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On singular integrals and Orlicz spaces*

by

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Abstract. This paper deals with the boundedness in $L^2(R^n)$ of singular integral operators with variable kernel when such kernel belongs to an Orlicz space L_Φ. We give necessary and sufficient conditions on Φ related with the continuity. On the other hand, we show a counterexample concerning with the result given in [2].

1. Introduction. In this paper we deal with singular integral operators of the form

$$Kf(x) = \text{p.v.} \int_{R^n} k(x, y) f(x-y)\, dy, \tag{1.1}$$

where f is a measurable function and $k(x, y)$ is defined for $x, y \in R^n$, $y \neq 0$ and satisfies the conditions:

$$\text{For } t > 0,\ k(x, ty) = t^{-n} k(x, y); \tag{1.2}$$

$$\int_{\Sigma_{n-1}} k(x, y')\, d\sigma(y') = 0 \quad \text{for each } x \in R^n. \tag{1.3}$$

Calderón and Zygmund studied the continuity in $L^2(R^n)$ of such operators and obtained that if the kernel $k(x, y)$ belongs to $L^q(\Sigma_{n-1})$ in the variable y and it has bounded norm as function of x, then K is bounded in $L^2(R^n)$ if and only if $q > 2(n-1)/n$ (see [1] and [2]).

We present here an improvement to this theorem in the case which the kernel belongs to an Orlicz space $L_\Phi(\Sigma_{n-1})$. This will be done developing the kernel in spherical harmonics as in [2] and then using an interpolation theorem given by M. Jodeit and A. Torchinsky [3].

On the other hand, we give a counterexample related with L^p-case, that is in [2] it has been shown that if $1 < p \leqslant 2$, then K is bounded

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in L^p if and only if $q > (n-1)p/n(p-1)$. We show that if $p > 1$ and $q < (n-1)p/n(p-1)$, then $f \in L^p$ doesn't imply $Kf \in L^p$ even locally.

2. Preliminaries and statement of results. In this section are given some definitions and properties concerning to the Orlicz spaces which we are going to use (see [3] and [4]).

A *generalized Young function* (GYF) Φ is a function defined in $[0, +\infty)$ such that:

(i) $\Phi(t) \geqslant 0$, $\Phi(0) = 0$;

(ii) Φ is left continuous;

(iii) $\Phi(x)/x$ is non-decreasing.

A convex GYF will be a *Young function*. Given a GYF Φ, its *regularization* Φ_0 is defined as

$$\Phi_0(x) = \int_0^x \Phi(t)/t\,dt, \tag{2.1}$$

Φ_0 is a Young function and it satisfies

$$\Phi_0(x) \leqslant \Phi(x) \leqslant \Phi_0(2x). \tag{2.2}$$

Let (X, μ) be a measure space and let Φ be a GYF. The *Orlicz space* $L_\Phi(X, \mu)$ consists of all the μ-measurable functions f (modulo the equivalence relation a.e.) such that

$$\int_X \Phi\big(\varepsilon|f(x)|\big)\,d\mu(x) < \infty$$

for some $\varepsilon > 0$ (depending on f). The norm

$$\|f\|_\Phi = \inf\Big\{\lambda > 0 : \int_X \Phi_0\big(|f(x)|/\lambda\big)\,d\mu(x) \leqslant 1\Big\}$$

turns L_Φ into a Banach space.

The *Young's complement* of a GYF Φ, denoted $\bar{\Phi}$, is given by $\bar{\Phi}(x) = \sup\limits_{y \geqslant 0}\big(xy - \Phi(y)\big)$. $\bar{\Phi}$ is a Young function and holds the inequality:

$$\int_X |f(x)g(x)|\,d\mu(x) \leqslant 2\|f\|_\Phi \|g\|_{\bar{\Phi}}. \tag{2.3}$$

The *inverse* of a GYF Φ is defined by $\Phi^{-1}(y) = \inf\{x\colon \Phi(x) > y\}$ $(\inf\emptyset = +\infty)$ and it satisfies

$$\Phi\big(\Phi^{-1}(x)\big) \leqslant x \leqslant \Phi^{-1}\big(\Phi(x)\big), \quad x > 0. \tag{2.4}$$

