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Hopf's theorem on invariant measures for a group of transformations

by

D. RAMACHANDRAN (Calcutta) and M. MISIUREWICZ (Warszawa)

Abstract. Let G be a group of measurable and nonsingular transformations on $(X, \mathscr{A}, P)$. Some results on the equivalence (by countable decomposition) of measurable sets are proved, using which Hopf's theorem on invariant measures for a group of transformations is derived.

1. Introduction. Let $(X, \mathscr{A}, P)$ be a probability space, and let G be a group of measurable and nonsingular transformations defined on $(X, \mathscr{A}, P)$. Hopf [4] gave a necessary and sufficient condition for the existence of a finite invariant measure μ on $\mathscr{A}$ equivalent to P for the case where G is a cyclic group. In this paper we prove that condition (H), the analogue of Hopf's condition, when G is a general group of transformations, is necessary and sufficient for the existence of a finite equivalent measure invariant under every g in G. The only known alternative proof (see Hajian and Ito [3]) of Hopf's theorem for a group of transformations is through the equivalence of condition (H) and the condition of nonexistence of weakly wandering sets of positive measure introduced by Hajian and Kakutani. While Hajian and Ito use functional analytic methods in their proof, we use a direct argument motivated by an idea of Tarski (see [6], Definition 1.25) regarding measures on semigroups. Our proof constructs the invariant measure and shows that it is the unique, finitely additive, G-invariant measure equivalent to P which equals P on almost invariant sets.

2. Definitions and notation. Let $(X, \mathscr{A}, P)$ be a probability space. A finitely additive measure μ on $\mathscr{A}$ is said to be *equivalent to* P if μ and P have the same sets of measure zero, that is, $\mu(A) = 0$ iff $P(A) = 0$ for every $A \in \mathscr{A}$. A measurable map g of X into itself is called *nonsingular* if, for every $A \in \mathscr{A}$, $P(A) > 0$ implies $P(g^{-1}A) > 0$. For $A, B \in \mathscr{A}$, $A \overset{P}{=} B$ iff $P(A \triangle B) = 0$, where $\triangle$ is the symmetric difference.

Suppose that G is a group of measurable and nonsingular transformations defined on $(X, \mathscr{A}, P)$, that is, each $g \in G$ is a 1-1 bimeasurable and nonsingular transformation defined on $(X, \mathscr{A}, P)$ and the multiplication in G is defined by composition. A finitely additive measure μ on $\mathscr{A}$ is said to be *G-invariant* if for every $g \in G$ and $E \in \mathscr{A}$ we have $\mu(gE) = \mu(E)$.

A set $E \in \mathscr{A}$ is called *almost invariant* if $P(gE - E) = 0$ for every $g \in G$. It is easy to check that the collection

$$\mathscr{A}^* = \{E \in \mathscr{A}: E \text{ is almost invariant}\}$$

is a sub σ-algebra of $\mathscr{A}$. Given $A \in \mathscr{A}$, there is a minimal almost invariant set $A^* \in \mathscr{A}^*$ containing A. Minimal means that $A_1^* \in \mathscr{A}^*$, $A_1^* \supset A$ implies $P(A^* - A_1^*) = 0$ (see Lemma 4). Note that as a consequence we have $(\bigcup_{n=1}^{\infty} A_n)^* \overset{P}{=} \bigcup_{n=1}^{\infty} A_n^*$.

We use the notation $A = C + D$, $A = \sum_{j=1}^{n} A_j$ and $A = \sum_{j=1}^{\infty} A_j$ to mean, respectively, that A is the union of the $\mathscr{A}$-measurable disjoint sets C and D, A is the union of pairwise disjoint sets $\{A_1, A_2, \ldots, A_n\} \subset \mathscr{A}$ and A is the union of pairwise disjoint sets $\{A_j: j = 1, 2, \ldots\} \subset \mathscr{A}$.

3. Equivalence of measurable sets and condition (H). Two $\mathscr{A}$-measurable sets E and F are said to be *equivalent*, in symbols $E \sim F$, if

$$E \overset{P}{=} E' = \sum_{j=1}^{\infty} E_j, \quad F \overset{P}{=} F' = \sum_{j=1}^{\infty} F_j$$

and there is a sequence $\{g_j\} \subset G$ such that

$$F_j = g_j E_j \quad \text{for every } j.$$

A set $E \in \mathscr{A}$ is called *bounded* if it is not equivalent to a measure-theoretically proper subset of itself, that is, $F \subset E$ and $E \sim F$ implies $E \overset{P}{=} F$. The group G is *bounded* if X is bounded. We now consider the following condition, introduced by Hopf for cyclic groups:

Condition (H): G is bounded.

