 2.6. The method of proof of this theorem can also be applied to non-positive operators (see [17]). In Section 3 we state a random version of the individual and strong ergodic theorems together with the identification of the limit function (Theorems 3.4 and 3.6). Finally, Section 4 contains an example connected with Theorem 3.6. It is shown that given a two-sided Bernoulli shift there exists an integrable function f with the property that the ergodic averages conditioned on the future are divergent almost everywhere. We note that a similar example was exhibited by Burkholder in [1] but it is not sufficient for our purposes.

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1. Preliminaries. Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and let $(S, \mathscr{B}, \lambda)$ be a probability space. By $(S^*, \mathscr{B}^*, \lambda^*)$ we denote a countable product of $(S, \mathscr{B}, \lambda)$: $S^* = S_1\times S_2\times \ldots$, $\mathscr{B}^* = \mathscr{B}_1\times\mathscr{B}_2\times \ldots$, $\lambda^* = \lambda_1\times\lambda_2\times\ldots$, where $S_i = S$, $\mathscr{B}_i = \mathscr{B}$, $\lambda_i = \lambda$ for $i = 1, 2, \ldots$ By ξ we shall denote the shift transformation on S^*, i.e. $(s_1, s_2, \ldots)\xi = (s_2, s_3, \ldots)$ for each $(s_1, s_2, \ldots)\in S^*$.

In this paper only real measurable functions are considered and they are denoted by f, g, h. For each $1\leqslant p\leqslant\infty$ by $L_p(X, \mathscr{A}, \mu) = L_p(\mu)$ we denote the usual Banach spaces and $L_p^+(\mu)$ denotes a cone consisting of all nonnegative elements from $L_p(\mu)$. All inequalities and limit operations appearing in this paper are understood almost everywhere, unless stated otherwise. Furthermore, we write $\lim\limits_{n\to\infty} f_n = f$ on A if $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for μ-almost all $x\in A\subseteq X$.

The analogous notation is applied to the measure space $(X\times S^*, \mathscr{A}\times\mathscr{B}^*, \mu\times\lambda^*)$ and to functions defined on $X\times S^*$. If we wish to regard $f(x; s^*)$ as a function of x defined on X for an s^* arbitrary fixed in S^*, we shall write $f_{(s^*)}(x)$ for $f(x; s^*)$.

Now, let $\mathscr{P} = \{P_s\colon s\in S\}$ be a family of sub-Markovian operators on $L_\infty(\mu)$, i.e. for each $s\in S$ P_s is a linear operator on $L_\infty(\mu)$ (whose value on $f\in L_\infty(\mu)$ is denoted by $P_s f$) with the following properties:

(i) $P_s 1\leqslant 1$ on X,

(ii) if $f\in L_\infty^+(\mu)$, then $P_s f\in L_\infty^+(\mu)$,

(iii) if $\lim\limits_{n\to\infty} 1_{A_n} = 0$ on X $(A_n\in\mathscr{A})$, then $\lim\limits_{n\to\infty} P_s 1_{A_n} = 0$ on X.

The conditions (i)–(iii) are equivalent to the statement that each operator $P_s\in\mathscr{P}$ is a positive weak*-continuous contraction on $L_\infty(\mu)$. Therefore, each operator $P_s\in\mathscr{P}$ is an adjoint to a positive contraction on $L_1(\mu)$, which is denoted by the same symbol P_s, but is written to the right of its variable. So,

$$\int_X fP_s\cdot g\,d\mu = \int_X f\cdot P_s g\,d\mu$$

for $f\in L_1(\mu)$, $g\in L_\infty(\mu)$.

1.1. Definition. The family $\mathscr{P} = \{P_s\colon s\in S\}$ of sub-Markovian operators on $L_\infty(\mu)$ is called *measurable* if for every $A\in\mathscr{A}$ there exists a $\mathscr{A}\times\mathscr{B}$-measurable function $g(x, s)$ and a set $S'\in\mathscr{B}$ with $\lambda(S') = 1$ such that for every $s\in S'$ $P_s 1_A(x) = g_{(s)}(x)$ on X.

Now, throughout the paper we fix a triple consisting of a σ-finite measure space $(X, \mathscr{A}, \mu)$, a probability space $(S, \mathscr{B}, \lambda)$ and a measurable family $\mathscr{P} = \{P_s\colon s\in S\}$ of sub-Markovian operators on $L_\infty(\mu)$.

1.2. Lemma. *For every $f\in L_\infty(\mu\times\lambda^*)$ there exists a $g\in L_\infty(\mu\times\lambda^*)$ and a set $S^{*\prime}\in\mathscr{B}^*$ with $\lambda^*(S^{*\prime}) = 1$ such that for each $s^* = (s_1, s_2, \ldots)\in S^{*\prime}$*

$$(1)\qquad (P_{s_1} f_{(s_2, s_3, \ldots)})(x) = g_{(s^*)}(x) \quad\text{on}\quad X.$$

The function $g(x; s^*)$ is determined uniquely mod $(\mu\times\lambda^*)$.

Proof. Without loss of generality we may assume that the measure μ is finite. First, let us consider a function $f\in L_\infty(\mu\times\lambda^*)$ of the form

$$(2)\qquad f(x; s^*) = \sum_{i,j=1}^{k} a_{ij} 1_{A_i}(x) 1_{B_j^*}(s^*),$$

where $A_1, \ldots, A_k$ $[B_1, \ldots, B_k]$ is a partition of X $[S^*]$ into mutually disjoint $\mathscr{A}$-measurable $[\mathscr{B}^*$-measurable$]$ sets and a_{ij} are some real numbers. By 1.1 there exist functions $g_i(x; s_1)\in L_\infty(\mu\times\lambda_1)$ and a set $S_1'\in\mathscr{B}_1$ with $\lambda_1(S_1') = 1$ such that for every $s_1\in S_1'$

$$(g_i)_{(s_1)}(x) = P_{s_1} 1_{A_i}(x) \quad\text{on}\quad X$$

$(i = 1, \ldots, k)$. Since for each $s^* = (s_1, s_2, \ldots)\in S_1'\times S_2\times S_3\times\ldots$, we have

$$(P_{s_1} f_{(s^*\xi)})(x) = \sum_{i,j=1}^{k} a_{ij} P_{s_1} 1_{A_i}(x) 1_{B_j^*}(s^*\xi),$$

we infer that the function

$$g(x; s^*) = \sum_{i,j=1}^{k} a_{ij} g_i(x; s_1) 1_{B_j^*}(s^*\xi)$$

satisfies (1).

Now, for an arbitrary function $f\in L_\infty(\mu\times\lambda^*)$ there exists a sequence $f_n(x; s^*)$ of functions of the form (2) such that $\lim\limits_{n\to\infty} f_n(x; s^*) = f(x; s^*)$ on $X\times S^*$ and that $|f_n(x; s^*)|\leqslant M<\infty$ on $X\times S^*$. Therefore there exists a set $S^{*\prime}\in\mathscr{B}^*$ with $\lambda^*(S^{*\prime}) = 1$ such that for every $s^*\in S^{*\prime}$

$$\lim_{n\to\infty} (f_n)_{(s^*\xi)}(x) = f_{(s^*\xi)}(x) \quad\text{on}\quad X$$

and that

$$(3)\qquad |(f_n)_{(s^*\xi)}(x)|\leqslant M \quad\text{on}\quad X.$$

Hence, for each $s^*\in S^{*\prime}$ $\lim\limits_{n\to\infty} (f_n)_{(s^*\xi)} = f_{(s^*\xi)}$ in the weak*-topology of $L_\infty(\mu)$ and, by the weak*-continuity of P_s,

$$(4)\qquad \lim_{n\to\infty} P_{s_1}(f_n)_{(s^*\xi)} = P_{s_1} f_{(s^*\xi)} \quad\text{in the weak*-topology of } L_\infty(\mu).$$

As we have seen, there exist $g_n(x; s^*) \in L_\infty(\mu\times\lambda^*)$ and a set $S^{*''} \in \mathscr{B}^*$ with $\lambda^*(S^{*''}) = 1$ such that for each $s^* = (s_1, s_2, \ldots) \in S^{*''}$ and each $n = 1, 2, \ldots$

(5) $$\big(P_{s_1}(f_n)_{(s^*\xi)}\big)(x) = (g_n)_{(s^*\xi)}(x) \quad \text{on} \quad X.$$

Moreover, since each operator $P_s \in \mathscr{P}$ is sub-Markovian, the inequality (3) implies

