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On domination and extensions of Banach algebras

by

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Abstract. The paper studies connections between the domination property in a Banach algebra and existence of an extension with given properties. Two examples of Banach algebras are exhibited giving negative answers to problems of Żelazko.

Introduction. The concept of domination in commutative Banach algebras was introduced and studied by W. Żelazko in [4], [5] and [6]. We say that u is *dominated by* $v_1, v_2, \ldots, v_n$ (where $u, v_1, \ldots, v_n$ are elements of an unital commutative Banach algebra A) if $|ux| \leqslant K \cdot \sum_{i=1}^{n} |v_i x|$ for some constant $K \geqslant 0$ and for every $x \in A$.

A unital Banach algebra B is said to be an *isometric extension of* A if there exists a unit preserving isometric isomorphism from A into B. In this case we consider A as a subalgebra of B and write $A \subset B$.

If $A \subset B$, $u, v_1, \ldots, v_n \in A$ and $u = \sum_{i=1}^{n} v_i b_i$ for some elements $b_i \in B$, then $|ux| \leqslant (\max_{1\leqslant i\leqslant n} |b_i|) \cdot \sum_{i=1}^{n} |v_i x|$, so that u is dominated by $v_1, \ldots, v_n$. The converse statement is true in some particular cases ([1], [5]) but not in general ([2]). In the present paper we extend the result of [2].

In Section I we exhibit an example of a commutative finite-dimensional Banach algebra A with $e_1, e_2, e_3 \in A$ such that e_1 is dominated by e_2, e_3 and $e_1 \notin e_2 B + e_3 B$ in any commutative algebra B without topology containing A as a subalgebra. This gives a negative answer to Problem 4 of [6] and also simplifies the example of [2].

In Section II we give an example of a unital commutative Banach algebra A with $u, v, w \in A$ such that u is dominated by v, w and dist $(u, vB + wB) = |u|_A$ in any isometric extension B of A. This gives a negative answer to Problem 6 of [6]. (Problem 6 of [6] was raised in this weaker form: Let A be a commutative Banach algebra, $x \in A$ and I an ideal in A. Let x be approximately dominated by I, i.e. for each $\varepsilon > 0$ there exist elements $x_1, \ldots, x_n \in I$ such that $xz \leqslant \sum_{j=1}^{n} |x_j z| + \varepsilon |z|$ for all $z \in A$.

Does this imply that $x \in \bar{I}_B$ in some isometric extension B of A, where I_B is the smallest ideal in B containing I?)

All algebras considered in this paper are commutative, complex and with unit.

I. In this section we exhibit an example of a unital commutative Banach algebra A with $e_1, e_2, e_3 \in A$ such that e_1 is dominated by e_2, e_3 and if C is any commutative algebra (without any topology) containing A as a subalgebra, there are no $b, c \in C$ satisfying $e_1 = be_2 + ce_3$.

CONSTRUCTION. Elements of A will be of the form $x = \sum_{j=0}^{10} \lambda_i e_i$, where $\lambda_0, \ldots, \lambda_{10}$ are complex numbers, $e_0, \ldots, e_{10}$ form a basis of A and e_0 is the unit. Multiplication in A is defined by

$$\Big(\sum_{j=0}^{10} \lambda_i e_i\Big)\Big(\sum_{j=0}^{10} \mu_j e_j\Big) = \sum_{i,j=0}^{10} \lambda_i \mu_j e_i e_j,$$

where the multiplication of the basis elements e_i is given by the table

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
e_1	e_1				e_7	e_8	e_9				
e_2	e_2		0		e_8	0	e_{10}		0		
e_3	e_3				e_9	$-e_{10}$	0				
e_4	e_4	e_7	e_8	e_9							
e_5	e_5	e_8	0	$-e_{10}$		0			0		
e_6	e_6	e_9	e_{10}	0							
e_7	e_7										
e_8	e_8		0			0			0		
e_9	e_9										
e_{10}	e_{10}										

Clearly A with the norm $\big|\sum_{i=0}^{10} \lambda_i e_i\big| = \sum_{i=0}^{10} |\lambda_i|$ is a Banach algebra.

Let $x = \sum_{i=0}^{10} \lambda_i e_i$. Then

$$|e_1 x| = |\lambda_0 e_1 + \lambda_4 e_7 + \lambda_5 e_8 + \lambda_6 e_9| = |\lambda_0| + |\lambda_4| + |\lambda_5| + |\lambda_6|,$$

$$|e_2 x| = |\lambda_0 e_2 + \lambda_4 e_8 + \lambda_6 e_{10}| = |\lambda_0| + |\lambda_4| + |\lambda_6|,$$

$$|e_3 x| = |\lambda_0 e_3 + \lambda_4 e_9 - \lambda_5 e_{10}| = |\lambda_0| + |\lambda_4| + |\lambda_5|,$$

hence $|e_1 x| \leqslant |e_2 x| + |e_3 x|$ for every $x \in A$ and e_1 is dominated by e_2, e_3.

