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On Morrey–Besov inequalities

by

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Abstract. The authors consider functions u which satisfy a local integrability condition of Morrey–Besov type. They prove that u is locally in a certain Lorentz space. Examples are given to show this result is best possible.

1. Introduction. Let x be a point in n dimensional Euclidean space $\boldsymbol{R}^n$, $|x|$ the Euclidean norm of x, dx Lebesgue measure on $\boldsymbol{R}^n$, and $|E|$ the Lebesgue measure of a measurable set $E \subset \boldsymbol{R}^n$. If Q_0 is a cube in $\boldsymbol{R}^n$, let $L^p(Q_0)$, $1 \leqslant p \leqslant \infty$, denote the usual Lebesgue space of functions that are pth power integrable on Q_0. Then in [8], Ross shows that any function $u \in L^p(Q_0)$ which satisfies

$$\int_Q |u(x+t)-u(x)|^p\,dx \leqslant A^p |t|^{\alpha p} |Q|^{1-\lambda/n}, \tag{1.1}$$

whenever Q and $Q+t = \{x+t\colon x \in Q\}$ are parallel subcubes of Q_0, also belongs to $L^r(Q_0)$ for all $r < p^* = \lambda p/(\lambda - \alpha p)$. Here $1 \leqslant p < \lambda/\alpha$, $0 < \alpha \leqslant 1$, $0 < \lambda \leqslant n$, and A is a positive constant independent of $t \in \boldsymbol{R}^n$ and Q. (1.1) can be viewed as a mixture of the usual Nikol'skii condition ([6], p. 153) when $\lambda = n$, and a Morrey type condition involving the αth difference quotient. When $\lambda = n$, it is known [4], that (1.1) implies $u \in$ weak-$L^{p^*}(Q_0)$, whereas when $\alpha = 1$, it is known [1] that condition (1.1) implies $u \in L^{p^*}(Q_0)$. Thus we are motivated to determine, in all cases, the "best integrability class" for functions satisfying (1.1) as well as for a related Morrey–Besov condition, (1.2) below. To be more specific, let $L(a, b)$, $1 \leqslant a < \infty$, $1 \leqslant b \leqslant \infty$, denote the Lorentz space of measurable functions f on $\boldsymbol{R}^n$ with

$$\|f\|_{a,b} = \Big(\int_0^\infty \big[s\cdot|\{x\colon |f(x)| > s\}|^{1/a}\big]^b s^{-1}\,ds\Big)^{1/b} < \infty.$$

Note that $L(a, a) = L^a(\boldsymbol{R}^n)$, $L(a, \infty) =$ weak-$L^a(\boldsymbol{R}^n)$, and $L(a, b_1) \subseteq L(a, b_2)$ when $b_1 \leqslant b_2$. Given a function u, let $\Delta_t u(x) = u(x+t) - u(x)$, whenever x and $x+t$ are in the domain of definition of u. Inductively, define $\Delta_t^k u$ for $k \geqslant 2$, a positive integer, by $\Delta_t^k u = \Delta_t(\Delta_t^{k-1} u)$. We then consider functions $u \in L^p(\boldsymbol{R}^n)$ which satisfy

$$(1.2)\qquad \Big[\int_{\boldsymbol{R}^n}\Big(\int_Q |\Delta_t^k u|^p\,dx\Big)^{q/p}|t|^{-(n+\alpha q)}\,dt\Big]^{1/q} \leqslant A\cdot|Q|^{(1-\lambda/n)/p},$$

for each cube Q in $\boldsymbol{R}^n$. Again A is independent of Q. Note that if $q = \infty$, then (1.2) is equivalent to (1.1).

In § 2, we prove:

THEOREM 1. *Let p, q, k, λ, and α be fixed numbers with $1 \leqslant p < \lambda/\alpha$, $1 < q \leqslant \infty$, $0 < \alpha < k$, and $0 < \lambda < n$. If $p^* = \lambda p/(\lambda-\alpha p)$ and $u \in L^p(\boldsymbol{R}^n)$ satisfies* (1.2), *then for each cube Q,*

$$(1.3)\qquad \|u\cdot\chi_Q\|_{p^*,qp^*/p} \leqslant c\cdot A\,|Q|^{(1-\lambda/n)/p^*},$$

where c is a positive constant independent of A, u, and Q.

In Theorem 1, χ_Q denotes the characteristic function of Q. In § 3, we prove a local version of Theorem 1 when $k = 1$:

THEOREM 2. *Let p, q, λ, and α be as in Theorem 1 and let Q_0 be a fixed cube of $\boldsymbol{R}^n$. Suppose that $u \in L^p(Q_0)$ and that* (1.2) *holds for $k = 1$ and each parallel subcube $Q \subset Q_0$ provided that the integral involving t is taken only over those values of t for which $Q+t \subset Q_0$. Then*

$$(1.4)\qquad \|(u-u_{Q_0})\cdot\chi_{Q_0}\|_{p^*,qp^*/p} \leqslant c\cdot A\,|Q_0|^{(1-\lambda/n)/p^*}.$$

Here u_Q denotes the average of u over the cube Q. Again, c is a constant independent of A, u, and Q_0.

Observe that if $q = \infty$ in Theorem 2, then $u \in \text{weak-}L^{p^*}(Q_0)$. This is an improvement of the theorem of Ross in [8]. In § 4, we show that Theorem 1 is best possible. We prove:

THEOREM 3. *There is a function u which satisfies the conditions of Theorem 1, which has compact support, but which does not belong to $L(p^*, r)$ for any $r < qp^*/p$.*

Finally in § 5, we make some remarks concerning possible extensions of Theorems 1 and 2.

