Approximation by Abel means and
Tauberian Theorems in sequence spaces*

by

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Abstract. Let $E$ be a topological sequence space. We say the Abel means of
a sequence $x \in E$ exist if for $0 < r < 1$ the sequences $A^r_x = (a_0, a_1, a_2, \ldots, \ldots, r^n a_n, 0, 0, \ldots)$ converge to the sequence $x = (x_0, x_1, x_2, \ldots, r^n x_n, 0, 0, \ldots)$ in the
topology of $E$. We say that $x$ can be approximated by Abel means if $A^r_x$ converges
to $x$ (as $r \to 1^-$) in the topology of $E$. Approximation by Abel means in topological
sequence spaces is investigated. It is a concept more general than Abel summability
since in the FK-space

$A = \{ x : \lim_{r \to 1^-} \sum_{k=0}^{\infty} x_k r^k \text{ exists} \}$

of all summable sequences, every sequence can be approximated by Abel means. Also
a sequence $x \in A$ has the property of sectional convergence (AK) if and only if it is in
the summability field $A_s = \{ x : \sum_{k=0}^{\infty} x_k \text{ exists} \}$. Yet the properties of approximation by
Abel means and AK apply also to spaces which are not summability fields. Thus
statements which give conditions under which approximation by Abel means implies
AK are generalizations of Tauberian Theorems for the Abel method. Several such
approximation statements are obtained which extend classical Tauberian Theorems.
Approximation by Abel means is useful in spaces of Fourier coefficients as well as
being of interest in the general theory of sequence spaces. As an application, a general-
ization of results of Ives and Gooe is obtained about the equivalence of absolute
Tauberian conditions.

1. Introduction. Section 2 contains definitions. In Section 3 we
investigate approximation by Abel means in topological sequence spaces.
It is shown that in the FK-space

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of Abel summable sequences, every sequence can be approximated by Abel

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means and a sequence \( c \) in \( \alpha \) has sectional convergence if and only if its series \( \sum_{k=1}^{\infty} c_k \) converges. We also give some criteria for approximation by Abel means in general topological sequence spaces. In Section 4 it is shown that many of the Tauberian Theorems for Abel summability such as Tauber's First and Second Theorem [16], the O-Tauberian Theorem of Littlewood [10] and the gap Tauberian Theorem of Hardy and Littlewood [7] can be generalized to approximation statements. As a further application of Abel means, in Section 5 we generalize results of Tietz [17] and Goes [5] on the equivalence of absolute Tauberian conditions.

2. Definitions. A \( K \)-space is a Hausdorff locally convex space of sequences \( \alpha = (\alpha_k) \) of real or complex coordinates with continuous coordinate functionals \( f_k : \alpha \rightarrow \mathbb{R} \) (respectively \( \mathbb{C} \)-space) is a \( K \)-space with a Fréchet (respectively Banach) space topology ([6], [18], [20]). For each \( k = 0, 1, 2, \ldots \), let \( \delta^k \) be the sequence with 1 in the \( k \)-th position and zero elsewhere and let \( \varphi \) be the space of all finite linear combinations of the \( \delta^k \)'s. All sequence spaces \( E \) and \( F \) considered will be assumed to be \( K \)-spaces which contain \( \varphi \).

Let \( x \in E \). We say that the \textit{Abel mean of} \( x \) \textit{exist if, for all} \( 0 < r < 1 \), the series \( A^r x = \sum_{k=0}^{\infty} a_k r^k \delta^k \) converges with respect to the topology of \( E \). We say \( x \) \textit{can be approximated by its Abel means if the Abel means of} \( x \) \textit{exist and limit} \( A^r x = x \), the limit being convergent with respect to the topology of \( E \).

Let \( T \) be an infinite matrix with rows \( c \) and columns \( \alpha \) converging to 1. A sequence \( x \) in \( E \) has \( T \)-sectional convergence (TK) if \( \lim_{n \to \infty} T^n x = x \), where \( T^n x = \sum_{k=0}^{\infty} t_{ik} a_k \delta^k \), convergence being with respect to the topology of \( E \). \( r^s x \) is called the \( s \)-th \( T \)-section of \( x \). For \( s \geq 0 \), \( C^s \)-sectional convergence is TK with respect to the triangular matrix given by \( t_{ik} = (r^{i-k})^{(k+1)} k \leq n \). Sectional convergence (AK) is the same as \( C^s \)-sectional convergence; the \( s \)-th sections of \( x \) being \( S^s x = \sum_{k=0}^{\infty} a_k \delta^k \). The \( C^s \)-sections are denoted

\[ \sigma^s x = \sum_{k=0}^{n} \frac{(1 - r^{k+1})}{1 - r} a_k \delta^k = \frac{1}{1 - r} \sum_{k=0}^{n} S^s x. \]

A sequence \( x \) in \( E \) has bounded sections if \( S^s x \) is a bounded subset of \( E \).

