[3] P. L. Butzer, R. J. Nessel and W. Trebels, *On summation processes of Fourier expansions in Banach spaces III: Jackson- and Zamansky-type inequalities for Abel-bounded expansions*, Tôhoku Math. J. 27 (1975), 213–223.

[4] D. J. H. Garling, *On topological sequence spaces*, Proc. Cambridge Philos. Soc. 63 (1967), 997–1019.

[5] G. Goes, *Summen von FK-Räumen; Funktional Abschnittskonvergenz und Umkehrsätze*, Tôhoku Math. J. (II) 26 (1974), 487–504.

[6] C. Goffman and G. Pedrick, *A First Course in Functional Analysis*, Prentice-Hall Inc., Englewood Cliffs 1965.

[7] G. H. Hardy and J. E. Littlewood, *A further note on the converse of Abel's Theorem*, Proc. London Math. Soc. (2) 25 (1926), 219–236.

[8] J. M. Hyslop, *A Tauberian Theorem for absolute summability*, J. London Math. Soc. 12 (1937), 176–180.

[9] J. Karamata, *Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjessche Transformation betreffen*, J. Reine Angew. Math. 164 (1931), 27–39.

[10] J. E. Littlewood, *The converse of Abel's theorem on power series*, Proc. London Math. Soc. (2), 9 (1910), 434–448.

[11] D. G. Northcott, *Abstract Tauberian Theorems with applications to power series and Hilbert series*, Duke Math. J. 14 (1947), 483–502.

[12] W. Orlicz, *Über k-fach montone Folgen*, Studia Math. 6 (1936), 149–159.

[13] J. Pfleger, *Über Reihen mit r-fach monton abnehmenden Gliedern*, Monatsh. Math. Phys. 41 (1934), 191–200.

[14] W. H. Ruckle, *An abstract concept of the sum of a numerical series*, Canad. J. Math. 22 (1970), 863–874.

[15] — *A universal topology for sequence spaces*, Math. Ann. 236 (1978), 43–48.

[16] A. Tauber, *Ein Satz aus der Theorie der unendlichen Reihen*, Monatsch. Math. 8 (1897), 273–277.

[17] H. Tietz, *Über absolute Tauber-Bedingungen*, Math. Z. 113 (1970), 136–144.

[18] A. Wilansky, *Functional Analysis*, Blaisdell Publ. Co., New York 1964.

[19] L. Włodarski, *Sur les methodes continues de limitation* (I) *and* (II), Studia Math. 14 (1955), 161–187, 188–199.

[20] K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. 53 (1951), 463–487.

[21] — *Abschnittskonvergenz in FK-Räumen*, ibid. 55 (1951), 55–70.

[22] — *Approximation in Wirkfeldern von Summierungsverfahren*, Arch. Math. 4 (1953), 425–431.

[23] — *Sur la méthode de sommation d'Abel*, C. R. Acad. Sci. Paris 236 (1953), 568–569.

[24] A. Zygmund, *Trigonometric Series*, Cambridge University Press, London 1959)

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Singular integrals supported on submanifolds

by

ROBERT S. STRICHARTZ* (Ithaca, N.Y.)

Abstract. Three results related to Calderón-Zygmund singular integrals in generalized contexts are established. In the first, Hilbert transforms supported on hypersurfaces of the form $\prod |x_j|^{b_j} = r$ where b_j are non-zero real numbers, are proved bounded on L^p for certain values of p (Nagel and Wainger proved L^2 boundedness). In the second, L^p boundedness is established for convolutions on $\boldsymbol{R}^2$ with kernels of the form $g(|x_1x_2|^{1/2})\operatorname{sgn} x_1$ by transference from known results about radial kernels. In the third, the "method of rotations" is carried through for the L^2 theory of Knapp–Stein singular integrals on 2-stage nilpotent Lie groups.

§ 1. Introduction. The Calderón–Zygmund theory of singular integrals has been extended in many directions in recent years. One major theme in these extensions is the "method of rotations" introduced in [1]. This leads to the study of various Hilbert transforms supported on curves or more general submanifolds. The reader is urged to consult the excellent exposition of these ideas in Stein and Wainger [10].

In this paper we present three contributions to this study. The first concerns Hilbert transforms supported on hypersurfaces in $\boldsymbol{R}^n$ defined by the equation $\prod_{j=1}^{n} |x_j|^{b_j} = r$ where $b_1, \ldots, b_n$ are non-zero real numbers. The L^2 boundedness of these operators was established by Nagel and Wainger [7]. We give an independent proof that also establishes L^p boundedness for certain values of p. We do not know if these values of p are best possible.

Our second contribution is a transference result relating convolutions on $\boldsymbol{R}^2$ with functions of the form $g(|x_1x_2|^{1/2})\operatorname{sgn} x_1$ to convolutions with the radial function $g(\sqrt{x_1^2+x_2^2})$. We show that L^2-boundedness is equivalent for these two operators, and the applicability of the Marcinkiewicz multiplier theorem is also equivalent.

Our third contribution is to the Knapp–Stein [5] theory of singular integrals on nilpotent Lie groups. For 2-stage groups we carry out the method of rotations for L^2-boundedness. We use the Euclidean Plancherel

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formula, and our proof is independent of the Cotlar lemma approach of [5].

These three contributions are presented in the next three sections, which can be read independently of each other. None our results are definitive; we indicate a number of open problems at the end of each section.

