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On Urbanik's characterization of Gaussian measures on locally compact abelian groups

by

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Abstract. It is shown that the following two conditions are equivalent for a locally compact abelian group X:

(i) any probability measure γ on X such that, for each character $y \in X^*$, $y(\gamma)$ is a Gaussian measure on the Unit Circle Group T is a Gaussian measure on X itself,

(ii) each non-trivial quotient group of the dual group X^* has an infinitely divisible element, except the case where the group of integers is a quotient of X^* by the subgroup of all compact elements in X^*.

Let X be a metric, separable, locally compact abelian group and let $Y = X^*$ be its dual group. (x, y) will denote the value of a character $y \in Y$ at an element $x \in X$, and δ_x the probability measure (distribution) on X which is concentrated at element x. The convolution of two measures μ and ν on X, the characteristic function of μ and the adjoint measure $\bar{\mu}$ are defined as usual:

$$\mu * \nu(A) = \int_X \mu(A-x)\,d\nu(x), \qquad \hat{\mu}(y) = \int_X (x, y)\,d\mu(x), \qquad \bar{\mu}(A) = \mu(-A).$$

If $\mu = \mu_1 * \mu_2$, then μ_1 and μ_2 are said to be *factors of* μ. A distribution of μ is called *infinitely divisible* if for each positive integer n there exist a distribution ν and an element $x \in X$ such that $\mu = \nu^{*n} * \delta_x$. If F is a finite measure on X, then $e(F)$ will denote:

$$e(F) = \exp\{-F(X)\}(\delta_0 + F + F^{*2}/2 + \ldots).$$

R, Z denote as usual the groups of real, and integer numbers, respectively and the Unit Circle Group T is the factor group R/Z.

DEFINITION 1 (Parthasarathy, Rao, Varadhan [3]). A distribution γ on X is called *Gaussian* if it fulfils the following two conditions:

(a) γ is infinitely divisible,

(b) if $\gamma = e(F) * a$ for some infinitely divisible a, then $F = \delta_0$.

It was proved in [3] that a distribution γ is Gaussian if and only if its characteristic function is of the form

$$\hat{\gamma}(y) = (x, y)\exp(-\varphi(y)),$$

where x is a fixed element and ψ fulfils:

$$\psi(y_1+y_2)+\psi(y_1-y_2) = 2[\psi(y_1)+\psi(y_2)] \tag{1}$$

for each y_1, y_2 in Y.

The class of Gaussian distributions defined above coincides with the usual one in the case of groups R^n and T^n (recall that Gaussian distributions on T^n are defined as images of Gaussian distributions on R^n by the canonical homomorphism R^n onto T^n).

Before the appearence of paper [3] Urbanik gave another definition of Gaussian distributions and investigated their properties.

DEFINITION 2 (Urbanik [4]). A distribution γ on X is said to be *Gaussian* if the following two conditions are satisfied:

(c) γ is infinitely divisible,

(d) for each $y \in Y$ the image $y(\gamma)$ is a Gaussian distribution on T.

Using the form of the characteristic function of Gaussian distributions, obtained in [3], it is easy to check that Definitions 1 and 2 are equivalent. In the same paper [4], Urbanik gave an example of a distribution on T^2 which fulfils condition (d) and which is not Gaussian. It is the aim of this paper to give a description of those groups on which such an example is impossible, i.e. on which each distribution fulfilling condition (d) is Gaussian.

For a precise statement of the results we have to recall some definitions.

Let G be an abelian topological group. An element $g \in G$ is called *divisible by an integer* n if there exists an element $x \in G$ such that $g = nx$. We will denote such an element x by g/n.

An element $g \in G$ different from 0 is called *infinitely divisible* if it is divisible by infinitely many integers.

An element $g \in G$ is said to be *compact* if the sequence ng is relatively compact in G.

THEOREM. *The following two conditions are equivalent:*

(i) *any probability measure γ on X such that for each character $y \in Y$ the distribution $y(\gamma)$ is Gaussian on T is a Gaussian distribution on X itself,*

(ii) *each non-trivial quotient group of Y has an infinitely divisible element, except the case where the group of integers is the quotient of Y by the subgroup of all compact elements in Y.*

Proof. (ii) ⇒ (i). Let γ be a distribution on X with the property that, for each $y \in Y$, $y(\gamma)$ is a Gaussian distribution on T. Replacing γ by $\gamma * \delta_x$ with a suitable $x \in X$, we can assume that 0 is in the support of γ. Then, for each character $y \in Y$ of finite order, the support of $y(\gamma)$ consists only of 0. Thus the support of γ is contained in $y^{-1}(0)$ for each character y

of finite order. Hence the support γ is contained in C — the component of 0 in G (cf. [2], Chapter 6).