Suppose now that there exists a commutative algebra B (without topology) containing A as a subalgebra and elements $b, c \in B$ such that $e_1 = e_2 b + e_3 c$. Then

$$\begin{aligned} 0 &= (e_1 - e_2 b - e_3 c)(e_4 + e_5 b + e_6 c) \\ &= e_1 e_4 + b(e_1 e_5 - e_2 e_4) + c(e_1 e_6 - e_3 e_4) - b^2 e_2 e_5 - c^2 e_3 e_6 - bc(e_2 e_6 + e_3 e_5) \\ &= e_7, \end{aligned}$$

a contradiction.

II. LEMMA. *There exists a unital commutative Banach algebra A with generators $u, v, w, a_{s,k}$ ($s = 0, 1, 2,\ k = 1, 2, \ldots$) such that*

(1) $|u| = |v| = |w| = |a_{0,k}| = 1$, $|a_{1,k}| = |a_{2,k}| = k$ *and* $|a_{0,k} u^k| = 1$ ($k = 1, 2, \ldots$);

(2) $a_{s,k} a_{t,m} = 0$ ($s, t = 0, 1, 2,\ k, m = 1, 2, \ldots$);

(3) $a_{s,k} u^r v^i w^j = 0$ ($s = 0, 1, 2,\ r+i+j > k$);

(4) *u is dominated by v, w;*

(5) $r a_{0,k} u^{r-1} v^i w^j - i a_{1,k} u^r v^{i-1} w^j - j a_{2,k} u^r v^i w^{j-1} = 0$ *for* $r, i, j \geqslant 1$, $r + i + j = k+1$,

$$\begin{aligned} r a_{0,k} u^{r-1} w^j - j a_{2,k} u^r w^{j-1} &= 0 \quad (r, j \geqslant 1,\ r+j = k+1), \\ r a_{0,k} u^{r-1} v^i - i a_{1,k} u^r v^{i-1} &= 0 \quad (r, i \geqslant 1,\ r+i = k+1), \\ -i a_{1,k} v^{i-1} w^j - j a_{2,k} v^i w^{j-1} &= 0 \quad (i, j \geqslant 1,\ i+j = k+1), \\ a_{2,k} w^k = 0, \quad a_{1,k} v^k &= 0. \end{aligned}$$

Remark. Notice that the formulas of (5) are of the same kind. We may state them as follows:

$$r a_{0,k} u^{r-1} v^i w^j - i a_{1,k} u^r v^{i-1} w^j - j a_{2,k} u^r v^i w^{j-1} = 0$$

for $r, i, j \geqslant 0$, $r+i+j = k+1$, $i+j \geqslant 1$ with the convention that whenever occurs an expression with a negative exponent, the corresponding term should be taken to be 0 (by the way the term is then multiplied by 0). We shall use this convention in the sequel.

Proof. Let S be the free commutative semigroup with unit 1 and zero element 0 and with generators $u, v, w, a_{s,k}$ ($s = 0, 1, 2,\ k = 1, 2, \ldots$) satisfying conditions (2), (3). Let B be the l^1 algebra over S with the norm defined by (1), i.e. B consists of formal linear combinations

$$x = \sum_{r,i,j=0}^{\infty} \lambda_{r,i,j} u^r v^i w^j + \sum_{s=0}^{2} \sum_{k=1}^{\infty} \sum_{0 \leqslant r+i+j \leqslant k} \mu_{k,r,i,j}^{(s)} a_{s,k} u^r v^i w^j,$$

where $\lambda_{r,i,j}$ and $\mu_{k,r,i,j}^{(s)}$ are complex numbers and

$$|x| = \sum_{r,i,j=0}^{\infty} |\lambda_{r,i,j}| + \sum_{k=1}^{\infty} \sum_{0 \leqslant r+i+j \leqslant k} \big(|\mu_{k,r,i,j}^{(0)}| + k|\mu_{k,r,i,j}^{(1)}| + k|\mu_{k,r,i,j}^{(2)}|\big) < \infty.$$

Clearly B with this norm is a commutative Banach algebra with unit. Let I be the closed ideal generated by elements which are on the left sides of formulas in condition (5). Denote $A = B/I$ and for $x \in B$ let $\bar{x} = x+I \in A$. We prove that A satisfies (1)–(5) (with $u, v, w, a_{s,k}$ replaced by $\bar{u}, \bar{v}, \bar{w}, \bar{a}_{s,k}$).