§ 2. Hilbert transforms on hypersurfaces. Let $b_1, \ldots, b_n$ be non-zero real numbers, $r > 0$, and let

$$M_r = \{x \in \mathbf{R}^n: \pi(x) = r\} \quad \text{where} \quad \pi(x) = \prod_{j=1}^{n} |x_j|^{b_j}.$$

Then M_r is a hypersurface with 2^n components (depending on the signs of the x_j); we will call the *principal component* the one for which all the x_j's are positive.

The *Hilbert transform supported on* M_r is the operator of convolution with the tempered distribution H_r supported on M_r which is odd in each variable and equal to $b_k^{-1} \prod_{j\neq k} \frac{dx_j}{x_j}$ in local coordinates (this is independent of k). If all the b_k's have the same sign then H_r is a locally finite measure, but in general H_r is defined by means of a principal value integral that depends on the oddness of the kernel. It is easily verified that this Hilbert transform commutes with the $n-1$-dimensional group D of diagonal matrices with positive diagonal entries that preserve $\pi(x)$, and this property characterizes H_r up to a constant multiple on each component of M_r. We also note the change of variable formula

$$\int_{\mathbf{R}^n} f(x)\pi(x)^a \prod_{j=1}^{n} \frac{dx_j}{x_j} = \int_0^\infty \langle H_s, f\rangle s^{a-1}\,ds \tag{2.1}$$

which is valid for test functions f supported away from the coordinate walls $\prod_{j=1}^{n} x_j = 0$.

Next we introduce an analytic family of distributions F_λ depending on the complex parameter λ via the identity

$$\langle F_\lambda, f\rangle = \Gamma(\lambda+1)^{-1} \int_{\mathbf{R}^n} f(x)\big(r-\pi(x)\big)_+^\lambda \pi(x) \prod_{j=1}^{n} \frac{dx_j}{x_j}. \tag{2.2}$$

The integral is convergent if $\operatorname{Re}\lambda \geqslant 0$ and f is a test function supported away from the coordinate walls. From (2.1) we see that

$$\langle F_\lambda, f\rangle = \Gamma(\lambda+1)^{-1} \int_0^r (r-s)^\lambda \langle H_s, f\rangle\,ds \tag{2.3}$$

which gives the analytic continuation of $\operatorname{Re}\lambda < 0$. Note then that $\langle F_{-1}, f\rangle = \langle H_r, f\rangle$. We will later show that $\langle F_\lambda, f\rangle$ can be defined without any restriction on the support of f, so we can conclude $F_{-1} = H_r$ as distributions.

Our goal is to compute the Fourier transform of F_λ, and to this

end we will use the method of Mellin analysis used in [11]. From the well-known Beta integral and the Mellin inversion formula we have

$$\Gamma(\lambda+1)^{-1}(r-t)_+^\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\sigma+i\varrho)}{\Gamma(\lambda+1+\sigma+i\varrho)}\, r^{\lambda+\sigma+i\varrho} t^{-\sigma-i\varrho}\,d\varrho \tag{2.4}$$

for any $\sigma > 0$ if $\operatorname{Re}\lambda > 0$, and these conditions on σ and λ will make the integral absolutely convergent. We may substitute $t = \pi(x)$ in (2.4) with $\sigma = 1$ and substitute the result into (2.2) to obtain

$$\langle F_\lambda, f\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} f(x)\pi(x)^{-i\varrho} \prod_{j=1}^{n} \frac{dx_j}{x_j}\, r^{1+\lambda+i\varrho} \frac{\Gamma(1+i\varrho)}{\Gamma(\lambda+2+i\varrho)}\,d\varrho. \tag{2.5}$$

But there is a well-known formula for the Fourier transform of $x_j^{-1}|x_j|^{-b_j-i\varrho}$, namely

$$i\pi^{-1/2+b_j i\varrho} \frac{\Gamma\big((1-b_j i\varrho/2)\big)}{\Gamma\big((2+b_j i\varrho)/2\big)} |x_j|^{b_j i\varrho} \operatorname{sgn} x_j$$

(see [9]), so we may compute

$$\langle F_\lambda, \hat f\rangle = \frac{i^n}{2\pi} \int_{-\infty}^{\infty} \Big(\int_{\mathbf{R}^n} f(x)\pi(x)^{i\varrho} \prod_{j=1}^{n} (\operatorname{sgn} x_j)\,dx\Big) \cdot \pi^{-n/2+\Sigma b_j i\varrho} \prod_{j=1}^{n} \frac{\Gamma\big((1-b_j i\varrho)/2\big)}{\Gamma\big((2+b_j i\varrho)/2\big)} \frac{\Gamma(1+i\varrho)}{\Gamma(\lambda+2+i\varrho)} r^{1+\lambda+i\varrho}\,d\varrho \tag{2.6}$$

assuming the text function $\hat f$ is supported away from the coordinate walls

Now if $\operatorname{Re}\lambda > -n/2$ then the right side of (2.6) is an absolutely convergent integral (using Stirling's formula to estimate the Γ factors), so we may write $\langle F_\lambda, \hat f\rangle = \int_{\mathbf{R}^n} f(x) m_\lambda(x)\,dx$ where

$$m_\lambda(x) = \frac{i^n r^{1+\lambda}}{2\pi^{1+n/2}} \prod_{j=1}^{n} (\operatorname{sgn} x_j) \int_{-\infty}^{\infty} \big(\pi(x) r \pi^{\Sigma b_j}\big)^{i\varrho} \cdot \frac{\Gamma(1+i\varrho)}{\Gamma(\lambda+2+i\varrho)} \prod_{j=1}^{n} \frac{\Gamma\big((1-b_j i\varrho)/2\big)}{\Gamma\big((2+b_j i\varrho)/2\big)}\,d\varrho. \tag{2.7}$$

Since $m_\lambda(x)$ is clearly a bounded function for $\operatorname{Re}\lambda > -n/2$ this enables us to simultaneously show that F_λ is a tempered distribution and compute its Fourier transform. Actually we can do slightly better, if $\sum b_j \neq 0$, allowing $\operatorname{Re}\lambda \geqslant -\big((n+1)/2\big)$, although the integral defining m_λ no longer will be absolutely convergent.

