PROPOSITION. *Si E et F sont des espaces localement quasi-convexes séparés, il existe alors sur $E\otimes F$ une topologie tensorielle localement quasi-convexe séparée.*

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Analytic functionals on fully nuclear spaces

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Abstract. In this paper we study bounded linear functionals on $(H(U),\tau)$ (analytic functionals), where U is an open polydisc in a fully nuclear space with a basis. Our main result is that $(H(U),\tau_0)$ is a bornological space whenever U is an open polydisc in a certain class of Fréchet nuclear spaces with a basis. Apart from this result, we show that certain spaces of holomorphic functions and analytic functionals have a basis and are nuclear. An interesting feature of our results is the direct correspondence which we find between known results not hitherto seen as being connected.

We use the notation of [8] for the theory of holomorphic functions on fully nuclear spaces. We refer to [14] and [16] for the general theory of locally convex spaces, to [13] and [21] for the theory of nuclear spaces, to [10], [11], [18] and [19] for the theory of holomorphic functions on locally convex spaces. Further results on the theory of holomorphic functions on nuclear spaces are to be found in [2], [3], [4], [5], [6] and [24].

We begin by recalling some definitions from [8].

A *fully nuclear space* is a locally convex space E such that E and E'_β (the strong dual of E) are both complete reflexive nuclear spaces. This implies that the strong dual of a fully nuclear space is fully nuclear. Fréchet nuclear spaces are fully nuclear and the product of a countable set of fully nuclear spaces is also fully nuclear. If E is fully nuclear and has a Schauder basis, then it has an equicontinuous and hence an absolute basis.

Let P denote a collection of non-negative sequences such that for each $r\in N$ there exists $(a_n)_n\in P$ where $a_r>0$.

The *sequence space* $\Lambda(P)$ is the set of all sequences of complex numbers, $(z_n)_n$, such that

$$\sum_n |z_n|\,a_n<\infty \quad \text{for all } a=(a_n)_{n=1}^\infty\in P.$$

We endow $\Lambda(P)$ with the topology generated by the semi-norms p_a, $a=(a_n)_n\in P$, where

$$p_a(\{z_n\}_n)=\sum_n |z_n|\,a_n.$$

Each element of P is called a *weight*. We let $[P]$ denote the set of all sequences of non-negative real numbers $(a_n)_n$ such that

$$\Big\{(z_n)_n \in E;\ \sum_{n=1}^{\infty} |z_n| a_n < 1\Big\} \quad \text{is an open subset of } \Lambda(P).$$

It is clear that $\Lambda(P) = \Lambda([P])$.

The following is the basic result concerning nuclear sequence spaces (Grothendieck–Pietsch).

Proposition 1. *The locally convex space $\Lambda(P)$ is nuclear if and only if for each $(a_n)_n \in P$ there exists $(u_n)_n \in l_1$ and $(a'_n)_n \in P$ such that $a_n \leqslant |u_n| \cdot a'_n$ for all n.*

Hence if $\Lambda(P)$ is nuclear, then

$$\Lambda(P) = \{(z_n)_{n=1};\ \sup_n |z_n a_n| < \infty \text{ all } (a_n)_n \in P\}$$

and its topology is generated by

$$\|(z_n)_n\|_{(a_n)_n} = \sup_n |z_n a_n|,$$

where $a = (a_n)_n$ ranges over P.

If E is fully nuclear with a Schauder basis (we shall say fully nuclear with a basis from now on), then E and E'_β may be identified with nuclear sequence spaces $\Lambda(P)$ and $\Lambda(P')$.

We fix once and for all such an identification and denote the duality between E and E'_β as follows:

$$w(z) = \langle w, z\rangle = \langle (w_n)_n, (z_n)_n\rangle = \sum_n w_n z_n,$$

where $z \in E$ and $w \in E'_\beta$.