(6) $$|g_n(x; s^*)| \leqslant M \quad \text{on} \quad X\times S^*.$$

Now, we shall prove that the sequence g_n is fundamental in the weak*-topology of $L_\infty(\mu\times\lambda^*)$. In view of (6) it is sufficient to check that for every function $h(x; s^*) \in L_1(\mu\times\lambda^*)$ of the form $h(x; s^*) = h_1(x)h_2(s^*)$ $\lim\limits_{n\to\infty}\iint\limits_{X\times S^*} hg_n\, d(\mu\times\lambda^*)$ exists. It follows from (4) and (5) that for each $s^*\in S^{*''}$

$$\lim_{n\to\infty}\int\limits_X h_1(x)g_n(x; s^*)\,d\mu(x)$$

exists and, by (6), that

$$\Big|\int\limits_X h_1(x)g_n(x; s^*)\,d\mu(x)\Big| \leqslant M\cdot\|h_1\|_{L_1(\mu)}.$$

Therefore $\lim\limits_{n\to\infty}\iint\limits_{X\times S^*} hg_n\,d(\mu\times\lambda^*)$ exists by the Lebesgue dominated convergence theorem. Since the weak*-fundamental sequences are weak*-convergent, there exists a function $g(x; s^*)\in L_\infty(\mu\times\lambda^*)$ such that $\lim\limits_{n\to\infty} g_n = g$ in the weak*-topology of $L_\infty(\mu\times\lambda^*)$. It follows from (4) and (5) that g satisfies (1), which proves the lemma.

Now, let $f\in L_\infty(\mu\times\lambda^*)$ and let g be a function determined by Lemma 1.2. We define a linear operator U on $L_\infty(\mu\times\lambda^*)$ by

$$Uf = g.$$

It is easy to check that U is a sub-Markovian operator on $L_\infty(\mu\times\lambda^*)$. In particular, U is an adjoint to some positive contraction on $L_1(\mu\times\lambda^*)$, which will be denoted by the same letter U, but will be written to the right of its variable. Hence

$$\iint\limits_{X\times S^*} fU\cdot g\,d(\mu\times\lambda^*) = \iint\limits_{X\times S^*} f\cdot Ug\,d(\mu\times\lambda^*)$$

for $f\in L_1(\mu\times\lambda^*)$, $g\in L_\infty(\mu\times\lambda^*)$.

The following lemma, which is easy to prove, describes the action of the iterates of the operator U.

1.3. Lemma. *Let $f\in L_\infty(\mu\times\lambda^*)$. Then there exists a set $S^{*'}\in\mathscr{B}^*$ with $\lambda^*(S^{*'}) = 1$ such that for every $s^* = (s_1, s_2, \ldots)\in S^{*'}$ and every $n = 1, 2, \ldots$*

$$(U^n f)_{(s^*)}(x) = P_{s_1}P_{s_2}\ldots P_{s_n} f_{(s_{n+1}, s_{n+2}, \ldots)}(x) \quad \textit{on} \quad X.$$

Now, for every $n = 0, 1, 2, \ldots$ let $E^{\mathscr{C}_n}$ denotes a sub-Markovian operator on $L_\infty(\mu\times\lambda^*)$ which sends to every function $f\in L_\infty(\mu\times\lambda^*)$ the conditional expectation $E^{\mathscr{C}_n}f$ with respect to the sub-σ-field

$$\mathscr{C}_n = \mathscr{A}\times\mathscr{B}_1\times\ldots\times\mathscr{B}_n\times\{\emptyset, S_{n+1}\}\times\{\emptyset, S_{n+2}\}\times\ldots$$

We have

1.4. Lemma *$U^m E^{\mathscr{C}_n} = E^{\mathscr{C}_{n+m}}U^m$ for every $m, n = 0, 1, 2, \ldots$*

Proof. It is sufficient to check that $UE^{\mathscr{C}_n}f = E^{\mathscr{C}_{n+1}}Uf$ for each function $f(x; s_1, s_2, \ldots) = 1_A(x)1_{B_1}(s_1)\ldots 1_{B_k}(s_k)$, where $A\in\mathscr{A}$, $B_i\in\mathscr{B}_i$ $(i = 1, \ldots, k)$. For such f we have

$$(E^{\mathscr{C}_n}f)(x; s^*) = 1_A(x)\prod_{i=1}^{n}1_{B_i}(s_i)\prod_{j=n+1}^{k}\lambda(B_j) \quad \text{on} \quad X\times S^*,$$

hence there exists a set $S^{*'}\in\mathscr{B}^*$ with $\lambda^*(S^{*'}) = 1$ such that for each $s^* = (s_1, s_2, \ldots)\in S^{*'}$

$$(UE^{\mathscr{C}_n}f)_{(s^*)}(x) = P_{s_1}1_A(x)\prod_{i=1}^{n}1_{B_i}(s_{i+1})\prod_{j=n+1}^{k}\lambda(B_j) \quad \text{on} \quad X.$$

On the other hand, there exists a set $S^{*''}\in\mathscr{B}^*$ with $\lambda^*(S^{*''}) = 1$ such that for each $s^* = (s_1, s_2, \ldots)\in S^{*''}$

$$(Uf)_{(s^*)}(x) = P_{s_1}1_A(x)\prod_{i=1}^{k}1_{B_i}(s_{i+1}) \quad \text{on} \quad X.$$

Therefore

$$(E^{\mathscr{C}_{n+1}}Uf)_{(s^*)}(x) = P_{s_1}1_A(x)\prod_{i=1}^{n}1_{B_i}(s_{i+1})\prod_{j=n+1}^{k}\lambda(B_j) \quad \text{on} \quad X,$$

which proves the lemma.

We say that a function f defined on $X\times S^*$ *does not depend essentially on parameters* s^* if f is equivalent $\mathrm{mod}(\mu\times\lambda^*)$ to some $\mathscr{C}_0$-measurable function.

For the sake of convenience, we will denote by T the operator $E^{\mathscr{C}_0}U$ throughout the remaining part of the paper.

1.5. Lemma. *If a function $f\in L_1(\mu\times\lambda^*)$ does not depend essentially on parameters s^*, then the functions fU and fT also do not depend essentially on parameters s^* and $fU = fT$.*

Proof. A function f does not depend essentially on parameters s^*, iff $fE^{\mathscr{C}_0} = f$. If $fE^{\mathscr{C}_0} = f$, then $fE^{\mathscr{C}_1} = f$ and, by Lemma 1.4, $fUE^{\mathscr{C}_0} = fE^{\mathscr{C}_1}U = fU$. Moreover, if $fE^{\mathscr{C}_0} = f$, then $fU = fE^{\mathscr{C}_0}U = fT$, which completes the proof.

In the sequel the restriction of the operator T to $L_1(\mu)$ will be denoted by the same letter T.

1.6. Remark. All the operators defined up to now and acting on $L_1(\mu)$ or $L_1(\mu\times\lambda^*)$ as well as their adjoints acting on $L_\infty(\mu)$ or $L_\infty(\mu\times\lambda^*)$, respectively, can be uniquely extended to the operators acting on arbitrary nonnegative functions defined on X or $X\times S^*$, respectively (see e.g. [5], p. 4). These extensions will be denoted by the same letters and, as it is easily seen, the Lemmas 1.3, 1.4 and 1.5 are still valid for them.

1.7. Example. Let $\Phi=\{\varphi_s\colon s\in S\}$ be a family of transformations $\varphi_s\colon X\to X$ and suppose that Φ is measurable, i.e. the mapping $(x,s)\to x\varphi_s$ is $\mathscr{A}\times\mathscr{B}$-$\mathscr{A}$-measurable. Then the formula

$$(x;s^*)\varphi^*=(x\varphi_{s_1};s_2,s_3,\ldots)$$

defines a measurable transformation φ^* on $(X\times S^*,\mathscr{A}\times\mathscr{B}^*)$. Moreover, φ^* preserves the measure $\mu\times\lambda^*$ if and only if for every $A\in\mathscr{A}$

$$\int_S\mu(A\varphi_s^{-1})\,d\lambda(s)=\mu(A).$$

If every transformation $\varphi_s\in\Phi$ is non-singular, then the family Φ determines a measurable family of sub-Markovian operators $\mathscr{P}=\{P_s\colon s\in S\}$ by the formula

$$(P_sf)(x)=f(x\varphi_s)\quad\text{for}\quad f\in L_\infty(\mu).$$

The operator U is now given by $(Uf)(x;s^*)=f\big((x;s^*)\varphi^*\big)$ and its iterates by

$$(U^nf)(x;s^*)=f(x\varphi_{s_1}\varphi_{s_2}\ldots\varphi_{s_n};s_{n+1},\ldots).$$