In [4], Herz proved that if u satisfies (1.2) with $\lambda = n$ and $u \in L^p(\boldsymbol{R}^n)$, then $u \in L(p^*, q)$, a space strictly smaller than $L(p^*, qp^*/p)$. Because of this, it might seem surprising that Theorem 1 is sharp, as we have claimed in Theorem 3. We remark, however, that there are other results in the literature where a discontinuity, such as this, occurs in passing from a cube to a noncube condition. For example, if $p^* = \lambda p/(\lambda-p)$, $u \in L^p(\boldsymbol{R}^n)$, and the gradient of u (∇u) satisfies

$$(1.5)\qquad \int_Q |\nabla u|^p\,dx \leqslant A^p\,|Q|^{1-\lambda/n},$$

$1 < p < \lambda < n$, for all cubes Q, then from [1] it follows that

$$(1.6)\qquad \|u\cdot\chi_Q\|_{p^*,p^*} \leqslant c\cdot A\,|Q|^{(1-\lambda/n)/p^*},$$

but $u\cdot\chi_Q \in L(p^*, p)$ when $\lambda = n$ (see [7]). Also, if $p = \lambda$, then (1.5) implies $\exp(b\,|u|) \in L^1(Q)$ for $\lambda < n$, and $\exp(b\,|u|^{p'}) \in L^1(Q)$ for $\lambda = n$ (see [10] and [11]). Here b is a constant depending on A in (1.5). Furthermore, (1.6) is sharp as can be seen by a simple modification of the example in Theorem 3 (take $\alpha = k = 1$ and $p = q$). The example $u(x) = \log|x|$, $x \in \boldsymbol{R}^n$, shows that the results involving $\exp(b\,|u|)$ are sharp as well.

Finally, we make some comments concerning the methods employed in proving Theorems 1–3. The argument of Ross is based on the methods of John–Nirenberg [5], i.e. cube decompositions. Our proof of Theorems 1 and 2, however, goes back to the familiar idea: the trace of a Sobolev function on a lower dimensional hyperplane is characterized by a Besov–Lipschitz class, i.e. by classes defined in terms of L^p-L^q norms of difference quotients (see [3] and [9]). In particular, we show that functions $u \in L^p(\boldsymbol{R}^n)$ satisfying (1.2) can be realized pointwise almost everywhere as the restriction of a Riesz potential on $\boldsymbol{R}^{2n}$ of a function f which in turn satisfies a mixed L^p-L^q growth condition. Using this representation of u, we get Theorem 1. Theorem 2 is obtained in a similar manner using local arguments. The example in Theorem 3 is constructed by defining u on widely spaced cubes in such a way that u grows sufficiently fast and (1.2) is satisfied. This construction is somewhat similar to the construction of sets of Cantor type.

2. Proof of Theorem 1. Let p, q, k, λ, α, and u be as in Theorem 1. In the sequel, c will denote a positive constant, not necessarily the same at each occurence, which may depend on $n, p, q, \lambda, \alpha, k$, but not on u, Q or A. Put $f(x, t) = |t|^{-\gamma}\Delta_t^k u(x)$, for $(x, t) \in \boldsymbol{R}^n\times\boldsymbol{R}^n$ and let $I_\gamma(y) = |y|^{\gamma-2n}$, for $y \in \boldsymbol{R}^{2n}$, $\gamma = \alpha+n/q$. Then by $I_\gamma f(x, t)$ we mean the usual convolution of I_γ with f over $\boldsymbol{R}^{2n}$. We claim that

$$(2.1)\qquad u(x) = c\cdot I_\gamma f(x, 0)$$

for almost every $x \in \boldsymbol{R}^n$. (2.1) is easily proved using elementary properties of the Fourier transform (see e.g. [4]), once we show that $I_\gamma|f|(x, 0)$ is locally integrable. In fact we prove, for each cube Q, that

$$(2.2)\qquad \|\chi_Q\cdot I_\gamma|f|(\cdot, 0)\|_{p^*,qp^*/p} \leqslant c\cdot A\,|Q|^{(1-\lambda/n)/p^*}.$$

Clearly (2.1) and (2.2) imply Theorem 1.

To prove (2.2), we first observe from (1.2), the definition of f, and Hölder's inequality that

$$(2.3)\qquad V_x(|f|, r) \equiv \iint_{|s|^2+|y|^2<r^2} |f(y+x, s)|\,dy\,ds \leqslant c\cdot A r^{2n-\lambda/p-n/q}$$

for all $r > 0$ and $x \in \mathbf{R}^n$. Next observe that

$$(2.4)\qquad I_\gamma |f|(x, 0) = (2n-\gamma)\int_0^\infty V_x(|f|, r) r^{\gamma-2n-1}\, dr$$

as follows from an integration by parts. Let Q be a cube in $\mathbf{R}^n$ of side length ϱ and center x_0. Set $F(x, t) = |f(x, t)|$, when $|x-x_0|^2+|t|^2 \leqslant 4n\varrho^2$, and $F(x, t) = 0$, elsewhere in $\mathbf{R}^{2n}$. Let $G(x, t) = |f(x, t)| - F(x, t)$. Then $I_\gamma|f| = I_\gamma F + I_\gamma G$ and for $x \in Q$,

$$(2.5)\qquad I_\gamma G(x, 0) \leqslant (2n-\gamma)\int_0^\infty r^{\gamma-2n-1} V_x(|f|, r)\, dr \leqslant c\cdot A\, |Q|^{-\lambda/(np^*)},$$

as follows from (2.3), (2.4) and the fact that $\gamma = \alpha + n/q$. Again from (2.3) and (2.4), we see that

$$(2.6)\qquad I_\gamma F(x, 0) = (2n-\gamma)\int_0^\infty r^{\gamma-2n-1} V_x(F, r)\, dr = (2n-\gamma)\Big[\int_0^\sigma \ldots + \int_\sigma^\infty \ldots\Big] \leqslant c\Big[\int_0^\sigma \ldots + A\sigma^{-\lambda/p^*}\Big].$$