Subsets of $\Lambda(P)$ which have either of the following forms

$$A = \{(z_n)_n \in \Lambda(P);\ \sup_n |z_n a_n| < 1\}$$

or

$$B = \{(z_n)_n \in \Lambda(P);\ \sup_n |z_n a_n| \leqslant 1\},$$

where $a_n \in [0, +\infty]$ all n and $a\cdot(+\infty) = +\infty$ if $a > 0$ and $0\cdot(+\infty) = 0$ are called *polydiscs*. A is open if and only if $(a_n)_n \in [P]$ and B is always closed. The multiplicative polar of a subset U of a fully nuclear space with a basis, $E \approx \Lambda(P)$, is defined as follows,

$$U^M = \{(w_n)_n \in E'_\beta;\ \sup_n |w_n z_n| \leqslant 1 \text{ all } (z_n)_n \in U\}.$$

If U is an open polydisc in E, then U^M is a compact polydisc in E'_β.

If E is a nuclear space with an absolute basis, then E can be identified with a subspace of $\Lambda(P)$ and the equicontinuous basis is mapped onto the unit vector basis of $\Lambda(P)$. With this notation, we have the following definition.

Definition 2. E is an *Λ-nuclear space* if there exists a sequence of positive real numbers, $(\delta_n)_n$, $\delta_n > 1$ all n and $\sum_n (1/\delta_n) < \infty$, such that $(a_n\delta_n)_n \in [P]$ whenever $(a_n)_n \in P$.

Proposition 3. (a) *The strong dual of an Λ-nuclear space is an Λ-nuclear space.*

(b) *A reflexive Λ-nuclear space is fully nuclear.*

(c) *A complete nuclear space with an equicontinuous basis, $E \approx \Lambda(P)$, is Λ-nuclear if and only if there exists a linear automorphism of E, φ, such that $\varphi(\{z_n\}_n) = \{\delta_n z_n\}_n$ and $\sum_n (1/\delta_n) < \infty$.*

(d) *The strong dual of a reflexive Λ-nuclear space is a reflexive Λ-nuclear space.*

Proof. If B is a subset of $\Lambda(P)$, we let

$$\tilde{B} = \{(z_n)_n \in \Lambda(P);\ \exists\, (z'_n)_n \in B \text{ and } |z_n| \leqslant |z'_n| \text{ all } n\}$$

If B is a bounded subset of $\Lambda(P)$, then $\tilde{B}$ is also bounded.

(a) Suppose E is an Λ-nuclear space. Then $E'_\beta \approx \Lambda(P)'_\beta$. If $T \in \Lambda(P)'$, then

$$T(\{z_n\}_n) = \sum_n z_n b_n$$

and the sequence $\{b_n\}_n$ uniquely determines T. Let $T_n(\{z_m\}_m) = z_n$. Then $T = \sum_n b_n T_n$ in the $\sigma\big(\Lambda(P)', \Lambda(P)\big)$ topology. If B is a bounded subset of $\Lambda(P)$, then

$$\delta\tilde{B} = \{(\delta_n z_n)_n;\ (z_n)_n \in \tilde{B}\}$$

and

$$\delta^2\tilde{B} = \{(\delta_n^2 z_n)_n;\ (z_n)_n \in \tilde{B}\}$$

are also bounded subsets of $\Lambda(P)$. Hence, for all n,

$$\|b_n T_n\|_B \leqslant (1/\delta_n)\|b_n T_n\|_{\delta\tilde{B}} \leqslant (1/\delta_n)\cdot\|T\|_{\delta\tilde{B}}$$

and thus

$$\|T\|_B \leqslant \sum_n |b_n|\cdot\|T_n\|_B \leqslant \Big(\sum_n (1/\delta_n)\Big)\cdot\|T\|_{\delta\tilde{B}}.$$

Hence $\{T_n\}_n$ is an absolute and equicontinuous basis for $\Lambda(P)'_\beta$. Moreover, since $\sum_n (\|T_n\|_{\delta\tilde{B}}/\|T_n\|_{\delta^2\tilde{B}}) \leqslant \sum_n (1/\delta_n) < \infty$ for all B it follows that $\Lambda(P)'_\beta$ is Λ-nuclear. This completes the proof of (a).

(b) If E is a reflexive Λ-nuclear space, then E is quasi-complete, and since it has an absolute basis, it is complete. The strong dual of E is nuclear by (a) and quasi-complete (since E is barrelled) and thus E'_β is a complete reflexive nuclear space. Thus E is fully nuclear.

(c) is obvious and (d) follows from (a).