Also holds the following inequality (Φ Young function):

$$x \leqslant \Phi^{-1}(x)(\bar{\Phi})^{-1}(x) \leqslant 2x. \tag{2.5}$$

We shall now state our results:

THEOREM 1. *Let $k(x,y)$ be a kernel which satisfies* (1.2) *and* (1.3) *and Φ a Young function such that $\Phi(x)/x^2$ is non-increasing and $\Phi(2x) \leqslant N\Phi(x)$ for some $N \geqslant 1$ and for each $x \geqslant 0$. Then if*

$$\sup_{x \in R^n} \|k(x,\cdot)\|_{L_\Phi(\Sigma_{n-1})} \leqslant M, \tag{2.6}$$

there exists a constant C depending on n and Φ only such that for each $f \in C_0^\infty(R^n)$ we have

$$\|Kf\|_2 \leqslant CM\|f\|_2,$$

provided that

$$\int_1^\infty \Phi(t^{n/2(n-1)})^{n-1}\,dt < \infty. \tag{2.7}$$

THEOREM 2. *Let Φ be a Young function such that for some $M > 0$*

$$\int_M^\infty \Phi(t^{1/2})^{-n/(n-1)}\,dt = +\infty. \tag{2.8}$$

Then there exist $k(x,y)$ satisfying (1.2) *and* (1.3), *and $f \in C_0^\infty(R^n)$ such that $\sup\limits_{x \in R^n} \|k(x,\cdot)\|_{L_\Phi(\Sigma_{n-1})} < \infty$ but $Kf \notin L^2(R^n)$.*

THEOREM 3. *If $n \geqslant 2$, $p > 1$ and $1 \leqslant q < (n-1)p/n(p-1)$, there exist $k(x,y)$ satisfying* (1.2), (1.3) *and $\sup\limits_{x \in R^n} \|k(x,\cdot)\|_{L_q(\Sigma_{n-1})} < \infty$ and $f \in L^p(R^n)$, such that $Kf \notin L^p(R^n)$ locally.*

3. Proof of Theorem 1. This will be done in three steps.

3.1. For each $m \geqslant 1$, by Y_{mj}, $1 \leqslant j \leqslant d_m$, we denote a complete orthonormal set of spherical harmonics of degree m in $L^2(\Sigma_{n-1})$.

Let us suppose that $k(x,y)$ has the form:

$$k(x,y) = \sum_{m \geqslant 1} \sum_{j=1}^{d_m} a_{mj}(x) Y_{mj}(y')|y|^{-n}, \quad y' = y/|y|, \tag{3.1.1}$$

where the sum is finite. As in [2], it may be seen that the integral in (1.1) exists for $f \in C_0^\infty(R^n)$. We write

$$\tilde{f}_{mj}(x) = \text{p.v.} \int_{R^n} Y_{mj}(y')|y|^{-n} f(x-y)\,dy$$

and set $a_m(x) = \big(\sum\limits_{j=1}^{d_m} a_{mj}^2(x)\big)^{1/2}$, $b_{mj}(x) = a_{mj}(x)/a_m(x)$ so that $\sum\limits_{j=1}^{d_m} b_{mj}^2(x) = 1$. Now, taking a sequence γ_m of positive numbers, we have

$$\begin{aligned}[Kf(x)]^2 &= \Big[\sum_m a_m(x) \sum_{j=1}^{d_m} b_{mj}(x)\tilde{f}_{mj}(x)\Big]^2 \\ &\leqslant \Big[\sum_m a_m^2(x)\gamma_m\Big]\Big[\sum_m \gamma_m^{-1} \sum_{j=1}^{d_m} \tilde{f}_{mj}^2(x)\Big].\end{aligned} \tag{3.1.2}$$

If we can select this sequence such that

(3.1.3) $$\sum_m a_m^2(x)\gamma_m \leqslant CM^2,$$

where C is a constant depending on n and Φ, then integrating (3.1.2) we obtain

$$\|Kf\|_2^2 \leqslant CM^2 \sum_m \gamma_m^{-1} \sum_{j=1}^{d_m} \|\tilde{f}_{mj}\|_2^2.$$

