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Weak approximate identities and multipliers

by

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Abstract. Let A be a Banach algebra and let Ω be a right Banach A-submodule of A^*. Then $\mathrm{Hom}_A(A, \Omega^*)$ is characterized in terms of dual spaces and approximate identities for certain choices of A and Ω having applications in harmonic analysis.

For example, let A be an L-algebra, let $\Omega \subseteq \mathfrak{W}$ (the space of all weakly almost periodic functionals in A^*) be a C^*-subalgebra of the commutative von Neumann algebra A^*, and suppose that Ω is a left A-module as well. Then $\Omega \cong C_0(\Gamma)$, where Γ is the maximal ideal space of Ω, and $\Omega^* \cong M(\Gamma)$. Now, if A has a right $\mathfrak{W}$-approximate identity (a.i.) bounded by one, then $\mathrm{Hom}_A(A, M(\Gamma))$ is isometrically algebra anti-isomorphic to a closed subalgebra of $M(\Gamma)$ via the map $T \mapsto \mu_T$, where the action of T is by generalized convolution on the right with μ_T. Moreover, the map $T \mapsto \mu_T$ is onto if and only if Ω is an essential right Banach A-module if and only if A has a bounded two-sided Ω-a.i.

It is also proved that if A is a convolution measure algebra and Ω (as above) contains the identity of the von Neumann algebra A^*, then A has a two-sided Ω-a.i. bounded by one if and only if the compact semigroup Γ has an indentity.

I. Introduction. Let A be an arbitrary Banach algebra and W a right Banach A-module. Then $\mathrm{Hom}_A(A, W^*) \cong (A \otimes_A W)^*$, where $A \otimes_A W$ is the (completed) A-tensor product of A and W ([16], [17]). Concrete realizations of $\mathrm{Hom}_A(A, W^*)$ have been given in terms of $(A \otimes_A \otimes_A W)^*$, $(A \circ W)^*$ (defined below), and W^* in cases where more information is available about A and/or W ([16], [17], [12], [10], [2], [21]). In this paper, we study $\mathrm{Hom}_A(A, W^*)$ primarily when $W = \Omega$ is a right Banach A-submodule of A^* and A possesses a right Ω-approximate identity.

In Section II, property $\mathrm{P2}(\Omega)$ is defined ($\mathrm{P2}(A^*)$ is Máté's property P2 ([11], [21])) and is shown to be equivalent to A possessing a certain kind of right Ω-approximate identity (Theorem 2.1). Applications of Theorem 2.1 are made to (noncommutative) Segal algebras, and to commutative Banach algebras A with $\Omega = \mathrm{cl}(\mathrm{sp}(\Delta A))$. In the latter case, $\mathrm{P2}(\Omega)$ is equivalent to $\mathrm{Hom}_A(A, \Omega^*) \cong (A \circ \Omega)^*$ (Theorem 2.5).

Section III is devoted to a discussion of $\mathrm{Hom}_A(A, \Omega^*)$ for algebras A possessing a bounded right Ω-approximate identity (when $\Omega \subsetneq A^*$, this assumption is weaker than assuming A has a bounded right approximate identity [7]). Such algebras have property $P2(\Omega)$ (Corollary 2.2). Typically in Section III, Ω is a Banach A-subbimodule of $\mathfrak{W}$, the space of weakly almost periodic functionals in A^*. If A has a bounded two-sided Ω-approximate identity, then Ω is essential (as a right Banach A-module) and $A\circ\Omega$ and Ω have equivalent norms; the converse holds provided A has property $P2(\Omega)$ (Theorem 3.4). This result (together with Theorem 2.5) for A commutative and $\Omega = \mathrm{cl}(\mathrm{sp}(\Delta A))$ yields Birtel's representation of the multiplier algebra of A when A has a bounded ΔA-approximate identity [1].

Now, if A is an L-algebra [14], then A^* is a commutative von Neumann algebra. Suppose that the Banach A-subbimodule $\Omega\subseteq\mathfrak{W}$ is a C^*-subalgebra of A^* with maximal ideal space Γ. Then $\mathrm{Hom}_A(A, M(\Gamma))$ is represented (isometrically and algebra anti-isomorphically) in the Banach algebra (under generalized convolution) $M(\Gamma)$ for algebras A possessing a right $\mathfrak{W}$-approximate identity bounded by one (Theorem 3.8), thereby extending previous representations of the right multiplier algebra $M_R(A)$ of A ([13], [9]). A special instance of Theorem 3.8 (see also Corollary 3.3) is the following. Let G be an arbitrary locally compact group, let $A = L^1(G)$, and let Ω be the C^*-algebra of (continuous) almost periodic functions on G. Then

$$\mathrm{Hom}_{L^1(G)}(L^1(G),\ M(\Gamma)) \cong M(\Gamma),$$

where Γ, the maximal ideal space of Ω, is the almost periodic (Bohr) compactification of G. The final result of Section III is another application of Theorem 3.4 and states that if A is a convolution measure algebra (CMA) and the C^*-algebra Ω contains the identity of A^*, then A has a two-sided Ω-approximate identity bounded by one if and only if the compact semigroup Γ has an identity. This result was proved in [8] for semisimple commutative CMA's A with $\Omega = \mathrm{cl}(\mathrm{sp}(\Delta A))$.

We close the introduction with some definitions, notation, and basic facts. The projective tensor product $A\hat{\otimes} W$ of A and W is the Banach space completion of the algebraic tensor product $A\otimes W$ with respect to the greatest cross-norm. Each tensor t in $A\hat{\otimes} W$ has a representation of the form

$$t = \sum_{k=1}^{\infty} a_k\otimes w_k,$$

where $\sum_{k=1}^{\infty}\|a_k\|\|w_k\| < +\infty$, and the norm of t in $A\hat{\otimes} W$ is

$$\inf\Big\{\sum_{k=1}^{\infty}\|a_k\|\|w_k\|:\ t = \sum_{k=1}^{\infty} a_k\otimes w_k\Big\}.$$

If W is an arbitrary right Banach A-module, then W^* is a left Banach A-module under the adjoint action of A (i.e., for $a\in A$, $w^*\in W^*$, define aw^* by $\langle w, aw^*\rangle = \langle wa, w^*\rangle$, all $w\in W$). The Banach space $\mathrm{Hom}_A(A, W^*)$ of all continuous left A-module homomorphisms (i.e., $T\in \boldsymbol{B}(A, W^*)$ such that $T(ab) = a(Tb)$, all $a, b\in A$) is isometrically isomorphic to the dual space $\big((A\hat{\otimes} W)/K\big)^*$, where

$$K = \mathrm{cl}(\mathrm{sp}\{ab\otimes w - b\otimes wa:\ a, b\in A,\ w\in W\})$$

in $A\hat{\otimes} W$ ([17], p. 72). In fact, if $\big((A\hat{\otimes} W)/K\big)^*$ is identified with the subspace $K^\perp$ of $(A\hat{\otimes} W)^*$, then this isometric isomorphism is simply the restriction to $\mathrm{Hom}_A(A, W^*)$ of the isometric isomorphism $T\mapsto g_T$ from $\boldsymbol{B}(A, W^*)$ onto $(A\hat{\otimes} W)^*$, where $g_T(a\otimes w) = \langle w, Ta\rangle$, $a\in A$, $w\in W$.