LEMMA 2.1. *If $\sum b_j \neq 0$ then the functions m_λ may be analytically* *continued to* $\operatorname{Re}\lambda \geqslant -\big((n+1)/2\big)$ *as* L^∞ *functions of* x, *with* $\|m_\lambda\|_\infty \leqslant A(\lambda)r^{1+\operatorname{Re}\lambda}$ *where* $A(\lambda)$ *has at worst exponential growth in* $\operatorname{Im}(\lambda)$.

Proof. Let $m_{\lambda,N}(x)$ be defined by the same formula as $m_\lambda(x)$ except that the integral is truncated to the domain $|\varrho| \leqslant N$. For fixed $x \neq 0$ and N this is clearly an entire function of λ, and we seek to estimate it independent of N.

Now Stirling's formula with remainder can be written (see [4])

$$\Gamma(z) = \sqrt{2\pi}\exp\big((z+1/2)\log z - z\big)\big(1+w(z)\big)$$

where $|w(z)| \leqslant c/|z|$ if $\operatorname{Re} z$ is bounded below and z does not lie on the negative real axis (the principal branch of the logarithm cut along the negative real axis is taken). From this we get the estimates

$$\frac{\Gamma(a_1-bi\varrho)}{\Gamma(a_2+bi\varrho)} = (e|b|)^{a_1-a_2}e^{i\varrho(2b-2b\log b)}\cdot e^{-\frac{\pi i}{2}(a_1+a_2+1)\operatorname{sgn}(b\varrho)}|\varrho|^{a_1-a_2-2bi\varrho} + O(|\varrho|^{a_1-a_2-1}),$$

$$\frac{\Gamma(a_1+i\varrho)}{\Gamma(a_2+i\varrho)} = e^{a_2-a_1}e^{\frac{\pi i}{2}(a_1-a_2)\operatorname{sgn}\varrho}|\varrho|^{a_1-a_2}+O(|\varrho|^{a_1-a_2-1}).$$

For λ real this allows us to estimate the integrand in (2.7) by $|\varrho|^{-\lambda-1-n/2}$, and so for $\lambda \geqslant -\big((n+1)/2\big)$ we can ignore all the remainder terms since they are at least of order $|\varrho|^{-3/2}$. Thus we need to establish the uniform boundedness (independent of N and t) of $\int_1^N e^{it\varrho}\varrho^{-c-Bi\varrho}d\varrho$ where $B = \sum_{j=1}^n b_j \neq 0$ and $c = \lambda+1+n/2 \geqslant 1/2$.

We establish this uniform boundedness by a Van der Corput Lemma argument. We break up the interval of integration according as $|t-B-B\log\varrho|$ is greater than or less than $\frac{1}{2}B$; this results in at most three components, so it sufficies to bound each separately. If $|t-B-B\log\varrho| \geqslant \frac{1}{2}B$ on the interval $[a, b] \subseteq [1, N]$ then

$$\int_a^b e^{it\varrho}\varrho^{-c-Bi\varrho}d\varrho = \int_a^b \varphi'(\varrho)\psi(\varrho)\,d\varrho = \varphi(b)\psi(b)-\varphi(a)\psi(a)-\int_a^b \varphi(\varrho)\psi'(\varrho)\,d\varrho$$

where $\varphi(\varrho) = e^{i(t-B\log\varrho)\varrho}$ and $\psi(\varrho) = -i\varrho^{-c}(t-B-B\log\varrho)^{-1}$. But φ and ψ are bounded on the interval and

$$\psi'(\varrho) = -ic\varrho^{-c-1}(t-B-B\log\varrho)^{-1}-iB\varrho^{-c-1}(t-B-B\log\varrho)^{-2}$$

is integrable since $-c-1 < -1$.

If $|t-B-B\log\varrho| \leqslant \frac{1}{2}B$ on the interval $[a, b] \subseteq [1, N]$ then also $|t-2B-B\log\varrho| \geqslant \frac{1}{2}B$ on that interval. Making the change of variable

$s = \log\varrho$ converts the integral to $\int_{\log a}^{\log b} e^{ig(s)}e^{(1-c)s}ds$ where $g(s) = (t-Bs)e^s$. Note that $g''(s) = (t-2B-Bs)e^s$ so $|g''(s)| \geqslant \frac{1}{2}Be^s$ on the interval $[\log a, \log b]$. By Van der Corput's Lemma ([14], vol. I, p. 197) the integral of $e^{ig(s)}$ over any subinterval is $\leqslant \text{const.}\, e^{-s/2}$, hence by integration by parts the integral is bounded provided $c \geqslant 1/2$ because the interval $[\log a, \log b]$ is of uniformly bounded length and $e^{(1/2-c)s}$ is bounded on $s \geqslant 0$.