Now let $E_s = \{x\colon I_\gamma F(x, 0) > s\}$ and put $\sigma = (c_0 s A^{-1})^{-p^*/\lambda}$. Then (2.6) gives

$$s\,|E_s| \leqslant \int_{E_s} I_\gamma F(x, 0)\, dx \leqslant c\int_{E_s}\Big(\int_0^\sigma \ldots\Big) + c_0 c\cdot s\,|E_s|.$$

Thus for c_0 sufficiently small,

$$s\,|E_s| \leqslant c\int_0^\sigma r^{\gamma-2n-1}\Big(\int_{E_s} V_x(F, r)\, dx\Big)\, dr$$
$$\leqslant c\int_0^\sigma r^{\gamma-2n-1}\Big[\int_{|t|<r}\int_{|y|<r}\Big(\int_{E_s} F(x+y, t)\, dx\Big)\, dy\, dt\Big]\, dr$$
$$\leqslant c\,|E_s|^{1-1/p}\int_0^\sigma r^{\gamma-n-1}\Big[\int_{|t|<r}\Big(\int F(x, t)^p\, dx\Big)^{1/p} dt\Big]\, dr.$$

Hence if $g(t) = (\int F(x, t)^p\, dx)^{1/p}$, then

$$(2.7)\qquad s\,|E_s|^{1/p} \leqslant c\int_0^\sigma r^{\gamma-n-1}\Big(\int_{|t|<r} g(t)\, dt\Big)\, dr \leqslant c\int_{|t|<\sigma} g(t)\Big(\int_{|t|}^\sigma r^{\gamma-n+\varepsilon-1}\cdot r^{-\varepsilon}\, dr\Big)\, dt \leqslant c\int_{|t|<\sigma} g(t)\,|t|^{-\varepsilon}\, dt\cdot\sigma^{\alpha+n/q-n+\varepsilon},$$

provided $\varepsilon > 0$ is chosen so that $n-n/q-\alpha < \varepsilon < n-n/q$. It is easily checked that $\sigma^{-\alpha}s = (c_0A^{-1})^{\alpha p^*/\lambda}\cdot s^{p^*/p}$ and that $d\sigma/\sigma = -p^*/\lambda\cdot ds/s$. Multiplying (2.7) by $(Ac_0^{-1})^{\alpha p^*/\lambda}\cdot\sigma^{-\alpha}$, raising the resulting inequality to the qth power and integrating with respect to ds/s, we get (for $q < \infty$)

$$\int_0^\infty (s^{p^*}|E_s|)^{q/p}\frac{ds}{s} \leqslant cA^{\alpha p^* q/\lambda}\int_0^\infty\Big(\sigma^{n/q-n+\varepsilon}\int_{|t|<\sigma} g(t)\,|t|^{-\varepsilon}\, dt\Big)^q\frac{d\sigma}{\sigma}$$
$$\leqslant cA^{\alpha p^* q/\lambda}\int g(t)^q\, dt \leqslant cA^{(1+\alpha p^*/\lambda)q}|Q|^{(1-\lambda/n)q/p} = cA^{p^*q/p}\,|Q|^{(1-\lambda/n)q/p},$$

by Hardy's inequality and (1.2). Thus

$$(2.8)\qquad \|I_\gamma F(\cdot, 0)\|_{p^*, qp^*/p} \leqslant c\cdot A\,|Q|^{(1-\lambda/n)/p^*}.$$

(2.8) also holds for $q = \infty$ as follows from (2.7). Hence from (2.8) and (2.5), we deduce (2.2) and the proof of Theorem 1 is complete.

3. Proof of Theorem 2. In what follows, we will denote the distance between two sets E and F by $\operatorname{dist}(E, F)$ and the boundary of E by ∂E. Let Q_0, u, p, q, λ, and α be as in Theorem 2. Given a cube $Q \subseteq Q_0$, let $s(Q)$ be the side length of Q and, as before, $u_Q = |Q|^{-1}\int_Q u\, dx$, the average of u on Q. We first show that if $Q \subseteq Q_0$, $s(Q_0) = 2^{n+3}\cdot\varrho$, and

$$(3.1)\qquad \varrho/4 \leqslant s(Q) \leqslant \varrho,$$

then

$$(3.2)\qquad |u_Q - u_{Q_0}| \leqslant c\,|Q_0|^{-\lambda/(np^*)}.$$

To prove (3.2), choose Q^* so that $s(Q^*) = \varrho$ and $\operatorname{dist}(Q^*, \partial Q_0) \geqslant 2^{n+1}\cdot\varrho$. Let Q be a cube satisfying (3.1) and put $G = \{t\colon Q+t \subseteq Q_0\}$. Observe that $Q^* \subseteq G+x$ for each $x \in Q$. Hence,

$$(3.3)\qquad |u_Q - u_{Q^*}| \leqslant c|Q_0|^{-2} \int_Q \Big[\int_{Q^*} |u(x)-u(y)|\,dy \Big]\,dx$$