Let $B\colon A\hat{\otimes} W\to W$ be the norm-decreasing linear map defined by $B(a\otimes w) = wa$, $a\in A$, $w\in W$, and let $A\circ W = (A\hat{\otimes} W)/\ker B$ with quotient norm. Then $(A\circ W)^* = (\ker B)^\perp$ in $(A\hat{\otimes} W)^*$. Further, since K is clearly contained in $\ker B$, it follows that

$$(A\circ W)^* = (\ker B)^\perp \subseteq K^\perp \cong \mathrm{Hom}_A(A, W^*).$$

II. Property $P2(\Omega)$. Let $A = (A, \|\cdot\|)$ be a Banach algebra, let Ω be a closed linear subspace of the dual $A^* = (A^*, \|\cdot\|)$ of A, and suppose that Ω is a right Banach A-submodule of A^* with respect to the pre-Arens product fa of $f\in A^*$, $a\in A$, defined by $\langle fa, b\rangle = \langle f, ab\rangle$, $b\in A$. Then A is said to *possess property* $P2(\Omega)$ if whenever

$$\sum_{k=1}^{\infty}\|f_k\|\|a_k\| < +\infty,$$

$f_k\in\Omega$, $a_k\in A$, and $\sum_{k=1}^{\infty} f_k a_k = 0$, then

$$\sum_{k=1}^{\infty}\langle f_k, a_k\rangle = 0.$$

In terms of the norm-decreasing map $B\colon A\hat{\otimes}\Omega\to\Omega\subseteq A^*$ defined above and the norm-decreasing evaluation map $\xi\colon A\hat{\otimes}\Omega\to \boldsymbol{C}$ defined by $\xi(a\otimes f) = \langle f, a\rangle$, property $P2(\Omega)$ states that $\ker B\subseteq\ker\xi$. Property $P2(A^*)$ is simply property P2 of Máté [11] and will be denoted as such. Our first characterizations of property $P2(\Omega)$ involve right Ω-approximate identities and the space $M(A)$ of (continuous) double multipliers of A [4].

THEOREM 2.1. *Let A be a Banach algebra and let Ω be a right Banach A-submodule of A^*. Then the following statements are equivalent:*

(1) *The algebra A has property* $P2(\Omega)$.

(2) *There exists a net $\{u_\lambda\}$ in A such that, if*

$$\sum_{k=1}^{\infty}\|a_k\|\|f_k\| < +\infty,$$

$a_k \in A$, $f_k \in \Omega$, then

$$\sum_{k=1}^{\infty} \langle f_k, a_k \rangle = \lim_\lambda \sum_{k=1}^{\infty} \langle f_k, a_k u_\lambda \rangle.$$

(3) *The map $(S, T) \mapsto g_T$ is a vector space homomorphism from $M(A)$ into $(A \circ \Omega)^*$.*

(4) *The evaluation map ξ is in $(A \circ \Omega)^* = (\ker B)^\perp$.*

Proof. ((1)⇔(4)). Immediate.

((2)⇒(3)). The map $(S, T) \mapsto T$ from $M(A)$ into $\mathrm{Hom}_A(A, \Omega^*)$ is a vector space homomorphism, and the map $T \mapsto g_T$ from $\mathrm{Hom}_A(A, \Omega^*)$ into $(A \hat{\otimes} \Omega)^*$ is an isometric isomorphism. Thus, it suffices to show that, if $(S, T) \in M(A)$, then $g_T \in (\ker B)^\perp$. However, if $t = \sum_{k=1}^{\infty} a_k \otimes f_k \in \ker B$, then, assuming (2),

$$\begin{aligned} g_T(t) &= \sum_{k=1}^{\infty} \langle f_k, Ta_k \rangle = \lim_\lambda \sum_{k=1}^{\infty} \langle f_k, (Ta_k) u_\lambda \rangle \\ &= \lim_\lambda \sum_{k=1}^{\infty} \langle f_k, a_k (S u_\lambda) \rangle = \lim_\lambda \Big\langle \sum_{k=1}^{\infty} f_k a_k, S u_\lambda \Big\rangle \\ &= \lim_\lambda \langle B(t), S u_\lambda \rangle = 0, \end{aligned}$$

establishing (3).

((3)⇒(4)). If $I: A \to A$ is the identity operator, then the image of the double multiplier (I, I) in $(A \circ \Omega)^* = (\ker B)^\perp$ is $g_I = \xi$, the evaluation map.

((4)⇒(2)). Since $B \in \boldsymbol{B}(A \hat{\otimes} \Omega, A^*)$, $B^* \in \boldsymbol{B}(A^{**}, (A \hat{\otimes} \Omega)^*)$; hence, $B^*(A) \subseteq (A \hat{\otimes} \Omega)^*$. Now, if $t \in A \hat{\otimes} \Omega$ and $a \in A$, then $\langle t, B^*(a) \rangle = \langle B(t), a \rangle$. Consequently, with respect to the dual pair $\langle A \hat{\otimes} \Omega, (A \hat{\otimes} \Omega)^* \rangle$ the polar (in $A \hat{\otimes} \Omega$) of $B^*(A)$ is $\ker B$. Thus, the bipolar (in $(A \hat{\otimes} \Omega)^*$) of $B^*(A)$ is $(\ker B)^\perp$, and so, by the bipolar theorem, $B^*(A)$ is weak*- (i.e., $\sigma((A \hat{\otimes} \Omega)^*, (A \hat{\otimes} \Omega))$) dense in $(\ker B)^\perp$. Therefore, since $\xi \in (\ker B)^\perp$ by hypothesis, there exists a net $\{u_\lambda\}$ in A such that $\xi = wk^* - \lim_\lambda B^*(u_\lambda)$ in $(A \hat{\otimes} \Omega)^*$, thereby proving (2). ∎

It is readily verified that, if A has no right annihilators and if Ω separates the points of A, then the map in statement (3) of Theorem 2.1 is a vector space isomorphism. Further, for an arbitrary Banach algebra A, the net $\{u_\lambda\}$ in statement (2) is in particular, a right Ω-approximate identity; that is, if $a \in A$, $f \in \Omega$, then $\langle f, au_\lambda \rangle \underset{\lambda}{\to} \langle f, a \rangle$. However, the existence of a right Ω-approximate identity does not seem to imply property P2(Ω) without some boundedness assumption on the approximate identity. A right Ω-approximate identity $\{u_\lambda\}$ for A is said to be *operator-bounded* if

$$\sup\{\|au_\lambda\|: a \in A, \|a\| \leqslant 1, \text{ all } \lambda\} < +\infty.$$

Corollary 2.2. *Let A be a Banach algebra and let Ω be a right Banach A-submodule of A^*. If A has an operator-bounded right Ω-approximate identity $\{u_\lambda\}$, then A has property* P2(Ω).