Let \( \alpha = \{ \alpha \}_{n=0}^{\infty} \) be the sequence of Abel summability field. \( \alpha \) is an \( F \)-space with seminorms given by:

\[ p_k(x) = \sup_{|a|=1} |x(a)|_{k}, \]

\[ p_j(x) = \sup_{n=0,1,2,\ldots} |x(a)|_{|a|=j}^{1/j}, \quad j = 1, 2, \ldots \]

This was proved in [19] (Theorem IV) (and stated earlier without proof in [23]). The seminorms \( p_j, j = 1, 2, 3, \ldots \) may be replaced by

\[ p_j(x) = \sup_{n=0}^{\infty} |x(a)|_{|a|=j}^{1/j} \]

for \( j = 1, 2, 3, \ldots \)

3. Approximation by Abel means. We start with a proposition about the existence of Abel means.

**Proposition 1.** Let \( E \) be a sequentially complete \( K \)-space and \( x \in E \).

The following statements are equivalent.

(a) The Abel means of \( x \) exist;
(b) Every \( 0 < r < 1, x \rightarrow r^s x \) as \( n \rightarrow \infty \);
(c) Every \( 0 < r < 1, (x, r^s \delta^k) \) is a bounded subset of \( E \);
(d) Every \( 0 < r < 1, r^s S_n^s x = r^s \sum_{k=0}^{n} a_k \delta^k \rightarrow 0 \) as \( n \rightarrow \infty \);
(e) Every \( 0 < r < 1, A^r x = (1-r) \sum_{k=0}^{\infty} r^k \delta^k \).

**Proof.** Since \( a^r \delta^k = \sum_{k=0}^{n} a_k r^k \delta^k - \sum_{k=0}^{n} a_k r^k \delta^k, (a) \Rightarrow (b) \) follows. (b) \Rightarrow (c) is trivial.

(c) \Rightarrow (d): Let \( 0 < r < 1 \) and let \( p \) be a continuous seminorm on \( E \). Then

\[ p(r^s S_n^s x) = \left( \frac{r^n}{1-r} \right) \sup_{k} \| x_k r^k \delta^k \| \sum_{k=0}^{n} f^{r-1} \sup_{k} (x_k r^k \delta^k) \frac{1-r^{k+1}}{1-r} \]

which tends to zero as \( n \rightarrow \infty \).

(d) \Rightarrow (e): Let \( 0 < r < 1 \) and let \( p \) be a continuous seminorm on \( E \).

Then

\[ p \left( \sum_{k=0}^{n} a_k r^k \delta^k \right) = \sup_k \| x_k r^k \delta^k \| \sum_{k=0}^{n} \left( \frac{r^n}{1-r} \right) \]

Since \( E \) is sequentially complete, \( \sum_{k=0}^{\infty} r^k \delta^k \) exists. The rest follows from the
observation that
\[ \sum_{n=0}^{N} a_n r^n \theta^n = (1 - r) \sum_{n=0}^{N} r^n \theta^n x + r^{N+1} \theta^n x. \]

Finally, (e) = (a) is trivial. \( \square \)

A sequence space \( E \) is tempered if for each \( x \in E \), there exists \( \alpha \) such that \( a - \theta^n \in \ell^\alpha \). All spaces of Fourier coefficients are tempered. If an\( \mathcal{F} \) space \( E \) contains the space \( \ell^\alpha \) of absolutely convergent series, then \( \{ \theta^n \}_{n=0}^{\infty} \) is a bounded subset of \( E \). It follows from (c) above that in a tempered \( \mathcal{F} \) space \( E \) containing \( \ell^\alpha \) the Abel means of every sequence exist.

The following shows that approximation by Abel means is more general than \( C_{\alpha} \)-sectional convergence.

**Proposition 2.** Suppose \( E \) is sequentially complete and \( \alpha \geq 0 \). If \( x \) has \( C_{\alpha} \)-sectional convergence, then \( x \) can be approximated by its Abel means. In particular, \( \mathcal{A}K \) implies approximation by Abel means.

**Proof.** Since \( C_{\alpha} \)-sectional convergence implies \( C_{\alpha} \)-sectional convergence for \( 0 \leq \alpha \leq \beta \), it is sufficient to assume that \( \alpha \) is a non-negative integer. Let \( p \) be a continuous semi-norm on \( E \) and suppose \( x \in E \) has \( C_{\alpha} \)-sectional convergence. Let
\[ T^k x = \sum_{n=0}^{\infty} t_n a_n \theta^n, \quad \text{where} \quad t_n = \binom{n}{n-k} \frac{(n+\alpha)}{n+\beta}. \]

Clearly \( p(t_n \theta^n) = O(k^\beta) \). By Proposition 1 (c), the Abel means of \( x \) exist. Further,
\[ A^k x = (1 - r)^{k+1} \sum_{n=0}^{\infty} \binom{k+1}{n} r^n T^n x. \]

If \( p(T^k x - x) < \varepsilon \) whenever \( k > N \), then
\[ p(A^k x - x) \leq (1 - r)^{k+1} \sum_{n=0}^{\infty} \binom{k+1}{n} r^n \sup_x p(T^n x - x) + \varepsilon, \]
which tends to \( \varepsilon \) as \( r \to 1 \). \( \square \)

The following two results show that for a general class of summability fields, approximation by Abel means (respectively, \( C_{\alpha} \)-sectional convergence) of a sequence is the same as Abel summability (resp. \( C_{\alpha} \)-summability). The following result is stated in [23] without proof. For completeness we give a proof.