From Plancherel's theorem and the following facts:

(i) $\hat{\tilde{f}}_{mj}(x) = C_m Y_{mj}(x')\hat{f}(x)$, where $C_m \leqslant C_n m^{-n/2}$; ($\hat{}$ denotes the Fourier transform),

(ii) $\sum_j Y_{mj}^2 = h_m \omega_{n-1}^{-1}$, where $h_m \leqslant C_n m^{n-2}$ (see [5]),

we get

$$\|Kf\|_2 \leqslant CM\Big(\sum_m \gamma_m^{-1} m^{-2}\Big)^{1/2} \|f\|_2.$$

In consequence we are going to show that there exists a sequence γ_m which satisfies (3.1.3) and

(3.1.4) $$\sum_m \gamma_m^{-1} m^{-2} < \infty.$$

3.2. Let us consider for each $m \geqslant 1$ a normalized spherical harmonic Y_m. If $g \in L^1(\Sigma_{n-1})$, we put

$$g(x) \sim \sum_{m\geqslant 1} a_m Y_m(x), \quad \text{with} \quad a_m = \int_{\Sigma_{n-1}} g(x) Y_m(x)\, d\sigma(x).$$

It is well known that the following inequalities are valid (see [5]):

(i) $\big(\sum_{m\geqslant 1} a_m^2\big)^{1/2} \leqslant \|g\|_{L^2(\Sigma_{n-1})}$;

(ii) $|a_m| \leqslant Cm^{(n-2)/2}\|g\|_{L^1(\Sigma_{n-1})}$, $m \geqslant 1$, C a constant depending on n.

We need to interpolate the two last inequalities, for which we use the following result of M. Jodeit and A. Torchinsky (see [3], p. 255):

(3.2.1) THEOREM. *Let (X, μ) and (Y, ν) be two measure spaces. Let T be a linear operator defined for all μ-measurable functions on X, which satisfies*

$$\|Tf\|_{L^2(Y,\nu)} \leqslant M_1\|f\|_{L^2(X,\mu)} \quad \text{and} \quad \|Tf\|_{L^\infty(Y,\nu)} \leqslant M_2\|f\|_{L^1(X,\mu)}.$$

Then if Φ is a GYF such that $\Phi(x)/x^2$ is non-increasing, we have $\|Tf\|_{L_{\bar\Phi^*}(Y,\nu)} \leqslant C\|f\|_{L_\Phi(X,\mu)}$, where $\bar\Phi^*$ is the GYF defined by

$$\bar\Phi^*(x) = \begin{cases} 0 & \text{for } x = 0, \\ \sup\limits_{y<x} \bar\Phi(y^{-1})^{-1} & \text{for } x \neq 0 \end{cases}$$

and C is a constant which depends on M_1, M_2 and Φ only.

Now, we put

$$k(x, .) \sim \sum_{m,j} a_{mj}(x) Y_{mj} = \sum_m a_m(x) Y_m$$

with $Y_m = \sum_{j=1}^{d_m} b_{mj}(x) Y_{mj}$, where $a_m(x)$ and $b_{mj}(x)$ are defined as in (3.1). Let us fix $x \in R^n$, we define $T_x g = \big(b_m(x) m^{-(n-2)/2}\big)_{m\geqslant 1}$, where $g \in L^1(\Sigma_{n-1})$ and $b_m(x)$ are the coefficients of g in the system Y_m defined before.

Taking on the set N of natural numbers the measure $\mu(m) = m^{n-2}$ and observing now that T_x satisfies the hypothesis of (3.2.1) and $T_x k(x, .) = \big(a_m(x) m^{-(n-2)/2}\big)_{m\geqslant 1}$ for each x, we have

(3.2.2) $$\big\|\big(a_m(x) m^{-(n-2)/2}\big)_{m\geqslant 1}\big\|_{L_{\bar\Phi^*}(N,\mu)} \leqslant C\|k(x, .)\|_{L_\Phi(\Sigma_{n-1})},$$

where C depends on n and Φ only.

