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The isomorphic problem of envelopes

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Abstract. It is shown that there is a separable Banach space X which has no separable isomorphic envelope, i.e. there is no separable space Y such that whenever Z is separable and finitely representable in X, Z embeds isomorphically into Y. This strengthens Stern's solution to the (isometric) problem of envelopes posed by Lindenstrauss and Pełczyński.

1. Introduction. The notion of an envelope of a Banach space was introduced by Lindenstrauss and Pełczyński in [6]: A Banach space Y is an *envelope* of a Banach space X if Y is finitely representable in X and each space Z, which is finitely representable in X and whose density character does not exceed that of Y, embeds isometrically into Y. Let us recall that, given $\lambda \geq 1$, a space Y is said to be *finitely λ-representable in* X if for each $\varepsilon > 0$ and each finite-dimensional subspace $F \subset Y$ there exists a subspace $E \subset X$ satisfiyng $d(E, F) < \lambda + \varepsilon$, where d denotes the Banach–Mazur distance. Y is called to be *finitely representable in* X if it is finitely 1-representable. The *density character* of a space X is the least cardinal $\varkappa$ such that X possesses a dense subset of power $\varkappa$.

Lindenstrauss and Pełczyński [6] proved that L_p is an envelope of l_p ($1 \leq p < \infty$) and posed the problem whether every separable Banach space has a separable envelope. This problem was solved by Stern [10] who showed that there exists an equivalent norm on l_2 arbitrarily close to the original one so that the resulting space X has no separable envelope.

Clearly, the notion of an envelope is an isometric one, but it has a natural isomorphic counterpart: Let us say that a Banach space Y is an *isomorphic envelope* of a space X if Y is finitely representable in X and each space Z which is finitely representable in X and whose density character does not exceed that of Y embeds isomorphically into Y.

Note that formally one could still replace the phrase "finitely representable" by "finitely λ-representable for some λ", but this would not lead to essential changes. Indeed, a Banach space Y which is finitely λ-representable in X is λ-isomorphic to a space which is finitely (1-)representable in X.

Roughly speaking, a separable envelope of X is an isometrically universal member within the class of all separable "local subspaces" of X, while a separable isomorphic envelope is an isomorphically universal member of the same class.

With the definition above, the isomorphic problem of envelopes arises: Does every separable Banach space have a separable isomorphic envelope? It follows from the results of [6] that each separable $\mathscr{L}_p$-space $(1 \leqslant p \leqslant \infty)$ has a separable isomorphic envelope. So do all separable subspaces of $\mathscr{L}_p$-spaces for $2 \leqslant p < \infty$ since each such space X is either isomorphic to l_2 or contains an isomorphic copy of l_p (cf. [4], [6]). In the latter case l_p is finitely representable in X ([5]) and L_p will be an isomorphic envelope of X.

The space constructed by Stern [10] has an isomorphic envelope, because it is isomorphic to a Hilbert space. Moreover, his method is based essentially on the metric geometry of l_2 and does not carry over to the isomorphic context.

The aim of this paper is to show that there exists a separable Banach space without separable isomorphic envelope. The space we present has a simple representation: It is an l_r-sum of l_p-spaces. This way we also get an alternative counter-example to the isometric problem. Our proof involves ultrapowers of Banach spaces, which enable us to describe the structure of a possible envelope (this was also used in [10]), and the local incomparability of L_p-spaces for different $p > 2$, established in [9].

We now introduce some notation. Given p, $1 \leqslant p \leqslant \infty$, a set I and a family of Banach spaces $(X_i)_{i\in I}$, we denote by $(\sum_{i\in I} X_i)_p$ the space of all families $(x_i)_{i\in I}$ with $x_i \in X_i$ and $\|(x_i)\| = (\sum_{i\in I} \|x_i\|^p)^{1/p} < \infty$ ($\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$ if $p = \infty$). If $I = N$ and $X_i \equiv X$, we use the notation $l_p(X)$. Furthermore, given a Banach space X and a σ-additive measure μ on a measure space (Ω, Σ), we denote by $L_p(\mu, X)$ the space of X-valued μ-measurable pth power integrable functions. We write $L_p(X)$ if μ is the Lebesgue measure on $[0,1]$.