2. The conservative part and invariant functions.

Let $f_0\in L_1(\mu)$ be a function such that $\{f_0>0\}=X$ and let

$$C_T=\Big\{x\in X\colon\sum_{k=0}^{\infty}(f_0T^k)(x)=\infty\Big\}.$$

The set C_T does not depend on the choice of $f_0\in L_1(\mu)$ with the property $\{f_0>0\}=X$ and is called the *conservative part* of the sub-Markovian operator T (see [5], Ch. II, 2.2). The set $D_T=X\setminus C_T$ is called the *dissipative part* of T. The function f_0 considered as a function of the variables $(x;s^*)$ has the following properties: $f_0\in L_1(\mu\times\lambda^*)$ and $\{f_0>0\}=X\times S^*$. Therefore the set

$$C_U=\Big\{(x;s^*)\in X\times S^*\colon\sum_{k=0}^{\infty}(f_0U^k)(x;s^*)=\infty\Big\}$$

is the conservative part of the sub-Markovian operator U. By Lemma 1.5 we have $f_0T^n=f_0U^n$ for $n=0,1,2,\ldots$ Hence

2.1. Theorem. *$C_U=C_T\times S^*$ on $X\times S^*$.*

A function f ($f\geqslant 0$ or $f\in L_\infty(\mu)$) defined on X is called *harmonic* on a set $C\in\mathscr{A}$ if $Tf=f$ on C. We say that a function f is *$\mathscr{P}$-invariant* on C if there exists a set $S'\in\mathscr{B}$ with $\lambda(S')=1$ such that for each $s\in S'$ $P_sf=f$ on C. Denote

$$\mathscr{I}_T=\{A\in\mathscr{A}\colon A\subseteq C_T\text{ and }T1_A=1_A\text{ on }C_T\}$$

and

$$\mathscr{I}_{\mathscr{P}}=\{A\in\mathscr{A}\colon A\subseteq C_T\text{ and }1_A\text{ is }\mathscr{P}\text{-invariant on }C_T\}.$$

2.2. Lemma. *Each function f ($f\geqslant 0$ or $f\in L_\infty(\mu)$) defined on X and harmonic on C_T is $\mathscr{P}$-invariant on C_T. In particular, $\mathscr{I}_T=\mathscr{I}_{\mathscr{P}}$.*

Proof. By ([13], Proposition V.5.2) if $0\leqslant f<\infty$, then the equality $Tf=f$ on C_T is equivalent to the $\mathscr{I}_T$-measurability of f on C_T. Moreover, proceeding as in the proof of Theorem A in ([5], Ch. III), we observe that the assumption of the finiteness of f is not necessary here. Therefore it is sufficient to prove the lemma for the functions of the form $f=1_A$, where $A\in\mathscr{A}$.

Let $T1_A=1_A$ on C_T. By Theorem 2.1 this means that

$$E^{\mathscr{C}_0}U1_A=1_A\quad\text{on}\quad C_U.$$

Multiplying the above equality by the $\mathscr{C}_0$-measurable function $1_{A^c}1_{C_U}$ ($A^c=X\setminus A$) we obtain

$$E^{\mathscr{C}_0}(U1_A\cdot 1_{A^c}1_{C_U})=0\quad\text{on}\quad X\times S^*,$$

so $U1_A\cdot 1_{A^c}=0$ on C_U. Since $U1_A\leqslant 1$ on $X\times S^*$, $U1_A\leqslant 1_A$ on C_U. By ([13], Proposition V.5.2), $U1_A=1_A$ on C_U, which proves the lemma.

The following example shows that in the above lemma C_T cannot be replaced by X, even if the family $\mathscr{P}$ is determined by a family Φ of measure preserving transformations.

2.3. Example. Let X be a free group with two free generators a and b and let μ be a counting measure on $(X,\mathscr{A})$, where $\mathscr{A}=2^X$. Let $S=\{a,b\}$ and $\lambda(\{a\})=\lambda(\{b\})=1/2$. For each $s\in S$ let $\varphi_s\colon X\to X$ be the right translation of X. Let $\mathscr{P}=\{P_s\colon s\in S\}$ be determined by the family of transformations $\Phi=\{\varphi_s\colon s\in S\}$ as in Example 1.7.

For each $u\in[0,1)$ and a fixed $d\in(0,1)$, $d\neq 1/2$ put

$$u\psi_a=\begin{cases}u/(2-2d) & \text{for}\quad 0\leqslant u<1-d,\\ (u+2d-1)/2d & \text{for}\quad 1-d\leqslant u<1,\end{cases}$$

$$u\psi_b=\begin{cases}u/2d & \text{for}\quad 0\leqslant u<d,\\ (u+1-2d)/(2-2d) & \text{for}\quad d\leqslant u<1.\end{cases}$$

Since ψ_a and ψ_b are the invertible transformations on $[0,1)$, we can define $\psi_{a^{-1}}=\psi_a^{-1}$, $\psi_{b^{-1}}=\psi_b^{-1}$.

Now, each element $x\in X$ can be written in the form $x=t_1t_2\ldots t_n$, where $t_i\in\{a,b,a^{-1},b^{-1}\}$. Denote $A=[1/2,1)$ and put

$$f(x)=f(t_1t_2\ldots t_n)=m(A\psi_{t_1}^{-1}\psi_{t_2}^{-1}\ldots\psi_{t_n}^{-1})$$

if $x = t_1 t_2 \dots t_n \in X$. Here m denotes the Lebesgue measure on $[0, 1)$. The function f does not depend on the representation of an element $x \in X$ and has the following properties:

1° $0 \leqslant f(x) \leqslant 1$ for each $x \in X$,

2° f is harmonic since for each $x = t_1 t_2 \dots t_n \in X$ we have

$$\tfrac{1}{2} f(x\varphi_a) + \tfrac{1}{2} f(x\varphi_b) = \tfrac{1}{2} m(A\psi_{t_1}^{-1} \dots \psi_{t_n}^{-1}\psi_a^{-1}) + \tfrac{1}{2} m(A\psi_{t_1}^{-1} \dots \psi_{t_n}^{-1}\psi_b^{-1}) = m(A\psi_{t_1}^{-1} \dots \psi_{t_n}^{-1}) = f(x),$$

3° f is not $\mathscr{P}$-invariant since for $x = e$ (the identity in X) we have

$$f(e) = f(aa^{-1}) = m(A\psi_a^{-1}\psi_{a^{-1}}^{-1}) = m(A) = 1/2$$

and

$$f(e\varphi_a) = f(a) = m(A\psi_a^{-1}) = d \neq 1/2.$$

We say that a function h ($h \geqslant 0$ or $h \in L_\infty(\mu \times \lambda^*)$) defined on $X \times S^*$ is *U-invariant* on a set $C \in \mathscr{A} \times \mathscr{B}^*$ if $Uh = h$ on C. Let

$$\mathscr{I}_U = \{D \in \mathscr{A} \times \mathscr{B}^* \colon D \subseteq C_U \text{ and } U1_D = 1_D \text{ on } C_U\}.$$

2.4. THEOREM. *Each function h ($h \geqslant 0$ or $h \in L_\infty(\mu \times \lambda^*)$) defined on $X \times S^*$ and U-invariant on C_U does not depend essentially on parameters s^* on C_U. In particular, $\mathscr{I}_U = \mathscr{I}_T \times \{\emptyset, S^*\} \bmod (\mu \times \lambda^*)$.*

Proof. Without loss of generality we may assume that the measure μ is finite (observe that the sets C_T and C_U do not change if we take a measure m equivalent to μ). Moreover, it is sufficient to prove the theorem for the functions $h \in L_\infty(\mu \times \lambda^*)$.