Let A be a GYF. From (2.3) we obtain

(3.2.3) $$\begin{aligned}\sum_m a_m^2(x)\gamma_m &= \sum_m \big(a_m(x) m^{-(n-2)/2}\big)^2 \gamma_m m^{n-2} \\ &\leqslant 2\big\|\big(a_m(x) m^{-(n-2)/2}\big)^2_{m\geqslant 1}\big\|_{L_A(N,\mu)} \|\gamma_m\|_{L_{\bar A}(N,\mu)}.\end{aligned}$$

Since $B(x) = 2A(x^2)$ is a GYF, we have as a consequence from the definition of the norm in L_A that

(3.2.4) $$\big\|\big(a_m(x) m^{-(n-2)/2}\big)^2_{m\geqslant 1}\big\|_{L_A(N,\mu)} = \big\|\big(a_m(x) m^{-(n-2)/2}\big)_{m\geqslant 1}\big\|^2_{L_B(N,\mu)}.$$

Now we choose $A(x) = \frac{1}{2}\bar\Phi^*(x^{1/2})$ (see lemma below), then using (3.2.2) and (3.2.4) in (3.2.3) we obtain

$$\sum_m a_m^2(x)\gamma_m \leqslant C\|k(x, .)\|^2_{L_\Phi(\Sigma_{n-1})} \|\gamma_m\|_{L_{\bar A}(N,\mu)}.$$

Hence is enough to show that there exists γ_m such that $\|\gamma_m\|_{L_{\bar A}(N,\mu)} < \infty$ and (3.1.4) is valid. In order to do this we need two lemmas.

(3.2.5) LEMMA. *Let Φ be a Young function such that $\Phi(x)/x^2$ is non-increasing, then*

(i) *$A(x) = \frac{1}{2}\bar\Phi^*(x^{1/2})$ is a GYF;*

(ii) $\frac{1}{4}x^{-1}\Phi^{-1}(x^{-1})^{-2} \leqslant (\bar A)^{-1}(x) \leqslant 2x^{-1}\Phi^{-1}(2^{-1}x^{-1})^{-2}$.

Proof. (i) $\bar\Phi$ is a Young function; then $A(x) = \frac{1}{2}\bar\Phi(x^{-1/2})^{-1}$ if $x \neq 0$. Hence it is sufficient to see that $g(x) = x\bar\Phi(x^{-1/2})$ is non-increasing. If $x > 0$, $\varepsilon > 0$, for each $y > 0$ we take $y' = \big((x+\varepsilon)/x\big)^{1/2} y$, then

$$y' > y \quad \text{and} \quad \Phi(y')/y'^2 \leqslant \Phi(y)/y^2.$$

Hence $(x+\varepsilon)\Phi(y) \geqslant x\Phi(y')$; then

$$(x+\varepsilon)^{1/2}y-(x+\varepsilon)\Phi(y)\leqslant x^{1/2}y'-x\Phi(y')\leqslant g(x).$$

Now taking sup over y we obtain $g(x+\varepsilon)\leqslant g(x)$.

(ii) Since Φ and $\bar{\Phi}$ are strictly increasing, $A^{-1}(x)=(\bar{\Phi})^{-1}(2^{-1}x^{-1})^{-2}$. Hence the estimates follow using (2.5) with A and Φ.

(3.2.6) LEMMA. *Let Φ be as in (3.2.5). Φ satisfies (2.7) if and only if there exists a sequence $\varepsilon_m>0$ such that*

$$S_1=\sum_{m\geqslant 1}\Phi(\varepsilon_m)^{-1}m^{n-2}<\infty \quad\text{and}$$

(3.2.7)

$$S_2=\sum_{m\geqslant 1}\Phi(\varepsilon_m)^{-1}\varepsilon_m^2 m^{-2}<\infty.$$

Proof. Condition (2.7) is equivalent to

(3.2.8) $$\int_1^\infty x^{n-2}\Phi(x^{n/2})^{-1}dx<\infty.$$

Hence if (2.7) holds, we put $\varepsilon_m=m^{n/2}$.