Thus we have established the uniform boundedness of $m_{\lambda,N}(x)$ for λ real and $\lambda \geqslant -\big((n+1)/2\big)$. But if λ is complex, say $\lambda = \lambda_0+is$, this just introduces a factor

$$\frac{\Gamma(a+i\varrho)}{\Gamma\big(a+i(\varrho+s)\big)}$$

where $a = \lambda_0+2$ which, using Stirling's formula, only increases the bound by a factor which grows exponentially in s. The same argument which establishes the uniform boundedness of $m_{\lambda,N}$ also shows the limit exists as $N\to\infty$, and the remainder of the proof is routine. ■

Thus we have the Hilbert transform $H_r * f$ embedded in an analytic family of operators $F_\lambda * f = \mathfrak{F}^{-1}(m_\lambda\hat{f})$ at $\lambda = -1$, and $F_\lambda * f$ is bounded on L^2 for $\operatorname{Re}\lambda \geqslant -\big((n+1)/2\big)$ if $\sum b_j \neq 0$ or $\operatorname{Re}\lambda > -n/2$ if $\sum b_j = 0$. To obtain L^p boundedness we will use interpolation.

THEOREM 2.2. $\|H_r * f\|_p \leqslant A_p\|f\|_p$ *provided*

$$\frac{4n}{3n-1} < p < \frac{4n}{n+1}$$

if $\sum b_j \neq 0$ *or*

$$\frac{4n}{3n-2} < p < \frac{4n}{n+2}$$

if $\sum b_j = 0$, *where* A_p *is independent of* r.

Proof. By Stein's interpolation theorem for analytic families of operators, it suffices to show that convolution with F_λ is a bounded operator on all L^p with $1 < p < \infty$ provided $\operatorname{Re}\lambda \geqslant (n-1)/2$ if $\sum b_j \neq 0$ or $\operatorname{Re}\lambda > n/2$ if $\sum b_j = 0$, where the operator norm is also bounded by $A(\lambda)r^{1+\operatorname{Re}\lambda}$. We will do this by appealing to the Marcinkiewcz multiplier theorem, since the multipliers m_λ are given explicitly by (2.7). The hypotheses of the Marcinkiewcz multiplier theorem will be satisfied if we can show that $x^\alpha\left(\frac{\partial}{\partial x}\right)^\alpha m_\lambda(x)$ is bounded for all multi-indices α with each $\alpha_j = 0$ or 1. But applying $x^\alpha\left(\frac{\partial}{\partial x}\right)^\alpha$ to $m_\lambda(x)$ merely introduces a factor of $\varrho^{|\alpha|}$ into (2.7), so the proof of the lemma applies if we increase λ by n to handle the additional factors. ■

If all the b_j's are 1 then the Hilbert transform satisfies some additional estimates.

THEOREM 2.3. *If* $b_1 = \ldots = b_n = 1$ *then* $\|H_r * f\|_q \leqslant A r^{(1-n)/(1+n)} \|f\|_p$ *for* $p = (n+1)/n$ *and* $q = n+1$.

Proof. From (2.2) we see that F_λ is a bounded function when $\operatorname{Re}\lambda = 0$, so $\|F_\lambda * f\|_\infty \leqslant A(\lambda)\|f\|_1$. The result follows again by interpolation of this with Lemma 2.1. ■

Remark. The results of this section remain valid if in place of H_r, which is odd in all variables, we take the distribution which is odd in all but one variable and even in the remaining variable. For this change means that one of the factors

$$\frac{\Gamma((1-b_j i\varrho)/2)}{\Gamma((2+b_j i\varrho)/2)}$$

in (2.7) is replaced by

$$\frac{\Gamma(-(b_j i\varrho)/2)}{\Gamma((2+b_j i\varrho)/2)}.$$

This change does not affect the estimates for large ϱ, but does introduce a singularity at $\varrho = 0$ due to $\Gamma(-b_j i\varrho/2)$. But this means the integrand in (2.7) is of the form $e^{it\varrho}\varrho^{-1}g(\varrho)$ near $\varrho = 0$ where g is analytic, and this has a bounded integral by the L^2 boundedness of the classical Hilbert transform.

It is not possible to allow evenness with respect to more than one variable, however, for then the singularity near $\varrho = 0$ in (2.7) would be too severe.

We do not know if the restrictions on p in the theorems are necessary, or if they are merely the result of the inadequacy of the method.

It seems likely that the method can be extended to deal with the Hilbert transforms supported on lower dimensional submanifolds discussed in Nagel and Wainger [7]. This would require a multi-dimensional generalization of the Van der Corput Lemma.

§ 3. Singular integrals in $\boldsymbol{R}^2$. In this section we specialize to the case $n = 2$, $b_1 = b_2 = 1$. We will study the convolution operators that commute with the unimodular dilation group of matrices

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \quad a > 0.$$

If g is any function of a single real variable let us write

$$G_1(x_1, x_2) = g(|x_1 x_2|^{1/2})\operatorname{sgn} x_1,$$
$$G_2(x_1, x_2) = g(|x_1 x_2|^{1/2})\operatorname{sgn} x_2,$$
$$G_0(x_1, x_2) = g(\sqrt{x_1^2 + x_2^2}).$$

Clearly convolution with G_1 or G_2 will commute with unimodular dilations, while convolution with G_0 commutes with rotations. The Fourier transform of G_0 is given by the well-known formula

$$\hat{G}_0(\xi_1, \xi_2) = 2\pi \int_0^\infty J_0(2\pi s|\xi|) g(s) s\, ds. \tag{3.1}$$