The product (finite) of A-nuclear spaces is A-nuclear. Most of the fully nuclear spaces with a basis that we encounter are reflexive A-nuclear spaces, e.g. S, $H(C)$, $\sum_N C$ and it is probable that every Fréchet nuclear space with a basis is a reflexive A-nuclear space.

$H(U)$ will denote the holomorphic functions on the open subset U of E, $H_{HY}(U)$ will denote the hypoanalytic (i.e. Gâteaux holomorphic and continuous on compact sets) functions on U.

τ_0 (resp. τ_ω) will denote the compact open (resp. the ported Nachbin) topology on $H_{HY}(U)$ and $H(U)$ (resp. $H(U)$). $\tau_{0,b}$ (resp. $\tau_{\omega,b}$) will denote the associated bornological topologies on $H(U)$ and τ_δ will denote the topology generated by the countable open covers of U (i.e. p, a semi-norm on $H(U)$, is τ_δ continuous if and only if for each increasing countable open cover of U, $(V_n)_{n=1}^\infty$, there exists an integer N and $C>0$ such that $p(f)\leqslant C\|f\|_{V_N}$ for all $f\in H(U)$). If U is balanced, then τ_δ is the barrelled topology associated with τ_ω ([20]).

If K is a compact subset of E, $H(K)$ will denote the germs of holomorphic functions on K. $H(K)$ is endowed with the inductive limit topology

$$\varinjlim_{\substack{V\supset K\\ V\text{ open}}} \big(H(V),\ \|\ \|_\infty\big)$$

and we also have ([8])

$$H(K) = \varinjlim_{\substack{V\supset K\\ V\text{ open}}} \big(H(V),\tau_\omega\big).$$

$H_{HY}(K)$ is the space of hypoanalytic germs about K, i.e.

$$H_{HY}(K) = \bigcup_{\substack{V\supset K\\ V\text{ open}}} H_{HY}(V)/\sim,$$

where $f\sim g$ if f and g coincide in some neighbourhood of K. We endow $H_{HY}(K)$ with the inductive limit topology $\varinjlim_{\substack{V\supset K\\ V\text{ open}}} \big(H_{HY}(V),\tau_0\big)$. Let

$$N^{(N)} = \{(m_i)_{i=1}^\infty;\ m_i\in N,\ m_i\geqslant 0 \text{ and } m_i \text{ is eventually zero}\}.$$

If $m\in N^{(N)}$ and $z=(z_n)_n\in\Lambda(P)$, we let $z^m=\prod_{i=1}^\infty z_i^{m_i}$, where $z^0=1$ for all $z\in C$. The mapping $z\in E\to z^m\in C$ called a *monomial* for all m in $N^{(N)}$.

PROPOSITION 4. *If U is an open polydisc in a fully nuclear space with a basis $E\approx\Lambda(P)$, then the monomials form an unconditional equicontinuous basis in $(H(U),\tau)$, where $\tau=\tau_0,\tau_{0,b},\tau_\omega,\tau_{\omega,b}$ or τ_δ.*

Proof. When $\tau=\tau_0$ or τ_ω this has already been shown in [8]. Let $f\in H(U)$; then $f(z)=\sum_{m\in N^{(N)}} a_m z^m$ for all z in U (see [8]). Let $V_n = \{z\in U;\ |\sum_{m\in J} a_m z^m|\leqslant n$ for every finite subset J of $N^{(N)}\}$ and let W_n denote the interior of V_n. If K is a compact subset of U, then (see [8]) we can find a neighbourhood V of 0 in E such that

$$\sum_{m\in N^{(N)}} \|a_m z^m\|_{K+V} < \infty.$$

Hence $(W_n)_{n=1}^\infty$ is an increasing countable open cover of U. Now if p is a τ_δ continuous semi-norm on $H(U)$, there exists $C>0$ and N such that $p(f)\leqslant C\|f\|_{W_N}$ for all $f\in H(U)$. Hence

$$\sup_{\substack{J\subset N^{(N)}\\ J\text{ finite}}} p\Big(\sum_{m\in J} a_m z^m\Big) < \infty.$$