Next recall the definition of an ultraproduct, introduced in [2]: Let U be an ultrafilter on a set I, and let $(X_i)_{i\in I}$ be a family of Banach spaces. Denote by N_U the closed subspace of $(\sum_{i\in I} X_i)_\infty$ which consists of all families (x_i) satisfying $\lim_U \|x_i\| = 0$. Then the *ultraproduct* $(X_i)_U$ is defined to be the quotient space $(\sum_{i\in I} X_i)_\infty / N_U$, equipped with the usual quotient norm. If $(x_i)_U$ denotes the equivalence class determined by (x_i), then the norm can be computed as $\|(x_i)_U\| = \lim_U \|x_i\|$. If all of the spaces X_i are identical with some X, we speak of an *ultrapower*, $(X)_U$. Given operators $T_i\colon X_i \to Y_i$ with $\sup_I \|T_i\| < \infty$, we can define their ultraproduct $(T_i)_U\colon (X_i)_U \to (Y_i)_U$ in a canonical way by setting $(T_i)_U (x_i)_U = (T_i x_i)_U$. Finally, we say that an ultrafilter U on a set I is *countably incomplete* if there exists a sequence of sets $D_n \in U$ with $\bigcap_{n=1}^{\infty} D_n = \emptyset$. For the basic facts concerning ultraproducts of Banach spaces we refer to [2], [11] and the survey [3].

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2. The counter-example. We first state the main result, which will be proved through a series of lemmas. Q denotes the set of rational numbers.

THEOREM. *Let $2 < a < b < r < \infty$ and let $X = (\sum_{q\in[a,b]\cap Q} l_q)_r$. Then X has no separable isomorphic envelope.*

Throughout this section a, b, r and X will be fixed as defined in the Theorem. The first two lemmas concern the local structure of those subspaces of X which correspond to subsets of the interval $[a, b]$. More precisely, we shall investigate the question whether or not l_p is finitely representable in these spaces. In this connection we shall frequently use a result of Krivine [5], stating that if, for some λ, l_p is finitely λ-representable in a Banach space Z, then l_p is finitely representable in Z. The first lemma is just a reduction result which will simplify the proof of Lemma 2.

LEMMA 1. *Let $A \subset [a, b]$ be a closed set and let $p \in [a, b]$. Assume that l_p is finitely representable in $(\sum_{q\in A} l_q)_r$. Then there exists an $s \in A$ such that l_p is finitely representable in $l_r(l_s)$.*

Proof. If l_p is finitely representable in a direct sum of two Banach spaces, then l_p is finitely representable in at least one of the summands. Indeed, it is a standard fact, that if a direct sum of two Banach spaces contains an isomorph of l_p, then so does one of the summands. By means of ultrapowers, this fact is easily localized (cf. [3], Ch. 6 for this kind of argument), and an application of Krivine's result yields the desired statement.

Using this, one can find a sequence (A_n) of closed subsets of A such that, for all n, $A_n \supseteq A_{n+1}$, diam $(A_n) < 1/n$, and l_p is finitely representable in $(\sum_{q\in A_n} l_q)_r$. Define $s \in A$ to be the unique common point of the sets A_n, i.e. $\{s\} = \bigcap A_n$.