The Fourier transform of G_1 and G_2 is given by a closely related formula. This can be deduced by the methods of the previous section (as in [11]), or more directly as follows:

$$\begin{aligned} \hat{G}_1(\xi_1, \xi_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(|x_1 x_2|^{1/2}) \operatorname{sgn} x_1 e^{2\pi i(x_1\xi_1 + x_2\xi_2)}\, dx_1 dx_2 \\ &= 8i \int_0^\infty \int_0^\infty g(s)\sin(2\pi x_1 \xi_1)\cos(2\pi s^2 \xi_2 x_1^{-1}) s\, ds\, x_1^{-1} dx_1 \\ &= 4\pi i \operatorname{sgn}\xi_1 \int_0^\infty J_0(4\pi s|\xi_1\xi_2|^{1/2}) g(s) s\, ds \end{aligned} \tag{3.2}$$

where we have used the well-known identity

$$J_0(x) = 2\pi^{-1}\int_0^\infty \sin(x\cosh t)\, dt$$

(see [13]) to obtain the last line. Obviously the formula for $\hat{G}_2$ is the same except that $\operatorname{sgn}\xi_1$ must be replaced by $\operatorname{sgn}\xi_2$. By comparing (3.2) and (3.1) we deduce immediately

THEOREM 3.1. *Convolution with G_1 (or G_2) is bounded on $L^2(\boldsymbol{R}^2)$ if and only if convolution with G_0 is bounded on $L^2(\boldsymbol{R}^2)$ with operator norm exactly twice as large.*

To obtain transference results from G_0 to G_1 (or G_2) for L^p boundedness we first recall the statement of the Marcinkiewicz multiplier theorem in $\boldsymbol{R}^2$:

If T is a distribution on $\boldsymbol{R}^2$ and $m = \hat{T}$ then the *Marcinkiewicz conditions* are satisfied if m is C^2 away from the coordinate walls and there exists a constant B such that for all integers j, k,

$$|m(\xi_1, \xi_2)| \leqslant B, \tag{3.3}$$

$$\sup_{\xi_2} \int_{\pm 2^k}^{\pm 2^{k+1}} \left|\frac{\partial m}{\partial \xi_1}(\xi_1, \xi_2)\right| d\xi_1 \leqslant B, \tag{3.4}$$

$$\sup_{\xi_1} \int_{\pm 2^k}^{\pm 2^{k+1}} \left|\frac{\partial m}{\partial \xi_2}(\xi_1, \xi_2)\right| d\xi_2 \leqslant B, \tag{3.5}$$

$$\int_{\pm 2^k}^{\pm 2^{k+1}} \int_{\pm 2^j}^{\pm 2^{j+1}} \left|\frac{\partial^2 m}{\partial \xi_1 \partial \xi_2}(\xi_1, \xi_2)\right| d\xi_1 d\xi_2 \leqslant B. \tag{3.6}$$

The Marcinkiewicz multiplier theorem ([9], p. 109) says that if these conditions are satisfied then convolution with T is bounded on L^p for $1 < p < \infty$ with bound depending only on p and B.

THEOREM 3.2. *G_1 (or G_2) satisfies the Marcinkiewicz condition* (3.3–3.6) *if and only if G_0 does.*

Proof. If we write

$$h(t) = 2\pi\int_0^\infty J_0(2\pi st)g(s)s\,ds$$

then we have

$$\hat{G}_0(\xi_1, \xi_2) = h(|\xi|)$$

and

$$\hat{G}_1(\xi_1, \xi_2) = 2i\operatorname{sgn}\xi_1 h(2|\xi_1\xi_2|^{1/2}).$$

To establish the theorem we will show that the Marcinkiewicz condition for either G_0 or G_1 is equivalent to the conditions

$$|h(t)| \leqslant A, \tag{3.7}$$

$$\int_{2^k}^{2^{k+1}} |h'(t)|\,dt \leqslant A, \tag{3.8}$$

$$\int_{2^k}^{2^{k+1}} |h''(t)|\,dt \leqslant 2^{-k}A \tag{3.9}$$

for all integers k.

Let us first consider the case of G_0. Clearly (3.3) and (3.7) are equivalent. We claim (3.4) and (3.5) together are equivalent to (3.8). To see this we compute

$$\frac{\partial\hat{G}_0}{\partial\xi_1} = \frac{\xi_1}{|\xi|}h'(|\xi|)$$

so

$$\int_{\pm 2^k}^{\pm 2^{k+1}} \left|\frac{\partial\hat{G}_0}{\partial\xi_1}(\xi_1, \xi_2)\right| d\xi_1 = \int_a^b |h'(t)|\,dt$$

where $a = (2^{2k}+\xi_2^2)^{1/2}$ and $b = (2^{2k+2}+\xi_2^2)^{1/2}$. Taking ξ_2 near zero we obtain (3.8) from (3.4) (with $A = B$), while (3.4) follows from (3.8) (with $B = 2A$) since the interval $[a, b]$ can always be contained in the union of two consecutive intervals of the form $[2^j, 2^{j+1}]$. The argument for (3.5) is almost identical. Finally to handle (3.6) and (3.9) we compute

$$\frac{\partial^2\hat{G}_0}{\partial\xi_1\partial\xi_2} = \frac{\xi_1\xi_2}{|\xi|^2}h''(|\xi|) - \frac{\xi_1\xi_2}{|\xi|^3}h'(|\xi|) \tag{3.10}$$

so if $j \leqslant k$

$$\int_{\pm 2^j}^{\pm 2^{j+1}}\int_{\pm 2^k}^{\pm 2^{k+1}} \left|\frac{\partial^2\hat{G}_0}{\partial\xi_1\partial\xi_2}(\xi_1, \xi_2)\right| d\xi_1 d\xi_2 \leqslant \int_{\pm 2^j}^{\pm 2^{j+1}}\int_a^b \left|\frac{\xi_2}{t}\right| |h''(t)|\,dt\,d\xi_2 + \int_{\pm 2^j}^{\pm 2^{j+1}}\int_a^b \left|\frac{\xi_2}{t^2}\right| |h'(t)|\,dt\,d\xi_2$$

and we can dominate the first term by (3.9) and the second by (3.8), establishing (3.6). Conversely, if we have (3.6) and also (3.8) we can establish (3.9) by taking $j = k$ in (3.6) and using (3.8) to handle the term

$$-\frac{\xi_1\xi_2}{|\xi|^2}h'(|\xi|)$$

in (3.10).