Let ε_m satisfy (3.2.7); we define

$$g_m=\frac{\Phi(\varepsilon_m)m^{n/2}}{\varepsilon_m\Phi(m^{n/2})}.$$

Then $g_m\leqslant 1$ if $\varepsilon_m<m^{n/2}$ and $g_m\leqslant\varepsilon_m m^{-n/2}$ if $\varepsilon_m\geqslant m^{n/2}$; in consequence

$$\sum_{m\geqslant 1}m^{n-2}\Phi(m^{n/2})^{-1}\leqslant\Big(\sum_{\{m:\varepsilon_m<m^{n/2}\}}+\sum_{\{m:\varepsilon_m\geqslant m^{n/2}\}}\Big)\frac{m^{n/2}\varepsilon_m}{m^2\Phi(\varepsilon_m)}\,g_m$$
$$\leqslant S_1^{1/2}S_2^{1/2}+S_2.$$

Hence (3.2.8) holds.

Now we define $\gamma_m=(\bar{A})^{-1}(\Phi(\varepsilon_m)^{-1})$ with ε_m as in (3.2.7). From (2.4) we get

$$\sum_m\bar{A}(\gamma_m)m^{n-2}\leqslant\sum_m\Phi(\varepsilon_m)^{-1}m^{n-2}=S_1<\infty$$

and then $\|\gamma_m\|_{L_{\bar{A}}(N,\mu)}<\infty$. And also from (3.2.5) we have

$$\sum_m m^{-2}\gamma_m^{-1}\leqslant 4\sum_m\Phi(\varepsilon_m)^{-1}\varepsilon_m^2m^{-2}=S_2<\infty.$$

This completes the proof of the theorem when $k(x,y)$ has the form (3.1.1).

3.3. We now prove the theorem in the general case. Let us first suppose that $k(x,y')$ is bounded in x and y'. If we take

$$k_N(x,y')=\sum_{m\leqslant N}a_{mj}(x)Y_{mj}(y'),$$

where $a_{mj}(x)$ are the coefficients of $k(x,y')$ in $\{Y_{mj}\}$, we have

(3.3.1) $$\|k_N(x,.)-k(x,.)\|_{L^2(\Sigma_{n-1})}\to 0,\quad N\to\infty,\ x\in R^n.$$

Since $\Phi(x)/x^2$ is non-increasing, there exist C_1 and C_2 such that

$$\|f\|_{L_\Phi(\Sigma_{n-1})}\leqslant C_1\|f\|_{L^2(\Sigma_{n-1})}\quad\text{and}\quad\|f\|_{L^1(\Sigma_{n-1})}\leqslant C_2\|f\|_{L_\Phi(\Sigma_{n-1})},$$

in consequence

(3.3.2) $$\|k_N(x,.)-k(x,.)\|_{L_\Phi(\Sigma_{n-1})}\to 0\quad\text{if}\quad N\to\infty.$$

Let s be a positive integer, $\varepsilon>0$; by Egorof's theorem there exists a measurable set $B_\varepsilon^s\subset\{|x|\leqslant s\}$ such that $|\{|x|\leqslant s\}-B_\varepsilon^s|<\varepsilon$ and (3.3.2) holds uniformly on B_ε^s. Hence if φ_ε^s is the characteristic function of B_ε^s, we have

(3.3.3) $$\|\varphi_\varepsilon^s(x)k_N(x,.)\|_{L_\Phi(\Sigma_{n-1})}\to\|\varphi_\varepsilon^s(x)k(x,.)\|_{L_\Phi(\Sigma_{n-1})}$$

uniformly, $N\to\infty$. We put

$$K_Nf(x)=\text{p.v.}\int_{R^n}k_N(x,y)f(x-y)\,dy,\qquad f\in C_0^\infty(R^n).$$

Suppose that $f(y)=0$; if $|y|>\varrho$; then

$$|K_Nf(x)-Kf(x)|\leqslant C\sup_y|\nabla f(y)|(|x|+\varrho)\|k_N(x,.)-k(x,.)\|_{L_\Phi(\Sigma_{n-1})}.$$

Now combining this with (3.3.2), we obtain $K_Nf\to Kf$ p.p.

Since the theorem holds for the kernel $\varphi_\varepsilon^s(x)k_N(x,y')$, then if $N\to\infty$, by Fatou's lemma and (3.3.3) we have

$$\|\varphi_\varepsilon^sKf\|_2\leqslant C\sup_x\|k(x,.)\|_{L_\Phi(\Sigma_{n-1})}\|f\|_2.$$

Hence if $\varepsilon\to 0$ and $s\to\infty$, we obtain the desired result.