Fix $m \in N$ and $\varepsilon > 0$. By a result of Pełczyński and Rosenthal [9], there exists a $\delta > 0$ (depending on m and ε only) such that the following holds: If $|q-s| < \delta$, then every m-dimensional subspace of l_q is $(1+\varepsilon)$-isomorphic to a subspace of l_s. Now choose n so that $1/n < \delta$ and let E

be an m-dimensional subspace of $(\sum_{q\in A_n} l_q)_r$ with $d(E, l_p^m) < 1+\varepsilon$. There exist m-dimensional subspaces $E_q \subset l_q$ such that $E \subset (\sum_{q\in A_n} E_q)_r$. Each of the spaces E_q is $(1+\varepsilon)$-isomorphic to some subspace of l_s, thus $(\sum_{q\in A_n} E_q)_r$, and in particular E, are $(1+\varepsilon)$-isomorphic to a subspace of $l_r(l_s)$. Consequently, $l_r(l_s)$ contains a $(1+\varepsilon)^2$-isomorph of l_p^m. Since ε and m were arbitrary, this concludes the proof.

LEMMA 2. *Let $A \subset [a,b]$ be a closed set and let $p \in [a,b]\setminus A$. Then l_p is not finitely representable in $(\sum_{q\in A} l_q)_r$.*

Proof. In view of Lemma 1 it suffices to show that for $p, q \in [a,b]$, $p \neq q$, l_p is not finitely representable in $l_r(l_q)$. For technical reasons, we shall actually prove that l_p is not finitely representable in $L_r(L_q)$.

We assume the contrary, i.e. for each n there exists a subspace $E_n \subset L_r(L_q)$ with $\dim E_n = n$ and

$$d(E_n, l_p^n) < 1+1/n. \tag{1}$$

The elements of E_n are vector-valued functions, so let us consider the set of "norm-functions" $S_n \subset L_r$:

$$S_n = \{f \in L_r:\ \text{There exists an } x \in E_n,\ x = x(t) \text{ with } \|x(t)\| = f(t) \text{ a.e.}\}.$$

We shall apply the method of weighted L_2-norms, developed by Pełczyński and Rosenthal in [9], to these sets S_n. Let Φ be the set of all measurable functions φ on $[0,1]$ satisfying $\varphi(t) > 0$ for all $t \in [0,1]$ and $\int \varphi(t)\,dt = 1$. First we show that, taken on S_n, the ratio of the L_r-norm to weighted L_2-norms tends to infinity with $n\to\infty$, more precisely

$$\lim_{n\to\infty} \inf_{\varphi\in\Phi} \sup_{f\in S_n\setminus\{0\}} \|f\|_{L_r}\cdot \|f\cdot\varphi^{(r-2)/2r}\|_{L_2}^{-1} = \infty. \tag{2}$$

Assume that this is not the case. Then there exist a constant C and, for each n, a function $\varphi_n \in \Phi$ so that for all $f \in S_n$,

$$\|f\|_{L_r} \leqslant C\|f\cdot\varphi_n^{(r-2)/2r}\|_{L_2}.$$

Let μ_n be the probability measure on $[0,1]$ defined by $d\mu_n = \varphi_n dt$. We get for $f \in S_n$,

$$\begin{aligned} C^{-1}\|f\|_{L_r} &\leqslant \|f\cdot\varphi_n^{(r-2)/2r}\|_{L_2} = \|f\cdot\varphi_n^{-1/r}\|_{L_2(\mu_n)} \\ &\leqslant \|f\cdot\varphi_n^{-1/r}\|_{L_q(\mu_n)} \leqslant \|f\cdot\varphi_n^{-1/r}\|_{L_r(\mu_n)} = \|f\|_{L_r}. \end{aligned}$$

Thus

$$C^{-1}\|f\|_{L_r} \leqslant \|f\cdot\varphi_n^{-1/r}\|_{L_q(\mu_n)} \leqslant \|f\|_{L_r} \quad (f \in S_n). \tag{3}$$

We now define an operator $T_n: E_n \to L_q(\mu_n, L_q)$ by setting $(T_n x)(t) = \varphi_n^{-1/r}(t)x(t)$, for $x \in E_n$, $t \in [0,1]$. Then (3) shows that

$$C^{-1}\|x\| \leqslant \|T_n x\| \leqslant \|x\| \quad (x \in E_n).$$

Since $L_q(\mu_n, L_q)$ is isometric to L_q, the last inequality together with (1) implies that l_p is finitely C-representable in L_q. But this is known to be impossible [9]. We have got a contradiction which proves (2).

