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DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE DUBLIN
BELFIELD, DUBLIN 4, IRELAND

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A generalization of the Banach–Stone theorem

by

KRZYSZTOF JAROSZ (Warszawa)

Abstract. We investigate the geometric properties of a Banach space X^* which give the following implication: There is an isomorphism between $C(S, X)$ and $C(S', X)$ with a small bound iff S and S' are homeomorphic.

Let X be a Banach space and let $C(S, X)$ $(C(S))$ denote the space of X-valued (scalar-valued) continuous functions on a compact Hausdorff space S provided with supremum norm. The classical Banach–Stone theorem states that the existence of an isometric isomorphism from $C(S)$ onto $C(S')$ implies that S and S' are homeomorphic. Cambern [3] proved that this property is stable: If Ψ is an isomorphism of $C(S)$ onto $C(S')$ such that $\|\Psi\|\,\|\Psi^{-1}\| < 2$ then S and S' are homeomorphic.

Several authors have considered a vector-valued generalization of the Banach–Stone theorem. Behrends [2] proved that if the centralizers of X and Y are one-dimensional then the existence of an isometric isomorphism between $C(S, X)$ and $C(S', Y)$ implies that S and S' are homeomorphic. (For the definition and properties of the centralizer see [1], [2]). Cambern [4] proved that if X is a finite-dimensional Hilbert space and if Ψ is an isomorphism of $C(S, X)$ onto $C(S', X)$ such that $\|\Psi\|\,\|\Psi^{-1}\| < \sqrt{2}$ then S and S' are homeomorphic. In this paper we investigate the relation between geometric properties of X^* and the stability of X-valued Banach–Stone theorem.

THEOREM 1. *Let S and S' be compact Hausdorff spaces and let X be a complex (real) Banach space. If there is an isomorphism Ψ of a complex (real) Banach space $C(S, X)$ onto $C(S', X)$ with $\|\Psi\|\,\|\Psi^{-1}\| \leq k$ and if*

$$\sup\{\|x_1^* - x_2^*\|:\ \|(x_1^*+x_2^*)/2\| = 1,\ \|x_1^*\| \leq k,\ \|x_2^*\| \leq k\} = a < 4/3,$$

then S and S' are homeomorphic.

We divide the proof of the theorem into a number of lemmas. Let K^* denote the set $\{x^* \in X^*:\ \|x^*\| = 1\}$ provided with the weak * topology. A Banach space $C(S, X)$ can be identified in a natural way with a subspace of $C(S \times K^*)$: $\Phi(f)(s, x^*) := x^*(f(s))$. Hence any $F \in (C(S, X))^*$ gives a Borel measure μ on $S\times K^*$ (usually μ is not uniquely determined). Let S_1 and S_2 be Borel subsets of S and denote by F_i the elements of $C^*(S, X)$ given by $\mu_i := \mu|_{S_i\times K^*}$, $i = 1, 2$.

LEMMA 1. *For all scalars α and β,*

(1) $\|\alpha F_1+\beta F_2\| = |\alpha|\,\|F_1\|+|\beta|\,\|F_2\|$,

(2) *if* $\operatorname{var}(\mu) = \|F\|$ *then* $\|F_i\| = \operatorname{var}(\mu_i)$, $i = 1, 2$.

Proof. (1) Fix $\varepsilon > 0$ and let

(a) $S_i = \bigcup_{j=1}^{\infty} A_j^i\cup B^i$ with A_j^i compact, $A_{j+1}^i \supset A_j^i$ and $\mu(B^i\times K^*) = 0$, $i = 1, 2$,

(b) $f_n \in C(S)$, $\|f_n\| = 1$, $f_n|A_n^1 = 1$, $f_n|A_n^2 = 0$,

(c) α', β' scalars such that $\alpha\alpha' = |\alpha|$, $\beta\beta' = |\beta|$,

(d) $g_i \in C(S, X)$, $\|g_i\| = 1$, $\|F_i\| < F_i(g_i)+\varepsilon$, $i = 1, 2$.