Proof. Let

$$M = \sup\{\|au_\lambda\|: a \in A, \|a\| \leqslant 1, \text{ all } \lambda\} < +\infty,$$

and suppose that

$$t = \sum_{k=1}^{\infty} a_k \otimes f_k \in \ker B \subseteq A \hat{\otimes} \Omega.$$

Then, for each $N > 0$,

$$\begin{aligned} |\xi(t)| &= |\xi(t) - \langle B(t), u_\lambda \rangle| = \Big| \sum_{k=1}^{\infty} \langle f_k, a_k - a_k u_\lambda \rangle \Big| \\ &\leqslant \Big| \sum_{k=1}^{N} \langle f_k, a_k - a_k u_\lambda \rangle \Big| + (M+1) \sum_{k=N+1}^{\infty} \|f_k\| \|a_k\|, \end{aligned}$$

for every λ; hence, it follows that

$$|\xi(t)| \leqslant (M+1) \sum_{k=N+1}^{\infty} \|f_k\| \|a_k\|.$$

Letting $N \to \infty$ yields $\xi(t) = 0$. Thus, $\xi \in (\ker B)^\perp$. ∎

If G is a compact group and $A = L^P(G)$, where $1 \leqslant P < +\infty$, then, since A has an operator-bounded (by 1) two-sided approximate identity, it follows from Corollary 2.2 that A has property P2 (and so P2(Ω), for every right Banach A-submodule Ω of A^*). Some of the strength of property P2 is revealed in its characterizations below.

Corollary 2.3. *For a Banach algebra A, the following statements are equivalent:*

(1) *A has property* P2.

(2) $\mathrm{Hom}_A(A, A^{**}) \cong (A \circ A^*)^*$.

(3) $\mathrm{Hom}_A(A, W^*) \cong (A \circ W)^*$, *for every right Banach A-module W.*

Proof. The equivalence (1)⇔(2) is proved in ([21], Thm. 2.3), and (3)⇒(2) is clear. To prove that (1)⇒(3), we first observe that statement (3) holds if and only if, for every $T \in \mathrm{Hom}_A(A, W^*)$, the corresponding g_T in $(A \hat{\otimes} W)^*$ is contained in $(\ker B)^\perp$. However, if $T \in \mathrm{Hom}_A(A, W^*)$, then $T^* \in \boldsymbol{B}(W^{**}, A^*)$. Thus, for each $t = \sum_{k=1}^{\infty} a_k \otimes w_k \in \ker B$,

$$g_T(t) = \sum_{k=1}^{\infty} \langle w_k, Ta_k \rangle = \sum_{k=1}^{\infty} \langle T^* w_k, a_k \rangle;$$

hence, since A has property P2 (see Theorem 2.1),

$$g_T(t) = \lim_\lambda \sum_{k=1}^{\infty} \langle T^* w_k, a_k u_\lambda\rangle = \lim_\lambda \sum_{k=1}^{\infty} \langle w_k, a_k(Tu_\lambda)\rangle$$
$$= \lim_\lambda \sum_{k=1}^{\infty} \langle w_k a_k, Tu_\lambda\rangle = \lim_\lambda \langle B(t), Tu_\lambda\rangle = 0.$$

Consequently, $g_T \in (\ker B)^\perp$ and (3) obtains. ∎

It is clear from the remarks made in the introduction that $\mathrm{Hom}_A(A, W^*) \cong (A\circ W)^*$ if and only if $K = \ker B$. Thus, A possesses property P2 if and only if $K = \ker B$ for the right Banach A-module A^* (equivalently, $K = \ker B$ for every right Banach A-module W). Similarly, if Ω is a right Banach A-submodule of A^*, then $\mathrm{Hom}_A(A, \Omega^*) \cong (A\circ\Omega)^*$ if and only if $K = \ker B$ and, in this case, A has property P2(Ω) (by Theorem 2.1, since ξ is clearly contained in $K^\perp$). However, the fact that A has property P2(Ω) does not, in general, seem to imply that $\mathrm{Hom}_A(A, \Omega^*) \cong (A\circ\Omega)^*$, although this implication does hold whenever $T^*\Omega \subseteq \Omega$, for all T in $\mathrm{Hom}_A(A, \Omega^*)$ (see the proof of Theorem 2.5 below).

If A is a symmetric Segal algebra in the sense of Reiter ([15], Sec. 4) with auxiliary norm $\|\cdot\|_S$, then Corollary 2.3 yields our next result, from which the noncommutative versions of several factorization theorems for (symmetric) Segal algebras (c.f. [10], Thm.'s 1, 2, 3) follow readily.

PROPOSITION 2.4. *Let G be a locally compact group, let A be a symmetric Segal algebra in $L^1(G)$, and let W be a right Banach $L^1(G)$-module. Then*

$$\mathrm{Hom}_{L^1(G)}(A, W^*) = \mathrm{Hom}_A(A, W^*) \quad \text{and} \quad A\otimes_{L^1(G)} W = A\circ W = A\otimes_A W.$$

Proof. It is immediate that W is a right Banach A-module and that $\mathrm{Hom}_{L^1(G)}(A, W^*) \subseteq \mathrm{Hom}_A(A, W^*)$. On the other hand, since A is dense in $L^1(G)$ and W^* is a left Banach $L^1(G)$-module, $\mathrm{Hom}_A(A, W^*) \subseteq \mathrm{Hom}_{L^1(G)}(A, W^*)$. Thus, $\mathrm{Hom}_{L^1(G)}(A, W^*) = \mathrm{Hom}_A(A, W^*)$ and so $A\otimes_{L^1(G)} W = A\otimes_A W$.

Finally, since A is a symmetric Segal algebra, it follows from [15], p. 34, that A possesses an operator-bounded approximate identity. Hence, A has property P2 by Corollary 2.2, and by Corollary 2.3 $\mathrm{Hom}_A(A, W^*) = (A\circ W)^*$, implying that $A\circ W = A\otimes_A W$. ∎

If A is a commutative Banach algebra, then one Ω of particular interest is $\mathrm{cl}(\mathrm{sp}(\Delta A))$, the closure in A^* of the linear span of the maximal ideal space ΔA of A. Note that Ω separates the points of A if and only if A is semisimple.

THEOREM 2.5. *Let A be a commutative Banach algebra with maximal ideal space ΔA and let $\Omega = \mathrm{cl}(\mathrm{sp}(\Delta A))$. Then A has property P2(Ω) if and only if $\mathrm{Hom}_A(A, \Omega^*) \cong (A\circ\Omega)^*$.*

Proof. ($\Leftarrow$). This follows immediately from Theorem 2.1, since $M(A) = M_R(A) \to \mathrm{Hom}_A(A, \Omega^*)$ is a vector space homomorphism.