We have

LEMMA 1. *Every measurable subset of a bounded set is bounded. In particular, if* (H) *holds, then every $A \in \mathscr{A}$ is bounded.*

LEMMA 2. *Let $E, F \in \mathscr{A}$. If $E \sim F$ and $F \neq \emptyset$, then there is a map h from E to F such that*

(i) *h is 1-1, onto and bimeasurable* a.e. $[P]$, *and*

(ii) *for every $E_1 \in \mathscr{A} \cap E$, $E_1 \sim h(E_1)$.*

Proof. Let

$$E \overset{P}{=} E' = \sum_{j=1}^{\infty} E_j, \quad F \overset{P}{=} F' = \sum_{j=1}^{\infty} F_j,$$

and let $\{g_j\} \subset G$ be such that $g_j E_j = F_j$ for all j. The function h on E de-

fined by

$$h = \begin{cases} g_j & \text{on } E_j \cap E, \\ x_0 \in F & \text{elsewhere} \end{cases}$$

has the required properties.

LEMMA 3. *If E is bounded and if $\{A_n: n \geqslant 1\} \subset \mathscr{A}$ is a sequence of pairwise disjoint, mutually equivalent subsets of E, then $P(A_n) = 0$ for all n.*

Proof. If E is bounded, then, by Lemma 1, since $\sum_{n=1}^{\infty} A_n \sim \sum_{n=2}^{\infty} A_n$, we have $P(A_1) = 0$. Since each $g \in G$ is nonsingular and $A_n \sim A_1$, $P(A_n) = 0$ for all n.

LEMMA 4. *Let $A \in \mathscr{A}$. There exists $A^* \in \mathscr{A}^*$ such that*

(i) $A \subset A^*$,

(ii) $A_1^* \in \mathscr{A}^*$, $A \subset A_1^* \Rightarrow P(A^* - A_1^*) = 0$, *and*

(iii) $A^* = \bigcup_{n=0}^{\infty} g_n A$ *for some* $\{g_n: n \geqslant 0\} \subset G$.

Proof. Take $g_0 =$ identity. Let $\alpha_1 = \sup_{g \in G}\{P(gA - A)\}$. Choose $g_1 \in G$ such that $P(g_1 A - A) \geqslant \alpha_1/2$. Use induction as follows: Let

$$\alpha_n = \sup_{g \in G} P\Big(gA - \bigcup_{j=0}^{n-1} g_j A\Big),$$

and choose g_n such that

$$P\Big(g_n A - \bigcup_{j=0}^{n-1} g_j A\Big) \geqslant \alpha_n/2.$$

Set $A^* = \bigcup_{n=0}^{\infty} g_n A$. Since $P(A^*) \geqslant P(A) + \sum_{n=1}^{\infty} \alpha_n/2$, we have $\lim_{n\to\infty} \alpha_n = 0$. For $g \in G$,

$$P(gA - A^*) \leqslant P\Big(gA - \bigcup_{j=0}^{n-1} g_j A\Big) \leqslant \alpha_n \quad \text{for every } n,$$

and so $P(gA - A^*) = 0$. Hence, for every $g \in G$,

$$P(gA^* - A^*) \leqslant \sum_{n=0}^{\infty} P(gg_n A - A^*) = 0,$$

and thus $A^* \in \mathscr{A}^*$. By construction, (i), (ii) and (iii) hold.

In the sequel we assume that (H) holds for G.

LEMMA 5. *Let $A, B \in \mathscr{A}$. Then there exist $A_B \subset A$, $B_A \subset B$, satisfying the following conditions:*

(a) $A_B^* \cap A \overset{P}{=} A_B$ *and* $B_A^* \cap B \overset{P}{=} B_A$.

(b) $A_B \sim B'$ *for some* $B' \subset B$ *and* $B_A \sim A'$ *for some* $A' \subset A$.

(c) $A_B^* \cup B_A^* \overset{P}{=} A^* \cap B^*$.

(d) A_B *and* B_A *are the sets with maximal P-measures having properties* (a) *and* (b).

Proof. Let $a_1 = \sup_{g\in G}\{P(A\cap gB)\}$, and choose $g_1 \in G$ such that $P(A\cap g_1B) \geqslant a_1/2$. Then set $A_1 = A\cap g_1 B$, $B_1 = g_1^{-1}A_1$. We proceed by induction: suppose that pairwise disjoint sets $A_1, A_2, \ldots, A_{n-1} \in \mathscr{A}$ and pairwise disjoint sets $B_1, B_2, \ldots, B_{n-1} \in \mathscr{A}$ have already been chosen; let

$$a_n = \sup_{g\in G}\Big\{P\Big(\Big(A-\sum_{j=1}^{n-1}A_j\Big)\cap g\Big(B-\sum_{j=1}^{n-1}B_j\Big)\Big)\Big\},$$

choose $g_n \in G$ such that

$$P\Big(\Big(A-\sum_{j=1}^{n-1}A_j\Big)\cap g_n\Big(B-\sum_{j=1}^{n-1}B_j\Big)\Big) \geqslant a_n/2,$$

and set

$$A_n = \Big(A-\sum_{j=1}^{n-1}A_j\Big)\cap g_n\Big(B-\sum_{j=1}^{n-1}B_j\Big), \qquad B_n = g_n^{-1}A_n.$$