Let a function $h \in L_\infty(\mu \times \lambda^*)$ satisfy $Uh = h$ on C_U. Then, using ([5], Ch. II, (2.7)), we obtain for every $m = 1, 2, \dots$

(1) $$U^m h = h \quad \text{on} \quad C_U.$$

Using Lemma 1.4, (1), and Theorem 2.1 we have for $g = E^{\mathscr{C}_0} h$:

$$1_{C_T} \cdot Tg = 1_{C_T} \cdot E^{\mathscr{C}_0} U E^{\mathscr{C}_0} h = 1_{C_T} \cdot E^{\mathscr{C}_0} E^{\mathscr{C}_1} Uh = 1_{C_T} \cdot E^{\mathscr{C}_0} Uh = E^{\mathscr{C}_0}(1_{C_U} \cdot Uh) = E^{\mathscr{C}_0}(1_{C_U} h) = 1_{C_T} g \quad \text{on} \quad X \times S^*,$$

i.e. $Tg = g$ on C_T. By Lemma 2.2 $Ug = g$ on C_U, hence for each $m = 1, 2, \dots$

(2) $$U^m g = g \quad \text{on} \quad C_U.$$

On the other hand, using Lemma 1.4, (1), and Theorem 2.1 we have for $m = 0, 1, 2, \dots$

$$1_{C_U} \cdot U^m g = 1_{C_U} \cdot U^m E^{\mathscr{C}_0} h = 1_{C_U} \cdot E^{\mathscr{C}_m} U^m h = E^{\mathscr{C}_m}(1_{C_U} \cdot U^m h) = E^{\mathscr{C}_m}(1_{C_U} h) = 1_{C_U} \cdot E^{\mathscr{C}_m} h \quad \text{on} \quad X \times S^*,$$

i.e. $U^m g = E^{\mathscr{C}_m} h$ on C_U. By the martingale convergence theorem

$\lim_{m\to\infty} E^{\mathscr{C}_m} h = h$ on $X \times S^*$ (see [12], Corollary II-2-12). Therefore

(3) $$\lim_{m\to\infty} U^m g = h \quad \text{on} \quad C_U.$$

Comparing (2) and (3) we have $h = g = E^{\mathscr{C}_0} h$ on C_U.

In particular, taking $h = 1_D$ ($D \in \mathscr{A} \times \mathscr{B}^*$) we obtain the equality $\mathscr{I}_U = \mathscr{I}_T \times \{\emptyset, S^*\} \bmod (\mu \times \lambda^*)$, which completes the proof.

In the above theorem C_U cannot be replaced by $X \times S^*$. This is the consequence of Example 2.3 and of the following

2.5. PROPOSITION. *The following conditions are equivalent:*

(i) *each harmonic function $f \in L_\infty(\mu)$ is $\mathscr{P}$-invariant,*

(ii) *each U-invariant function $h \in L_\infty(\mu \times \lambda^*)$ does not depend essentially on parameters s^*.*

Proof. Without loss of generality we may assume that the measure μ is finite.

To obtain the implication (i)$\Rightarrow$(ii) one proceeds as in the proof of Theorem 2.4, replacing C_U and C_T by $X \times S^*$ and X, respectively, and using the assumption (i) instead of Lemma 2.2.

For the proof of the converse implication assume that $Tf = f$ for a $f \in L_\infty(\mu)$. Using Lemma 1.4 we obtain for $m = 0, 1, 2, \dots$

$$E^{\mathscr{C}_m} U^{m+1} f = E^{\mathscr{C}_m} U^m Uf = U^m E^{\mathscr{C}_0} Uf = U^m Tf = U^m f \quad \text{on} \quad X \times S^*,$$

which means that the sequence $U^m f$ forms a martingale with respect to the increasing sequence of sub-σ-fields $\mathscr{C}_m$. Moreover, the functions $U^m f$ are uniformly bounded in $L_\infty(\mu \times \lambda^*)$, and since the measure μ is finite, also in $L_1(\mu \times \lambda^*)$. By the martingale convergence theorem there exists a function $h \in L_\infty(\mu \times \lambda^*)$ such that

(4) $$\lim_{m\to\infty} U^m f = h \quad \text{on} \quad X \times S^*$$

(see [12], Theorem IV-1-2). Moreover, the convergence in (4) holds in the norm topology of $L_1(\mu \times \lambda^*)$ and in the weak*-topology of $L_\infty(\mu \times \lambda^*)$. This implies that the martingale $U^m f$ is regular (see [12], Proposition IV-2-3), hence

(5) $$U^m f = E^{\mathscr{C}_m} h \quad \text{for} \quad m = 0, 1, 2, \dots$$

and, by the weak*-continuity of the operator U, that $Uh = h$. The assumption (ii) implies that $h = E^{\mathscr{C}_0} h$ and from (5) applied to $m = 0$ we obtain $f = E^{\mathscr{C}_0} h = h$. Therefore the function f is $\mathscr{P}$-invariant, which completes the proof.

Now, we turn to the investigation of functions h defined on $X \times S^*$ and satisfying the equation $hU = h$.

The following result generalizes the first part of Satz 4 in [11].

2.6. THEOREM. *Each function $h \in L_1(\mu \times \lambda^*)$ satisfying the equation $hU = h$ does not depend essentially on parameters s^*.*

Proof. If $h \in L_1(\mu \times \lambda^*)$ and $hU = h$, then, using Lemma 1.4, we have

$$hE^{\mathscr{C}_0} - h = hU^m E^{\mathscr{C}_0} - hU^m = (hE^{\mathscr{C}_m} - h)\, U^m.$$

Since for each $m = 1, 2, \ldots$ U^m is a contraction on $L_1(\mu \times \lambda^*)$, hence

$$\|hE^{\mathscr{C}_0} - h\|_{L_1(\mu\times\lambda^*)} \leqslant \|hE^{\mathscr{C}_m} - h\|_{L_1(\mu\times\lambda^*)}.$$

By the $L_1(\mu)$-valued martingale convergence theorem, $\lim\limits_{m\to\infty} hE^{\mathscr{C}_m} = h$ in the norm of $L_1(\mu \times \lambda^*)$ (see [3], Theorem 2.1 (a)). Therefore $hE^{\mathscr{C}_0} = h$, which completes the proof.

2.7. Remark. For each finite signed measure ϱ defined on $\mathscr{A} \times \mathscr{B}^*$ and absolutely continuous with respect to $\mu \times \lambda^*$ ($\varrho \ll \mu \times \lambda^*$) the formula

$$(\varrho U)(D) = \iint\limits_{X\times S^*} U 1_D \, d\varrho \qquad (D \in \mathscr{A} \times \mathscr{B}^*)$$

defines a finite signed measure $\varrho U \ll \mu \times \lambda^*$. Moreover, if $h \in L_1(\mu \times \lambda^*)$ and $\varrho(D) = \iint\limits_D h\, d(\mu \times \lambda^*)$, then $hU = d(\varrho U)/d(\mu \times \lambda^*)$ on $X \times S^*$, where $d(\varrho U)/d(\mu \times \lambda^*)$ denotes the Radon-Nikodym derivative.

The action of the operator T on a measure ν defined on $\mathscr{A}$, $\nu \ll \mu$, is defined similarly.

Theorem 2.6 formulated in the language of measures says that each finite signed measure ϱ defined on $\mathscr{A} \times \mathscr{B}^*$, $\varrho \ll \mu \times \lambda^*$ and invariant for the operator U (i.e. satisfying the equation $\varrho U = \varrho$) is of the form $\varrho = \nu \times \lambda^*$ for a finite signed measure ν on $\mathscr{A}$. Moreover, we have $\nu \ll \mu$ and $\nu T = \nu$, which follows from Lemma 1.5.

2.8. THEOREM. *Suppose that μ is σ-finite and subinvariant for the operator T (i.e. $1T \leqslant 1$ on X). Then each function $h \in L_p(\mu \times \lambda^*)$ with $1 \leqslant p < \infty$ satisfying the equation $hU = h$ does not depend essentially on parameters s^*.*

Proof. If $1T \leqslant 1$ on X, then $1U \leqslant 1$ on $X \times S^*$ by Lemma 1.5. Therefore the operator U, which is a contraction on $L_1(\mu \times \lambda^*)$, is simultaneously a contraction on $L_\infty(\mu \times \lambda^*)$. By the interpolation theorem (see [4], Theorem VI.10.11) U is a contraction on each $L_p(\mu \times \lambda^*)$ with $1 \leqslant p \leqslant \infty$.

Now, let $hU = h$ for a $h \in L_p(\mu \times \lambda^*)$, $1 \leqslant p < \infty$. Analogously as in the proof of Theorem 2.6 we obtain

$$\|hE^{\mathscr{C}_0} - h\|_{L_p(\mu\times\lambda^*)} \leqslant \|hE^{\mathscr{C}_m} - h\|_{L_p(\mu\times\lambda^*)}.$$

By the $L_p(\mu)$-valued martingale convergence theorem, $\lim\limits_{m\to\infty} hE^{\mathscr{C}_m} = h$

in the norm of $L_p(\mu \times \lambda^*)$ (see [3], Theorem 2.1 (a)). Therefore $hE^{\mathscr{C}_0} = h$, which completes the proof.

2.9. Remark. Theorem 2.6 is not true if h is not integrable (and only positive) and Theorem 2.8 is false for $p = \infty$. To see this let us consider the family $\mathscr{P} = \{P_a, P_b\}$ defined in Example 2.3. As we have seen, there exists a function $h_0 \in L_\infty^+(\mu \times \lambda^*)$ such that $Uh_0 = h_0$ and h_0 depends essentially on parameters s^*. Since the measure $\mu \times \lambda^*$ in Example 2.3 is φ^*-invariant, it follows that the measure $d\varrho = h_0\, d(\mu \times \lambda^*)$ is also φ^*-invariant and so $h_0 U = h_0$.