($\Rightarrow$). It suffices to show that, if $T \in \mathrm{Hom}_A(A, \Omega^*)$ then g_T in $K^\perp$ is also in $(\ker B)^\perp$. Toward this end, fix T in $\mathrm{Hom}_A(A, \Omega^*)$. Now given χ in ΔA, $\chi a = \langle\chi, a\rangle\chi$, for every a in A. Choose e in A so that $\langle\chi, e\rangle = 1$; then $\chi e = \chi$, and

$$\langle T^*\chi, a\rangle = \langle\chi, Ta\rangle = \langle\chi e, Ta\rangle = \langle\chi, e(Ta)\rangle$$
$$= \langle\chi, T(ea)\rangle = \langle\chi, T(ae)\rangle = \langle\chi, a(Te)\rangle$$
$$= \langle\chi a, Te\rangle = \langle\chi, a\rangle\langle\chi, Te\rangle,$$

for all a in A. Hence, $T^*\chi = \langle\chi, Te\rangle\chi \in \mathrm{sp}(\Delta A)$, and so $T^*(\Delta A) \subseteq \mathrm{sp}(\Delta A)$; whence, by the linearity and continuity of T^*, $T^*(\Omega) \subseteq \Omega$. Next, suppose that $t = \sum_{k=1}^{\infty} a_k\otimes f_k \in \ker B$; then

$$g_T(t) = \sum_{k=1}^{\infty} \langle f_k, Ta_k\rangle = \sum_{k=1}^{\infty} \langle T^* f_k, a_k\rangle,$$

and since A has P2(Ω) (see Theorem 2.1),

$$g_T(t) = \lim_\lambda \sum_{k=1}^{\infty} \langle T^* f_k, a_k u_\lambda\rangle = \lim_\lambda \sum_{k=1}^{\infty} \langle f_k, a_k(Tu_\lambda)\rangle$$
$$= \lim_\lambda \sum_{k=1}^{\infty} \langle f_k a_k, Tu_\lambda\rangle = \lim_\lambda \langle B(t), Tu_\lambda\rangle = 0.$$

Therefore, $g_T \in (\ker B)^\perp$ as desired. ∎

In the setting of Theorem 2.5, it is more convenient to work with ΔA rather than Ω. Indeed, every commutative Banach algebra A possesses a ΔA-approximate identity, directed by the finite subsets of ΔA, and composed of the quasi-product of elements e_i, where $\langle\chi_i, e_i\rangle = 1$, and $\{\chi_i\colon i = 1, \ldots, n\}$ is a finite subset of ΔA. Using standard estimates, it follows that an operator-bounded ΔA-approximate identity is also an operator-bounded Ω-approximate identity, where $\Omega = \mathrm{cl}(\mathrm{sp}(\Delta A))$. Hence, a commutative Banach algebra possessing an operator-bounded ΔA-approximate identity has property P2(Ω).

III. Norm-bounded Ω-approximate identities. The presence of an operator-bounded right Ω-approximate identity guarantees that A has property P2(Ω) (Corollary 2.2) and, therefore, that $M(A)$ can be represented in $(A\circ\Omega)^*$. In this section, we investigate $(A\circ\Omega)^*$ as a Banach algebra (under Arens product) for certain right Banach A-submodules Ω of A^* when A possesses a norm-bounded right Ω-approximate identity. Arens products on A^{**} are defined in the following way. If $a \in A$, $f \in A^*$, and $\varphi \in A^{**}$, then $f^a, f^\varphi, f_\varphi$ are those elements of A^* defined by: $\langle f^a, b\rangle$

$=\langle f, ba\rangle, b\in A$; $\langle f^\varphi, a\rangle = \langle fa, \varphi\rangle$, $a\in A$; $\langle f_\varphi, a\rangle = \langle f^a, \varphi\rangle$, $a\in A$. Finally, if $\varphi, \psi\in A^{**}$, then the Arens products $\varphi\circ\psi$ and $\varphi\psi$ are those elements of A^{**} defined by: $\langle f, \varphi\circ\psi\rangle = \langle f_\varphi, \psi\rangle$, and $\langle f, \varphi\psi\rangle = \langle f^\psi, \varphi\rangle$, $f\in A^*$. The above definitions for Arens products are taken from [14], as are the following definitions. An element f in A^* is *weakly almost periodic* (resp., *almost periodic*) if the set $\{f^a\colon a\in A,\ \|a\|\leqslant 1\}$ (equiv., the set $\{fa\colon a\in A,\ \|a\|\leqslant 1\}$) is relatively compact in the weak (resp., norm) topology on A^*. The linear subspace $\mathfrak{W}$ (resp., $\mathfrak{A}$) of all weakly almost periodic (resp., almost periodic) functionals in A^* is norm-closed.

Throughout this section, unless expressly stated otherwise, Ω will be assumed to be a closed subspace of $\mathfrak{W}$. Further, in the spirit of [14], it is assumed that Ω is a Banach A-subbimodule of $\mathfrak{W}$ under the two pre-Arens products fa and f^a, $a\in A$, $f\in\Omega$; this is equivalent ([14], Thm. 3.1) to f_φ and f^φ belonging to Ω, for all $f\in\Omega$, $\varphi\in A^{**}$. Under these assumptions, $\Omega^\perp$ is a two-sided ideal in A^{**} in each of the Arens products, and the two Arens products coincide on $\Omega^* = A^{**}/\Omega^\perp$ ([14], Thm.'s 3.2, 3.4). Thus the product of two elements in Ω^* is obtained by extending them to A^*, computing either Arens product of the extensions in A^{**}, and then restricting to Ω. For notational convenience, the same symbol will sometimes be used for a functional in Ω^* and an extension of it in A^{**}.

If Ω is an arbitrary right Banach A-submodule of A^*, then $\Omega_e = \mathrm{cl}(\mathrm{sp}\{fa\colon f\in\Omega,\ a\in A\})$ is called the *essential part of* Ω and is again a right Banach A-submodule of A^* ([16], Def. 3.5). If $\Omega = \Omega_e$, then Ω is said to be an *essential right Banach A-module*. The following proposition and corollaries interpret this concept in our setting.

PROPOSITION 3.1. *Let A be a Banach algebra and let Ω be an arbitrary right Banach A-submodule of A^* which is also a left A^{**}-module (i.e., $f^\varphi\in\Omega$, all $f\in\Omega$, $\varphi\in A^{**}$). If A possesses a bounded right Ω-approximate identity $\{u_\lambda\}$, then Ω is essential if and only if $fu_\lambda\to f$ weakly, for all f in Ω.*

Proof. ($\Leftarrow$). This is clear, since $\{fu_\lambda\}\in\Omega_e$, and Ω_e is norm (hence, weakly) closed.