Letting

$$A_0 = \sum_{n=1}^{\infty}A_n, \qquad B_0 = \sum_{n=1}^{\infty}B_n,$$

we have $A_0 \sim B_0$, and since

$$P(A_0) = \sum_{n=1}^{\infty}P(A_n) \geqslant \sum_{n=1}^{\infty}a_n/2,$$

we have $\lim a_n = 0$. For $g\in G$, $P\big((A-A_0)\cap g(B-B_0)\big) \leqslant a_n$ for all n, and so

$$P\big((A-A_0)\cap g(B-B_0)\big) = 0.$$

By Lemma 4 (iii), $P\big((A-A_0)^*\cap(B-B_0)^*\big) = 0$. If we now define $A_B = A_0-(A-A_0)^*$, $B_A = B_0-(B-B_0)^*$, then it is easy to check (a) and (b); (c) follows from Lemma 4 (iii) and the construction of A_B and B_A, and (d) follows from (H) and the construction of A_0, B_0, A_B and B_A.

We remark that Lemma 5 contains as a special case Lemma 6 of [2].

Now for every $n = 1, 2, \ldots$ let

$$X_n = \{1, 2, \ldots, n\}\times X$$

(we have $X = X_1 \subset X_2 \subset \ldots$),

$$\mathscr{A}_n = \Big\{\sum_{k=1}^{n}\{k\}\times A_k:\ A_k\in\mathscr{A} \text{ for } k = 1, 2, \ldots, n\Big\},$$

$$P_n\Big(\sum_{k=1}^{n}\{k\}\times A_k\Big) = \frac{1}{n}\sum_{k=1}^{n}P(A_k).$$

Then $\sigma_n\times G$, where σ_n is the group of permutations of $\{1, 2, \ldots, n\}$, is a group of nonsingular transformations on $(X_n, \mathscr{A}_n, P_n)$, and hence the preceding definitions and lemmas apply for $(X_n, \mathscr{A}_n, P_n, \sigma_n\times G)$. We have

LEMMA 6. *Let* $A, B, D \in \mathscr{A}_r$ *be such that* $A\cap B = \emptyset = A\cap D$, *and let* A *be bounded. Suppose* $A+B$ *is equivalent to a subset* E *of* $A+D$. *Then* B *is equivalent to a subset of* D.

Proof. Let h be the map from $A+B$ to E given by Lemma 2. Let:

$$\begin{aligned}
D_1 &= h(B)\cap D, & E_1 &= h^{-1}(D_1),\\
D_2 &= h^2(B-E_1)\cap D, & E_2 &= h^{-2}(D_2),\\
&\ldots\ldots\ldots\ldots & &\ldots\ldots\ldots\\
D_n &= h^n\Big(B-\sum_{j=1}^{n-1}E_j\Big)\cap D, & E_n &= h^{-n}(D_n),\\
&\ldots\ldots\ldots\ldots & &\ldots\ldots\ldots
\end{aligned}$$

Since h is 1-1, onto from $A+B$ to E (a.e. [P]), it is easy to verify that $D_1, D_2, \ldots$ is a sequence of pairwise disjoints subsets of D and that $E_1, E_2, \ldots$ is a sequence of pairwise disjoint subsets of B. By Lemma 2 (ii), $E_n \sim h^n(E_n) = D_n$ for every $n \geqslant 1$. Let $B_1 = B-\sum_{j=1}^{\infty}E_j$. Clearly, $B-B_1 \sim \sum_{j=1}^{\infty}D_j \subset D$. By the definition of h, we conclude that $\{h^n(B_1):\ n\geqslant 1\}$ is a sequence of pairwise disjoint subsets of A each equivalent to $h(B_1)$. By Lemma 3, $P(B_1) = 0$.

LEMMA 7. *Condition* (H) *holds for* $\sigma_n\times G$.

Proof. We use induction. Clearly, (H) holds for $\sigma_1\times G$. Suppose (H) holds for $\sigma_{n-1}\times G$ but not for $\sigma_n\times G$. Then for some $E\subset X_n$, $E\sim X_n$ and $P_n(X_n-E) > 0$. There is k such that $P_n(\{k\}\times X-E) > 0$. The set $A = X_n-(\{k\}\times X)$ is bounded, and we can apply Lemma 6 with $B = \{k\}\times X$, $D = E\cap(\{k\}\times X)$. We get $B\sim F$ for some $F\subset D$, which is impossible since (H) holds for G.