Then for all n

$$\|\alpha' f_n g_1+\beta'(1-f_n)g_2\| \leqslant 1$$

and we get

$$\begin{aligned}\|\alpha F_1+\beta F_2\| &\geqslant \lim_n \big|\big[(\alpha F_1+\beta F_2)\big(\alpha' f_n g_1+\beta'(1-f_n)g_2\big)\big]\big|\\ &= \lim_n \Big|\Big[|\alpha|\int f_n g_1 d\mu_1+\alpha\beta'\int(1-f_n)g_2 d\mu_1+\beta\alpha'\int f_n g_1 d\mu_2+\\ &\qquad +|\beta|\int(1-f_n)g_2 d\mu_2\Big]\Big| = |\alpha|\int g_1 d\mu_1+|\beta|\int g_2 d\mu_2\\ &\geqslant |\alpha|(\|F_1\|-\varepsilon)+|\beta|(\|F_2\|-\varepsilon).\end{aligned}$$

(2) If $\operatorname{var}(\mu) = \|F\|$ then

$$\begin{aligned}\operatorname{var}(\mu) = \|F\| &= \|F_1+F_2\| \leqslant \|F_1\|+\|F_2\|\\ &\leqslant \operatorname{var}(\mu_1)+\operatorname{var}(\mu_2) = \operatorname{var}(\mu);\end{aligned}$$

hence we obtain $\|F_i\| = \operatorname{var}(\mu_i)$.

LEMMA 2. *Let $s_0' \in S$ and $y_0^* \in K^*$, and let G denote the functional on $C(S, X)$ defined by $G(g) := y_0^*(g(s_0'))$; then $F := \Psi^* G$ is of the form*

$$F(f) = a x_0^*\big(f(s_0)\big)+\Delta F(f)$$

where $x_0^* \in K^*$, $\Delta F \in C^*(S, X)$, $a+\|\Delta F\| = \|F\|$ and $a > \frac{2}{3}\|F\|$.

Proof. Let μ be a measure on $S\times K^*$ given by F with $\operatorname{var}(\mu) = \|F\|$ and let $S = S_1\cup S_2$ be a partition of S into two disjoint Borel subsets such that $\operatorname{var}(\mu_1) \geqslant \operatorname{var}(\mu_2) > 0$ where $\mu_i = \mu|_{S_i}$, $i = 1, 2$. We have $F = \frac{1}{2}F_1+\frac{1}{2}F_2$, where

$$F_1 := \frac{\|F\|}{\operatorname{var}(\mu_1)}\mu_1, \quad F_2 := \left(2-\frac{\|F\|}{\operatorname{var}(\mu_1)}\right)\mu_1+2\mu_2$$

and by Lemma 1 $\|F_i\| = \|F\|$.

Let $G_i := (\Psi^*)^{-1}F_i$ and let ν_i be measures on $S'\times K^*$ given by G_i with $\operatorname{var}(\nu_i) = \|G_i\| \leqslant \|\Psi^{-1}\|\,\|F\|$. We have

$$\nu_i = a_i\delta_{(s_0', y_i^*)}+\xi_i,$$

where

$$\xi_i = \nu_i|_{(S'-\{s_0'\})\times K^*}, \qquad a_i y_i^*(y) = \nu_i|_{\{s_0'\}\times K^*}(\underline{y}), \qquad a_i \geqslant 0$$

and $y_i^* \in K^*$. Since $G = \frac{1}{2}G_1+\frac{1}{2}G_2$, we get from Lemma 1

$$y_0^* = \tfrac{1}{2}a_1 y_1^*+\tfrac{1}{2}a_2 y_2^*,$$

$$a_i+\|\xi_i\| = \|G_i\|,$$

$$-\xi_1 = \xi_2 \quad \text{(as a functional on } C(S', X)).$$

Let us assume $a_1 \geqslant a_2$; then we have

$$y_0^* = \frac{1}{2}\left(a_1 y_1^*+(k-a_1)\frac{a_1 y_1^*-y_0^*}{\|a_1 y_1^*-y_0^*\|}\right)+\frac{1}{2}\left(a_2 y_2^*+(k-a_1)\frac{a_2 y_2^*-y_0^*}{\|a_2 y_2^*-y_0^*\|}\right)$$