($\Rightarrow$). First, if $f\in\Omega$, $a\in A$, then

$$\langle (fa)u_\lambda, \varphi\rangle = \langle f, (au_\lambda)\varphi\rangle = \langle f^\varphi, au_\lambda\rangle \to \langle f^\varphi, a\rangle = \langle fa, \varphi\rangle,$$

for all $\varphi\in A^{**}$; thus, $(fa)u_\lambda\to fa$ weakly. It follows immediately that $gu_\lambda\to g$ weakly, for every $g\in\mathrm{sp}\{fa\colon f\in\Omega,\ a\in A\}$. For an arbitrary f in $\Omega = \Omega_e$, we argue as follows. Let $\varepsilon>0$ and $\varphi\in A^{**}$ be given, and let

$$M = (\sup\{\|u_\lambda\|\colon \lambda\}+1)(\|\varphi\|+1) < +\infty.$$

There exists a g in $\mathrm{sp}\{fa\colon f\in\Omega,\ a\in A\}$ such that $\|f-g\|<\varepsilon/2M$ and, for this g, there exists an index λ_ε such that $\lambda>\lambda_\varepsilon$ implies that $|\langle gu_\lambda - g, \varphi\rangle| < \varepsilon/2$. For each such λ,

$$|\langle fu_\lambda - f, \varphi\rangle| \leqslant |\langle (f-g)u_\lambda - (f-g), \varphi\rangle| + |\langle gu_\lambda - g, \varphi\rangle| < \|f-g\|(\|u_\lambda\|+1)\|\varphi\| + \varepsilon/2 < \varepsilon/2+\varepsilon/2 = \varepsilon. \ \blacksquare$$

COROLLARY 3.2. *Under the hypotheses of Proposition 3.1, if $\Omega = \Omega_e$, then $\{u_\lambda\}$ is a (bounded) left Ω-approximate identity as well.*

COROLLARY 3.3. *Let A be a Banach algebra and let Ω be a Banach A-subbimodule of $\mathfrak{W}$. Then the following statements are equivalent.*

(1) *A has a bounded right Ω-approximate identity $\{u_\lambda\}$, and Ω is essential.*

(2) *A has a bounded two-sided Ω-approximate identity $\{u_\lambda\}$.*

Proof. ((1)$\Rightarrow$(2)). Corollary 3.2.

((2)$\Rightarrow$(1)). If $f\in\Omega$, then, since $\Omega\subseteq\mathfrak{W}$, the net $\{fu_\lambda\}$ has a weakly convergent subnet with limit g in Ω_e. However, the fact that $\{u_\lambda\}$ is a left Ω-approximate identity implies that $\{fu_\lambda\}$ converges weak* to f. Thus, $f = g\in\Omega_e$. $\blacksquare$

Now, $A\circ\Omega = (A\hat{\otimes}\Omega)/\ker B$ is isometrically isomorphic to the image Z of B in Ω equipped with the quotient norm $|\cdot|$; more precisely,

$$Z = \Big\{h\in\Omega\colon h = \sum_{k=1}^{\infty} f_k a_k = B(t),\ t = \sum_{k=1}^{\infty} a_k\otimes f_k \in A\hat{\otimes}\Omega\Big\}$$

and

$$|h| = \inf\Big\{\sum_{k=1}^{\infty}\|a_k\|\,\|f_k\|\colon h = \sum_{k=1}^{\infty} f_k a_k\in Z\Big\},$$

for h in Z. With respect to the norm $\|\cdot\|$ that Z inherits from Ω (or A^*), $\|h\|\leqslant|h|$, for all $h\in Z$.

If A possesses a bounded right approximate identity $\{u_\lambda\}$ ($\|u_\lambda\|\leqslant M$, for all λ) and if $\Omega = A^*_e = \mathrm{cl}(\mathrm{sp}\{fa\colon a\in A,\ f\in A^*\})$, then by the Hewitt–Cohen factorization theorem, $\Omega = \{fa\colon a\in A,\ f\in\Omega\}$; in particular, Ω is essential. Hence, $\{u_\lambda\}$ is a (bounded) two-sided Ω-approximate identity (Corollary 3.2). Moreover, since $|fau_\lambda - fa|\leqslant\|f\|\,\|au_\lambda - a\|\underset{\lambda}{\to}0$ and $|fau_\lambda|\leqslant M\|fa\|$, $f\in\Omega$, $a\in A$, it follows that $\|fa\|\leqslant|fa|\leqslant M\|fa\|$, for all $f\in\Omega$, $a\in A$; thus, $A\circ\Omega\cong(Z, |\cdot|)$ is topologically isomorphic to Ω. At the same time, it is easily seen [12] that $A\circ A^*$ is topologically isomorphic to Ω. Hence, by Corollaries 2.2 and 2.3, $\mathrm{Hom}_A(A, A^{**})\cong(A\circ A^*)^*$ and $\mathrm{Hom}_A(A, \Omega^*)\cong(A\circ\Omega)^*$ are topologically isomorphic to each other and to Ω^*.

It is a natural problem then to consider the equivalence of the two norms $\|\cdot\|$ and $|\cdot|$ on Z for other choices of Ω.

THEOREM 3.4. *Let A be a Banach algebra and let Ω be a Banach A-sub-bimodule of $\mathfrak{W}$. Then the statements:*

(1a) *A possesses a bounded two-sided Ω-approximate identity.*

(1b) *Ω^* has a two-sided identity.*

are equivalent, as are the statements:

(2a) *$Z = \Omega$, as sets.*

(2b) *Ω is essential, and $A \circ \Omega$ and Ω have equivalent norms.*

Moreover, (1) ⇒ (2) *and, if A has property* P2(Ω), *then* (2) ⇒ (1).

Proof. ((1a) ⇒ (1b)). Let $\{u_\lambda\}$ be a bounded two-sided Ω-approximate identity for A, and let $\|u_\lambda\| \leqslant M$, for all λ. Then $\{u_\lambda\}$ is a bounded net in A^{**} and, as such, has a $\sigma(A^{**}, A^*)$-convergent subnet (still written $\{u_\lambda\}$) with $\sigma(A^{**}, A^*)$-limit E in A^{**}, $\|E\| \leqslant M$. Since $\Omega \subseteq A^*$, $E \in \Omega^*$ and $\|E\|_{\Omega^*} \leqslant \|E\| \leqslant M$. Further, if $\varphi \in \Omega^*$, then

$$\langle f, E\varphi\rangle = \langle f^\varphi, E\rangle = \lim_\lambda \langle f^\varphi, u_\lambda\rangle = \lim_\lambda \langle fu_\lambda, \varphi\rangle = \langle f, \varphi\rangle,$$

for all f in Ω (see Proposition 3.1), so $E\varphi = \varphi$ in Ω^*. On the other hand, if $f \in \Omega$, then

$$\langle f^E, a\rangle = \langle fa, E\rangle = \lim_\lambda \langle fa, u_\lambda\rangle = \lim_\lambda \langle f, au_\lambda\rangle = \langle f, a\rangle,$$

for all a in A; hence, $f^E = f$, and so

$$\langle f, \varphi E\rangle = \langle f^E, \varphi\rangle = \langle f, \varphi\rangle,$$

for all φ in Ω^*.