Thus $\{\sum_{m\in J} a_m z^m\}_{\substack{J\subset N^{(N)}\\ J\text{ finite}}}$ is a τ_δ-bounded subset of $H(U)$. Since the monomials form an absolute basis for $(H(U),\tau_\omega)$, Corollary 1.2 of [10] implies that they form a Schauder basis in $(H(U),\tau_\delta)$. Since $(H(U),\tau_\delta)$ is barrelled ([10], [20]), it follows that the monomials form an equicontinuous basis for $(H(U),\tau_\delta)$. Since $\tau_\delta\geqslant\tau_{\omega,b}\geqslant\tau_{0,b}$, this also shows that the monomials form a Schauder basis in $(H(U),\tau_{\omega,b})$ and $(H(U),\tau_{0,b})$.

If p is a τ_0 (resp. τ_ω) continuous semi-norm on $H(U)$, then Theorems 11 and 15 of [8] imply that $p'(f)=\sum_{m\in N^{(N)}} p(a_m z^m)$ is also a τ_0 (resp. τ_ω) continuous semi-norm on $H(U)$. Hence if $(f_a)_{a\in A}$ is a τ_0 (resp. τ_ω) bounded subset of $H(U)$ and $f_a(z)=\sum_{m\in N^{(N)}} a_m^a z^m$ all $a\in A$, then

$$\Big\{\sum_{m\in J} a_m^a z^m\Big\}_{\substack{a\in A\\ J\subset N^{(N)}\\ J\text{ finite}}}$$

is also a τ_0 (resp. τ_ω) bounded subset of $H(U)$. Hence if p is a $\tau_{0,b}$ (resp. $\tau_{\omega,b}$) continuous semi-norm on $H(U)$, then

$$p'(f) = \sup_{\substack{J\subset N^{(N)}\\ J\text{ finite}}} p\Big(\sum_{m\in J} a_m z^m\Big)$$

is bounded on τ_0 (resp. τ_ω) bounded subsets of $H(U)$ and thus is $\tau_{0,b}$ (resp. $\tau_{\omega,b}$) continuous. This shows that the monomials form an equicontinuous basis in $(H(U),\tau_{0,b})$ and $(H(U),\tau_{\omega,b})$. The basis is obviously unconditional in all cases.

COROLLARY 5. *If U is an open polydisc in a fully nuclear space with a basis, then τ_δ is the barrelled topology associated with τ_0 on $H(U)$.*

Proof. Since $\tau_0\leqslant\tau_\delta$ and τ_δ is barrelled, it follows that $\tau_0\leqslant\tau\leqslant\tau_\delta$, where τ is the barrelled topology associated with τ_0. If p is a τ_δ continuous semi-norm on $H(U)$, then

$$p'(f) = \sup_{\substack{J\subset N^{(N)}\\ J\text{ finite}}} p\Big(\sum_{m\in J} a_m z^m\Big)$$

is also τ_δ continuous and $p' \geqslant p$. For each finite subset J of $N^{(N)}$ the semi-norm $p_J(f) = p\left(\sum_{m\in J} a_m z^m\right)$ is τ_0 continuous and hence p' is τ continuous. Hence $\tau \geqslant \tau_\delta$ and $\tau = \tau_\delta$.

COROLLARY 6. *If U is an open polydisc in a fully nuclear space with a basis $E \approx \Lambda(P)$, then the following are equivalent for $\tau = \tau_0, \tau_\omega, \tau_{\omega,b}, \tau_{0,b}$ or τ_δ:*

(a) *$(H(U), \tau)$ is complete,*

(b) *$(H(U), \tau)$ is quasi-complete,*

(c) *$(H(U), \tau)$ is sequentially complete,*

(d) *if $\{a_m\}_{m\in N^{(N)}}$ is a set of complex numbers and $\left\{\sum_{m\in J} a_m z^m\right\}_{J\subset N^{(N)},\ J \text{ finite}}$ is a τ-Cauchy net, then $\sum_{m\in N^{(N)}} a_m z^m \in H(U)$.*

Moreover, for $\tau = \tau_0$ (resp. τ_ω) the above are all equivalent to

(e) *if $\{a_m\}_{m\in N^{(N)}}$ is a set of complex numbers and $\sum_{m\in N^{(N)}} p(a_m z^m) < \infty$ for every τ_0 (resp. τ_ω) continuous semi-norm p, then $\sum_{m\in N^{(N)}} a_m z^m \in H(U)$.*

Proof. Since $(H(U), \tau)$ has an equicontinuous basis, we may apply the theorem and corollary of [15] to obtain the equivalence of (a), (b), (c) and (d). (d) and (e) are equivalent since $(H(U), \tau)$ has an absolute basis in each of these cases ([8]).