$$\leqslant c|Q_0|^{-2} \int_Q \Big[\int_G |u(x)-u(x+t)|\,dt \Big]\,dx$$

$$\leqslant c|Q_0|^{-2}|Q_0|^{(\alpha q+n)/(nq)} \int_G |t|^{-(\alpha+n/q)} \Big(\int_Q |\Delta_t u|\,dx \Big)\,dt$$

$$\leqslant c|Q_0|^{-\lambda/(np^*)},$$

as follows from (1.2) and Hölder's inequality. To complete the proof of (3.2), divide Q_0 into $2^{n(n+3)}$ congruent subcubes $\{Q_i\}$ of side length ϱ by the bisection method. Since $u_{Q_0} = 2^{-n(n+3)} \sum_{i=1}^{2^{n(n+3)}} u_{Q_i}$, it follows from (3.3) that

$$|u_Q - u_{Q_0}| \leqslant |u_{Q_0} - u_{Q^*}| + |u_{Q^*} - u_Q| \leqslant 2^{-n(n+3)} \sum |u_{Q_i} - u_{Q^*}| + |u_{Q^*} - u_Q|$$

$$\leqslant c|Q_0|^{-\lambda/(np^*)},$$

whenever Q satisfies (3.1).

Now fix i, $1 \leqslant i \leqslant 2^{n(n+3)}$, and let Q_i be as above. We assume, as we may, that

$$Q_i = \{y = (y_1, \ldots, y_n):\ 0 \leqslant y_j \leqslant \varrho,\ 1 \leqslant j \leqslant n\} = Q(\varrho),$$

and $Q(5\varrho) \subseteq Q_0$ since otherwise we change coordinate systems. Given $x \in Q(\varrho)$, let

$$Q'(\delta, x) = \{x+y:\ \delta/2 \leqslant y_j \leqslant \delta,\ 1 \leqslant j \leqslant n\}.$$

Then by Lebesgue's theorem, we have, for almost every $x \in Q(\varrho)$,

$$\int_\delta^{\varrho/2} [u_{Q'(s,x)} - u_{Q'(2s,x)}]\frac{ds}{s} = \int_\delta^{2\delta} u_{Q'(s,x)}\frac{ds}{s} - \int_{\varrho/2}^{\varrho} u_{Q'(s,x)}\frac{ds}{s}$$

which tends to

$$\big(u(x) - u_{Q_0}\big)\log 2 - \int_{\varrho/2}^{\varrho} [u_{Q'(s,x)} - u_{Q_0}]\frac{ds}{s}$$

as $\delta \to 0$. Thus using (3.2), it follows that

$$(3.4)\qquad |u(x) - u_{Q_0}| \leqslant c\Big(\int_0^{\varrho/2} |u_{Q'(s,x)} - u_{Q'(2s,x)}|\frac{ds}{s} + |Q_0|^{-\lambda/(np^*)} \Big).$$

Next observe that for $0 < s < \varrho/2$,

$$|u_{Q'(s,x)} - u_{Q'(2s,x)}| \leqslant cs^{-2n} \int_{Q'(s,0)} \Big(\int_{Q'(2s,0)} |u(x+y) - u(x+t)|\,dt \Big)\,dy$$

$$\leqslant cs^{-2n} \int_{Q'(s,0)} \Big(\int_{Q(2s)} |u(x+y) - u(x+y+t)|\,dt \Big)\,dy$$

$$\leqslant cs^{\gamma-2n+1} g(s),$$

where

$$g(s) = s^{-(1+\gamma)} \int_{Q(2s)} \int_{Q(2s)} |u(x+y) - u(x+y+t)|\,dt\,dy.$$

Using the above inequality and integrating by parts, we get

$$(3.5)\qquad \int_0^{\varrho/2} |u_{Q'(s,x)} - u_{Q'(2s,x)}|\frac{ds}{s} \leqslant c \int_0^{\varrho} s^{\gamma-2n-1} \Big(\int_0^s g(\tau)\,d\tau \Big)\,ds.$$

Moreover,

$$(3.6)\qquad \int_0^s g(\tau)\,d\tau \leqslant \int_0^s \int_{Q(2s)} \Big[\int_{Q(2\tau)} |u(x+y) - u(x+y+t)|\,\tau^{-(1+\gamma)}\,dt \Big]\,dy\,d\tau$$

$$\leqslant c \int_{Q(2s)} \int_{Q(2s)} |t|^{-\gamma} |\Delta_t u|(x+y)\,dt\,dy,$$

as follows from interchanging the order of integration. Now let $f(x,t) = |t|^{-\gamma}\Delta_t u(x)$, when $(x,t) \in Q(3\varrho) \times Q(2\varrho)$, and $f(x,t) = 0$ otherwise, and set

$$V_x(|f|, s) = \int_{Q(2s)} \int_{Q(2s)} |f(x+y, t)|\,dy\,dt.$$

Then from (3.6), (3.5), and (3.4), we deduce that

$$|u(x) - u_{Q_0}| \leqslant c\Big[\int_0^\infty s^{\gamma-2n-1} V_x(|f|, s)\,ds + |Q_0|^{-\lambda/(np^*)} \Big],$$

for almost every $x \in Q_i$. Using (1.2) it is easily checked that (2.3) holds for $x \in Q_i$. From (2.3), the above inequality, and an argument as in the proof of Theorem 1, we obtain (1.4) with χ_{Q_0} replaced by χ_{Q_i}. Since Q_i is an arbitrary cube of the subdivision, it follows that Theorem 2 is valid.