Now we proceed with the application of the argument from [9]. The proof of Proposition 3.1 of [9] combined with (2) yields the following: For each $\delta > 0$ and $m \in \mathbf{N}$ we can find an $n_0 \in \mathbf{N}$ such that whenever $n > n_0$, there exist functions $f_1, \ldots, f_m \in S_n$ of norm one and measurable sets $A_1, \ldots, A_m \subset [0,1]$ satisfying

$$\int_{A_j} f_j^r\,dt > 1-\delta, \qquad \int_{A_j} f_i^r\,dt < \delta \qquad (i<j,\ j = 1, \ldots, m). \tag{4}$$

Define $B_j = A_j \setminus \bigcup_{i>j} A_i$, choose $x_1, \ldots, x_m \in E_n$ so that $\|x_j(t)\| = f_j(t)$ (a.e.) and put $y_j = \chi_{B_j} x_j$. The y_j's are disjointly supported vector-valued functions in $L_r(L_q)$, consequently, they span a subspace isometric to l_r^m. On the other hand, it follows from (4) that

$$\|x_j - y_j\|_{L_r(L_q)} = \|f_j - \chi_{B_j} f_j\|_{L_r} < (m\delta)^{1/r}.$$

If now δ is chosen small enough ($\delta < (4m)^{-2r}$ will suffice), span $(x_j)_{j=1}^m$ is $(1+1/m)$-isomorphic to span $(y_j)_{j=1}^m$, hence to l_r^m. Recalling (1) again, we derive that l_r is finitely representable in l_p, contradicting e.g. the fact [8] that l_p and all spaces which are finitely representable in it are of cotype p while l_r is not $(2 < p < r$, by assumption). This accomplishes the proof of Lemma 2.

Let U be a non-trivial ultrafilter on $\mathbf{N}$. In the next lemma, which is the crucial part of the proof of the Theorem, we shall study isomorphic embeddings of l_p into $(X)_U$. We certainly cannot expect a full description of these embeddings, but we will get some information about the position of l_p-isomorphs with respect to some natural decomposition of $(X)_U$. For this, let us first introduce some more notation. Given a set $A \subset [a,b]$, we denote by P_A the canonical projection of X onto the subspace corresponding to the sum $(\sum_{q\in A\cap \mathbf{Q}} l_q)_r$. The ultrapower of P_A will be denoted by Q_A, thus $Q_A = (P_A)_U$. Furthermore, if (A_n) is a sequence of subsets of $[a,b]$, we define $Q_{(A_n)}$ to be the ultraproduct $(P_{A_n})_U$ of the sequence of projections (P_{A_n}). It is easy to check that Q_A and $Q_{(A_n)}$ are projections, acting in $X_{(U)}$.

LEMMA 3. *Let $p \in (a,b)$, let U be a non-trivial ultrafilter on $\mathbf{N}$, and let Z be a subspace of $(X)_U$ isomorphic to l_p. Then there exists an element*

$z \in Z$ *and a sequence of open intervals* (I_n) *with* $I_n \subset [a, b]$, $p \in I_n$ *and* $\lim_U \operatorname{diam}(I_n) = 0$, *such that* $Q_{(I_n)} z \neq 0$.

Proof. We shall show that the following statement $(*)$ holds:

$(*)$ There exists a $z \in Z$ and an $\varepsilon > 0$ such that for each open interval $I \subset [a, b]$ with $p \in I$, $\|Q_I z\| > \varepsilon$.