3. Ergodic theorems.

In general the restriction of the σ-finite measure μ to the σ-field $\mathscr{I}_T$ of subsets of C_T is not σ-finite. However

3.1. LEMMA (see [5], Ch. III, (3.7)). *The conservative part of the operator T decomposes uniquely* mod μ *into a disjoint union*

$$C_T = C_T^1 \cup C_T^2,$$

where $C_T^1 \in \mathscr{I}_T$, $C_T^1 = \bigcup\limits_{n=1}^{\infty} A_n$ for an increasing sequence of sets $A_n \in \mathscr{I}_T$ with $\mu(A_n) < \infty$, and $\mu(A) = 0$ or ∞ *for every set $A \in \mathscr{I}_T$, $A \subseteq C_T^2$. Let $\mathscr{I}_T^1$ be the restriction of the σ-field $\mathscr{I}_T$ to C_T^1. Then $(C_T^1, \mathscr{I}_T^1, \mu)$ is a σ-finite measure space.*

Given a function $f \in L_1(\mu)$ supported on C_T^1, $E^{\mathscr{I}_T^1} f$ will denote the unique $\mathscr{I}_T^1$-measurable function defined on C_T^1 and satisfying for every $A \in \mathscr{I}_T^1$ the equation $\int\limits_A f\, d\mu = \int\limits_A E^{\mathscr{I}_T^1} f\, d\mu$.

Analogously, we have

3.2. LEMMA. *The conservative part of the operator U decomposes uniquely* mod$(\mu \times \lambda^*)$ *into a disjoint union*

$$C_U = C_U^1 \cup C_U^2,$$

where $C_U^1 \in \mathscr{I}_U$, $C_U^1 = \bigcup\limits_{n=1}^{\infty} D_n$ for an increasing sequence of sets $D_n \in \mathscr{I}_U$ with $(\mu \times \lambda^*)(D_n) < \infty$, and $(\mu \times \lambda^*)(D) = 0$ or ∞ for every set $D \in \mathscr{I}_U$, $D \subseteq C_U^2$. Let $\mathscr{I}_U^1$ be the restriction of the σ-field $\mathscr{I}_U$ to C_U^1. Then $(C_U^1, \mathscr{I}_U^1, \mu \times \lambda^*)$ is a σ-finite measure space.

Given a function $h \in L_1(\mu \times \lambda^*)$ supported on C_U^1, $E^{\mathscr{I}_U^1} h$ denotes the unique $\mathscr{I}_U^1$-measurable function defined on C_U^1 and satisfying for every $D \in \mathscr{I}_U^1$ the equation

$$\iint\limits_D h\, d(\mu \times \lambda^*) = \iint\limits_D E^{\mathscr{I}_U^1} h\, d(\mu \times \lambda^*).$$

3.3. THEOREM. (1) $C_U^1 = C_T^1 \times S^*$ *on* $X \times S^*$,

(2) $C_U^2 = C_T^2 \times S^*$ *on* $X \times S^*$,

(3) $\mathscr{I}_U^1 = \mathscr{I}_T^1 \times \{\emptyset, S^*\} \bmod (\mu \times \lambda^*)$.

Proof. Let A_n be the sets appearing in Lemma 3.1. We have $A_n \times S^* \in \mathscr{I}_T \times \{\emptyset, S^*\}$ and $\mathscr{I}_T \times \{\emptyset, S^*\} = \mathscr{I}_U \bmod (\mu \times \lambda^*)$, by Theorem 2.4. Moreover, $(\mu \times \lambda^*)(A_n \times S^*) = \mu(A_n) < \infty$. Therefore

$$C_T^1 \times S^* = \bigcup_{n=1}^{\infty} A_n \times S^* \subseteq C_U^1 \quad \text{on} \quad X \times S^*.$$

On the other hand, if D_n are the sets appearing in Lemma 3.2, then $D_n \in \mathscr{I}_U = \mathscr{I}_T \times \{\emptyset, S^*\} \bmod (\mu \times \lambda^*)$, by Theorem 2.4. Therefore $D_n = B_n \times S^*$ on $X \times S^*$ for an increasing sequence of sets $B_n \in \mathscr{I}_T$. Moreover, $\mu(B_n) = (\mu \times \lambda^*)(D_n) < \infty$. Hence

$$C_U^1 = \bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} B_n \times S^* \subseteq C_T^1 \times S^* \quad \text{on} \quad X \times S^*,$$

which proves (1). Furthermore, (2) follows from (1) and from Theorem 2.1, and (3) follows from (1) and from Theorem 2.4, which completes the proof.

The following theorem generalizes the random ergodic theorem of Ryll-Nardzewski ([15], Theorem 1) and of Nawrotzki ([11], Satz 4). In the formulation of this theorem the symbol P_{s_k} denotes the extension of the operator $P_{s_k} \in \mathscr{P}$ to the class of all nonnegative functions defined on X (see Remark 1.6).

3.4. THEOREM. *Suppose that μ is σ-finite and subinvariant for the operator T. Then for each function $f(x; s^*) \in L_p^+(\mu \times \lambda^*)$ with $1 \leqslant p < \infty$ there exist a function $f^*(x) \in L_p^+(\mu)$ and a set $S^{*\prime} \in \mathscr{B}^*$ with $\lambda^*(S^{*\prime}) = 1$ such that for each $s^* = (s_1, s_2, \ldots) \in S^{*\prime}$*

$$\text{(4)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_{s_1} P_{s_2} \ldots P_{s_k} f_{(s_{k+1}, \ldots)}(x) = f^*(x) \quad \text{on} \quad X,$$

and if $p > 1$,

$$\text{(5)} \qquad \lim_{n\to\infty} \left(\int_X \left| \frac{1}{n} \sum_{k=0}^{n-1} P_{s_1} P_{s_2} \ldots P_{s_k} f_{(s_{k+1}, \ldots)}(x) - f^*(x) \right|^p d\mu(x) \right)^{1/p} = 0.$$

Moreover, the limit function satisfies

$$\text{(6)} \qquad f^* = \begin{cases} 0 & \text{on} \quad (D_T \cup C_T^2) \times S^*, \\ E^{\mathscr{I}_T^1 \times \{\emptyset, S^*\}}(1_{C_T^1} f) & \text{on} \quad C_T^1 \times S^*. \end{cases}$$

Proof. From the assumption that $1T \leqslant 1$ on X and from Lemma 1.5 it follows that $1U \leqslant 1$ on $X \times S^*$. By the pointwise ergodic theorem applied to the operator U it follows that for each function $f \in L_p^+(\mu \times \lambda^*)$ there

exists a function $f^* \in L_p^+(\mu \times \lambda^*)$ such that

$$\text{(7)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f = f^* \quad \text{on} \quad X \times S^*$$

and

$$\text{(8)} \qquad f^* = \begin{cases} 0 & \text{on} \quad D_U \cup C_U^2, \\ E^{\mathscr{I}_U^1}(1_{C_U^1} f) & \text{on} \quad C_U^1 \end{cases}$$

(for $f \in L_1^+(\mu \times \lambda^*)$—see [5], Ch. VII, Theorem B; for $f \in L_p^+(\mu \times \lambda^*)$ (7) holds by [4], Theorem VIII.6.6 and (8) follows from the just mentioned theorems).

Using Theorem 2.1, Lemma 3.3 and (8) we conclude that the limit function f^* does not depend essentially on parameters s^* and has the form (6). (The $\mathscr{C}_0$-measurability of f^* can also be deduced from Theorem 2.8.)

By the Fubini theorem, Lemma 1.3 and (7) it follows that (4) holds for λ^*-almost all s^*. Finally, (5) follows from (4), the Fubini theorem and the fact that for $f \in L_p^+(\mu \times \lambda^*)$ with $1 < p < \infty$ we have

$$\sup_{n=1,2,\ldots} \frac{1}{n} \sum_{k=0}^{n-1} U^k f \in L_p^+(\mu \times \lambda^*)$$

(see [4], Theorem VIII.6.8). The proof of Theorem 3.4 has hereby been completed.

The following result is analogous to Theorem 2.8.