4. Proof of Theorem 3. For fixed p, q, k, λ, α, as in Theorem 1, we construct $u(x)$ satisfying the conditions of Theorem 3 with $u \notin L(p^*, r)$ for $r < p^* q/p$. To do this, let $\langle a \rangle$ denote the greatest integer less than or equal to a. Let

$$m_j = \max\{2, \langle j(\log j)^2 \rangle^{p/nq}\}, \quad j = 1, 2, \ldots$$

Put $s_0 = 1$ and for $j \geqslant 1$, put

$$s_j = (s_{j-1})(m_j)^{-n/(n-\lambda)}, \quad t_j = (s_{j-1} - s_j)/(m_j - 1).$$

If $J = [a, b]$ is an interval with $b - a = s_{j-1}$, let $\Phi(J, m_j)$ be the m_j equally spaced closed intervals of length s_j contained in J defined by

$$\Phi(J, m_j) = \{[a+(i-1)t_j, a+(i-1)t_j+s_j]: 1 \leqslant i \leqslant m_j\}.$$

Define G_j inductively as follows:

$$G_0 = \{[0, 1]\}, \quad G_j = \{\Phi(J, m_j): J \in G_{j-1}\}, \quad j \geqslant 1.$$

From the definition of t_j and s_j, it is easily seen that

$$(4.1) \qquad c_0 s_j \leqslant t_j - s_j$$

for some constant c_0 independent of j. (4.1) implies that if $J \in G_{j-1}$, then the distance between successive intervals in $\Phi(J, m_j)$ is at least c_0 times the length of any one interval.

Let H_j, $j = 0, 1, 2, \ldots$, be the family of cubes defined by

$$H_j = \{Q = J_1 \times \ldots \times J_n: J_i \in G_j, 1 \leqslant i \leqslant n\}.$$

If $Q \in H_{l-1}$, $l \geqslant 1$, we note that

$$(4.2) \qquad \sum_{\substack{T \in H_j \\ T \subseteq Q}} |T|^{1-\lambda/n} = |Q|^{1-\lambda/n} = s_{l-1}^{n-\lambda},$$

when $j \geqslant l-1$. We now define u as follows: if $T \in H_{j-1}$, $j \geqslant 1$, choose a cube T', with sides parallel to the coordinate axes, and satisfying

$$(4.3) \qquad \begin{aligned} &T' \subset T - \bigcup_{Q \in H_j} Q, \\ &|T'| = (c_1 s_j)^n, \\ &\operatorname{dist}\Big(T', \bigcup_{Q \in H_j} Q\Big) \geqslant \sqrt{n}\, c_1 s_j \end{aligned}$$

for some positive constant $c_1 < 1$. That this choice is possible is clear from (4.1). Let $u \cdot \chi_{T'} \in C_0^\infty(T')$, with

$$(4.4) \qquad \|D^m(u \cdot \chi_{T'})\|_\infty \leqslant c\,|T'|^{-(mp^*+\lambda)/(np^*)}$$

for $m = 0, 1, 2, \ldots, k+1$, and

$$(4.5) \qquad u(x) = |T'|^{-\lambda/(np^*)}, \quad x \in T^*$$

where T^* is a subcube of T' with $|T^*| = 2^{-n}|T'|$. In (4.4), D^m represents an arbitrary mth derivative. We do this for each $T \in H_{j-1}$, $j \geqslant 1$, and set $u = 0$ elsewhere. For a set E, card E will denote the number of elements of E. Note from (4.2), if $j \geqslant l \geqslant 1$ and $Q \in H_{l-1}$, then

$$(4.6) \qquad \begin{aligned} \sum_{\substack{T \in H_{j-1} \\ T \subseteq Q}} |T'|^{1-\lambda/n} &= c s_j^{n-\lambda} \cdot \operatorname{card}\{T \in H_{j-1}: T \subseteq Q\} \\ &= c s_j^{n-\lambda} m_j^{-n} \cdot \operatorname{card}\{T \in H_j: T \subseteq Q\} \\ &= c m_j^{-n} |Q|^{1-\lambda/n}. \end{aligned}$$

We first show that $u \notin L(p^*, r)$ for $r < p^* q/p$. Let $\lambda_j = s_j^{-\lambda/p^*}$ for $j \geqslant 1$. Then from (4.3), (4.5), and (4.6) for $l = 1$, it follows that

$$\lambda_j^{p^*} |\{x: u(x) > \lambda_j\}| \geqslant c s_j^{-\lambda} \sum_{T \in H_{j-1}} |T'| = c \sum_{T \in H_{j-1}} |T'|^{1-\lambda/n} = c m_j^{-n}.$$

Thus for $r < qp^*/p$,

$$\sum \lambda_j^r |\{x: u(x) > \lambda_j\}|^{r/p^*} \geqslant c \sum m_j^{-nr/p^*} = +\infty.$$

This implies $u \notin L(p^*, r)$ since $\lambda_{j+1}/\lambda_j \geqslant c > 1$ for all $j \geqslant 1$.

Next we show that $u \in L^p(\mathbf{R}^n)$ and satisfies (1.2). In proving (1.2), we assume, as we may, that all cubes have sides parallel to the coordinate axes. We begin by showing

$$(4.7) \qquad \int_Q u^p\, dx \leqslant c s_l^{n-\lambda p/p^*}, \quad Q \in H_{l-1}, \quad l \geqslant 1,$$

$$(4.8) \qquad \int_Q u^p\, dx \leqslant c|Q|^{1-\lambda p/(np^*)}, \quad Q = \text{cube}.$$

Indeed, if $Q \in H_{l-1}$, $l \geqslant 1$, then from (4.4), we have

$$\int_Q u^p\, dx \leqslant c s_l^{n-\lambda p/p^*} + c \sum_{j=l+1}^{\infty} \Big(\sum_{\substack{T \in H_{j-1} \\ T \subseteq Q}} |T'|^{1-\lambda p/(np^*)} \Big),$$

and the expression inside the parentheses above does not exceed

$$c s_j^{\lambda(1-p/p^*)} \sum |T'|^{1-\lambda/n} \leqslant c s_j^{\lambda(1-p/p^*)} m_j^{-n} |Q|^{1-\lambda/n},$$

by (4.6). But upon using the inequalities: $s_{j+1}/s_j \leqslant c$, $m_j/m_{j+1} \leqslant c^{-1}$, $j \geqslant 0$ $c < 1$, and the definition of s_j, we easily get (4.7). Note that (4.7), with $l = 1$, implies $u \in L^p(\mathbf{R}^n)$.