LEMMA 8. *Let* $F\in\mathscr{A}$. *Suppose that for every* n *there are pairwise disjoint sets* $F_1^n, \ldots, F_n^n \in\mathscr{A}$, *equivalent to* F. *Then* $P(F) = 0$.

Proof. We use induction to find a sequence $\{F_n:\ n\geqslant 1\}$ of pairwise disjoint subsets of A, each equivalent to F. Then we can use Lemma 3.

We set $F_1 = F_1^1$. Suppose we have already defined $F_1, \ldots, F_{n-1}$. Denote $\sum_{k=1}^{n-1}F_k = K$, $\sum_{k=1}^{n-1}F_k^n = L$. Since $K\sim L$, we may use Lemma 6 with $r = 1$, $A = K\cap L$, $B = K-L$, $D = L-K$ (here we advise the reader to start drawing a picture). We find that B is equivalent to some subset of D. Hence, $B\cap F_n^n \sim M$ for some $M\subset D$. Since F_n^n is disjoint from L and $M\subset D\subset L$, we have $(F_n^n-B)+M \sim F_n^n \sim F$. Thus, we set $F_n = (F_n^n-B)+M$; clearly F_n defined in this way is also disjoint from K.

4. Hopf's theorem on invariant measures for a group of transformations.

We are now ready to prove our main result.

THEOREM 1. *There is a finite, G-invariant measure equivalent to P iff* (H) *holds.*

Proof. The necessity part is easy to verify. To prove the sufficiency part we note that it suffices to show the existence of a finite, *finitely additive*, G-invariant measure μ equivalent to P (see Friedman [1], Theorem 3.13).

Let $A \in \mathscr{A}$, and let A_n denote $\{1, 2, \ldots, n\} \times A \subset X_n$ for $n = 1, 2, \ldots$ For $B \in \mathscr{A}_n$ let $B^+ = B^* \cap X \in \mathscr{A}^*$. For $1 \leqslant k \leqslant n$ let $A_n^k = ((X_k)_{A_n})^+$ (see Lemma 5), $\bar{A}_n^k = A_n^k - A_n^{k+1}$. Define the $\mathscr{A}^*$-measurable step function $f_n(A)$ on X by

$$f_n(A) = \sum_{k=1}^{n} 1_{A_n^k} = \sum_{k=1}^{n} k 1_{\bar{A}_n^k}.$$

f_n is well defined a.e. $[P]$. Using Lemma 4 (d) and (c) it can be checked that $(\bar{A}_n^k)^* \cap X_k \sim$ a subset of $(\bar{A}_n^k)^* \cap A_n$ and $(\bar{A}_n^k)^* \cap A_n \sim$ a subset of $(\bar{A}_n^k)^* \cap X_{k+1}$ for $k = 1, 2, \ldots, n-1$. From this it is straightforward to establish

(1) $$f_{n+m}(A) - 1 \leqslant f_n(A) + f_m(A) \leqslant f_{n+m}(A) \text{ a.e. } [P],$$

and

(2) $$f_n(A+B) - 1 \leqslant f_n(A) + f_n(B) \leqslant f_n(A+B) \text{ a.e. } [P],$$

for $A, B \in \mathscr{A}$, m, n positive integers. From (1) it follows that the sequence $\{f_n(A) + 1\}_{n=1}^{\infty}$ is subadditive a.e. $[P]$, and so

$$f(A) = \lim_{n\to\infty} \frac{1}{n} f_n(A)$$

exists a.e. $[P]$. The function $f(A)$ is $P|_{\mathscr{A}^*}$-integrable, and so we define $\mu(A) = \int f(A)\,dP$. From (2) it follows that μ is finitely additive. If $P(A) = 0$, then $f(A) = 0$ a.e. $[P]$, and so $\mu(A) = 0$. If $P(A) > 0$, then, by Lemma 8, there exists $B \in \mathscr{A}^*$ with $P(B) > 0$ such that $f_n(A)|_B \geqslant 1$ for some n; hence, by (1), $f_{kn}(A)|_B \geqslant k$ for $k = 1, 2, \ldots$, and so

$$\mu(A) \geqslant \frac{1}{n} P(B) > 0.$$

$A \sim B$ implies $f(A) = f(B)$ a.e. $[P]$, and so μ is G-invariant, and the proof is complete.

From our proof it follows also that if (H) holds, then μ is the unique, finitely additive, G-invariant measure on $\mathscr{A}$ equivalent to P such that $\mu = P$ on $\mathscr{A}^*$.

It is possible to obtain a proof of Theorem 1 by using Tarski's theory of measures on semigroups (see [5]), thereby showing Hopf's theorem to be a special case of Tarski's general theory, but the present proof is short, direct and shows the uniqueness of the invariant measure obtained.

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