**Theorem 1.** Every element of \( \mathcal{A} \) can be approximated by its Abel means.

**Proof.** Let \( x \in \mathcal{A} \). We show (a): \( p_j(x - A^j x) \to 0 \) (as \( r \to 1^- \)) for \( j = 1, 2, 3, \ldots \); and (b): \( p_j(A^j x - A^j x) \to 0 \) (as \( r_1, r_2 \to 1 \)).

(a): Let \( r > 0, \ j \neq 0 \). Choose \( N \) such that
\[ |s_m| \left( \frac{j^{N+1}}{j+1} \right)^N < \varepsilon \text{ for } m > N. \]

For \( r \) sufficiently large,
\[ |s_m| \left( \frac{j^{N+1}}{j+1} \right)^N (1 - r^m) < \varepsilon \quad \text{for } m = 1, 2, \ldots, N. \]

Then
\[ p_j(x - A^j x) = \sup_{\alpha \leq 1} |s_m| \left( \frac{j^{N+1}}{j+1} \right)^N (1 - r^m) < \varepsilon. \]

(b): Let \( r > 0 \). Since \( x \in \mathcal{A} \), there exists \( j \) such that
\[ \left| \sum \sum a_{k} r_{k} \theta^{k} \right| < \varepsilon \quad \text{whenever} \quad \left( \frac{j^{N+1}}{j+1} \right)^N < \varepsilon. \]

By the equivalence of the seminorms \( (p_j) \) and \( (p_j^p) \) we have
\[ p_j(A^j x - A^j x) < \varepsilon \quad \text{for } r < r_1, r_2 < 1 \]

sufficiently close to 1. Let, further, \( \left( \frac{j^{N+1}}{j+1} \right)^N < \varepsilon \). For \( t > \frac{j^{N+1}}{j+1} \), we have
\[ \left| \sum \sum a_{k} r_{k} \theta^{k} \right| = \left| \sum \sum a_{k} r_{k} \theta^{k} \right| < \varepsilon \]

since \( \left( \frac{j^{N+1}}{j+1} \right)^N < \varepsilon \). For \( t < \frac{j^{N+1}}{j+1} \), we have
\[ \left| \sum \sum a_{k} r_{k} \theta^{k} \right| = \left| \sum \sum a_{k} r_{k} \theta^{k} \right| < \varepsilon. \]

Thus
\[ p_j(A^j x - A^j x) = \sup_{\alpha \leq 1} \left| s_m \theta^{m} \right| < \varepsilon. \]

Let \( A \) be a summability method given by \( A - \sum a_{k} = \lim_{r \to 1^-} \sum_{k=0}^{\infty} a_{k} \theta^{k} \) satisfying:

(1) \( a_{k} = \theta^{k} \) is continuous for \( 0 < r < r_{\infty} < \infty, \ k = 0, 1, 2, \ldots \); and some fixed \( r_{\infty} \);

(2) for any \( x \in \mathcal{A} \), the convergence of \( \sum a_{k} \theta^{k} \) implies uniform convergence of \( \sum a_{k} \theta^{k} \) in the interval \( 0 \leq r \leq r_{1} \);

(3) for all \( n = 0, 1, 2, \ldots, \) if \( x = \theta^{n} \), then \( A - \sum a_{k} = 1 \).

The summability field \( \mathcal{A} \) is \( [A : \sum a_{k} \theta^{k} \exists \] of such a method is an \( \mathcal{F} \)-space [19]. All summability methods defined by a matrix \( A - \sum a_{k} = \lim_{n \to \infty} \sum a_{nk} \theta^{k} \) satisfy (1) and (2) when \( r_{\infty} = \infty, \ a_{k}(n) = a_{nk} \) and \( a_{k}(r) \) is
linear between $a_n(r)$ and $a_n(n+1)$. Condition (3) states that the method $\Delta$ sums finite sequences to their usual sum.

**Theorem 2.** Let $a_n$ be the summability field of a method satisfying (1)-(3).

(a) If $x \in A_\delta$ can be approximated by its Abel means, then $x \in A_\delta$.

(b) If $x \in A_\delta$ has $C_\delta$-sectional convergence, then $x$ is $C_\delta$-summable.

**Proof.** By standard equicontinuity arguments on FK-spaces, one can show that $\lambda(x) = A - \sum a_r x_r$ is a continuous linear functional on $\mathcal{A}$. If $x = \lim A^n x$, then

$$f(x) = \lim_{n \to \infty} f(A^n x) = \lim_{n \to \infty} \sum_{r=0}^{\infty} a_r x_r = \lim_{n \to \infty} \sum_{r=0}^{\infty} a_r x_r.$$

Thus $x \in A_\delta$. Similarly, if $x$ has $C_\delta$-sectional convergence, then $f(x) = \ell' = -\sum a_r$. ■

**Remark.** Suppose $c_j$ satisfies (1)-(3). If $A_\delta \subset c_\delta$, the inclusion map $\mathcal{A} \to \mathcal{C}_\delta$ is continuous. Then a sequence $x \in c_\delta$ can be approximated by Abel means if and only if $x \in A_\delta$. Such a statement can also be made about $C_\delta$-sections ($\alpha \geq 0$) if the summability fields of the $c_\delta$ methods have $C_\delta$-sectional convergence [23]. In particular, a sequence $x \in A_\delta$ has the property $\mathcal{A} = \mathcal{A}_\delta$ if and only if $\sum a_r x_r$ exists.