Turning to G_1, we again have the obvious equivalence of (3.3) and (3.7), while since

$$\frac{\partial}{\partial\xi_1}\hat{G}_1 = 2i\left|\frac{\xi_2}{\xi_1}\right|^{1/2} h'(2|\xi_1\xi_2|^{1/2})\operatorname{sgn}\xi_1$$

we have

$$\int_{\pm 2^k}^{\pm 2^{k+1}} \left|\frac{\partial}{\partial\xi_1}\hat{G}_1(\xi_1, \xi_2)\right| d\xi_1 = 2\int_c^d |h'(t)|\,dt$$

for $c = 2^{(k+2)/2}|\xi_2|^{1/2}$ and $d = 2^{(k+3)/2}|\xi_2|^{1/2}$ which shows the equivalence of (3.4) and (3.5) with (3.8). Finally we compute

$$\frac{\partial^2\hat{G}_1}{\partial\xi_1\partial\xi_2} = 2ih''(2|\xi_1\xi_2|^{1/2})\operatorname{sgn}\xi_1 + i|\xi_1\xi_2|^{-1/2}h'(2|\xi_1\xi_2|^{1/2})\operatorname{sgn}\xi_1 \tag{3.11}$$

so

$$\int_{\pm 2^j}^{\pm 2^{j+1}}\int_{\pm 2^k}^{\pm 2^{k+1}} \left|\frac{\partial^2\hat{G}_1}{\partial\xi_1\partial\xi_2}(\xi_1, \xi_2)\right| d\xi_1 d\xi_2 \leqslant 2\int_{\pm 2^j}^{\pm 2^{j+1}}\int_c^d \left|\frac{t}{\xi_2}\right| |h''(t)|\,dt\,d\xi_2 + \int_{\pm 2^j}^{\pm 2^{j+1}}\int_c^d |\xi_2|^{-1}|h'(t)|\,dt\,d\xi_2$$

and the first term can be dominated by (3.9) while the second can be dominated by (3.8), establishing (3.6). For the converse we take $j = 1$ in (3.6) and use (3.8) to handle the

$$i|\xi_1\xi_2|^{-1/2}h'(2|\xi_1\xi_2|^{1/2})\operatorname{sgn}\xi_1$$

term in (3.11). ∎

Remarks. There are many known examples of radial distributions satisfying the Marcinkiewicz conditions. For example, $(1+|x|^2)^{-1+it}$ for $t \neq 0$ has a Fourier transform which behaves like $|\xi|^{-2it}$ near $\xi = 0$ and decays exponentially at infinity. Thus convolution with $(1+|x_1 x_2|)^{-1+it}$ $\operatorname{sgn} x_1$ is bounded on L^p, $1 < p < \infty$, for $t \neq 0$.

It is vexing that the methods of this section are limited to $n = 2$ and even there we must take distributions that are even with respect to one of the variables. The problem with extending the results to the more general situation is that we no longer have an exact formula like (3.2). We can obtain a Fourier transform formula using the methods of the previous section which will have a Bessel function integral plus an error term; the question is then what to do with the error term.

§ 4. Nilpotent Lie groups. We consider a 2-stage nilpotent Lie group G, connected and simply connected. Such a group consists of a Euclidean space $G = \boldsymbol{R}^{n+m}$ endowed with an associative product

$$(x, y)\circ(x', y') = \big(x+x', y+y'+B(x, x')\big)$$

where x varies in $\boldsymbol{R}^n$, y varies in $\boldsymbol{R}^m$, and

$$B(x, x')_i = \sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijk} x_j x'_k \quad \text{for} \quad i = 1, \dots m,$$

with B skew-symmetric in x and x'. An example is the Heisenberg group where n is even, $m = 1$, and B is a non-degenerate skew-symmetric bilinear form.

There is a natural group of automorphisms for such a group, namely $\delta(t)(x, y) = (tx, t^2 y)$, and classes of convolution operators that commute with these dilations have been studied by Knapp and Stein [5] and others. The basic building blocks of these operators are the Hilbert trransforms

$$H_{(x',y')} f(x, y) = \text{P.V.} \int_{-\infty}^{\infty} f\Big((x, y)\circ\big(\delta(t)(x', y')\big)^{-1}\Big)\frac{dt}{t}.$$

Our first goal is to prove the boundedness of $H_{(x',y')}$ on L^2, independent of (x', y'). We will use a method introduced by the author in [12], which involves using the Euclidean Fourier transform and the fact that the group multiplication is affine in the (x, y) variables.

THEOREM 4.1. *There exists a universal constant A such that*

$$\|H_{(x',y')} f\|_2 \leqslant A \|f\|_2$$

for all (x', y').