Let $c_2 = (4nk)^{-1}\min\{c_0, c_1\}$, where c_0 and c_1 are as in (4.1) and (4.3), respectively. If $s(Q) \geqslant c_2$, then (4.8) follows from (4.7) with $l = 1$. If

$c_2 s_l \leqslant s(Q) < c_2 s_{l-1}$, $l \geqslant 1$, then Q meets at most one cube of H_{l-1} and at most one of the cubes T' used to define u, where $s(T') \geqslant c_1 s_{l-1}$. But we have (4.7) on cubes of H_{l-1}, and (4.4) implies that $u \leqslant c|Q|^{-\lambda/np^*}$ on such a T'. Hence, we conclude that (4.8) is valid.

Fix a cube Q. Note that to prove (1.2) it suffices to show

$$(4.9)\qquad \Big[\int\limits_{|t|\leqslant m(Q)}\Big(\int\limits_{Q}|\Delta_t^k u|^p\,dx\Big)^{q/p}|t|^{-(n+\alpha q)}dt\Big]^{1/q} \leqslant c|Q|^{(1-\lambda/n)/p},$$

where $m(Q) = c_2\cdot\min(s(Q),1)$, since otherwise (i.e. if $|t| \geqslant m(Q)$) (4.8) gives the result. To prove (4.9) we first show that for $q < \infty$

$$(4.10)\qquad \Big(\int\limits_{|t|<c_2 s_{l-1}}\sup\Big[\int\limits_{aQ}|\Delta_t^k u|^p\,dx\Big]^{q/p}|t|^{-(n+\alpha q)}dt\Big)^{1/q} \leqslant c s_{l-1}^{(n-\lambda)/p},$$

where the supremum is over $Q \in H_{l-1}$, and aQ is the cube with the same center as Q and a-times the side length. Here we take $a = 1+2kc_2$. Set b_j equal to the integral in (4.10) over the set where $c_2 s_j \leqslant |t| < c_2 s_{j-1}$, $j \geqslant l$. From (4.7), (4.1), and (4.3), it follows that

$$\int\limits_{aQ}|\Delta_t^k u|^p\,dx \leqslant c\int\limits_{Q}|u|^p\,dx \leqslant c s_l^{n-\lambda p/p^*}$$

when $Q \in H_{l-1}$, $c_2 s_l \leqslant |t| < c_2 s_{l-1}$. Hence,

$$(4.11)\qquad b_l \leqslant c s_l^{(n-\lambda p/p^*)q/p} s_l^{-\alpha q} \leqslant c s_{l-1}^{(n-\lambda)q/p}.$$

Now if $c_2 s_j \leqslant |t| < c_2 s_{j-1}$, $j \geqslant l+1$, and $T \in H_r$, $l-1 \leqslant r \leqslant j-2$, then (4.3), (4.4), and Taylor's theorem imply

$$(4.12)\qquad |\Delta_t^k u(x)|^p \leqslant c|T'|^{-(kp^*+\lambda)p/(np^*)}|t|^{kp},$$

whenever $x \in 2T'$. Let

$$L = \bigcup_{r=l-1}^{j-2}\Big(\bigcup_{\substack{T\in H_r\\ T\subseteq Q}} 2T'\Big)$$

and

$$M = \bigcup_{\substack{T\in H_{j-1}\\ T\subseteq Q}} aT$$

for $Q \in H_{l-1}$. For $c_2 s_j \leqslant |t| < c_2 s_{j-1}$, we note that $\Delta_t^k u(x)\cdot\chi_{aQ\setminus L}(x)$ is supported by M. Thus

$$\int\limits_{aQ}|\Delta_t^k u|^p\,dx = \int\limits_{L}|\Delta_t^k u|^p\,dx + \int\limits_{M}|\Delta_t^k u|^p\,dx.$$

And from (4.12) and (4.6), we get

$$(4.13)\qquad \int\limits_{L}|\Delta_t^k u|^p\,dx \leqslant c\sum_{r=l-1}^{j-2}\Big(\sum_{T\in H_r}|T'|^{1-(kp^*+\lambda)p/(np^*)}\Big)|t|^{kp}$$
$$\leqslant c\Big[\sum_{r=l-1}^{j-2}s_{r+1}^{(\alpha-k)p}m_{r+1}^{-n}\Big]|Q|^{1-\lambda/n}|t|^{kp}$$
$$\leqslant c m_{j-1}^{-n}s_{j-1}^{(\alpha-k)p}|Q|^{1-\lambda/n}|t|^{kp},$$

where again we have used the inequality $s_{l+1}/s_l \leqslant c \leqslant 1$, $l \geqslant 0$. And from (4.1), (4.3), (4.6), and (4.7), we get

$$(4.14)\qquad \int\limits_{M}|\Delta_t^k u|^p\,dx = \sum_{\substack{T\in H_{j-1}\\ T\subseteq Q}}\int\limits_{aT}|\Delta_t^k u|^p\,dx$$
$$\leqslant c\sum\int\limits_{T}u^p\,dx \leqslant c s_j^{\alpha p}m_j^{-n}|Q|^{1-\lambda/n}.$$

From (4.13) and (4.14), we then have for $j \geqslant l+1$,

$$b_j \leqslant c[m_j^{-nq/p} + m_{j-1}^{-nq/p}]s_{l-1}^{(n-\lambda)q/p}.$$

Hence (4.10) follows upon summing the b_j, $j \geqslant l$. If $q = \infty$, (4.10) also holds, as easily follows from (4.11)–(4.14).