A sequence $\lambda$ is a multiplier from $E$ to $F$ if $x \cdot \lambda = (a_n \lambda_n) \in F$ for all $x \in E$. The space of all multipliers from $E$ to $F$ is denoted by $(E,F)$. For example $(F,F) = E$ where $1 \leq p \leq q \leq \infty$. There are multipliers $x \to x \lambda$ between FK-spaces that are continuous [20]. If $E$ is an FK-space and $\lambda$ is a multiplier from $E$ to $A_\delta$, it follows that the linear functional $f(x) = \lambda x \lambda = f(\lambda)$ is continuous on $E$. We define $E^{(\delta)}$ as the space of all continuous linear functionals on $E$.

**Proposition 3.** Let $E$ be an FK-space containing $\lambda$. $(E,A_\delta) = E^{(\delta)}$ if and only if $\sum \lambda x \lambda f(\lambda)$ exists for every continuous linear functional $f$ on $E$ and every $x \in E$.

**Proof.** ($\Rightarrow$): Suppose $(E,A_\delta) = E^{(\delta)}$. Let $f$ be a continuous linear functional on $E$ and $f(\lambda) = f(\lambda_\delta)$. Then $\lambda : x \in E$ for every $x \in E$. That is, $\sum a_r x_r = f(\lambda)$ exists for every $x \in E$. Therefore $\lambda x \lambda = f(\lambda)$. Conversely, let $\lambda$ be a continuous linear functional on $E$ and let $\lambda_\delta = f(\lambda)$. Then for every $x \in E$, $\lim_{n \to \infty} \sum a_r x_r f(\lambda_\delta)$ exists. Thus $\lambda : x \in A_\delta$ for every $x \in E$ and $\lambda \in E^{(\delta)}$. ■

**Theorem 3.** Let $E$ be a $K$-space containing $\lambda$. If every element of $E$ can be approximated by Abel means, then every continuous functional $f$ on $E$ is of the form

$$f(x) = \lim_{n \to \infty} \sum a_r x_r f(\lambda_\delta)$$

for some multiplier $\lambda$ from $E$ to $A_\delta$. If $E$ is an FK-space, the converse is also true.

The proof of omitted. The first statement can be proved easily for $\lambda_\delta = f(\lambda)$. The converse uses an equicontinuity argument as in [21], Satz 3.4, and (4), Proposition 1.

In particular, $A_\delta = (\mathcal{A},A_\delta)$ by Theorem 1. A multiplier $\lambda$ from $\mathcal{A}$ to $A_\delta$ is of the form $\lambda_\delta = \int t \, dg(t) + O(\tau^k)$, where $0 \leq r < 1$ and $g$ is of bounded variation [23], [19].

In the theory of Fourier series, approximation by Abel means is a natural concept. Some examples of spaces of sequences of Fourier coefficients in which all sequences can be approximated by Abel means (but not by Cesaro sections) are given in [3]. Although T-sections are finite sequences whereas Abel means are not, many of the results for T-sectional convergence carry over to approximation by Abel means without major changes in proofs. Below we give three such results.

The following can be proved using equicontinuity arguments similar to [21], Satz 3.3, and (4), Proposition 1.

**Theorem 4.** Suppose $E$ is a barreled $K$-space containing $\lambda$. Every element of $E$ can be approximated by Abel means if and only if $\lambda$ is a dense subset of $E$ and for each $x \in E$ the Abel sections $(\mathcal{A}_\delta)_{\delta < \infty}$ form a bounded subset of $E$.

The proof of the following is similar to [15], Theorem 4.4.

**Theorem 5.** Let $E$ be an FK-space containing $\lambda$. Every element of $E$ can be approximated by Abel means if and only if $\lambda$ is a dense subset of $E$ and every multiplier from $\mathcal{A}$ to $\mathcal{A}_\delta$ is a multiplier from $E$ to $E$.

The proof of the following is similar to [1], Theorem 4 and [1], Proposition 2.

**Theorem 6.** Let $E$ be a BK-space containing $\lambda$. If every element of $E$ can be approximated by Abel means, then the space $(E,A_\delta)$ is a BK-space in which for each multiplier $\lambda \in (E,A_\delta)$ the set of Abel means $(\lambda_\delta)_{\delta < \infty}$ is bounded.
4. Tauberian theorems as approximation statements. Here we consider statements giving conditions under which approximation by Abel means or $C_1$-sectional convergence implies sectional convergence. When applied to the FK-space $\mathcal{V}$ or
\[ C_1 = \left\{ x : \lim_{n \to \infty} \sum_{k=0}^{n} \left( 1 - \frac{k}{n+1} \right) x_k \text{ exists} \right\}, \]
such statements become Tauberian Theorems in the classical sense because every element of $\mathcal{V}$ can be approximated by Abel means (Theorem 1), every $C_1$ summable sequence has $C_1$-sectional convergence ([22], Satz 5) and a sequence in these spaces has sectional convergence if and only if it has a convergent series (Theorem 2). Some Tauberian Theorems for Cesàro sections are given in [2].