and

$$\left\|a_i y_i^*+(k-a_1)\frac{a_i y_i^*-y_0^*}{\|a_i y_i^*-y_0^*\|}\right\| \leqslant k;$$

by the assumption about the spaces X^* we get

$$\begin{aligned}a &\geqslant \left\|a_1 y_1^*+(k-a_1)\frac{a_1 y_1^*-y_0^*}{\|a_1 y_1^*-y_0^*\|}-a_2 y_2^*-(k-a_1)\frac{a_2 y_2^*-y_0^*}{\|a_2 y_2^*-y_0^*\|}\right\|\\ &= \|a_1 y_1^*-a_2 y_2^*\|+2(k-a_1),\end{aligned}$$

and hence

$$\begin{aligned}\|F_1-F_2\| &\leqslant \|\Psi\|\,\|G_1-G_2\| = \|\Psi\|(\|a_1 y_1^*-a_2 y_2^*\|+2\|\xi_i\|)\\ &\leqslant \|\Psi\|\big(a-2(k-a_1)+2(\|G_1\|-a_1)\big)\\ &\leqslant \|\Psi\|\big(a-2(k-\|\Psi^{-1}\|\,\|F\|)\big).\end{aligned}$$

From the definition of F_i and Lemma 1 it is easy to check that $\|F_1-F_2\| = 4\operatorname{var}(\mu_2)$, and finally we obtain

$$\begin{aligned}\operatorname{var}(\mu_2) &\leqslant \frac{\|\Psi\|}{4}\big(a-2(k-\|\Psi^{-1}\|\,\|F\|)\big)\\ &\leqslant \frac{1}{3}\|F\|+\|\Psi\|\left(\frac{a}{4}-\frac{k}{2}+\frac{\|\Psi\|\,\|\Psi^{-1}\|}{2}-\frac{1}{3}\right) < \frac{1}{3}\|F\|.\end{aligned}$$

Now we shall use the following

Proposition. *Let μ be a regular measure on S with total variation 1. If for every partition of S into two Borel subsets the measure of the smaller one is less than $1/3$, then there is an $s_0 \in S$, $\mu(\{s_0\}) > 2/3$.*

To end the proof we define

$$a x_0^*(x) := \mu|_{\{s_0\}\times K^*}(\underline{x}), \qquad x_0^* \in K^*, \qquad a > 2/3$$

and

$$\Delta F(f) := \mu|_{(S-\{s_0\})\times K^*}(f).$$

It follows from Lemma 1 that the point s_0 in Lemma 2 is unique; thus, for each $y_0^* \in K^*$, Lemma 2 defines a function $\varphi\colon S' \to S$, $\varphi(s_0') = s_0$. Similarly for each $x_0^* \in K^*$ we get a function $\psi\colon S \to S'$, $\psi(s_0) = s_0'$ where s_0' is determined by

$$(\Psi^{-1})^*(\delta_{(s_0, x_0^*)})(g) = \beta y_0^*(g(s_0')) + \Delta G(g),$$

$$\beta > \tfrac{2}{3}\|(\Psi^{-1})^*(\delta_{(s_0, x_0^*)})\|.$$

We shall prove that φ is a homeomorphism of S' onto S and $\varphi^{-1} = \psi$.

Lemma 3. *φ and ψ are continuous.*

Proof. Let $\varphi(s_0') = s_0$ and let U be a neighbourhood of s_0. We shall find a function $f \in C(S, X)$ such that

(1) $$y_0^*(\Psi(f)(s_0')) > \tfrac{1}{3}\|\Psi\|$$

and

(2) $$|y_0^*(\Psi(f)(s'))| < \tfrac{1}{3}\|\Psi\| \quad \text{for} \quad s' \notin \varphi^{-1}(U).$$

The function $s' \to y_0^*(\Psi(f)(s'))$ is continuous; hence $\varphi^{-1}(U)$ is a neighbourhood of s_0' and φ is continuous.