If $k(x,y')$ is not bounded, we put $k^{(N)}(x,y')=k(x,y')$ if $|k(x,y')|\leqslant N$ and $k^{(N)}(x,y')=0$ otherwise and define

$$k_N(x,y)=k^{(N)}(x,y')|y|^{-n}-|y|^{-n}\frac{1}{\omega_{n-1}}\int_{\Sigma_{n-1}}k^{(N)}(x,y')\,d\sigma(y').$$

Since $|k^{(N)}(x,y')|\leqslant|k(x,y')|$, from Lebesgue's theorem we have

(3.3.4) $$k_N(x,y')\to k(x,y')\ \text{p.p.}$$

Furthermore, $|k_N(x,y')|\leqslant|k(x,y')|+CM\omega_{n-1}^{-1}=g(x,y')$.

We will prove that $k_N(x,y')\to k(x,y')$ in $L_\Phi(\Sigma_{n-1})$. In fact, using now that there exists $C\geqslant 1$ such that $\Phi(2x)\leqslant C\Phi(x)$, if $0<\varepsilon<2$, we can take $r \geqslant 1$ integer such that $2/\varepsilon < 2^r$ and we obtain

$$\int_{\Sigma_{n-1}} \Phi\left(\frac{|k_N(x, y') - k(x, y')|}{\varepsilon \|g(x, .)\|_{L_\Phi(\Sigma_{n-1})}}\right) d\sigma(y')$$

$$\leqslant \int_{\Sigma_{n-1}} \Phi\left(\frac{2\,|g(x, y')|}{\varepsilon \|g(x, .)\|_{L_\Phi}}\right) d\sigma(y') \leqslant \int_{\Sigma_{n-1}} \Phi\left(2^r \frac{|g(x, y')|}{\|g(x, .)\|_{L_\Phi}}\right) d\sigma(y')$$

$$\leqslant C^r \int_{\Sigma_{n-1}} \Phi\left(\frac{|g(x, y')|}{\|g(x, .)\|_{L_\Phi}}\right) d\sigma(y') \leqslant C^r.$$

Then from (3.3.4) we have

$$\|k_N(x, .) - k(x, .)\|_{L_\Phi(\Sigma_{n-1})} \leqslant \varepsilon \|g(x, .)\|_{L_\Phi(\Sigma_{n-1})}$$

for large N. Now if we argue as in the case where k is bounded, we obtain the desired result.

4. Proof of Theorem 2. Let f be in $C_0^\infty(R^n)$ such that $f(x) = 1$ if $|x| \leqslant 1$, $f(x) = 0$ if $|x| \geqslant 2$ and $0 \leqslant f(x) \leqslant 1$. We define

$$k(x, y') = \begin{cases} \Phi_0^{-1}(|x|^{n-1}) & \text{if} \quad |x' - y'| \leqslant |x|^{-1}, \\ -\Phi_0^{-1}(|x|^{n-1}) & \text{if} \quad |x' + y'| \leqslant |x|^{-1} \end{cases}$$

for $|x|$ sufficiently large and $k(x, y') = 0$ otherwise.

Then for large $|x|$ and $\varepsilon < 1$ we have

$$Kf(x) = \int_{R^n} k(x, y) f(x - y)\, dy = \int_{|x' - y'| \leqslant |x|^{-1}} k(x, y) f(x - y)\, dy$$

$$\geqslant C_n \Phi_0^{-1}(|x|^{n-1})\, |x|^{-n} \int_{\substack{|x' - y'| \leqslant |x|^{-1}, \\ |x - y| \leqslant 1}} dy$$

$$= C_n' \Phi_0^{-1}(|x|^{n-1})\, |x|^{-n}.$$

Furthermore from (2.4) we have

$$\int_{\Sigma_{n-1}} \Phi_0(|k(x, y')|)\, d\sigma(y') \leqslant C \quad \text{for each } x \in R^n.$$