Sectional boundedness can often be deduced from Tauberian conditions using the equivalence of boundedness and weak boundedness along with a classical Tauberian Theorem on $\mathcal{V}$. This technique is used in Theorems 8 and 10. The following theorem uses sectional boundedness to reduce a Tauberian Theorem for Abel means to one for $C_1$-sections. The proof is a modification of a Tauberian Theorem of Karamata [9]. It has been adapted to an approximation statement instead of a summability statement and also differs from the original by using the uniform approximation by polynomials of a continuous function $g$ instead of almost everywhere approximation by polynomials of an integrable function. For Banach spaces, the result was also obtained in [11] using another modification of Karamata’s argument.

**Theorem 7.** Let $E$ be a sequentially complete $K$-space containing $\mathcal{V}$. If a sequence $x$ in $E$ can be approximated by Abel means and has bounded sections, then it has $C_1$-sectional convergence.

**Proof.** Let $\tau = e^{\lambda^2}$. Since $\lim_{\lambda \to \infty} (1 - e^{\lambda^2}) \lambda^2 = 1$ and
\[ A^{\lambda^2} x = (1 - \tau) \sum_{k=0}^{\infty} \tau^k S^k x, \quad \text{as} \quad \tau \to 1, \]
we have
\[ \lim_{\tau \to 1} \sum_{k=0}^{\infty} e^{-t} S^k x = x. \]
For $m = 0, 1, 2, \ldots$ we have
\[ \lim_{t \to \infty} \sum_{k=0}^{\infty} e^{-t} (e^{-t})^k S^k x = \frac{1}{m+1} \lim_{t \to \infty} (m+1) t \sum_{k=0}^{\infty} e^{-m(t+1)+t} S^k x = x/m + 1 = x \frac{1}{\zeta} y^m dy. \]
and hence for each polynomial $P$, we have
\[ \lim_{t \to \infty} \sum_{k=0}^{\infty} e^{-t} P(e^{-t}) S^k x = x \int_0^1 P(y) dy. \]
Let $\varepsilon > 0$, $N > 1/\varepsilon$ and
\[ g(x) = \begin{cases} e^{N+1} x^N & \text{if} \quad 0 \leq x \leq \varepsilon^{-1}, \\ e^{-x} & \text{if} \quad \varepsilon^{-1} < x \leq 1. \end{cases} \]
Since $g$ is continuous, we can find a polynomial $P$ such that $|g(x) - P(x)| < \varepsilon$ for $0 \leq x \leq 1$. Let $p$ be a continuous seminorm on $E$. We have
\[ p(e^{\lambda^2} x - x) \leq p \left( \frac{1}{n+1} \sum_{k=0}^{\infty} \delta^k x - \sum_{k=n+1}^{\infty} e^{-t} P(e^{-t}) S^k x \right) + \varepsilon \int_0^1 P(y) dy + p \left( x \int_0^1 P(y) dy - x \right). \]
There exists $\delta > 0$ such that
\[ p \left( \sum_{k=0}^{\infty} e^{-t} P(e^{-t}) S^k x - x \int_0^1 P(y) dy \right) < \varepsilon \]
whenever $0 < t < \delta$. Let $t = \frac{1}{n+1} < \delta$ and $M = \sup_{x} (e^{\lambda^2} x)$. Then
\[ p(e^{\lambda^2} x - x) \leq M t \sum_{k=0}^{\infty} (1 - e^{-t} P(e^{-t} x)) + M t \sum_{k=n+1}^{\infty} e^{-t} P(e^{-t} x) + e + \varepsilon \int_0^1 P(y) dy - 1 \]
and hence for each polynomial $P$, we have
\[ \lim_{t \to \infty} \sum_{k=0}^{\infty} e^{-t} (e^{-t})^k S^k x = x \frac{1}{\zeta} y^m dy. \]
and $\varepsilon \int_0^1 P(y) dy < \varepsilon$. This can be made arbitrarily small since $Me \left( \frac{1}{1 - e^{-\lambda^2}} \right) \to Me \sim \infty (n \to \infty)$ and $Me \left( \frac{1}{1 - e^{-\lambda^2}} \right) \to Me (n \to \infty)$. \[ \blacksquare \]
Theorem 8 (Tate's second Theorem for Abel means). Let \( E \) be a sequentially complete \( K \)-space containing \( q \). Suppose that a sequence \( x \in E \) can be approximated by its Abel means. Let \( a^*_x = \frac{1}{m+1} \sum_{k=0}^m kx_k \delta^k \). Then \( x \) has sectional convergence if and only if \( a^*_x \to 0 \) \((n \to \infty)\).