((1b) ⇒ (1a)). Let E be a two-sided identity in Ω^* with $\|E\|_{\Omega^*} = M$, and let E' be a norm-preserving extension of E to A^*. Then there exists a net $\{u_\lambda\}$ in A, $\|u_\lambda\| \leqslant M$, for all λ, such that E' is the $\sigma(A^{**}, A^*)$-limit of $\{u_\lambda\}$. It follows immediately that $\{u_\lambda\}$ is a two-sided Ω-approximate identity for A.

((2a) ⇒ (2b)). Since $Z \subseteq \Omega_e$ always, (2a) implies that Ω is essential. In addition, the map $A \circ \Omega \hookrightarrow \Omega$ is a norm-decreasing vector space isomorphism which is onto by (2a); hence, $A \circ \Omega$ and Ω have equivalent norms by the open mapping theorem.

((2b) ⇒ (2a)). Since $A \circ \Omega$ and Ω have equivalent norms, Z is closed in Ω and, since Z is dense in Ω_e, $Z = \Omega$.

((1) ⇒ (2)). Let E be an identity for Ω^*, $\|E\| = M$, and let E' be an norm-preserving extension of E to A^*. Now, if $T \in \boldsymbol{B}(A, \Omega^*)$, then $T^{**}(E') \in \Omega^{***}$ and, by restriction to $\Omega \subseteq \Omega^{**}$, defines a unique element φ_T in Ω^*. The resulting map $T \mapsto \varphi_T$ is linear and continuous with $\|\varphi_T\| \leqslant \|T^{**}(E')\| \leqslant M\|T\|$. Moreover, if $\varphi \in \Omega^*$ and R_φ in $\boldsymbol{B}(A, \Omega^*)$ is defined by $R_\varphi(a) = a\varphi$, then $\varphi_{R_\varphi} = \varphi$. Indeed, it is immediate that $R_\varphi^*(f) = f^\varphi$, for $f \in \Omega$, and so $\langle f, R_\varphi^{**}(E')\rangle = \langle f, E\varphi\rangle = \langle f, \varphi\rangle$, for all f in Ω. Therefore, the map R: $\Omega^* \to \boldsymbol{B}(A, \Omega^*)$ defined by $\varphi \mapsto R_\varphi$ is a norm-decreasing linear isomorphism with $\|R_\varphi\| \leqslant \|\varphi\| = \|\varphi_{R_\varphi}\| \leqslant M\|R_\varphi\|$; hence, R imbeds Ω^* in $\boldsymbol{B}(A, \Omega^*)$ as a closed subspace with an equivalent norm.

Now, since $A \circ \Omega \cong (Z, |\cdot|) \subseteq (\Omega, \|\cdot\|)$ and $\|h\| \leqslant |h|$, for all $h \in Z$, statement (2) holds if $(A \circ \Omega)^*$ and Ω^* are topologically isomorphic. However, if $\varphi \in \Omega^*$, then it is routine to verify that $g_{R_\varphi} = B^*(\varphi)$ in $(\ker B)^\perp = (A \circ \Omega)^*$. Further, B^* is a norm-decreasing vector space isomorphism (if $B^*(\varphi) = B^*(\psi)$, then $\varphi = \psi$ on Z, and Z is dense in Ω by Corollary 3.3) from Ω^* onto a $\sigma = \sigma\big((A \hat{\otimes} \Omega)^*, A \hat{\otimes} \Omega\big)$ – dense subspace $B^*(\Omega^*)$ of $(\ker B)^\perp = (A \circ \Omega)^*$. Thus, if $B^*(\Omega^*)$ is σ-closed, then B^* is onto, and $(A \circ \Omega)^*$ and Ω^* are topologically isomorphic by the open mapping theorem.

By the Krein–Smulian theorem ([18], Cor. to Thm. 6.4, p. 152), $B^*(\Omega^*)$ is σ-closed in $(A \hat{\otimes} \Omega)^*$ if $B^*(\Omega^*) \cap S_1$ is σ-closed, where S_1 is the closed unit ball in $(A \hat{\otimes} \Omega)^*$. But, if $\{B^*(\varphi_\gamma)\}_\gamma$ is a σ-Cauchy net in $B^*(\Omega^*) \cap S_1$, then, for each $f \in \Omega$, $a \in A$, the net $\{\langle fa, \varphi_\gamma\rangle\}_\gamma$ converges. Since $\{\varphi_\gamma\}_\gamma$ is bounded in Ω^*, with $\|\varphi_\gamma\| \leqslant M$ for all γ, the formula: $\langle fa, \varphi\rangle = \lim_\gamma \langle fa, \varphi_\gamma\rangle$, $f \in \Omega$, $a \in A$, defines an element φ of Ω^* with $\|\varphi\| \leqslant M$. It follows immediately that $B^*(\varphi) = \sigma - \lim_\gamma B^*(\varphi_\gamma)$ in $(A \hat{\otimes} \Omega)^*$ and, since S_1 is σ-closed, $B^*(\varphi) \in B^*(\Omega^*) \cap S_1$. Thus, the proof of ((1) ⇒ (2)) is complete.

((2) ⇒ (1); A has P2(Ω)). Since A has property P2(Ω), the evaluation map $\xi \in (A \circ \Omega)^*$; hence, by (2), there is a unique element E in Ω^* such that $\xi = B^*(E)$. Because $\langle f^E, a\rangle = \langle fa, E\rangle = \xi(a \otimes f) = \langle f, a\rangle$, for all a in A, $f^E = f$, for all f in Ω; consequently, $\varphi E = \varphi$, for all φ in Ω^*. On the other hand, if $\varphi \in \Omega^*$, then $\langle fa, E\varphi\rangle = \langle f^\varphi, aE\rangle = \langle f^\varphi, a\rangle = \langle fa, \varphi\rangle$, for all $f \in \Omega$, $a \in A$; thus, since Ω is essential, $E\varphi = \varphi$. ∎

Theorem 3.4 provides an interpretation of a well-known multiplier representation theorem. If A is a commutative semisimple Banach algebra with a bounded ΔA-approximate identity $\{u_\lambda\}$, then $\{u_\lambda\}$ is a (bounded) approximate identity for $\Omega = \mathrm{cl}(\mathrm{sp}(\Delta A))$. Hence, Theorem 3.4, together with Theorem 2.5, implies that $\mathrm{Hom}_A(A, \Omega^*)$ is topologically isomorphic to Ω^*. Further, since A is semi-simple, the canonical map $M(A) \hookrightarrow \mathrm{Hom}_A(A, \Omega^*)$ is a norm-decreasing vector space isomorphism. Thus, $M(A) \hookrightarrow \Omega^*$ and this map is the continuous algebra isomorphism (Ω^* is a commutative Banach algebra) described by Birtel ([1], Sec. 3). In addition, Theorem 3.4 implies that such a representation of $M(A)$ exists, if and only if A has a bounded ΔA-approximate identity, showing Birtel's assumption is best possible.