3.5. COROLLARY. *Suppose that μ is σ-finite and subinvariant for the operator T and $1 \leqslant p < \infty$. Then*

(i) *each U-invariant function $h \in L_p^+(\mu \times \lambda^*)$ does not depend essentially on parameters s^* and*

(ii) *each harmonic function $f \in L_p^+(\mu)$ is $\mathscr{P}$-invariant.*

Proof. (i) follows from Theorem 3.4. Now, if $f \in L_p^+(\mu)$ for $1 \leqslant p < \infty$ and $Tf = f$, then

$$f = \begin{cases} 0 & \text{on} \quad D_T \cup C_T^2, \\ E^{\mathscr{I}_T^1}(1_{C_T^1} f) & \text{on} \quad C_T^1 \end{cases}$$

(see [5], Ch. VII, Theorem B). Therefore the $\mathscr{P}$-invariance of f follows from the formula (6) in Theorem 3.4 and from the fact that the limit function f^* in Theorem 3.4 is $\mathscr{P}$-invariant.

The part (i) of the above corollary can also be deduced directly from Theorem 2.8.

Observe that by Lemma 1.2, Lemma 1.5 and by the definitions of the operators U and T it follows that

$$\int_X fT\cdot g\,d\mu = \int_S\int_X fP_s\cdot g\,d\mu\,d\lambda(s)$$

for every $f \in L_1(\mu)$, $g \in L_\infty(\mu)$. Hence, if $A \in \mathscr{A}$, then

$$\mu T(A) = \int_X 1T\cdot 1_A\,d\mu = \int_S\int_X 1P_s\cdot 1_A\,d\mu\,d\lambda(s) = \int_S \mu P_s(A)\,d\lambda(s).$$

In particular, the measure μ is invariant for the operator T if for every $A \in \mathscr{A}$ we have

$$\int_S \mu P_s(A)\,d\lambda(s) = \mu(A).$$

Let us introduce a stronger condition. Namely, we say that μ is *$\mathscr{P}$-invariant* if for every $A \in \mathscr{A}$ we have

$$\mu P_s(A) = \mu(A) \quad \lambda\text{-almost everywhere.}$$

Now, we shall formulate a particular case of Theorem 3.4, when the measure μ is finite and T-invariant and $p = 1$.

3.6. THEOREM. *Suppose that μ is finite and invariant for the operator T. Then for each function $f(x; s^*) \in L_1^+(\mu\times\lambda^*)$ there exist a function $f^*(x) \in L_1^+(\mu)$ and a set $S^{*\prime} \in \mathscr{B}^*$ with $\lambda^*(S^{*\prime}) = 1$ such that for every $s^* = (s_1, s_2, \ldots) \in S^{*\prime}$*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} P_{s_1}P_{s_2}\ldots P_{s_k} f_{(s_{k+1},\ldots)}(x) = f^*(x) \quad \textit{on} \quad X, \tag{9}$$

and if

$$\int\int_{X\times S^*} f\log^+ f\,d(\mu\times\lambda^*) < \infty \tag{10}$$

(where $\log^+ a = \max(0, \log a)$) or if μ is $\mathscr{P}$-invariant, then

$$\lim_{n\to\infty}\int_X\left|\frac{1}{n}\sum_{k=0}^{n-1} P_{s_1}P_{s_2}\ldots P_{s_k} f_{(s_{k+1},\ldots)}(x) - f^*(x)\right| d\mu(x) = 0. \tag{11}$$

Moreover, the limit function satisfies

$$f^* = E^{\mathscr{I}_T\times\{\emptyset, S^*\}} f \quad \textit{on} \quad X\times S^*. \tag{12}$$

Proof. The convergence of the ergodic averages (9) was proved in Theorem 3.4. Since μ is finite and invariant for T, so $C_T^1 = C_T = X$ and formula (12) follows from (6). If f satisfies (10), then

$$\sup_{n=1,2,\ldots}\frac{1}{n}\sum_{k=0}^{n-1} U^k f \in L_1^+(\mu\times\lambda^*)$$

(see [4], Theorem VIII.6.8) which, analogously as in the proof of (5) in Theorem 3.4, gives (11).

Now, suppose that μ is $\mathscr{P}$-invariant. Then, as it is easily seen, we have $E^{\mathscr{E}}Uf = UE^{\mathscr{E}}f$ for each nonnegative function f, where $\mathscr{E} = \{\emptyset, X\}\times\mathscr{B}^*$. Therefore, for each $f \in L_1^+(\mu\times\lambda^*)$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} E^{\mathscr{E}}U^k f = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} U^k E^{\mathscr{E}} f = E^{\mathscr{I}_T\times\{\emptyset,S^*\}}E^{\mathscr{E}}f = E^{\mathscr{E}}f^*$$

on $X\times S^*$. Consequently, there exists a set $S^{*\prime\prime} \in \mathscr{B}^*$ with $\lambda^*(S^{*\prime\prime}) = 1$ such that for each $s^* = (s_1, s_2, \ldots) \in S^{*\prime\prime}$

$$\lim_{n\to\infty}\int_X\frac{1}{n}\sum_{k=0}^{n-1} P_{s_1}P_{s_2}\ldots P_{s_k} f_{(s_{k+1},\ldots)}(x)\,d\mu(x) = \int_X f^*(x)\,d\mu(x). \tag{13}$$

The formulas (9) and (13) imply (11), which completes the proof.

3.7. Remark. The additional assumption (10) in the above theorem is essential even if f does not depend essentially on parameters s^*. A suitable example will be given in Section 4.

Applying Theorem 3.6 and the Banach–Steinhaus theorem we obtain

3.8. COROLLARY. *Suppose that μ is finite and $\mathscr{P}$-invariant and let the σ-field $\mathscr{A}$ be countably generated. Then there exists a set $S^{*\prime} \in \mathscr{B}^*$ with $\lambda^*(S^{*\prime}) = 1$ such that for every $s^* = (s_1, s_2, \ldots) \in S^{*\prime}$ and for every $f \in L_1(\mu)$ we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} P_{s_1}P_{s_2}\ldots P_{s_k} f = E^{\mathscr{I}_T} f \quad \textit{in the norm of } L_1(\mu) \tag{14}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} fP_{s_1}P_{s_2}\ldots P_{s_k} = E^{\mathscr{I}_T} f \tag{15}$$

in the weak topology of $L_1(\mu)$.

The following example shows that the above corollary is false if the assumption of the $\mathscr{P}$-invariance of μ is replaced by the T-invariance.

3.9. EXAMPLE. Let $(X, \mathscr{A}, \mu)$ be the interval $[0, 1)$ with the Borel σ-field $\mathscr{A}$ and the Lebesgue measure μ and let $S = \{a, b\}$, $\lambda(\{a\}) = \lambda(\{b\}) = 1/2$. Let φ_a and φ_b denote the transformations on X given by $x\varphi_a = x/2$ and $x\varphi_b = x/2 + 1/2$ for $x \in X$ and let $\mathscr{P} = \{P_a, P_b\}$ be determined by the family $\Phi = \{\varphi_a, \varphi_b\}$ as in Example 1.7. Obviously, the measure μ is T-invariant. Moreover, as it is easily seen, for each $s^* = (s_1, s_2, \ldots) \in S^*$ we have

$$\lim_{n\to\infty} 1P_{s_1}P_{s_2}\ldots P_{s_n}(x) = 0 \quad \text{on} \quad X,$$

which implies that in (14) the convergence does not hold even in the weak topology of $L_1(\mu)$ and that (15) is false even for a fixed function $f \equiv 1$.

Using the random ergodic theorem we can give another proof of the following well-known result (see [10], Ch. VIII, Theorem 7 and [14], Ch. 5, Lemma 1).

3.10. Corollary. *Suppose that μ is finite and invariant for the operator T. If $\mathscr{P}$ is a commutative family of operators, i.e. if*

$$(16) \qquad P_s P_t = P_t P_s \quad \text{for all} \quad s, t \in S,$$

then μ is $\mathscr{P}$-invariant.

Proof. Fix a set $A \in \mathscr{A}$. The assumption (16) implies that for every $s^* = (s_1, s_2, \ldots) \in S^*$ and every $n = 0, 1, 2, \ldots$ we have

$$(17) \qquad \langle 1P_{s_1}P_{s_2}\ldots P_{s_n}, 1_A\rangle = \langle 1P_{s_n}P_{s_{n-1}}\ldots P_{s_1}, 1_A\rangle,$$

where $\langle f, g\rangle = \int_X fg\,d\mu$ for $f \in L_1(\mu)$, $g \in L_\infty(\mu)$. Since $1T = 1$, Theorem 3.6 implies that there exists a set $S^{*\prime} \in \mathscr{B}^*$ with $\lambda^*(S^{*\prime}) = 1$ such that

$$(18) \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\langle 1P_{s_1}P_{s_2}\ldots P_{s_k}, 1_A\rangle = \langle 1, 1_A\rangle$$

for every $s^* = (s_1, s_2, \ldots) \in S^{*\prime}$.