**Proof.** Suppose that \( x \) has sectional convergence. Then
\[
\begin{align*}
\hat{a}^*_x &= \frac{1}{n+1} \left( m^n x_n - \sum_{k=0}^{n-1} k^n x_k \right) \\
&= \frac{1}{n+1} \left( m^n x_n - \sum_{k=0}^{n-1} (k^n x_k - x_k) \right) \\
&= \frac{n}{n+1} \left( S^n x_n - x_n \right) - \frac{1}{n+1} \sum_{k=0}^{n-1} (S^k x_k - x_k)
\end{align*}
\]
each term of which tends to zero. Conversely let \( f \) be a continuous linear functional on \( E \), and let \( \lambda = f(\delta^t) \). Since \( x \) can be approximated by Abel means it follows that \( \lambda \cdot x \) is an element of \( A \). Clearly \( f(\hat{a}^*_x) = \frac{1}{n+1} \sum_{k=0}^n \lambda k x_k \)
tends to zero. By the second Theorem of Tate, \( \sum \lambda x_k \) converges. In particular \( f(\hat{a}^*_x) = \sum \lambda x_k \) is bounded for every continuous linear functional and hence \( \hat{a}^*_x \) is a bounded subset of \( E \). By Theorem 7, \( x \) has \( C_1 \)-sectional convergence. But \( S^n x - \hat{a}^*_x \to 0 \) \(n \to \infty\). Since \( \hat{a}^*_x \) converges to \( x \), so does \( S^n x \). \( \blacksquare \)

**Lemma 1** (Hardy's Tauberian Theorem for Cesàro sections). Let \( E \) be a \( K \)-space containing \( q \). Suppose that a sequence \( x \in E \) has \( C_1 \)-sectional convergence. If \( \{x_n, \delta^t\}_{n=1}^\infty \) is a bounded subset of \( E \), then \( x \) has the property \( \text{AK} \).

**Proof.** It is sufficient to show that \( S^n x - \sigma^n x \to 0 \) \((n \to \infty)\) under the conditions of the hypothesis. We have
\[
S^n x - \sigma^n x = \frac{1}{n+1} \sum_{k=0}^n kx_k \delta^k
\]
and for \( m > n \), we have
\[
\sigma^m x - \sigma^n x = \frac{m-n}{(m+1)(m+1)} \sum_{k=0}^m kx_k \delta^k + \frac{m-n}{m+1} \sum_{k=m+1}^n (m+1-k) x_k \delta^k.
\]
Substituting we get
\[
S^n x - \sigma^n x = \frac{m-n}{m+1} \left( S^m x - \sigma^n x \right) + \frac{m-n}{m+1} \sum_{k=m+1}^n (m+1-k) x_k \delta^k.
\]

For each continuous seminorm \( p \) and for \( m \leq n \), we have
\[
p(S^n x - \sigma^n x) \leq \left(1 + \frac{n+1}{m+1} \right) p(S^m x - \sigma^n x) + \frac{m-n}{m+1} \sup_k p(kx_k \delta^k).
\]

Let \( f \) be any positive integer and choose \( N \) so that \( N > 2f \) and \( p(S^n x - \sigma^n x) < 2^{-f} \) whenever \( m > n > N \). Let \( m \) be the smallest positive integer for which \( \left(1 + \frac{n+1}{m+n} \right) < 2^{-f} \). Then \( m > n \) and \( \left(1 + \frac{n+1}{m+n} \right) < 2^{-f} \) and hence
\[
p(S^n x - \sigma^n x) \leq (1 + 2^f) 2^{-f} + 2^{-f+1} \sup_k p(kx_k \delta^k) < 2^{-f+1} \left(1 + \sup_k p(kx_k \delta^k)\right).
\]

**Theorem 9** (Littlewood's Tauberian Theorem for Abel means). Let \( E \) be a sequentially complete \( K \)-space containing \( q \). Suppose a sequence \( x \) in \( E \) can be approximated by its Abel means and \( \{x_n, \delta^t\}_{n=1}^\infty \) is a bounded subset of \( E \). Then \( x \) has sectional convergence.

**Proof.** By Theorem 7 and Lemma 1 it is sufficient to show that \( x \) has bounded sections. Let \( p \) be a continuous seminorm on \( E \), let \( r = \frac{n}{m+1} \) and let \( M = \sup_k p(kx_k \delta^k) \). Then
\[
p(S^n x) \leq p(S^n x - S^m A^n x) + p(S^m A^n x)
\]
\[
\leq \sum_{k=1}^n (1-r) p(a_0 \delta^0) + \sum_{k=1}^n r^k p(a_k \delta^k)
\]
\[
\leq \frac{1}{n+1} \sum_{k=1}^n \left( \frac{1-r^k}{1-r} \right) p(a_0 \delta^0) + \frac{1}{n+1} \sum_{k=1}^n r^k M
\]
\[
= \frac{1}{n+1} \sum_{k=1}^n \left( 1 + r + r^2 + \ldots + r^{k-1} \right) p(a_0 \delta^0) + M \left(1 - r^{n+1}\right)
\]
\[
\leq \frac{1}{n+1} \sum_{k=0}^n kp(a_0 \delta^0) + M \leq 2M. \ \blacksquare
\]

**Corollary.** Let \( E \) be a sequentially complete \( K \)-space containing \( q \). Suppose a sequence \( x \) in \( E \) can be approximated by its Abel means and \( x_k \delta^k = O(k^{-1}) \) for every \( k \in (E-x^0) \). Then \( x \) has sectional convergence.