Finally, we prove (4.9). If $s(Q) \geqslant c_2$, (4.9) follows from (4.10) with $l = 1$. Suppose that $c_2 s_l \leqslant s(Q) < c_2 s_{l-1}$, $l \geqslant 1$. We consider two cases: If $Q\cap T = \{\emptyset\}$ for all $T \in H_{l-1}$, then from (4.4)

$$\int\limits_{Q}|\Delta_t^k u|^p\,dx \leqslant c|Q|^{1-(kp^*+\lambda)p/(np^*)}|t|^{kp},$$

for $|t| \leqslant c_2 s(Q)$. Using this estimate, one easily gets (4.9). If $Q\cap T \neq \{\emptyset\}$ for some $T \in H_{l-1}$, then for $c_2 s_l \leqslant |t| \leqslant c_2 s(Q)$, it follows from (4.1), (4.3), and (4.7) that

$$(4.15)\qquad \int\limits_{Q}|\Delta_t^k u|^p\,dx \leqslant c\int\limits_{T}u^p\,dx \leqslant c s_l^{n-\lambda p/p^*}.$$

If $|t| < c_2 s_l$, then $\Delta_t^k u\cdot\chi_Q$ has its support contained in $2T'\cup\big(\bigcup_{P\in W} aP\big)$, where $W = \{P \in H_l\colon aP\cap Q \neq \emptyset\}$. From (4.10) with $l-1$ replaced by l and (4.15), it follows that

$$(4.16)\qquad \Big[\int\limits_{|t|\leqslant c_2 s(Q)}\Big(\int\limits_{Q}|\Delta_t^k u|^p\,dx\Big)^{q/p}|t|^{-(n+\alpha q)}dt\Big]^{1/q}$$
$$\leqslant c s_l^{(n-\lambda)/p} + c\Big[\int\limits_{|t|\leqslant c_2 s_l}\Big(\int\limits_{2T'}|\Delta_t^k u|^p\,dx\Big)^{q/p}|t|^{-(n+\alpha q)}dt\Big]^{1/q} +$$
$$+\,c(\operatorname{card}W)^{1/p}\Big[\int\limits_{|t|\leqslant c_2 s_l}\sup\Big(\int\limits_{aP}|\Delta_t^k u|^p\,dx\Big)^{q/p}|t|^{-(n+\alpha q)}dt\Big]^{1/q}$$
$$\leqslant c s_l^{(n-\lambda)/p} + c[(\operatorname{card}W)\cdot s_l^{n-\lambda}]^{1/p}.$$

So to complete the proof of (4.9), it remains to show

$$(4.17)\qquad (\operatorname{card} W)\cdot s_l^{n-\lambda} \leqslant c|Q|^{1-\lambda/n}.$$

To do this note from the definition of H_l that if $s(Q) = \gamma t_l + s_l$, $0 < \gamma \leqslant m_l$, then $\operatorname{card} W \leqslant (\gamma+3)^n$. Now

$$\begin{aligned}[s(Q)]^{1-\lambda/n} &= \big[\big((m_l-1-\gamma)s_l+\gamma s_l\big)/(m_l-1)\big]^{1-\lambda/n} \\ &\geqslant [(m_l-1-\gamma)s_l^{1-\lambda/n}+\gamma s_{l-1}^{1-\lambda/n}]/(m_l-1) \\ &= (\gamma+1)s_l^{1-\lambda/n}.\end{aligned}$$

Here we have used the concavity of the function $x \to x^{1-\lambda/n}$, the fact that $s_{l-1}^{1-\lambda/n} = m_l s_l^{1-\lambda/n}$, and the definition of t_l. Thus (4.17) is true. Using (4.17) in (4.16), we get (4.9). The proof of Theorem 3 is now complete.

5. Remarks. Theorem 2 can be extended to include values of $a > 1$, $a \neq$ integer as follows. Suppose $u(x)$ is as in Theorem 2 with now $k > 1$, then it can be shown that u has derivatives of order $\leqslant \langle a\rangle$ which belong to $L^p(Q_0)$. Let P be the unique polynomial of order $\langle a\rangle$ which satisfies

$$\int_{Q_0} (D^j u - D^j P)\,dx = 0$$

$0 \leqslant j \leqslant \langle a\rangle$, where D^j represents any jth derivative. Then Theorem 2 holds with u_{Q_0} replaced by P.