COROLLARY 7. (a) *If τ_1 and $\tau_2 \in \{\tau_0, \tau_{0,b}, \tau_\omega, \tau_{\omega,b}, \tau_\delta\}$ and $\tau_1 \geqslant \tau_2$ on $H(U)$, then $(H(U), \tau_1)$ is complete if $(H(U), \tau_2)$ is complete.*

(b) *If $(H(U), \tau_0)$ is complete, then $\tau_{0,b} = \tau_\delta$.*

(c) *If $(H(U), \tau_\omega)$ is complete, then $\tau_{\omega,b} = \tau_\delta$.*

COROLLARY 8. *If $B = \left\{\sum_{m\in N^{(N)}} a_m^\lambda z^m\right\}_{\lambda\in\Lambda}$ is a bounded subset of $(H(U), \tau)$, $\tau \in \{\tau_0, \tau_\omega, \tau_\delta\}$, then $\tilde{B} = \left\{\sum_{m\in J} a_m^\lambda z^m\right\}_{J\subset N^{(N)},\ J \text{ finite},\ \lambda\in\Lambda}$ is also a bounded subset of $(H(U), \tau)$.*

We now discuss nuclearity of $(H(U), \tau)$. For this we must assume $U = E$, where E is a reflexive Λ-nuclear space. The mapping T in Lemma 3 is an isomorphism, hence it preserves open sets, compact sets, countable covers of E and neighbourhood systems of compact sets. Hence if ${}^tT$: $H(E) \to H(E)$ is the mapping which takes f to $f\circ T$, we immediately obtain the following result.

LEMMA 9. *If E is a complete Λ-nuclear space, then ${}^tT$ is a linear isomorphism from $(H(E), \tau)$ onto $(H(E), \tau)$ for $\tau \in \{\tau_0, \tau_\omega, \tau_{0,b}, \tau_{\omega,b}, \tau_\delta\}$.*

PROPOSITION 10. *If E is a reflexive Λ-nuclear space, then $(H(E), \tau)$ is an Λ-nuclear space for $\tau \in \{\tau_0, \tau_\omega, \tau_{0,b}, \tau_{\omega,b}, \tau_\delta\}$.*

Proof. (1) $\tau = \tau_0$. $(H(E), \tau_0)$ has an absolute basis ([8]) with system of weights $(\|z^m\|_K)_{m\in N^{(N)}}$ K ranging over all compact subsets of E. If $(\delta_n)_n$ is the sequence occurring in the definition of Λ-nuclearity, then δK is also compact and $\sum_{m\in N^{(N)}} (\|z^m\|_K / \|z^m\|_{\delta K}) = \sum_{m\in N^{(N)}} (1/\delta^m) < \infty$. Hence $(H(E), \tau_0)$ is Λ-nuclear.

(2) $\tau = \tau_\omega$. $(H(E), \tau_\omega)$ has an absolute basis ([8]) with systems of weights $\left\{\left(p(z^m)\right)_{m\in N^{(N)}}\right\}$ where p ranges over the τ_ω continuous semi-norms on $H(E)$. If p is a τ-continuous semi-norm ported by K, then it is easily seen that the weight $\left\{\left(\delta^m p(z^m)\right)_{m\in N^{(N)}}\right\}$ is ported by the compact set δK. Hence $(H(E), \tau_\omega)$ is also Λ-nuclear.