The dual group of C is isomorphic to the quotient group of Y by the subgroup of all compact elements in Y (cf. [2]). If this quotient group is isomorphic to Z, then C is isomorphic to T and thus for obvious reasons γ is a Gaussian distribution on X.

Thus we can assume that each non-trivial quotient group of Y has infinitely divisible elements. This assumption implies in particular that Z is not a quotient group of Y or, by duality, that T is not isomorphic to a closed subgroup of X. The last property, as was shown in [1], shows that a factor of a Gaussian distribution is Gaussian itself. Thus γ is a Gaussian distribution if $\nu = \gamma * \bar{\gamma}$ is such. Clearly, ν shares with γ the property that the image by each character is a Gaussian distribution on T. Therefore we have

$$\hat{\nu}(y) = \exp(-\psi(y)) \quad \text{for each } y \in Y,$$

where ψ is a non-negative function fulfilling the condition:

$$\psi(ky) = k^2\psi(y) \quad \text{for each } y \in Y \text{ and } k \in Z. \tag{2}$$

Let D be the set of all elements $a \in Y$ for which there exists a sequence a_n of elements in Y such that a_n is divisible by n and $\psi(a-a_n) \to 0$.

LEMMA. *D is a subgroup of Y and the quotient group Y/D has no infinitely divisible elements.*

Proof of the Lemma. Let $a, b \in D$ and let a_n, b_n be sequences of elements as in the definition of the set D. By the inequality for characteristic functions

$$|\hat{\nu}(y_1)-\hat{\nu}(y_2)| \leqslant \sqrt{2}\,|1-\mathrm{Re}(\hat{\nu}(y_1-y_2))|^{1/2} \quad \text{for} \quad y_1, y_2 \in Y \tag{3}$$

we obtain

$$|\hat{\nu}(a-b-a_n+b_n)-1| \leqslant |\hat{\nu}(a-b-a_n+b_n)-\hat{\nu}(b-b_n)|+|\hat{\nu}(b-b_n)-1|$$
$$\leqslant |\hat{\nu}(b-b_n)-1|+\sqrt{2}\,|1-\hat{\nu}(a-a_n)|^{1/2}.$$

Since for each sequence $y_n \in Y$, $\psi(y_n) \to 0$ if and only if $\hat{\nu}(y_n) \to 1$, we conclude that $\psi(a-b-a_n+b_n) \to 0$. Clearly, $a_n - b_n$ is divisible by n. So $a-b \in D$, and this proves that D is an algebraic subgroup of Y.

D is a closed subgroup. This follows from the uniform continuity of ψ and from the fact that if $a \in D$, then for each $\varepsilon > 0$ and a positive integer n there exists an element b which is divisible by n and such that $\psi(a-b) < \varepsilon$.

If Y/D has an infinitely divisible element, then there exists an element $a \notin D$ and an unbounded sequence p_n of integers together with a sequence x_n of elements in Y such that $a - p_n x_n \in D$ for positive $n \in Z$. Without loss of generality we can assume that $p_n \geqslant n^2$ and therefore there exists a sequence q_n of integers such that $nq_n/p_n \to 1$. It follows from the definition of the set D that there exists an element $y_n \in Y$ which is divisible by p_n and such that $\psi(a - p_n x_n - y_n)$ is arbitrarily close to 0. As a result, there exists a sequence $c_n \in Y$ such that c_n is divisible by p_n and $\psi(a - c_n) \to 0$. Let us put $a_n = nq_n c_n/p_n$ for $n = 1, 2, \ldots$ We have

$$|1-\hat{\nu}(a-a_n)| \leqslant |1-\hat{\nu}(a-c_n)| + |\hat{\nu}(a-a_n) - \hat{\nu}(a-c_n)|$$
$$\leqslant |1-\hat{\nu}(a-c_n)| + \sqrt{2}\,|1-\hat{\nu}(a_n-c_n)|^{1/2}.$$