Retain the notation of Lemma 2. Let $\varepsilon = a - \frac{2}{3}\|F\| > 0$ and let $\varkappa$ be a real continuous function on S such that

(a) $\sup\{|\varkappa(s)|\colon s \in S\} = \varkappa(s_0) = 1$,

(b) $\operatorname{supp}\varkappa \subset U$,

(c) $|\mu|((\operatorname{supp}\varkappa - \{s_0\}) \times K^*) < \varepsilon/2$.

Let $x_0 \in X$ be such that $\|x_0\| = 1$ and $x_0^*(x_0) > 1 - \varepsilon/4$. Set $f := \varkappa \underline{x}_0$. We have

(1) $$y_0^*(\Psi(f)(s_0')) = G(\Psi(f)) = F(f) = a x_0^*(x_0) + \Delta F(\varkappa \underline{x}_0)$$
$$> a(1-\varepsilon/4) - \mu|_{(S-\{s_0\})\times K^*}(\varkappa \underline{x}_0) > a(1-\varepsilon/4) - \varepsilon/2 > \tfrac{2}{3}\|F\| > \tfrac{1}{3}\|\Psi\|.$$

(2) For $s' \notin \varphi^{-1}(U)$ let us consider the functional $\tilde{G} \in C^*(S', X)$, $\tilde{G}(g) := y_0^*(g(s'))$. By Lemma 2

$$\tilde{F}(f) := \Psi^* \tilde{G}(f) = \beta x_1^*(f(s)) + \Delta \tilde{F}(f),$$

where $\|\Delta \tilde{F}\| < \frac{1}{3}\|\tilde{F}\|$ and $s \notin U$. Hence we get

$$y_0^*(\Psi f(s')) = \tilde{G}(\Psi f) = \tilde{F}(f)$$
$$= \beta x_1^*(\varkappa(s)\underline{x}_0) + \Delta\tilde{F}(\varkappa \underline{x}_0) \leqslant \|\Delta \tilde{F}\| < \tfrac{1}{3}\|\Psi\|.$$

The proof that ψ is continuous follows upon changing the notation.

Lemma 4. *φ is a homeomorphism S' onto S.*

Proof. Let F, G, a be as in Lemma 2, and put $G' := (\Psi^{-1})^* F'$ where $F' \in C^*(S, X)$ is defined by $F'(f) := x_0^*(f(s_0))$. By Lemma 2

$$G'(g) = \beta y_1^*(g(s_1')) + \Delta G'(g), \quad \text{where} \quad \beta > \tfrac{2}{3}\|G'\|, \quad \beta + \|\Delta G'\| = \|G'\|.$$

We have $G = aG' + G''$, where $G'' = (\Psi^{-1})^*(\Delta F)$. This means that, for $g \in C(S', X)$, $y_0^*(g(s_0')) = a\beta y_1^*(g(s_1')) + a\Delta G'(g) + G''(g)$. Hence $y_0^*(g(s_0')) - a\beta y_1^*(g(s_1')) = a\Delta G'(g) + G''(g)$.

Assume that $s_0' \neq s_1'$; then the norm of the functional on the left is equal to $1 + a\beta$, and hence $a\|\Delta G'\| + \|G''\| \geqslant 1 + a\beta$. Since $\|\Delta G'\| = \|G'\| - \beta$ and $\beta > \frac{2}{3}\|G'\|$, we have

$$\|G''\| + a\|G'\| \geqslant 1 + 2a\beta \geqslant 1 + \tfrac{4}{3} a\|G'\|.$$

Hence

$$\tfrac{1}{3}k \geqslant \tfrac{1}{3}\|\Psi^{-1}\|\,\|F\| \geqslant \|\Psi^{-1}\| \cdot \|\Delta F\|$$
$$\geqslant \|G''\| \geqslant 1 + \tfrac{1}{3} a\|G'\|$$
$$\geqslant 1 + \tfrac{1}{3}\tfrac{2}{3}\|F\|\,\|G'\| \geqslant 1 + 2/(9k).$$

Hence $k > 3{,}207\ldots$, but this is imposible for $a < 4/3$. This means that $s_0' = s_1'$ and $\psi \circ \varphi = \mathrm{id}_{S'}$. Similarly $\varphi \circ \psi = \mathrm{id}_S$ and φ is a homeomorphism.