If Ω satisfies statement (1) of Theorem 3.4, then A has property P2(Ω) (Corollary 2.2), and by Theorem 3.4, $(\ker B)^\perp = (A \circ \Omega)^*$ is topologically isomorphic to Ω^*. In the case discussed above, $(\ker B)^\perp = K^\perp$; thus, $\mathrm{Hom}_A(A, \Omega^*) \cong (A \circ \Omega)^*$ and a representation of $\mathrm{Hom}_A(A, \Omega^*)$ in Ω^* follows. In general, however, it seems that only the containment $(\ker B)^\perp \subseteq K^\perp$ obtains. Consequently, a description of the elements of

$\mathrm{Hom}_A(A, \Omega^*)$ which are represented by elements of Ω^* or, equivalently, of the subspace of $\mathrm{Hom}_A(A, \Omega^*)$ corresponding to $(A \circ \Omega)^*$ is in order.

PROPOSITION 3.5. *Let A be a Banach algebra, and let Ω be a Banach A-subbimodule of $\mathfrak{W}$. If A possesses a bounded right Ω-approximate identity, then an element T in $\mathrm{Hom}_A(A, \Omega^*)$ is of the form R_φ, for some φ in Ω^*, if and only if $T^*\Omega \subseteq \Omega$.*

Proof. Since $R_\varphi^*(f) = f^\varphi$, for all $f \in \Omega$, the implication ($\Rightarrow$) is clear. For the reverse implication ($\Leftarrow$), let $\varphi = \varphi_T$, where φ_T is as defined in the proof of Theorem 3.4 $\big((1)\Rightarrow(2)\big)$. Then for each $a \in A$, $f \in \Omega$,

$$\begin{aligned}\langle f, R_\varphi(a)\rangle &= \langle fa, T^{**}(E')\rangle = \langle T^*(fa), E'\rangle = \langle (T^*f)a, E'\rangle \\ &= \lim_\lambda \langle T^*f, au_\lambda\rangle = \langle T^*f, a\rangle = \langle f, Ta\rangle;\end{aligned}$$

hence, $T = R_\varphi$. ∎

If Ω is an arbitrary Banach A-subbimodule of $\mathfrak{W}$, then it is not to be expected that $\mathrm{Hom}_A(A, \Omega^*) \cong (A \circ \Omega)^*$, even when A has property P2(Ω). However, for the special cases $\Omega = \mathfrak{W}, \mathfrak{A}$, property P2$(\Omega)$ does suffice.

PROPOSITION 3.6. *Let A be a Banach algebra and let $\Omega = \mathfrak{W}, \mathfrak{A}$. Then A has property* P2(Ω) *if and only if $\mathrm{Hom}_A(A, \Omega^*) \cong (A \circ \Omega)^*$.*

Proof. It suffices to prove that $T^*\Omega \subseteq \Omega$, for all $T \in \mathrm{Hom}_A(A, \Omega^*)$. Let $f \in \Omega$ and let $O_L(f) = \{fa\colon a \in A,\ \|a\| \leqslant 1\}$. Then

$$\begin{aligned}O_L(T^*f) &= \{(T^*f)a\colon a \in A,\ \|a\| \leqslant 1\} = \{T^*(fa)\colon a \in A,\ \|a\| \leqslant 1\} \\ &= T^*\big(O_L(f)\big).\end{aligned}$$

Thus, if $f \in \mathfrak{W}$ ($\mathfrak{A}$), then $T^*f \in \mathfrak{W}$ ($\mathfrak{A}$) ([14], Thm.'s 2.1, 2.2). ∎

COROLLARY 3.7. *If A is a Banach algebra possessing* P2$(\mathfrak{W})$, *then $\mathrm{Hom}_A(A, \Omega^*) \cong (A \circ \Omega)^*$, for every Banach A-subbimodule Ω of $\mathfrak{W}$.*

Proof. Let $T \in \mathrm{Hom}_A(A, \Omega^*)$, and let g_T be the corresponding element of $K^\perp$. It suffices to show that $g_T \in (\ker B)^\perp$. Now, from the previous proposition, $T^*\Omega \subseteq \mathfrak{W}$. Hence, if $t = \sum_{k=1}^{\infty} a_k \otimes f_k \in \ker B \subseteq A \hat{\otimes} \Omega$, then, since A has P2$(\mathfrak{W})$,

$$\begin{aligned}g_T(t) &= \sum_{k=1}^{\infty} \langle f_k, Ta_k\rangle = \sum_{k=1}^{\infty} \langle T^*f_k, a\rangle \\ &= \lim_\lambda \sum_{k=1}^{\infty} \langle T^*f_k, a_k u_\lambda\rangle = \lim_\lambda \sum_{k=1}^{\infty} \langle f_k, a_k(Tu_\lambda)\rangle \\ &= \lim_\lambda \langle B(t), Tu_\lambda\rangle = 0,\end{aligned}$$

where $\{u_\lambda\}$ is the net of Theorem 2.1. ∎

If $T^*\Omega \subseteq \Omega$, for all $T \in \mathrm{Hom}_A(A, \Omega^*)$, then $\mathrm{Hom}_A(A, \Omega^*)$ becomes a Banach algebra under the product

$$\langle f, (ST)a\rangle = \langle S^*f, Ta\rangle, \quad f \in \Omega,\ a \in A,$$

where S, $T \in \mathrm{Hom}_A(A, \Omega^*)$ (see [6] for more details). Using this fact, we obtain a generalization of ([9], Thm. 3.1) and ([13], Thm. 2.1) for L-algebras (i.e., Banach algebras which are complex L-spaces ([14], [20])). Now, the dual A^* of an L-algebra A is a commutative von Neumann algebra and, in the following theorem, Ω will be a C^*-subalgebra of A^*; hence, $\Omega \cong C_0(\Gamma)$, where Γ is the maximal ideal space of Ω equipped with the usual Gelfand topology, and so $\Omega^* \cong M(\Gamma)$. The Banach space $M(\Gamma)$ is a Banach algebra under a generalized convolution product ([14], Eqn. 5.2). Let $\pi'\colon A \to M(\Gamma)$ be the canonical map given in ([14], Thm. 5.1).

THEOREM 3.8. *Let A be an L-algebra and let Ω be a Banach A-subbimodule of $\mathfrak{W}$ which is a $*$-subalgebra of A^*. Suppose that A possesses a right $\mathfrak{F}$-approximate identity bounded by one, where $\mathfrak{F} = \mathfrak{A}$ if $\Omega = \mathfrak{A}$ and $\mathfrak{F} = \mathfrak{W}$ if $\Omega \neq \mathfrak{A}$.*

Then $\mathrm{Hom}_A\big(A, M(\Gamma)\big)$ is isometrically algebra anti-isomorphic to closed subalgebra of $M(\Gamma) \cong \Omega^*$ via the map $T \mapsto \mu_T$, where T and μ_T satisfy $Ta = \pi'(a) * \mu_T$, $a \in A$. The map $T \mapsto \mu_T$ is onto $M(\Gamma)$ if and only if Ω is an essential right Banach A-module.