Next we indicate some extensions of Theorem 1. As we saw there, any function in $L^p(\boldsymbol{R}^n)$ which satisfies (1.2) can be written as the trace of a Riesz potential over $\boldsymbol{R}^{2n}$. This motivates us to consider simply Riesz potentials of functions that satisfy mixed L^p-L^q Morrey type conditions and the corresponding integrability classes. To be more specific (and in turn considering the case $n = 1$ for simplicity) consider the general mixed Morrey condition for non-negative measurable $f(x, t)$ on $\boldsymbol{R}^2$

$$(5.1)\qquad \sup\Big[r^{\mu-1}\int_{|s-t|<r}\Big\{r^{\lambda-1}\int_{|y-x|<r} f(x,t)^p\,dx\Big\}^{q/p} dt\Big]^{1/q} < \infty,$$

where the supremum is over all $r > 0$ and all $y, s \in \boldsymbol{R}^1$. Here $0 < \mu, \lambda \leqslant 1$, $1 < p, q \leqslant \infty$. Functions f satisfying (5.1) will belong to the space $T^{q;\mu}X^{p;\lambda}$ and have norm $T^{q;\mu}X^{p;\lambda}(f)$ given by (5.1). (Using this notation, the function $f(x, t)$ in (2.1) belongs to $T^{q;1}X^{p;\lambda}$ for $n = 1$.) We seek estimates of the form

$$(5.2)\qquad T^{q^*;\mu}X^{p^*;\lambda}(I_a f) \leqslant c\cdot T^{q;\mu}X^{p;\lambda}(f)$$

for some constant c independent of f. To accomodate the use of Lorentz spaces as in Theorem 1, we write (5.2) more generally as

$$(5.3)\qquad T^{q^*;\mu}X^{(p^*,\bar{p});\lambda}(I_a f) \leqslant c\,T^{q;\mu}X^{p;\lambda}(f).$$

The left side of (5.3) is now

$$\sup\Big[r^{\mu-1}\int_{|s-t|<r}\Big\{\int_0^\infty (\sigma^{p^*} r^{\lambda-1}|E_\sigma(r)|)^{\bar{p}/p^*}\frac{d\sigma}{\sigma}\Big\}^{q^*/\bar{p}} dt\Big]^{1/q^*},$$

where $E_\sigma(r) = \{x\colon |y-x| < r \,\&\, I_a f(x,t) > \sigma\}$. In this notation, inequality (2.2) can be stated as

$$T^{\infty;1}X^{(p^*,qp^*/p);\lambda}(I_{a+1/q}f) \leqslant c\cdot T^{q;1}X^{p;\lambda}(f).$$

Now in general from (5.2), we expect the parameters to be related by:

$$a = \lambda(1/p - 1/p^*) + \mu(1/q - 1/q^*)$$

with $1 < q \leqslant q^* \leqslant \infty$, $1 < p \leqslant p^* \leqslant \infty$. This follows by replacing $f(x, t)$ by $f(\delta x, \delta t)$, $\delta > 0$, in (5.2). Using this relationship between the parameters and the techniques of [1] and [2], we can also get the following estimates in the spirit of (5.2):

$$(5.4)\qquad T^{q^*;\mu}X^{p^*;1}(I_a f) \leqslant c\cdot T^{q;\mu}X^{p;1}(f),$$

whenever $p/p^* < q/q^*$, $\mu \leqslant 1$,

$$(5.5)\qquad T^{q^*;1}X^{p^*;\lambda}(I_a f) \leqslant c\cdot T^{q;1}X^{p;\lambda}(f),$$

whenever $p/p^* > q/q^*$, $\lambda \leqslant 1$, and $q \leqslant p$. Furthermore, when $p/p^* = q/q^*$, both λ and $\mu \leqslant 1$ are permitted in (5.4) and (5.5). We won't present the details of these estimates here since they are not in the mainstream of this paper and, because of the restriction $q \leqslant p$ in (5.5), they are incomplete. We mention here only that the key idea is to estimate the potential $I_a f$ in each case by iterated maximal functions of f in the two directions.

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Hopf's theorem on invariant measures for a group of transformations

by

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Abstract. Let G be a group of measurable and nonsingular transformations on $(X, \mathscr{A}, P)$. Some results on the equivalence (by countable decomposition) of measurable sets are proved, using which Hopf's theorem on invariant measures for a group of transformations is derived.

1. Introduction. Let $(X, \mathscr{A}, P)$ be a probability space, and let G be a group of measurable and nonsingular transformations defined on $(X, \mathscr{A}, P)$. Hopf [4] gave a necessary and sufficient condition for the existence of a finite invariant measure μ on $\mathscr{A}$ equivalent to P for the case where G is a cyclic group. In this paper we prove that condition (H), the analogue of Hopf's condition, when G is a general group of transformations, is necessary and sufficient for the existence of a finite equivalent measure invariant under every g in G. The only known alternative proof (see Hajian and Ito [3]) of Hopf's theorem for a group of transformations is through the equivalence of condition (H) and the condition of nonexistence of weakly wandering sets of positive measure introduced by Hajian and Kakutani. While Hajian and Ito use functional analytic methods in their proof, we use a direct argument motivated by an idea of Tarski (see [6], Definition 1.25) regarding measures on semigroups. Our proof constructs the invariant measure and shows that it is the unique, finitely additive, G-invariant measure equivalent to P which equals P on almost invariant sets.

2. Definitions and notation. Let $(X, \mathscr{A}, P)$ be a probability space. A finitely additive measure μ on $\mathscr{A}$ is said to be *equivalent to* P if μ and P have the same sets of measure zero, that is, $\mu(A) = 0$ iff $P(A) = 0$ for every $A \in \mathscr{A}$. A measurable map g of X into itself is called *nonsingular* if, for every $A \in \mathscr{A}$, $P(A) > 0$ implies $P(g^{-1}A) > 0$. For $A, B \in \mathscr{A}$, $A \overset{P}{=} B$ iff $P(A \triangle B) = 0$, where $\triangle$ is the symmetric difference.

Suppose that G is a group of measurable and nonsingular transformations defined on $(X, \mathscr{A}, P)$, that is, each $g \in G$ is a 1-1 bimeasurable and