Hence for $M_1 > 0$ sufficiently large we obtain

$$\int_{|x| \geqslant M_1} |Kf(x)|^2 dx \geqslant C \int_{M_1}^{\infty} t^{-(n+1)} \Phi_0^{-1}(t^{n-1})^2\, dt = I.$$

Now if we set $t = (\Phi_0(u))^{1/(n-1)}$, from (2.2) and (2.4), we get

$$I \geqslant C \int_{M_1}^{\infty} u \Phi(u) \Phi_0(u)^{-(2n-1)/(n-1)}\, du$$

$$\geqslant C \int_{M_2}^{\infty} u \Phi_0(u)^{-n/(n-1)}\, du \geqslant C \int_{M_3}^{\infty} \Phi(u^{1/2})^{n/(n-1)}\, du = +\infty.$$

5. Proof of Theorem 3. Let $\varphi(t) = t^{-\alpha}$ with $\alpha > 0$, $t > 0$. We define $f(x) = \varphi(|x|)$ if $|x| < 1$ and $f = 0$ otherwise, and

$$k(x, y') = \begin{cases} |x|^{-\beta} |x' - y'|^{-\gamma} & \text{if} \quad |x' - y'| \leqslant |x|,\ |x| < 1, \\ -|x|^{-\beta} |x' + y'|^{-\gamma} & \text{if} \quad |x' + y'| \leqslant |x|,\ |x| < 1, \\ 0 & \text{if} \quad |x| \geqslant 1, \end{cases}$$

where $\gamma, \beta > 0$ such that $\gamma + \beta = (n-1)/q$. Since

(5.1) $$\int_{|x' - y'| \leqslant |x|} |x' - y'|^{-\delta} d\sigma(y') \simeq C |x|^{n-1-\delta}, \qquad \delta < n-1,$$

we have

$$\int_{\Sigma_{n-1}} |k(x, y')|^q d\sigma(y') = 2|x|^{-q\beta} \int_{|x' - y'| \leqslant |x|} |x' - y'|^{-\gamma q} d\sigma(y') \leqslant C.$$

If we set $\Sigma_x^+ = \{y' \in \Sigma_{n-1}\colon k(x, y') \geqslant 0\}$, then

$$K_\varepsilon f(x) = \int_{|y| \geqslant \varepsilon} k(x, y) f(x - y)\, dy = \int_{\Sigma_x^+} k(x, y') \int_{|t| \geqslant \varepsilon} f(x - ty')/t\, dt\, dy'.$$

Now denoting by (x', y') the angle between x' and y' and putting $d = |x| \sin(x', y')$, $\mu = |x| \cos(x', y')$, we have for $|x| < 1$ that $f(x - ty') = \varphi_d(\mu - t)\chi_d(\mu - t)$, where $\varphi_d(t) = \varphi\big((t^2 + d^2)^{1/2}\big)$ and χ_d denote the characteristic function of the interval $I_d = [-(1-d^2)^{1/2}, (1-d^2)^{1/2}]$. If we denote by H and H_ε the Hilbert transform and the truncated Hilbert transform, respectively, we can write

$$K_\varepsilon f(x) = \pi \int_{\Sigma_x^+} k(x, y') H_\varepsilon(\varphi_d \chi_d)(\mu)\, d\sigma(y')$$

and we have $H_\varepsilon(\varphi_d \chi_d)(z) \nearrow H(\varphi_d \chi_d)(z)$; $z > 0$, if $\varepsilon \to 0$, and then we obtain

(5.2) $$Kf(x) = \pi \int_{\Sigma_x^+} k(x, y') H(\varphi_d \chi_d)(\mu)\, d\sigma(y').$$

If $y' \in \Sigma_x^+$, $|x| < 1$, we have $\cos(x', y') > 0$ and $\mu \in I_d$, then as φ_d is even, it follows

(5.3) $$H(\varphi_d \chi_d)(\mu) \geqslant H(\varphi_d)(\mu).$$

Furthermore, $H\varphi_d(x) = d^{-\alpha} H\varphi_1(d^{-1}x)$; then from (5.2) and (5.3) we get

(5.4) $$Kf(x) \geqslant \pi \int_{\Sigma_x^+} k(x, y') d^{-\alpha} H\varphi_1(\mu/d)\, d\sigma(y').$$