Conditions (2), (3) and (5) are evident. Let x be an arbitrary element of B. Then

$$(1) \qquad x = x_0 + \sum_{k=1}^{\infty} \sum_{1 \leqslant r+i+j \leqslant k+1} x_{k,r,i,j},$$

where

$$x_0 = \sum_{r,i,j=0}^{\infty} \lambda_{r,i,j} u^r v^i w^j,$$

$$x_{k,r,i,j} = \mu^{(0)}_{k,r-1,i,j} a_{0,k} u^{r-1} v^i w^j + \mu^{(1)}_{k,r,i-1,j} a_{1,k} u^r v^{i-1} w^j + \mu^{(2)}_{k,r,i,j-1} a_{2,k} u^r v^i w^{j-1}$$

(the λ's and μ's are complex numbers; if some exponent is negative, the corresponding term is taken to be 0). It is clear from the definition of the ideal I that

$$|\bar{x}|_A = |\bar{x}_0|_A + \sum_{k=1}^{\infty} \sum_{1 \leqslant r+i+j \leqslant k+1} |\bar{x}_{k,r,i,j}|_A,$$

$$|\bar{x}_0|_A = \sum_{r,i,j=0}^{\infty} |\lambda_{r,i,j}|;$$

$$|\bar{x}_{k,r,i,j}|_A = |\mu^{(0)}_{k,r-1,i,j}| + k|\mu^{(1)}_{k,r,i-1,j}| + k|\mu^{(2)}_{k,r,i,j-1}| \qquad (r+i+j \leqslant k)$$

and

$$|\bar{x}_{k,k+1,0,0}|_A = |\mu^{(0)}_{k,k,0,0}|.$$

This implies condition (1).

We prove now that $\bar{u}$ is dominated by $\bar{v}, \bar{w}$ with the constant $K = 3$. Let us notice that we have for x of form (1)

$$|\bar{x}\bar{u}|_A = |\bar{x}_0 \bar{u}|_A + \sum_{k,r,i,j} |\bar{x}_{k,r,i,j} \bar{u}|_A,$$

$$|\bar{x}\bar{v}|_A = |\bar{x}_0 \bar{v}|_A + \sum_{k,r,i,j} |\bar{x}_{k,r,i,j} \bar{v}|_A$$

and

$$|\bar{x}\bar{w}|_A = |\bar{x}_0 \bar{w}|_A + \sum_{k,r,i,j} |\bar{x}_{k,r,i,j} \bar{w}|_A.$$

Further,

$$|\bar{x}_0 \bar{u}|_A = |\bar{x}_0 \bar{v}|_A = |\bar{x}_0 \bar{w}|_A = |\bar{x}_0|_A = \sum_{r,i,j=0}^{\infty} |\lambda_{r,i,j}|.$$

Thus it is sufficient to prove $|\bar{y}\bar{u}|_A \leqslant 3(|\bar{y}\bar{v}|_A + |\bar{y}\bar{w}|_A)$ for $y = x_{k,r,i,j}$, $1 \leqslant r+i+j \leqslant k+1$.

Consider the following cases:

(1) If $r+i+j = k+1$, then $yu = yv = yw = 0$.

(2) If $r+i+j \leqslant k-1$, then $|\bar{y}\bar{u}|_A = |\bar{y}\bar{v}|_A = |\bar{y}\bar{w}|_A = |\bar{y}|_A = |\mu^{(0)}_{k,r-1,i,j}| + k|\mu^{(1)}_{k,r,i-1,j}| + k|\mu^{(2)}_{k,r,i,j-1}|$.

(3) Let $r+i+j = k$, $i+j \geqslant 1$. Then $y = \alpha a_{0,k} u^{r-1} v^i w^j + \beta a_{1,k} u^r v^{i-1} w^j + \gamma a_{2,k} u^r v^i w^{j-1}$ (α, β, γ are complex numbers; if $r = 0$ ($i = 0$, $j = 0$) we put $\alpha = 0$ ($\beta = 0$, $\gamma = 0$)). We have

$$|\bar{y}\bar{u}|_A = \inf_{\nu \in C} \big(|\alpha - (r+1)\nu| + k|\beta + i\nu| + k|\gamma + j\nu|\big),$$

$$|\bar{y}\bar{v}|_A = \inf_{\delta \in C} P_1(\delta),$$

where

$$P_1(\delta) = |\alpha - r\delta| + k|\beta + (i+1)\delta| + k|\gamma + j\delta|,$$

$$|\bar{y}\bar{w}|_A = \inf_{\varepsilon \in C} P_2(\varepsilon),$$

where

$$P_2(\varepsilon) = |\alpha - r\varepsilon| + k|\beta + i\varepsilon| + k|\gamma + (j+1)\varepsilon|.$$