Once $(*)$ is verified, the proof of the lemma is completed without difficulty. Indeed, take a sequence of open intervals $J_k \subset [a, b]$ with $p \in J_k$ and $\operatorname{diam}(J_k) < 1/k$ for all k. Let (x_n) be a sequence representing $z \in (X)_U$, i.e. $z = (x_n)_U$. By $(*)$, $\|(P_{J_k} x_n)_U\| > \varepsilon$. This means that for each k there exists a set $D_k \subseteq N$, $D_k \in U$ so that

$$\|P_{J_k} x_n\| > \varepsilon \qquad (n \in D_k). \tag{5}$$

We may assume without loss of generality that the sequence (D_k) is decreasing. Since U, as a non-trivial ultrafilter on N, is countably incomplete, we may further assume that $\bigcap_{k=1}^{\infty} D_k = \emptyset$. Now we define $I_n = (a, b)$ if $n \in N \setminus D_1$ and $I_n = J_k$ if $n \in D_k \setminus D_{k+1}$. Then (5) implies $\|P_{I_n} x_n\| > \varepsilon$ $(n \in D_1)$, and we conclude

$$\|Q_{(I_n)} z\| = \|(P_{I_n} x_n)_U\| = \lim_U \|P_{I_n} x_n\| \geqslant \varepsilon.$$

Finally, it follows that $\lim_U \operatorname{diam}(I_n) = 0$, because $\operatorname{diam}(I_n) < 1/k$ for all $n \in D_k$.

We now turn to the proof of statement $(*)$. Assume that $(*)$ does not hold, i.e. for each $z \in Z$ and each $\varepsilon > 0$ we can find an open interval $I \subset [a, b]$ with $p \in I$ and $\|Q_I z\| \leqslant \varepsilon$. Let T be an isomorphism from l_p onto Z, let (z_n) be the image under T of the unit vector basis of l_p, and put $\varepsilon = (8\|T\|\,\|T^{-1}\|)^{-1}$. We shall construct by induction a decreasing sequence of relatively open in $[a, b]$ intervals (I_j) with $p \in I_j$ for all j, and a normalized block-basis (u_j) of (z_n) such that for all j,

$$\|Q_{[a,b]\setminus I_j} u_j\| \leqslant \varepsilon \cdot 2^{-j} \tag{6}$$

and

$$\|Q_{I_{j+1}} u_j\| \leqslant \varepsilon \cdot 2^{-j}. \tag{7}$$

Put $u_1 = z_1/\|z_1\|$ and $I_1 = [a, b]$. Suppose now $(I_j)_{j=1}^k$ and $(u_j)_{j=1}^k$ have been found. By our assumption above, there exists an open interval $I_{k+1} \subset I_k$ with $p \in I_{k+1}$ and $\|Q_{I_{k+1}} u_k\| \leqslant \varepsilon \cdot 2^{-k}$. Choose m so that $u_1, \ldots, u_k \in \operatorname{span}\{z_n\colon 1 \leqslant n \leqslant m\}$. Suppose that we could not find a u_{k+1}, or equivalently, that for all $u \in \operatorname{span}\{z_n\colon n > m\}$

$$\|Q_{[a,b]\setminus I_{k+1}} u\| > \varepsilon \cdot 2^{-(k+1)} \|u\|. \tag{8}$$

Then the restriction of $Q_{[a,b]\setminus I_{k+1}}$ to the closure of $\operatorname{span}\{z_n\colon n > m\}$ is an isomorphism, which means that l_p embeds isomorphically into $\operatorname{Im} Q_{[a,b]\setminus I_{k+1}}$. It is readily checked that

$$\operatorname{Im} Q_{[a,b]\setminus I_{k+1}} = (\operatorname{Im} P_{[a,b]\setminus I_{k+1}})_U.$$

Therefore l_p is finitely λ-representable for some λ (and hence finitely representable) in

$$\operatorname{Im} P_{[a,b]\setminus I_{k+1}} = \Big(\sum_{q \in ([a,b]\setminus I_{k+1}) \cap Q} l_q\Big)_r,$$

contradicting Lemma 2. This shows that (8) cannot hold and we can find u_{k+1} as required. This completes the induction.