Proof. By the Euclidean Plancherel formula it suffices to estimate $\|(H_{(x',y')} f)^{\wedge}\|_2$ where $^{\wedge}$ denotes the Euclidean Fourier transform. Now

$$H_{(x',y')} f^{\wedge}(\xi, \eta) = \text{P.V.} \int_{-\infty}^{\infty} \int_{\boldsymbol{R}^{n+m}} e^{2\pi i((x+tx')\cdot\xi+(y+t^2y'+tB(x,x'))\cdot\eta)} f(x, y)\, dx\, dy \,\frac{dt}{t}$$

$$= \text{P.V.} \int_{-\infty}^{\infty} \hat{f}\big(\xi+\tilde{B}(x, \eta), \eta\big) e^{2\pi i(tx'\cdot\xi+t^2y'\cdot\eta)} \frac{dt}{t}$$

where

$$\tilde{B}(x', y)_j = \sum_{i=1}^{m}\sum_{k=1}^{n} b_{ijk} x'_k \eta_i.$$

For each fixed η we write $u = \tilde{B}(x', \eta) c = y'\cdot\eta$ and $g(\xi) = \hat{f}(\xi, \eta)$, and it clearly suffices to prove the $L^2(\boldsymbol{R}^n)$ boundedness of the operator

$$Tg(\xi) = \text{P.V.} \int_{-\infty}^{\infty} g(\xi+tu) e^{2\pi i(tx'\cdot\xi+ct^2)} \frac{dt}{t}$$

independent of u, c and x'.

Now by rotating ξ and dilating t we can arrange to take $u = (1, 0, \dots, 0)$. Also we can write

$$e^{2\pi i t x'_1 \xi_1} = e^{-\pi i \xi_1^2 x'_1} e^{\pi i(\xi_1+t)^2 x'_1} e^{-\pi i t^2 x'_1}.$$

The first factor has absolute value one, the second factor can be absorbed into g by setting $g_1(\xi) = e^{\pi i x'_1 \xi_1^2 g(\xi)}$, and the last factor can be absorbed into the term $e^{2\pi i c t^2}$ by changing c. Thus

$$\text{Tg}(\xi) = e^{-\pi i \xi_1^2 x'_1} \text{P.V.} \int_{-\infty}^{\infty} g_1(\xi_1+t, \xi_2, \dots, \xi_n)\cdot e^{2\pi i t(x'_2\xi_2+\dots+x'_n\xi_n)} e^{2\pi i c t^2} \frac{dt}{t}.$$

We now fix the variables $\xi_2, \dots, \xi_n$. To prove the L^2 boundedness of T it suffices to show that

$$S\varphi(\xi_1) = \text{P.V.} \int_{-\infty}^{\infty} \varphi(\xi_1+t) e^{i(at+ct^2)} \frac{dt}{t}$$

is bounded on $L^2(\boldsymbol{R}^1)$ independent of a and c.

Now S is a one-dimensional convolution operator so it suffices to prove the boundedness of the Fourier transform of the kernel $e^{i(at+ct^2)}t^{-1}$. But this is a special case of Theorem 3A in Stein and Wainger [10] (it is also possible to prove this special case by appealing to the fact that the Fourier transform of e^{ict^2} is known). ∎

The proof of course also shows that if we set (suppressing the dependence on (x', y'))

$$H_{\varepsilon,N}f(x, y) = \int_{\varepsilon<|t|<n} f\big((x, y)\circ(\delta(t)(x', y'))^{-1}\big)\frac{dt}{t}$$

then $H_{\varepsilon,N}$ is uniformly bounded and $H_{\varepsilon,N}f\to Hf$ in L^2 as $\varepsilon\to 0$ and $N\to\infty$. It does not, however, give any estimate for the maximal operator

$$\sup_{\varepsilon,N} |H_{\varepsilon,N}f(x, y)|.$$

Now let us turn to the general singular integrals. For simplicity of exposition we deal with scalar-valued singular integrals, but there is no difficulty in extending these results to Hilbert-space valued functions. Let $K(x, y)$ be a measurable complex-valued function which satisfies the homogeneity condition

(4.1) $$K\big(\delta(t)(x, y)\big) = t^{-n-2m}K(x, y)$$

and the integrability condition

(4.2) $$K|\Sigma \in L^1(\Sigma)$$

where Σ denotes the unit sphere in $\boldsymbol{R}^{n+m}$ (this is much weaker than the Dini condition required in previous work). Let us first suppose that K is odd in the x-variables,

(4.3) $$K(-x, y) = -K(x, y).$$

To define principal value convolution operators with kernel K we define annular regions

$$A_{\varepsilon,N} = \{(x, y)\in\boldsymbol{R}^{n+m}:\ \delta(t)(x, y) \in \Sigma \text{ for some } t \text{ in } \varepsilon \leqslant t \leqslant N\}$$

and then set

$$K_{\varepsilon,N}*f(x, y) = \int\int_{A_{\varepsilon,N}} f\big((x, y)\circ(x', y)^{-1}\big)K(x', y')\,dx'\,dy'$$

and

$$\text{P.V.}K*f = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} K_{\varepsilon,N}*f$$

if the limit exists. However, in view of the polar coordinates formula

$$\int_{\boldsymbol{R}^{n+m}} F(x, y)\,dx\,dy = \int_0^\infty\int_\Sigma F\big(\delta(t)\sigma\big)\psi(\sigma)t^{n+2m-1}\,d\sigma\,dt$$

where ψ is a smooth function bounded away from 0 and ∞, we have

$$K_{\varepsilon,N}*f = \tfrac{1}{2}\int_\Sigma H_{\varepsilon,N,\sigma}fK(\sigma)\psi(\sigma)\,d\sigma$$

so we obtain immediately

COROLLARY 4.2. *Under conditions* (4.1–4.3) *the operators* $K_{\varepsilon,N}*f$ *are uniformly bounded on* L^2 *and the limit* P.V. $K*f$ *exists in* L^2 *norm for each* $f\in L^2$. *The operator norm depends only on the* L^1 *norm of* K *on* Σ.