Choose $\delta, \varepsilon \in C$. Putting $\nu = \delta$ we get $|\bar{y}\bar{u}|_A \leqslant P_1(\delta) + (k+1)|\delta|$. Analogously $|\bar{y}\bar{u}|_A \leqslant P_2(\varepsilon) + (k+1)|\varepsilon|$. This gives

$$|\bar{y}\bar{u}|_A \leqslant \tfrac{1}{2}\big(P_1(\delta) + P_2(\varepsilon)\big) + \tfrac{1}{2}(k+1)(|\delta| + |\varepsilon|)$$

for every $\delta, \varepsilon \in C$.

Assume that $|\delta| \geqslant |\varepsilon|$. Then

$$|\beta + (i+1)\delta| + |\beta + i\varepsilon| \geqslant (i+1)|\delta| - i|\varepsilon| \geqslant |\delta| \geqslant \tfrac{1}{2}(|\delta| + |\varepsilon|),$$

hence $P_1(\delta) + P_2(\varepsilon) \geqslant \frac{1}{2}k(|\delta| + |\varepsilon|)$.

The same inequality can be proved also for $|\delta| \leqslant |\varepsilon|$. Thus

$$|\bar{y}\bar{u}|_A \leqslant \tfrac{1}{2}\big(P_1(\delta) + P_2(\varepsilon)\big) + \tfrac{1}{2}(k+1)(2/k)\big(P_1(\delta) + P_2(\varepsilon)\big)$$
$$\leqslant 3\big(P_1(\delta) + P_2(\varepsilon)\big)$$

for each $\delta, \varepsilon \in C$, hence $|\bar{y}\bar{u}|_A \leqslant 3(|\bar{y}\bar{v}|_A + |\bar{y}\bar{w}|_A)$.

(4) If $i = j = 0$, $y = \alpha a_{0,k} u^{k-1}$, then $|\bar{y}\bar{u}|_A = |\alpha|$ and it can be proved easily that

$$|\bar{y}\bar{v}|_A = |\alpha|\, |\bar{a}_{0,k} \bar{u}^{k-1} \bar{v}|_A = |\alpha| (1/k) |\bar{a}_{1,k} \bar{u}^k|_A = |\alpha| = |\bar{y}\bar{w}|_A.$$

This completes the proof.

THEOREM. *Let A be a unital commutative Banach algebra satisfying conditions* (1)–(5) *of the previous Lemma. Let C be any isometric extension of A. Then there are no $b, c \in C$ such that $|u - bv - cw|_C < 1 = |u|_A$.*

Proof. Suppose on the contrary that C is an isometric extension of A, $b, c \in C$ and $|u-bv-cw|_C = d < 1$. Choose an integer k such that $k \geqslant \max(|b|_C, |c|_C)$ and $d^k(1+2k^2) < 1$. Then we have (we use the convention about negative exponents)

$$\begin{aligned}
&(u-bv-cw)^k(a_{0,k}+a_{1,k}b+a_{2,k}c)\\
&= \sum_{0\leqslant i+j\leqslant k+1} \Big[b^ic^j(-1)^{i+j}\binom{k}{i}\binom{k-i}{j} a_{0,k}u^{k-i-j}v^iw^j +\\
&\quad +(-1)^{i+j-1}\binom{k}{i-1}\binom{k-i+1}{j} a_{1,k}u^{k-i-j+1}v^{i-1}w^j +\\
&\quad +(-1)^{i+j-1}\binom{k}{i}\binom{k-i}{j-1} a_{2,k}u^{k-i-j+1}v^iw^{j-1}\Big]\\
&= \sum_{0\leqslant i+j\leqslant k+1} b^ic^j \frac{k!(-1)^{i+j}}{i!j!(k-i-j+1)!} [(k-i-j+1) a_{0,k}u^{k-i-j}v^iw^j -\\
&\quad -ia_{1,k}u^{k-i-j+1}v^{i-1}w^j - ja_{2,k}u^{k-i-j+1}v^iw^{j-1}]\\
&= a_{0,k}u^k.
\end{aligned}$$

Thus

$$\begin{aligned}
|a_{0,k}u^k|_C &\leqslant |u-bv-cw|_C^k \cdot |a_{0,k}+a_{1,k}b+a_{2,k}c|_C\\
&\leqslant d^k(1+2k^2) < 1.
\end{aligned}$$

At the same time $|a_{0,k}u^k|_C = |a_{0,k}u^k|_A = 1$, a contradiction.

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