Next we define $v_j = Q_{I_j \setminus I_{j+1}} u_j$. According to (6) and (7), we have

$$\|u_j - v_j\| \leqslant \varepsilon \cdot 2^{(j-1)}. \tag{9}$$

As a block-basis of (z_n), the sequence (u_j) is equivalent to the unit vector basis of l_p. Moreover, its basis constant does not exceed $\|T\|\,\|T^{-1}\|$. By (9), the choice of ε, and a well-known perturbation result ([7], 1.a.9), (v_j) is equivalent to (u_j), consequently to the unit vector basis of l_p.

On the other hand, the projections Q_A form an "l_r-decomposition" of $(X)_U$. Precisely, if $A_1, \ldots, A_m$ are mutually disjoint subsets of $[a, b]$ and $x_1 = (x_{1,i})_U, \ldots, x_m = (x_{m,i})_U$ are elements of $(X)_U$, then

$$\Big\|\sum_{k=1}^{m} Q_{A_k} x_k\Big\| = \lim_U \Big\|\sum_{k=1}^{m} P_{A_k} x_{k,i}\Big\|$$
$$= \lim_U \Big(\sum_{k=1}^{m} \|P_{A_k} x_{k,i}\|^r\Big)^{1/r} = \Big(\sum_{k=1}^{m} \|Q_{A_k} x_k\|^r\Big)^{1/r}.$$

A look at the definition of (v_j) shows now that it must be equivalent to the unit vector basis of l_r, which is a contradiction. This completes the proof of $(*)$ and thus of the lemma.

Proof of the Theorem. Assume that Y is a separable isomorphic envelope of X. Let U be a non-trivial ultrafilter on N. By definition, Y is finitely representable in X, therefore it is isometric to a subspace of $(X)_U$ (cf. [3], Th. 6.3). In the sequel we thus assume $Y \subset (X)_U$. It is easily seen that for each $p \in (a, b)$, l_p is finitely representable in X. Hence Y contains a subspace isomorphic to l_p. An application of Lemma 3 shows that there exists an element $z_p \in Y$ and a sequence of open intervals $(I_{n,p})$ with $p \in I_{n,p}$, $I_{n,p} \subset [a, b]$, $\lim_U \operatorname{diam}(I_{n,p}) = 0$ and $Q_{(I_{n,p})} z_p \neq 0$. We denote for simplicity $Q_{(I_{n,p})} = Q_p$.

Let (y_n) be a sequence which is dense in Y. Then we can find for each $p \in (a, b)$ an $n \in N$ such that $Q_p y_n \neq 0$. Since the set (a, b) is uncountable, there must be an $n \in N$ and an uncountable subset $A \subset (a, b)$ such that $Q_p y_n \neq 0$ for all $p \in A$. Moreover, using the same argument again, we may also assume the existence of an $\varepsilon > 0$ such that

$$\|Q_p y_n\| \geqslant \varepsilon \quad (p \in A). \tag{10}$$

Let us now have a closer look at the projections Q_p. For $p_1, \ldots, p_m \in (a, b)$ with $p_i \neq p_j$ $(i \neq j)$ and for $x = (x_n)_U \in (X)_U$ we have

$$\Big\|\sum_{i=1}^{m} Q_{p_i} x\Big\| = \Big\|\sum_{i=1}^{m} Q_{(I_{n,p_i})} x\Big\| = \lim_U \Big\|\sum_{i=1}^{m} P_{I_{n,p_i}} x_n\Big\|.$$