On the other hand, the equation $1T = 1$ means that the sequence $\langle 1P_{s_n}P_{s_{n-1}}\ldots P_{s_1}, 1_A\rangle$ forms a martingale with respect to the increasing sequence of σ-fields $\mathscr{B}_1 \times \mathscr{B}_2 \times \ldots \times \mathscr{B}_n \times \{\emptyset, S_{n+1}\} \times \{\emptyset, S_{n+2}\} \times \ldots$ Moreover, this martingale is uniformly bounded in $L_\infty(\lambda^*)$. By the martingale convergence theorem there exists a function $h_A \in L_1^+(\lambda^*)$ and a set $S^{*\prime\prime} \in \mathscr{B}^*$ with $\lambda^*(S^{*\prime\prime}) = 1$ such that for every $s^* = (s_1, s_2, \ldots) \in S^{*\prime\prime}$

$$(19) \qquad \lim_{n\to\infty}\langle 1P_{s_n}P_{s_{n-1}}\ldots P_{s_1}, 1_A\rangle = h_A(s_1, s_2, \ldots)$$

and

$$(20) \qquad \langle 1P_{s_n}P_{s_{n-1}}\ldots P_{s_1}, 1_A\rangle = E^{\mathscr{C}_n}h_A(s_1, s_2, \ldots)$$

for $n = 0, 1, 2, \ldots$ (see [12], Theorem IV-1-2 and Proposition IV-2-3).

Using (17), (18) and (19) we obtain $h_A = \text{const} = \langle 1, 1_A\rangle$ λ^*-a.e. By (20) $\langle 1P_{s_1}, 1_A\rangle = \langle 1, 1_A\rangle$ for λ_1-almost all $s_1 \in S_1$, which completes the proof.

4. Example. In this section we shall construct a probability space $(X, \mathscr{A}, \mu)$, a measurable family $\mathscr{P} = \{P_s\colon s \in S\}$ of sub-Markovian operators on $L_\infty(\mu)$ with $\mu T = \mu$ and a function $f \in L_1^+(\mu)$ such that

$$(1) \qquad \limsup_{n\to\infty} E^{\mathscr{E}}\left(\frac{1}{n}U^n f\right) = +\infty \quad \mu\times\lambda^*\text{-a.e.},$$

where $\mathscr{E} = \{\emptyset, X\} \times \mathscr{B}^*$. This gives

$$\limsup_{n\to\infty} E^{\mathscr{E}}\left(\frac{1}{n}\sum_{k=0}^{n-1}U^k f\right) = +\infty \quad \mu\times\lambda^*\text{-a.e.},$$

and shows that the additional assumption (10) in Theorem 3.6 is essential.

Let $(S, \mathscr{B}, \lambda)$ be a nontrivial probability space and let

$$X = \prod_{i=-\infty}^{0} S_i, \quad \mathscr{A} = \prod_{i=-\infty}^{0}\mathscr{B}_i, \quad \mu = \prod_{i=-\infty}^{0}\lambda_i,$$

where $S_i = S$, $\mathscr{B}_i = \mathscr{B}$, $\lambda_i = \lambda$ for $i = 0, -1, -2, \ldots$ Let φ_s be a transformation on X given by

$$(\ldots, s_{-2}, s_{-1}, s_0)\varphi_s = (\ldots, s_{-1}, s_0, s),$$

and let $\mathscr{P} = \{P_s\colon s \in S\}$ be determined by the family of transformations $\Phi = \{\varphi_s\colon s \in S\}$ as in Example 1.7. (Note that if $\mathscr{B}$ contains all the sets of the form $\{s\}$, where $s \in S$ and if $\lambda(\{s\}) > 0$ for every $s \in S$, then each $\varphi_s \in \Phi$ is a nonsingular transformation on $(X, \mathscr{A}, \mu)$.) Observe that $\varphi^*\colon X \times S^* \to X \times S^*$ is simply a two-sided Bernoulli shift and $\mathscr{E}$ consists of those events which depend only on the future. Since φ^* preserves the measure $\mu\times\lambda^*$, we have $\mu T = \mu$.

We shall construct a function $f \in L_1^+(X\times S^*, \mathscr{A}\times\mathscr{B}^*, \mu\times\lambda^*)$ such that f is $\mathscr{A}\times\{\emptyset, S^*\}$-measurable and

$$(2) \qquad (\mu\times\lambda^*)(\{E^{\mathscr{E}}U^n f \geqslant n\log n \text{ i.o.}\}) = 1.$$

Obviously such a function f satisfies (1).

Choose a $B \in \mathscr{B}$ with $0 < \lambda(B) < 1$ and denote $p = \lambda(B)$, $q = 1-p$. Let $\{a_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of positive real numbers. Assume that there exists a strictly increasing sequence of nonnegative integers $\{b_n\}_{n=0}^{\infty}$ with $b_0 = 0$ such that

$$(3) \qquad \sum_{n=1}^{\infty}(a_{b_n-1} - p\,a_{b_{n+1}-1})^- p^n < \infty,$$

$$(4) \qquad \lim_{n\to\infty} a_{b_n-1}p^n = 0,$$

$$(5) \qquad \sum_{n=1}^{\infty} b_n p^n = \infty,$$

$$(6) \qquad \liminf_{n\to\infty}\frac{b_{n+1}}{b_n} > 1,$$

where $(x)^-$ denotes the negative part of x. We shall prove that under these assumptions there exists an $f \in L_1(\mu\times\lambda^*)$ such that

$$(7) \qquad (\mu\times\lambda^*)(\{E^{\mathscr{E}}U^n f \geqslant a_n \text{ i.o.}\}) = 1.$$

Let $A_0 = X\times S^*$, $A_n = \bigcap_{i=1}^{n} B_{b_i-1}$ $(n = 1, 2, \ldots)$ and $C_n = A_{n-1}\setminus A_n$ where $B_n = \ldots \times S_{-n-2}\times S_{-n-1}\times B\times S_{-n+1}\times S_{-n+2}\times \ldots \subseteq X\times S^*$ $(n = 0, 1, \ldots)$. Denote $a_n = a_{b_{n+1}-1}$ $(n = 0, 1, 2, \ldots)$ and $\beta_n = q^{-1}(a_{n-1}-pa_n)$ $(n = 1, 2, \ldots)$. Let

$$f = \sum_{n=1}^{\infty}\beta_n 1_{C_n}.$$

Since $\sum_{n=1}^{N}\beta_n p^{n-1}q = a_0 - a_N p^N$, we infer from (3) and (4) that f is integrable and $\int\int_{X\times S^*} f\,d(\mu\times\lambda^*) = a_{b_1-1}$. Denote

$$\mathscr{E}_n = \ldots\times S_{-n-1}\times S_{-n}\times \mathscr{B}_{-n+1}\times\mathscr{B}_{-n+2}\times\ldots \subseteq \mathscr{A}\times\mathscr{B}^*.$$

Since $\left|\sum_{n=1}^{N}\beta_n 1_{C_n}\right| \leqslant |f|$, we have by the Lebesgue theorem

$$E^{\mathscr{E}_m} f = \sum_{n=1}^{\infty}\beta_n E^{\mathscr{E}_m}1_{C_n} \quad \text{for} \quad m\geqslant 0. \tag{8}$$

Observe that

$$E^{\mathscr{E}_{b_m}}1_{C_n} = \begin{cases} 1_{C_n} & \text{for} \quad n\leqslant m, \\ p^{n-m-1}q1_{A_m} & \text{for} \quad n > m, \end{cases} \tag{9}$$

for $m\geqslant 0$. Using (8), (9) and (4) we obtain

$$E^{\mathscr{E}_{b_m}} f = \sum_{n=1}^{m}\beta_n 1_{C_n} + a_m 1_{A_m} \quad \text{for} \quad m\geqslant 0. \tag{10}$$

Clearly,

$$E^{\mathscr{E}_{b_m+k}} f = E^{\mathscr{E}_{b_m}} f \quad \text{for} \quad k = 0, 1, \ldots, b_{m+1}-b_m-1,\ m\geqslant 0. \tag{11}$$

Now, denote $D_k = \{E^{\mathscr{E}_k} f\geqslant a_k\}$ $(k = 0, 1, 2, \ldots)$. We shall prove that

$$D_k = A_m \quad \text{for} \quad b_m\leqslant k\leqslant b_{m+1}-1,\ m\geqslant 0. \tag{12}$$

Fix $m\geqslant 0$ and $b_m\leqslant k\leqslant b_{m+1}-1$. Using (11), (10) and the monotonicity of a_n, we have

$$E^{\mathscr{E}_k} f = E^{\mathscr{E}_{b_m}} f = a_m = a_{b_{m+1}-1}\geqslant a_k \quad \text{on} \quad A_m.$$

Thus $A_m\subseteq D_k$. For the proof of the converse inclusion observe that

$$\beta_n = q^{-1}(a_{n-1}-pa_n) < a_{n-1} = a_{b_n-1} < a_{b_n} \quad \text{for} \quad n\geqslant 1. \tag{13}$$

Using (11), (10) and (13), we obtain

$$E^{\mathscr{E}_k} f = E^{\mathscr{E}_{b_m}} f = \sum_{i=1}^{m}\beta_i 1_{C_i} \leqslant \max_{i=1,\ldots,m}\beta_i < a_{b_m}\leqslant a_k \quad \text{on} \quad A_m^c.$$

Thus $A_m^c\subseteq D_k^c$, which proves (12). (The case $m = 0$ in the last inclusion is trivial.)