Since $\psi(a-c_n)\to 0$, we obtain $\hat{\nu}(a-c_n)\to 1$ and hence by inequality (3) we have $\hat{\nu}(c_n)\to\hat{\nu}(a)$ or, what is the same, $\psi(c_n)\to\psi(a)$. On the other hand by property (2) of ψ we have

$$\psi(a_n-c_n) = \psi\big((nq_n-p_n)c_n/p_n\big) = (nq_n/p_n-1)^2\psi(c_n)\to 0.$$

Thus $\hat{\nu}(a_n-c_n)\to 1$ and therefore, by the preceding inequalities, we get $\hat{\nu}(a-a_n)\to 1$. This shows that $\psi(a-a_n)\to 0$. Since a_n is divisible by n, this leads to a contradiction with the fact that $a \notin D$.

Thus the Lemma is proved, and let us return to the proof of the implication (ii) ⇒ (i). By the Lemma and by the already made assumption on Y we deduce that $Y = D$. Let $a, b \in Y$. Thus we can find sequences a_n, b_n such that a_n and b_n are divisible by n and $\psi(a-a_n)\to 0$, $\psi(b-b_n)\to 0$. As we have seen in the proof of the Lemma, these imply that $\psi(a_n)\to\psi(a)$, $\psi(b_n)\to\psi(b)$, $\psi(a_n+b_n)\to\psi(a+b)$, $\psi(a_n-b_n)\to\psi(a-b)$. Since $\exp\big(-\psi(y)\big)$ is a positively defined function on Y, we have

$$\sum_{i,j=1}^{k}\exp\big(-\psi(y_i-y_j)\big)\xi_i\bar{\xi}_j \geqslant 0$$

for $y_1, \ldots, y_k \in Y$ and $\xi_1, \ldots, \xi_k$ — complex numbers. Substituting in the above $k = 4$, $y_1 = -y_2 = a_n/n$, $y_3 = -y_4 = b_n/n$, $\xi_1 = \xi_2 = -\xi_3 = -\xi_4 = n$, we conclude that

$$\big[2\exp\big(-(4/n^2)\psi(a_n)\big) + 2\exp\big(-(4/n^2)\psi(b_n)\big) - 4\exp\big(-(1/n^2)\psi(a_n-b_n)\big) -$$
$$- 4\exp\big(-(1/n^2)(a_n+b_n)\big) + 4\big]n^2 \geqslant 0.$$

Taking the limit when $n\to\infty$, we get from the above inequality

$$-4\psi(a+b) - 4\psi(a-b) + 8\psi(a) + 8\psi(b) \geqslant 0,$$

or equivalently

$$2[\psi(a)+\psi(b)] \geqslant \psi(a+b)+\psi(a-b).$$

Replacing in this inequality a by $a+b$ and b by $a-b$ and taking into account property (2) of ψ, we get

$$\psi(a+b)+\psi(a-b) \leqslant 2[\psi(a)+\psi(b)].$$

Thus we have the equality

$$\psi(a+b)+\psi(a-b) = 2[\psi(a)+\psi(b)].$$

Hence ψ fulfils condition (1) and this proves that ν is a Gaussian distribution on X.

(i) ⇒ (ii). Assume that condition (ii) is not satisfied. Then there exists a non-trivial quotient group $A = Y/U$ without infinitely divisible elements. Let us consider the case where A is not isomorphic to Z. For each element $a \in A$, $a \neq 0$, there exist, unique up to a change of sign, an integer n and element $\bar{a} \in A$ which is not divisible by any integer $m > 1$ and is such that $a = n\bar{a}$. Let us define $\psi(a) = \beta(\bar{a})n^2$ for $a \neq 0$ and $\psi(0) = 0$. The numbers $\beta(\bar{a})$ are positive and such that

$$1 \geqslant \sum_{a\neq 0,\, a\in A}\exp\big(-\beta(\bar{a})n^2\big). \tag{4}$$

Such a choise of the numbers $\beta(\bar{a})$ is possible because A is a discrete and countable group, as follows from the general facts about locally compact, metric, separable, abelian groups. Now let γ be the measure on A^* given by the following density f with respect to the Haar measure on A^*

$$f(x) = \sum_{a\in A}\exp\big(-\psi(a)\big)(x, a).$$

It follows, by inequality (4), that γ is a probability measure on A^*. The characteristic function of γ is given by

$$\hat{\gamma}(a) = \exp\big(-\psi(a)\big).$$

It is easy to see that ψ fulfils condition (2). Thus, for each $a \in A$ the distribution $a(\gamma)$ is Gaussian on T. Moreover, since A is not isomorphic to Z, it is possible to find the numbers $\beta(\bar{a})$ such that, additionally, the function ψ does not satisfy condition (1). So, the distribution γ is not Gaussian. Since the dual A^* may be identified with a subgroup of X, we see that condition (i) is not fulfilled for X.