The proof follows from Theorem 9 using the equivalence of boundedness and weak boundedness.

**Lemma 2** (Lucassen's Tauberian Theorem for Cesàro sections). Let \( E \) be a \( K \)-space containing \( q \). Suppose that \( x \in E \) has \( C_1 \)-sectional convergence.
If \( a_n = 0 \) except perhaps for \( n = n_k \), where \( (n_{k+1}/n_k) \geq r > 1 \) for \( k = 0, 1, 2, \ldots \), then \( x \) has the property \( \Delta K \).

Proof (cf. [24], Vol. I, p. 79). Since \( (n_{k+1} - n_k) p_n = n_{k+1} p(n_{k+1} - 1) x - n_k p(n_k - 1) x \), we have for each continuous seminorm \( p \),

\[
p((n_{k+1} - n_k) x - x) \leq \frac{n_{k+1} - n_k}{n_{k+1} - n_k} p(n_{k+1} - 1) x + \frac{n_k}{n_{k+1} - n_k} p(n_k - 1) x \leq \frac{1}{r-1} p(n_{k+1} - 1) x + \frac{1}{r-1} p(n_k - 1) x
\]

which can be made arbitrarily small for sufficiently large \( n_k \).

Theorem 10 (Lacunary Tauberian Theorem for Abel means). Let \( E \) be a sequentially complete \( K \)-space containing \( p \). Suppose that \( x \in E \) can be approximated by Abel means and \( a_n = 0 \) except perhaps for \( n = n_k \), where \( (n_{k+1}/n_k) \geq r > 1 \) for \( k = 0, 1, 2, \ldots \). Then \( x \) has sectional convergence.

Proof. By Theorem 7 and Lemma 2 it is sufficient to show that \( x \) has bounded sections. Let \( f \) be a continuous linear functional on \( E \) and let \( \lambda_f = \text{a.e.} \). Clearly, \( \lambda \) is a lacunary sequence in \( \mathcal{A} \). By the gap Tauberian Theorem of Hardy and Littlewood [7], \( f((n_{k+1}) = \sum a_k x_k \) converges as \( n \to \infty \). In particular the set \( \{ n_{k+1} \} \) is weakly bounded and therefore bounded.

5. An Application of Abel Means to Absolute Tauberian Conditions.

For each sequence \( \lambda = (\lambda_k) \) and \( \alpha \geq 0 \), let

\[
\lambda^* \lambda' = \sum_{j=1}^{\infty} \left( \frac{j-n_{k-1}}{j} \right) \lambda_j.
\]

Then \( \lambda^* \lambda' = \lambda_k' \) and \( \lambda^{(1)} \lambda = \lambda_k' - \lambda^* \lambda_{k+1} \). The sequence \( \lambda \) is monotonic of order \( \alpha \) if \( \lambda^* \lambda' \geq 0 \) for all \( k = 0, 1, 2, \ldots \) and it is of bounded variation of order \( \alpha \) if it is the difference of two bounded monotonic sequences of order \( \alpha \). The set of sequences of bounded variation of order \( \alpha \) form a BK-space which we denote by \( \text{bv} \). We write \( \text{bv} = \text{bv}^1 \). If \( \alpha < \beta \), then \( \text{bv}^\alpha \subset \text{bv}^\beta \) ([19], [23]).

We call a sequence fully monotonic if it is monotonic of all orders \( \alpha = 1, 2, 3, \ldots \). It is quasi-fully monotonic if it is the difference of two bounded fully monotonic sequences. The space of quasi-fully monotonic sequences is denoted by \( \text{bv}^\infty \). We have

\[
\text{bv}^\infty \subset \bigcap_{\alpha=1}^\infty \text{bv}^\alpha.
\]

For each sequence \( x = (x_k) \), let \( d_k = \frac{1}{n+1} \sum_k k x_k \) and \( d = (d_k) \).

For each sequence space \( E \), let \( \int E = \{ \frac{1}{k} a_k : x \in E \} \).

Hyslop [8] has shown that the condition \( d \in \text{bv} \) is an Absolute Tauberian Condition (ATC) for the Abel method. That is, if the Abel transform of a sequence \( x \) is of bounded variation and \( d \in \text{bv} \), then \( x \in \mathcal{F} \). Tietz [17] has shown that if \( V \) is an absolutely permanent and additive summability method, then \( d \in \text{bv} \) is an ATC for \( V \) if and only if \( a \in \mathcal{F} \) is an ATC for \( V \). Goes [5] has further shown the equivalence of \( d \in \text{bv} \), \( x \in \mathcal{F} \) and \( a \in \mathcal{F} \) as ATC. We extend these results.

Theorem 11. Let \( V \) be an absolutely permanent and additive summability method and let \( k \) be any number such that \( 1 \leq k \leq \infty \). Then \( d \in \text{bv} \) is an ATC for \( V \) if and only if \( a \in \mathcal{F} \) is an ATC for \( V \).