In Theorem 1 the assumption about X is stronger than: $\dim(Z(X)) = 1$. The natural question arises whether the spaces with trivial centralizers are stable. The answer is negative:

Theorem 2. *There is a real Banach space X such that*

(1) $$\dim(Z(X)) = 1$$

and

(2) $$\inf\{\|\Psi\|\,\|\Psi^{-1}\|\colon \Psi\colon C(S, X) \to C(S', X)\} = 1,$$

where S is a one-point set and S' a two-point set.

Proof. X will be the space of infinite, bounded sequences with the special norm defined below.

Let (k_n) be a sequence of positive integers such that $\sum_{n=1}^{\infty} 1/k_n = 1$.

By induction we define an increasing sequence of norms on l^∞. To this end let $a = (a_1, a_2, \ldots) \in l^\infty$,

$$\|a\|_0 := \sup\{|a_k|:\ k \in N\},$$

$$\|a\|_1 := \sup\{a_k^1:\ k \in N\},$$

where $a_k^1 := \sqrt[k_1]{|a_{2k-1}|^{k_1} + |a_{2k}|^{k_1}}$,

$$\|a\|_n := \sup\{a_k^n:\ k \in N\},$$

where $a_k^n := \sqrt[k_n]{(a_{2k-1}^{n-1})^{k_n} + (a_{2k}^{n-1})^{k_n}}$.

Now define $X := (l^\infty, \|\ \|)$, where $\|\ \| = \sup\limits_{n\in N} \|\ \|_n$. For $a \in l^\infty$ we have $\|a\|_n \leqslant \|a\|_{n+1} \leqslant \sqrt[k_{n+1}]{2}\,\|a\|_n$; hence

$$\|a\|_0 \leqslant \|a\| \leqslant \prod_{j=1}^{\infty} \sqrt[k_j]{2}\,\|a\|_0 \leqslant 2\|a\|_0.$$

Thus $\|\ \|$ is well defined and X is complete. For $n \in N$ denote by I_n the identity isomorphism from $(l^\infty, \|\ \|)$ onto $(l^\infty, \|\ \|_n)$. We have $\|I_n\| = 1$ and $\|I_n^{-1}\| \xrightarrow[n\to\infty]{} 1$. Notice that $(l^\infty, \|\ \|_n)$ is a direct sum, with the norm sup, of infinitely many spaces isometric with the subspace X_n of X: $X_n := \{a \in l^\infty:\ a_k = 0 \text{ for } k > 2^n\}$. Let us now consider the sequence of isomorphisms

$$\begin{aligned} C(S, X) &\overset{\Phi_1}{\cong} X \overset{I_n}{\cong} (l^\infty, \|\ \|_n) \overset{\Phi_2}{\cong} \bigotimes_{k=1}^{\infty}{}_{\infty} X_n \\ &\overset{\Phi_3}{\cong} \bigotimes_{k=1}^{\infty}{}_{\infty} X_n \otimes_\infty \bigotimes_{k=1}^{\infty}{}_{\infty} X_n \\ &\overset{\Phi_4}{\cong} (l^\infty, \|\ \|_n) \otimes_\infty (l^\infty, \|\ \|_n) \overset{I_n^{-1} \oplus I_n^{-1}}{\cong} X \otimes X \overset{\Phi_S}{\cong} C(S', X), \end{aligned}$$

where Φ_i, $i = 1, 2, 3, 4, 5$ are isometries and $\|I_n\|\,\|I_n^{-1} \otimes I_n^{-1}\| \xrightarrow[n\to\infty]{} 1$. This gives (2).

Let $T \in Z(X)$. It is easy to check that T_n: $X_n \to X_n$ defined by $T_n a := \pi_n(Ta)$ where $\pi_n: X \to X_n$, $\pi_n(a_1, a_2, \ldots) := (a_1, a_2, \ldots, a_{2^n}, 0, 0, \ldots)$, is in $Z(X_n)$ but $\|\ \|$ on X_n is strictly convex, and so $\dim Z(X_n) = 1$ and consequently $\dim Z(X) = 1$.

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