Finally we consider kernels that are even in the x-variables,

(4.4) $$K(-x, y) = K(x, y)$$

and satisfy the mean-value zero condition

(4.5) $$\int_\Sigma K(\sigma)\psi(\sigma)\,d\sigma = 0.$$

In place of the integrability condition (4.2) we require a slightly stronger condition

(4.6) $$\int_\Sigma |K(\sigma)|(1+\log^+|K(\sigma)|)\,d\sigma < \infty.$$

THEOREM 4.3. *Let* K *satisfy* (4.1), (4.4), (4.5) *and* (4.6). *Then the operators* $K_{\varepsilon,N}*f$ *are uniformly bounded in* L^2 *and the limit* P.V. $K*f$ *exists in* L^2 *norm for each* $f\in L^2$. *The operator norm depends only on the value of the integral in* (4.6).

Proof. The proof is a modification of the original argument of Calderón and Zygmund [1], pp. 298–304. First we require the analogue of the Riesz transforms. Let G_0 be the smallest closed subgroup generated by the elements $(x, 0)$. If G_0 is not all of G then we can find a complementary subspace G_1 so that $G = G_0\oplus G_1$ as Lie groups and G_1 is abelian. We can essentially ignore the G_1 component in what follows, so we shall assume $G = G_0$ to keep the notation simple.

Let

$$D_k = \frac{\partial}{\partial x_k} + \sum_{l=1}^{m} b_{ijk}x_j\frac{\partial}{\partial y_i}$$

be the left-invariant vector fields associated with the x-variables, and let $\Delta = -\sum_{k=1}^{n} D_k^2$ be the sub-Laplacian. The hypothesis $G = G_0$ is equivalent to the fact that the D_k and their first brackets generate all vector fields on G, so by a theorem of Hörmander Δ is hypoelliptic. Folland [3] has shown how to construct complex powers of Δ, so that $\Delta^{-1/2}D_k$ and $D_k\Delta^{-1/2}$ are operators of the form P.V.R_k*f and P.V.S_k*f where R_k and S_k are homogeneous of degree $-n-2m$ and C^∞ away from the origin, and

$$\sum_{k=1}^{n}(\Delta^{-1/2}D_k)(D_k\Delta^{-1/2}) = I.$$

Since the D_k are odd in the x-variables the same is true for R_k and S_k.

Furthermore, because these kernels are smooth we have the full Calderón–Zygmund theory available (see Koranyi and Vagi [6]), in particular that convolution with them is a bounded operator from $L\log^+ L$ to L^1.

Now given K as in the statement of the theorem, we write

$$K_{\varepsilon,N}*f = \sum_{k=1}^{n} R_k * S_k * K_{\varepsilon,N} * f.$$

Clearly it suffices to show that the operators $S_k * K_{\varepsilon,N} * f$ are uniformly bounded in L^2 and converge in L^2 norm as $\varepsilon \to 0$, $N \to \infty$ for each $f \in L^2$. But $S_k * K_{\varepsilon,N}$ is odd in the x-variables, so that we can hope to apply Corollary 4.2 to the kernel

$$S_k * K = \lim_{\substack{\varepsilon\to 0\\ N\to 0}} S_k * K_{\varepsilon,N}$$

if this limit exists. Now a straight-forward modification of the argument in [1], pp. 298–304 shows that $S_k * K$ exists and satisfies the hypotheses of Corollary 4.2, and the difference $S_k * K_{\varepsilon,N} - (S_k * K)_{\varepsilon,N}$ is uniformly bounded on L^2. We omit the details. ∎

It would be interesting to know whether the theorems remain true for L^p-boundedness; obviously different methods would be needed. It seems plausible that the method can be extended to all groups which have an affine multiplication as in [12]. It also seems likely that the $L\log^+ L$ condition in Theorem 4.3 can be replaced by an H^1 condition, as is the case in the classical theory ([2] and [8]).

References

[1] A. P. Calderón and A. Zygmund, *On singular integrals*. Amer. J. Math. 78 (1956), 289–309.

[2] W. Connett, *Singular integrals near L^1*, Proc. Symp. Pure Math. 35 part 1 (1979), 163–165.

[3] G. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. 13 (1975), 161–207.

[4] E. Hille, *Analytic Function Theory*, vol. I, Ginn and Company, 1959.

[5] A. W. Knapp and E. M. Stein, *Intertwining operators for semi-simple Lie groups*, Ann. of Math. (2) 93 (1971), 489–578.

[6] A. Korányi and S. Vági, *Singular integrals in homogeneous spaces and some problems of classical analysis*, Ann. Scuola Norm. Sup. Pisa 25 (1971), 575–648.

[7] A. Nagel and S. Wainger, *L^2 boundedness of Hilbert transforms along surfaces and convolution operators homogeneous with respect to a multiple parameter group*, Amer. J. Math. 99 (1977), 761–785.

[8] F. Ricci and G. Weiss, *A characterization of $H^1(\Sigma_{n-1})$*, Proc. Symp. Pure Math. 35 part 1 (1979), 289–294.

[9] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, N. J. 1970.

[10] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. 84 (1978), 1239–1295.

[11] R. S. Strichartz, *Fourier transforms and non-compact rotation groups*, Ind. Univ. Math. J. 24 (1974), 499–526.

[12] — *Singular integrals on nilpotent Lie groups*, Proc. Amer. Math. Soc. 53 (1975), 367–374.

[13] N. Ja. Vilenkin, *Special functions and the theory of group representations*, Amer. Math. Soc. Transl. 22 (1968).

[14] A. Zygmund, *Trigonometric series*, vol. I, Cambridge Univ. Press, London 1959.

MATHEMATICS DEPARTMENT
CORNELL UNIVERSITY
Ithaca, N.Y. 14853

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