Since the sequences (I_{n,p_i}) converge to p_i with respect to U, we can find a set $D \in U$ such that for all $n \in D$ the intervals $I_{n,p_1}, \ldots, I_{n,p_m}$ are mutually disjoint. Hence

$$\lim_U \Big\|\sum_{i=1}^{m} P_{I_{n,p_i}} x_n\Big\| = \lim_U \Big(\sum_{i=1}^{m} \|P_{I_{n,p_i}} x_n\|^r\Big)^{1/r}$$
$$= \Big(\sum_{i=1}^{m} \|(P_{I_{n,p_i}} x_n)_U\|^r\Big)^{1/r} = \Big(\sum_{i=1}^{m} \|Q_{p_i} x\|^r\Big)^{1/r}.$$

It follows similarly that $\|\sum_{i=1}^{m} Q_{p_i}\| = 1$.

Returning to (10), we get now

$$\Big\|\sum_{i=1}^{m} Q_{p_i} y_n\Big\| \leqslant \|y_n\| \quad (p_i \in A,\ p_i \neq p_j \text{ for } i \neq j)$$

and, on the other hand,

$$\Big\|\sum_{i=1}^{m} Q_{p_i} y_n\Big\| = \Big(\sum_{i=1}^{m} \|Q_{p_i} y_n\|^r\Big)^{1/r} \geqslant m^{1/r} \varepsilon,$$

which is a contradiction, because m was arbitrary. This shows that X cannot have a separable isomorphic envelope.

3. Remarks. In contrast to the main result of the previous section, it should be pointed out that for $1 \leqslant q, r < \infty$ the space $l_r(l_q)$ has a separable isomorphic envelope, namely $L_r(L_q)$. This can be shown by using uniformity results from [9].

There are some general positive results for higher cardinals which are, however, connected with additional set-theoretic assumptions: An easy modification of the proof of the Theorem (use ultrafilters on bigger sets) yields that each isomorphic envelope of X is actually of density character $\geqslant 2^{\omega}$. This together with the result of Stern ([10], Th. 2) shows that the following three statements are equivalent (in ZFC):

(1) The Continuum Hypothesis.

(2) Each Banach space of density character $\leqslant \omega_1$ has an isomorphic envelope of density character $\leqslant \omega_1$.

(3) Each Banach space of density character $\leqslant \omega_1$ has an (isometric) envelope of density character $\leqslant \omega_1$.

Stern proved (1)$\Leftrightarrow$(3). The implication (1)$\Rightarrow$(3) was derived from a model-theoretic result of Keisler (cf. [10], Th. 5 and [1], Ch. 6). In exactly the same manner it can be deduced from [1] (combine Prop. 5.1.6 (vi) and Th. 5.1.16) that if we assume the Generalized Continuum Hypothesis, then each non-separable Banach space X has an envelope Y of the same density character as X.

References

[1] C. C. Chang, H. J. Keisler, *Model theory*, North-Holland, Amsterdam 1973.

[2] D. Dacunha-Castelle, J. L. Krivine, *Applications des ultraproduits à l'étude des espaces et algébres de Banach*, Studia Math. 41 (1972), 315–334.

[3] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. 313 (1980), 72–104.

[4] M. I. Kadec, A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. 21 (1962), 161–176.

[5] J. L. Krivine, *Sous espaces de dimension finie des espaces de Banach réticulés*, Ann. of Math. 104 (1976), 1–29.

[6] J. Lindenstrauss, A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$ spaces and their applications*, Studia Math. 29 (1968), 275–326.

[7] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces, I, Sequence spaces*, Springer, Berlin–Heidelberg–New York 1977.

[8] B. Maurey, G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. 58 (1976), 45–90.

[9] A. Pełczyński, H. P. Rosenthal, *Localization techniques in L_p spaces*, Studia Math. 52 (1975), 263–289.

[10] J. Stern, *The problem of envelopes for Banach spaces*, Israel J. Math. 24 (1976), 1–15.

[11] — *Ultraproducts and local properties of Banach spaces*, Trans. Amer. Math. Soc. 240 (1978), 231–252.

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