Proof. Since \( \text{bv}^\alpha \subset \text{bv}^\beta \subset \text{bv}^\infty \) for \( 1 \leq \alpha \leq \infty \), and since the result of Tietz is the case \( k = 1 \), it is sufficient to show that if \( x \in \mathcal{F} \) is an ATC for \( V \), then \( x \in \mathcal{F} \) is an ATC for \( V \). By the 0-Tauberian Theorem of Littlewood,

\[
\mathcal{A} \cap \{ f, v = cs \} = \{ x : \sum_k k x_k \text{ exists} \}.
\]

Thus \( \{ (a = \text{bv}) \cap \{ f, v = cs \} \} = \{ (a = \text{bv}) \cap \{ f, v = cs \} \} \). We have \( \{ (a = \text{bv}) \cap \{ f, v = cs \} \} \) has the property \( \Delta K \) [4]. By Proposition 3, it follows that \( \{ (a = \text{bv}) \cap \{ f, v = cs \} \} \). By [14] (Theorem 3.1), \( \{ (a = \text{bv}) \} \). Thus

\[
\text{bv} \subset \text{bv}^1 \cap \{ f, v \beta \} = \text{bv}^1 \cap \{ f, v \beta \}.
\]

Every element of \( \{ (a = \text{bv}) \} \) is of the form \( x^2 + x_0 \), where \( x \in \text{bv}^\infty \) and \( x_0 = 0 \) for \( 0 < x \in \mathcal{F} \), [23]. Every element of \( \{ f, v \beta \} \) is of the form \( (k x_k) \), where \( x \in \mathcal{F} \) since by Theorem 3 and [5], Proposition 1.9, we have

\[
\{ (a = \text{bv}) \cap \{ f, v \beta \} \} = \{ (a = \text{bv}) \cap \{ f, v \beta \} \}.
\]

Every element of \( \{ a \} \) is of the form \( x = x^2 + x_0 \), where \( x \in \text{bv}^\infty \) and \( x_0 = 0 \) for \( 0 < x \in \mathcal{F} \), [23]. Every element of \( \{ f, v \beta \} \) is of the form \( (k x_k) \), where \( x \in \mathcal{F} \) since by Theorem 3 and [5], Proposition 1.9, we have

\[
\{ (a = \text{bv}) \cap \{ f, v \beta \} \} = \{ (a = \text{bv}) \cap \{ f, v \beta \} \}.
\]

Thus every element \( \lambda \) of \( f \) is of the form \( \lambda = x + w \), where \( x \in \text{bv}^\infty \) and \( w \in \mathcal{F} \). If \( \lambda \) is absolutely \( V \) summable, then \( x \) is absolutely \( V \) summable since \( V \) is absolutely permanent. If \( x \in \text{bv}^\infty \) is an ATC for \( V \), then

\[
\lambda = x + w \in \mathcal{F}.
\]

Then \( x \in \text{bv}^\infty \) is an ATC for \( V \).
Singular integrals supported on submanifolds

by

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Abstract. Three results related to Calderón-Zygmund singular integrals in generalized contexts are established. In the first, Hilbert transforms supported on hyper-surfaces of the form \( \prod |x|^b = r \) where \( b_\lambda \) are non-zero real numbers, are proved bounded on \( L^p \) for certain values of \( p \) (Nagel and Wainger proved \( L^p \) boundedness). In the second, \( L^p \) boundedness is established for convolutions on \( \mathbb{R}^n \) with kernels of the form \( g(x_\lambda x_\sigma^{1/2}) \) for small \( x \) by transference from known results about radial kernels. In the third, the "method of rotations" is carried through for the \( L^p \) theory of Knapp-Stein singular integrals on \( 2 \)-stage nilpotent Lie groups.

§ 1. Introduction. The Calderón-Zygmund theory of singular integrals has been extended in many directions in recent years. One major theme in these extensions is the "method of rotations" introduced in [1]. This leads to the study of various Hilbert transforms supported on curves or more general submanifolds. The reader is urged to consult the excellent exposition of these ideas in Stein and Wainger [10].

In this paper we present three contributions to this study. The first concerns Hilbert transforms supported on hypersurfaces in \( \mathbb{R}^n \) defined by the equation \( \prod |x|^b = r \) where \( b_\lambda \) are non-zero real numbers. The \( L^p \) boundedness of these operators was established by Nagel and Wainger [7]. We give an independent proof that also establishes \( L^p \) boundedness for certain values of \( p \). We do not know if these values of \( p \) are best possible.

Our second contribution is a transference result relating convolutions on \( \mathbb{R}^n \) with functions of the form \( g(x^{1/2}) \) to convolutions with the radial function \( g(r^2 + s^2) \). We show that \( L^p \)-boundedness is equivalent for these two operators, and the applicability of the Marcinkiewicz multiplier theorem is also equivalent.

Our third contribution is to the Knapp-Stein [5] theory of singular integrals on nilpotent Lie groups. For 2-stage groups we carry out the method of rotations for \( L^p \)-boundedness. We use the Euclidean Plancherel

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