Thus it remains to consider the case where A is isomorphic to Z. In this case U is complemented in Y, i.e. we can write $Y = U \oplus Z$. Since we have assumed that condition (ii) is not satisfied, U contains non-compact elements in Y. Hence, by duality, we infer that X contains a subgroup isomorphic to $V \oplus T$, where V is a connected, non-trivial group. Let μ, ν be two different Gaussian measures on V such that $2\nu - \mu$ is a positive measure on V and let ϱ be a Gaussian measure on T such that $2\varrho - \lambda$ is a positive measure, where λ is the Haar measure on T.

Let us define a measure γ on $V \oplus T$ by

$$\gamma = \nu\otimes\varrho + \mu\otimes\lambda - \nu\otimes\lambda.$$

Since $\gamma = \nu\otimes(\varrho-\lambda/2)+(\mu-\nu/2)\otimes\lambda$, we see that γ is a probability measure. It is a matter of simple calculations on characteristic functions to check that the characteristic function of γ is of the form $\hat{\gamma}(a) = \exp(-\psi(a))\cdot(b, a)$ and that ψ fulfils condition (2) and does not fulfil condition (1). Thus, for each character $a\in(V\oplus T)^*$ the image measure $a(\gamma)$ is Gaussian on T and γ is not a Gaussian distribution on $V\oplus T$. Since the last group is isomorphic to a subgroup of X, it follows that X does not fulfil condition (i). This ends the proof of the implication (i) ⇒ (ii).

Remark. A locally compact abelian group X fulfils condition (ii) if its dual group has a maximal, independent system of infinitely divisible elements, or the component of 0 in X is isomorphic to T.

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Fractional integration on Hardy spaces

by

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Abstract. The classical fractional integration theorems for Riesz potentials on L^p spaces are extended to the real variable Hardy classes, $0<p\leqslant 1$. It is further shown that the Riesz potentials can be replaced by a large class of convolution operators. Finally, one obtains results for certain operators which are not of convolution type by using a theory of local Hardy spaces.

§ 0. Introduction. Let $K_a\colon \boldsymbol{R}^n\setminus\{0\}\to\boldsymbol{R}$ be given by

$$K_a(x) = \gamma_{a,n}|x|^{a-n},\quad 0<a<n,$$

where $\gamma_{a,n}\equiv\big(\pi^{n/2}2^a\Gamma(a/2)\big)/\Gamma\big((n-a)/2\big)$. It is an old theorem of Hardy and Littlewood (for $n=1$) and of Sobolov (for $n>1$) that the operator

$$I_a(f) = f*K_a$$

is a bounded linear map from $L^p(\boldsymbol{R}^n)$ to $L^q(\boldsymbol{R}^n)$, $1<p<n/a$, $1/q = 1/p-a/n$ (see [10]).

For $0<a+\beta<n$, one can compute (see [11]) that $I_{a+\beta} = I_a\circ I_\beta = I_\beta\circ I_a$. This motivates the definition

$$I_a = I_{a/k}\circ I_{a/k}\circ\ldots\circ I_{a/k}\quad (k\text{ times})$$

when $a<kn$, and one checks that the definition is unambiguous.

Now let $H^p(\boldsymbol{R}^n)$ denote the generalized Hardy classes defined and developed in [11], [10], [3]. These will be considered in detail below. In [11] the following result is proved:

(0.1) $$I_a\colon H^p(\boldsymbol{R}^n)\to H^q(\boldsymbol{R}^n)$$

boundedly, provided $(n-1)/n<p\leqslant n/(n+a)$, $a>0$, and $1/q = 1/p-a/n$. Also

(0.2) $$I_a\colon H^p(\boldsymbol{R}^n)\to L^q(\boldsymbol{R}^n)$$

boundedly, provided $n/(n+a)<p<n/a$, $a>0$, and $1/q = 1/p-a/n$. It is known that in case $p = n/a$, the appropriate target space is BMO

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