Let us suppose for a moment that there exists $C_\alpha > 0$ such that

(5.5) $$H\varphi_1(x) \geqslant C_\alpha \frac{1}{x} \quad \text{if} \quad x \to +\infty;$$

then from (5.4) we obtain for small $|x|$

$$Kf(x) \geqslant C_a \int_{\Sigma_x^+} k(x, y') d^{-a+1} \mu^{-1} d\sigma(y') \geqslant C'_a |x|^{-\beta-a} \int_{|x'-y'|\leqslant|x|} |x'-y'|^{-(\gamma+a-1)} d\sigma(y').$$

If $a+\gamma < n$, from (5.1) the last term of the inequality above exceeds $C''_a |x|^{-(\beta+\gamma+2a-n)} = C''_a |x|^{-((n-1)/q+2a-n)}$. Now as a consequence of the hypothesis made on q the interval $I = [n/p+n-(n-1)/q, 2n/p)$ is non-empty, hence taking $a > 0$ such that $2a \in I$, we obtain $|Kf(x)| \geqslant C|x|^{-n/p}$ for small $|x|$.

Finally, we will show (5.5). As φ_1 is continuous, bounded and $\varphi_1 \in L^p(R)$ for some $p > 1$ we have

$$I_\varepsilon(x) = \int_{-\infty}^{+\infty} \varphi_1(t) Q(\varepsilon, x-t)\, dt \underset{\varepsilon\to 0}{\to} H\varphi_1(x) \quad \text{for each } x \in R,$$

where

$$Q(\varepsilon, t) = \frac{1}{\pi} \frac{t}{\varepsilon^2+t^2}.$$

Changing variables we obtain

$$I_\varepsilon(x) = \int_0^{+\infty} Q(\varepsilon, t) \int_{-\infty}^{+\infty} -\varphi'_1(s) \chi_{[x-t,x+t]}(s)\, ds\, dt = \int_0^{+\infty} Q(\varepsilon, t) \int_0^{+\infty} -\varphi'_1(s) [\chi_{[x-t,x+t]}(s) - \chi_{[x-t,x+t]}(-s)]\, ds\, dt,$$

where $\chi_{[x-t,x+t]}$ denote the characteristic function of the interval $[x-t, x+t]$. If in the integral above we integrate on $0 < t < x$ and $t > x$, separately, then we can change the order of integration and we get

$$I_\varepsilon(x) = \int_0^{+\infty} -\varphi'_1(s) \int_0^x Q(\varepsilon, t) \chi_{[x-t,x+t]}(s)\, dt\, ds + \int_0^{+\infty} -\varphi'_1(s) \int_x^{+\infty} Q(\varepsilon, t) \chi_{[-(x-t),x+t]}(s)\, dt\, ds$$

$$= \int_0^x -\varphi'_1(s) \int_{x-s}^x Q(\varepsilon, t)\, dt\, ds + \int_x^{2x} -\varphi'_1(s) \int_{s-x}^x Q(\varepsilon, t)\, dt\, ds + \int_0^{2x} -\varphi'_1(s) \int_x^{x+s} Q(\varepsilon, t)\, dt\, ds + \int_{2x}^{+\infty} -\varphi'_1(s) \int_{x-s}^{x+s} Q(\varepsilon, t)\, dt\, ds$$

$$= \frac{1}{\pi} \int_0^{+\infty} -\varphi'_1(s) \log\left(\frac{\varepsilon^2+(x+s)^2}{\varepsilon^2+(x-s)^2}\right) ds.$$

Now if $\varepsilon \to 0$, using Fatou's lemma we obtain

$$H\varphi_1(x) \geqslant \frac{1}{\pi} \int_0^x -\varphi'_1(s) \log\left|\frac{x+s}{x-s}\right| ds$$

but if $0 < s < x$, we have

$$\log\left|\frac{x+s}{x-s}\right| \geqslant \frac{2s}{x+s};$$

then

$$H\varphi_1(x) \geqslant \frac{1}{\pi x} \int_0^x -\varphi'_1(s) s\, ds \geqslant C_a \frac{1}{x} \quad \text{if} \quad x \to +\infty.$$

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