Proof. Let $\{u_\lambda\}$ be a right $\mathfrak{F}$-approximate identity, $\|u_\lambda\| \leqslant 1$ for all λ, and let E be the $\sigma(A^{**}, A^*)$ limit of a subnet (still written $\{u_\lambda\}$) of $\{u_\lambda\}$ ($\|E\| \leqslant 1$). Then it is routine to show that $f^E = f$ for all $f \in \mathfrak{F}$ and so $\langle f, \varphi E\rangle = \langle f, \varphi\rangle$ for all $f \in \mathfrak{F}$, $\varphi \in A^{**}$. Let $T \in \mathrm{Hom}_A(A, \Omega^*)$, and let φ_T be the restriction $T^{**}(E)$ to Ω. Clearly, $\|\varphi_T\| \leqslant \|T^{**}(E)\| \leqslant \|T\|$. Moreover, if $f \in \Omega \subseteq \mathfrak{F}$, then, since $T^*\mathfrak{F} \subseteq \mathfrak{F}$ by Proposition 3.6,

$$\begin{aligned}\langle f, a\varphi_T\rangle &= \langle fa, T^{**}(E)\rangle = \langle T^*(fa), E\rangle \\ &= \langle T^*f, aE\rangle = \langle T^*f, a\rangle = \langle f, Ta\rangle, \quad a \in A.\end{aligned}$$

Hence, $Ta = a\varphi_T$ for all $a \in A$, whence $\|T\| \leqslant \|\varphi_T\|$ and $T = R_{\varphi_T}$; so by Proposition 3.5, $T^*\Omega \subseteq \Omega$ for all $T \in \mathrm{Hom}_A(A, \Omega^*)$. It follows that the isometry $T \mapsto \varphi_T$ from $\mathrm{Hom}_A(A, \Omega^*)$ into Ω^* is an algebra anti-isomorphism. For if S, $T \in \mathrm{Hom}_A(A, \Omega^*)$, then for all $f \in \Omega$, $a \in A$,

$$\begin{aligned}\langle (ST)^*f, a\rangle &= \langle f, (ST)a\rangle = \langle S^*f, Ta\rangle \\ &= \langle S^*f, a\varphi_T\rangle = \langle (S^*f)^{\varphi_T}, a\rangle,\end{aligned}$$

and

$$\langle S^*f, a\rangle = \langle f, Sa\rangle = \langle f, a\varphi_S\rangle = \langle f^{\varphi_S}, a\rangle;$$

hence,

$$\langle f, a\varphi_{ST}\rangle = \langle fa, (ST)^{**}(E)\rangle = \langle (ST)^* f, aE\rangle$$
$$= \langle (ST)^* f, a\rangle = \langle (S^* f)^{\varphi_T}, a\rangle = \langle (f^{\varphi_S})^{\varphi_T}, a\rangle = \langle f, a\varphi_T\varphi_S\rangle.$$

Further, the map $\varphi_T \mapsto \mu_T$ from Ω^* into $M(\Gamma)$ is an isometric algebra isomorphism by ([14], Thm. 5.1) and hence the composite map $T \mapsto \varphi_T \mapsto \mu_T$ is an isometric algebra anti-isomorphism of $\mathrm{Hom}_A(A, \Omega^*)$ into $M(\Gamma)$ with $Ta = a\varphi_T \mapsto \pi'(a) * \mu_T$, $a \in A$.

Now, if Ω is an essential right Banach A-module, then the hypotheses imply that $\mathrm{Hom}_A(A, \Omega^*) \cong (A \circ \Omega)^* \cong \Omega^* \cong M(\Gamma)$ (the fact that the $\mathfrak{F}$-approximate identity is bounded by one yields $A \circ \Omega \cong \Omega$) as Banach spaces, with the isometry implemented as above (for $T \in \mathrm{Hom}_A(A, \Omega^*)$, the φ_T defined above is the φ_T defined in the proof of Theorem 3.4 ((1)$\mapsto$(2))). Hence, the mapping $T \mapsto \mu_T$ from $\mathrm{Hom}_A(A, \Omega^*)$ to $M(\Gamma)$ is onto. Conversely, if $\mathrm{Hom}_A(A, \Omega^*) \cong M(\Gamma) \cong \Omega^*$, then $(A \circ \Omega)^* \cong \Omega^*$, and $Z = \Omega$, as sets (see proof of Theorem 3.4 ((1)$\Rightarrow$(2))). Thus, Ω is essential by Theorem 3.4. ∎

If, in addition to satisfying the hypotheses of Theorem 3.8, Ω separates the points of A, then A is isometrically imbedded in $M(\Gamma)$ via the canonical map π': $A \to M(\Gamma)$ ([14], Thm. 5.1). Hence, the norm-decreasing algebra homomorphism from $M_R(A)$ into $\mathrm{Hom}_A(A, M(\Gamma))$ is an isometry, and so the map $T \mapsto \mu_T$ from $M_R(A)$ into $M(\Gamma)$ is also an isometry. In this manner, Theorem 3.8 is seen to be an extension of ([13], Thm. 2.1, Cor. 2.2) in which A is assumed to possess a right norm approximate identity bounded by one; moreover, $\{u_\lambda\}$ is a left Ω-approximate identity as well if and only if Ω is essential and $\mathrm{Hom}_A(A, M(\Gamma)) \cong M(\Gamma)$.

It should be noted that if it is known that $T^*\Omega \subseteq \Omega$, for all $T \in \mathrm{Hom}_A(A, \Omega^*)$, then the $\mathfrak{F}$-approximate identity assumption can be weakened to assuming only an Ω-approximate identity.

When A is a convolution measure algebra (CMA), both $\mathfrak{W}$ and $\mathscr{A}$ are $*$-subalgebras of A^* containing the identity ([20], Lemma 3.2). In addition, if Ω is a Banach A-subbimodule of $\mathfrak{W}$ which is a C^*-subalgebra of $\mathfrak{W}$ containing the identity of A^*, then Γ is a compact semigroup ([14], Cor. 5.3). As an easy consequence of Theorem 3.4, the existence of an identity in Γ is characterized (cf. [20], Sec. 4, Remark and [8], Cor. 3.2).

PROPOSITION 3.9. *Let A be a convolution measure algebra and Ω be a Banach A-subbimodule of $\mathfrak{W}$ which is a C^*-subalgebra of A^* containing the identity of A^*. Then A possesses a two-sided Ω-approximate identity bounded by one if and only if Γ has a two-sided identity.*

Proof. From Theorem 3.4, A has a two-sided Ω-approximate identity bounded by one if and only if $\Omega^* = M(\Gamma)$ has a two-sided identity of norm one, which is equivalent to Γ having an identity ([18], Prop. 1.6.6, [3], Lemma V.8.6). ∎

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