It follows from (12) that

$$\sum_{k=0}^{\infty}(\mu\times\lambda^*)(D_k) = \sum_{n=0}^{\infty}(b_{n+1}-b_n)(\mu\times\lambda^*)(A_n) = \sum_{n=0}^{\infty}(b_{n+1}-b_n)p^n. \tag{14}$$

However, we infer from (6) that there exists $\varepsilon > 0$ such that $b_{n+1}-b_n \geqslant \varepsilon b_n$ for $n\geqslant 0$. Thus (14) and (5) imply

$$\sum_{k=0}^{\infty}(\mu\times\lambda^*)(\{E^{\mathscr{E}_k} f\geqslant a_k\}) = \sum_{k=0}^{\infty}(\mu\times\lambda^*)(D_k) = \infty. \tag{15}$$

Denote $F_n = \{E^{\mathscr{E}}U^n f\geqslant a_n\}$. It is easy to check that $E^{\mathscr{E}}U^n = U^n E^{\mathscr{E}_n}$ for $n\geqslant 0$. Thus $F_n = \{U^n E^{\mathscr{E}_n} f\geqslant a_n\} = \varphi^{*-n}D_n$. Since φ^* is measure-preserving, (15) implies

$$\sum_{n=0}^{\infty}(\mu\times\lambda^*)(F_n) = \infty. \tag{16}$$

We shall need the following

LEMMA 1. *Let $G = \{b_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers and let $\psi\colon N\to N$ be the shift transformation on positive integers. Denote by $\psi^n G$ the image of G under ψ^n and by $|\cdot|$ the number of elements of a given set. If the sequence $G = \{b_n\}_{n=1}^{\infty}$ satisfies the Hadamard gap condition, i.e. if there exists an $\varepsilon > 0$ such that*

$$\frac{b_{n+1}}{b_n}\geqslant 1+\varepsilon \quad \textit{for} \quad n = 1, 2, \ldots, \tag{17}$$

then

$$\sup_{n\geqslant 1}|G\cap\psi^n G| < 2 - \frac{\log\varepsilon}{\log(1+\varepsilon)} < \infty.$$

Proof. Suppose that for some $n\geqslant 1$ and $N\geqslant 1$ we have

$$b_{k_i}+n = b_{l_i} \quad (i = 1, 2, \ldots, N), \tag{18}$$

where $l_1 < l_2 < \ldots < l_N$. Denote $x = b_{l_N-1}$. Since $l_N > k_N$, we have $b_{l_N-1}\geqslant b_{k_N}$. Thus, using (18) and (17) we obtain $b_{l_1}-b_{k_1} = n = b_{l_N} - b_{k_N}\geqslant b_{l_N}-b_{l_N-1}\geqslant \varepsilon b_{l_N-1} = \varepsilon x$. In particular,

$$b_{l_1} > \varepsilon x. \tag{19}$$

Using (17) and (19) we have $b_{l_{k+1}}\geqslant b_{l_1+k}\geqslant b_{l_1}(1+\varepsilon)^k > \varepsilon(1+\varepsilon)^k x$ for

$k = 0, 1, \ldots, N-1$. Hence $b_{l_{N-1}} > \varepsilon(1+\varepsilon)^{N-2}x$. On the other hand, $b_{l_{N-1}} \leqslant b_{l_{N-1}} = x$. Therefore $\varepsilon(1+\varepsilon)^{N-2} < 1$, i.e.

$$N-2 < -\frac{\log \varepsilon}{\log(1+\varepsilon)},$$

which proves the lemma.

Using (12) it is easy to check that

$$(\mu\times\lambda^*)(F_i \cap F_{i+j}) = p^{-k_{i,j}}(\mu\times\lambda^*)(F_i)(\mu\times\lambda^*)(F_{i+j}),$$

where $k_{i,j} = |G\cap\psi^j G\cap\{1, 2, \ldots, i+j\}|$, $G = \{b_1, b_2, \ldots\}$, $j \geqslant 1$, $i \geqslant 0$. Thus (6) and Lemma 1 imply that there exists a $K < \infty$ such that

$$(\mu\times\lambda^*)(F_m\cap F_n) \leqslant K(\mu\times\lambda^*)(F_m)(\mu\times\lambda^*)(F_n) \tag{20}$$

for all $m, n \geqslant 0$, $m \neq n$.

We have

LEMMA 2 (see [9]). *Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\{F_n\}_{n=1}^{\infty}$ be a sequence of events satisfying $\sum_{n=1}^{\infty} P(F_n) = \infty$. If for some constant $K < \infty$ and for all m and n with $m \neq n$ we have $P(F_m\cap F_n) \leqslant KP(F_m)P(F_n)$, then $P(F_n$ i.o.$) > 0$.*

Now, (16), (20) and Lemma 2 imply that $(\mu\times\lambda^*)(F_n$ i.o.$) > 0$. Finally, we shall use the following

LEMMA 3 (see [16], Corollary 2 and the footnote on p. 140). *Let $(\Omega, \mathscr{F}, P, T)$ be an ergodic dynamical system. If $\{D_n\}_{n=1}^{\infty}$ is a non-increasing sequence of events, then $P(T^{-n}D_n$ i.o.$) = 0$ or 1.*

Since $F_n = \varphi^{*-n}D_n$ and, by (12), D_n is a non-increasing sequence of sets we can apply Lemma 3 which gives $(\mu\times\lambda^*)(F_n$ i.o.$) = 1$. This proves (7).

Now, let $a_n = (n+2)\log(n+2)$ $(n = 0, 1, 2, \ldots)$ and let

$$b_0 = 0, \quad b_n = \left[\frac{1}{p^{N+n}(N+n)\log(N+n)}\right] - 1 \quad (n = 1, 2, \ldots),$$

where $[x]$ denotes the integral part of x. Here N is a natural number so large that b_n is strictly increasing. We have

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n} = p^{-1} > 1,$$

which gives (6). Next, it is easy to check that (4) and (5) hold. Now, we shall verify (3). Since

$$\frac{1}{p^n n\log n} - 1 \geqslant \frac{1}{p^{n-1/n} n\log n}$$

for sufficiently large n and since the function

$$h(x) = \frac{1}{x\log x}\log\frac{1}{p^x x\log x}$$

is decreasing for sufficiently large x, we have

$$\left[\frac{1}{p^n n\log n}\right]\log\left[\frac{1}{p^n n\log n}\right] - p\left[\frac{1}{p^{n+1}(n+1)\log(n+1)}\right]\times$$
$$\times\log\left[\frac{1}{p^{n+1}(n+1)\log(n+1)}\right] \geqslant \left(\frac{1}{p^n n\log n} - 1\right)\log\frac{1}{p^{n-1/n} n\log n} -$$
$$-\frac{1}{p^n n\log n}\log\frac{1}{p^n n\log n} = -\frac{\log(1/p)}{p^n n^2\log n} - \log\frac{1}{p^{n-1/n} n\log n}$$

for sufficiently large n Since.

$$\sum_n\left(\frac{\log 1/p}{p^n n^2\log n} + \log\frac{1}{p^{n-1/n} n\log n}\right)p^{n-N} < \infty,$$

we obtain (3). Therefore we have

$$(\mu\times\lambda^*)(\{E^{\mathscr{S}}U^n f \geqslant n\log n \text{ i.o.}\})$$
$$\geqslant (\mu\times\lambda^*)(\{E^{\mathscr{S}}U^n f \geqslant (n+2)\log(n+2) \text{ i.o.}\}) = 1.$$

Taking $|f|$ instead of f, we obtain (2) .

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