(3) If $\tau \in \{\tau_{0,b}, \tau_{\omega,b}, \tau_\delta\}$, then $(H(E), \tau)$ is a bornological space and has an equicontinuous basis. Let p denote a τ-continuous semi-norm on $H(E)$ and let $B = \left\{\sum_{m\in N^{(N)}} a_m^\lambda z^m\right\}_{\lambda\in\Lambda}$ denote a τ-bounded subset of $H(E)$. By Corollary 8 and Lemma 9

$$\delta\tilde{B} = \{\delta^m a_m^\lambda z^m\}_{\substack{\lambda\in\Lambda\\ m\in N^{(N)}}}$$

and

$$\delta^2\tilde{B} = \{(\delta^2)^m a_m^\lambda z^m\}_{\substack{\lambda\in\Lambda\\ m\in N^{(N)}}}$$

are also τ-bounded. Hence

$$\sum_{m\in N^{(N)}} \sup_\lambda p(a_m^\lambda z^m) \leqslant M\cdot \sum_{m\in N^{(N)}} (1/\delta^m) < \infty,$$

where $M = \sup_{f\in\delta\tilde{B}} p(f)$.

Since $p\left(\sum_{m\in N^{(N)}} a_m z^m\right) \leqslant \sum_{m\in N^{(N)}} p(a_m z^m) = p'\left(\sum_{m\in N^{(N)}} a_m z^m\right)$ and p' is bounded on τ-bounded sets, it follows that $(H(E), \tau)$ has an absolute basis. Moreover

$$\sum_{m\in N^{(N)}} \delta^m \sup_\lambda p(a_m^\lambda z^m) \leqslant C'\cdot \sum_{m\in N^{(N)}} (1/\delta^m) \leqslant \infty$$

where

$$C' = \sup_{f\in\delta^2\tilde{B}} p(f).$$

and

$$p''\left(\sum_{m\in N^{(N)}} a_m z^m\right) = \sum_{m\in N^{(N)}} \delta^m p(a_m z^m)$$

is a τ-continuous semi-norm on $H(E)$. Since

$$\sum_{m\in N^{(N)}} p'(z^m)/p''(z^m) = \sum_{m\in N^{(N)}} (1/\delta^m) < \infty,$$

it follows that $(H(E), \tau)$ is an Λ-nuclear space.

COROLLARY 11. *If E is a reflexive Λ-nuclear space, then $(H(E), \tau_\delta)$ is a reflexive Λ-nuclear space if and only if it is complete.*

The above proposition gives certain information about $(H(E), \tau)'_\beta$. For example if E is a reflexive Λ-nuclear space, then $(H(E), \tau)'_\beta$ is nuclear and has an absolute basis.

We now look at $H(U)$, U an open polydisc, where E is not necessarily Λ-nuclear.

PROPOSITION 12. *Let U denote an open polydisc in a hereditary Lindelöf fully nuclear space with a basis $E \approx \Lambda(P)$. Then the monomials form an absolute basis for $(H(U), \tau_\delta)$ if $(H(U), \tau_\delta)$ is complete.*

Proof. It suffices, since $(H(U), \tau_\delta)$ is barrelled, to show $\sum_{n\in N^{(N)}} p(a_m z^m) < \infty$ for every τ_δ continuous semi-norm p and every $\sum_{m\in N^{(N)}} a_m z^m \in H(U)$. Let $\mathscr{V} = (V)$ denote the set of all open subsets of U such that $\sum_{m\in N^{(N)}} \|a_m z^m\|_V < \infty$. $\mathscr{V}$ is closed under finite unions and forms an open cover of U. Hence it contains an increasing countable subcover, $(V_n)_{n=1}^{\infty}$, of U.

If p is τ_δ continuous, then there exists $c > 0$ and N a positive integer such that

$$p(f) \leqslant c\|f\|_{V_N} \quad \text{all } f \in H(U).$$

Hence $\sum_{m\in N^{(N)}} p(a_m z^m) \leqslant c \cdot \sum_{m\in N^{(N)}} \|a_m z^m\|_{V_N} < \infty$. This completes the proof.

We now consider linear functionals on $(H(U), \tau)$. An element of $(H(U), \tau)'$ is called an *analytic functional*. We shall always assume that U is an open polydisc in a fully nuclear space with a basis. In this case, the monomials form an unconditional equicontinuous basis for $(H(U), \tau)$ and an element T of $(H(U), \tau)'_\beta$ is uniquely determined by the scalars $\{T(z^m)\}_{m\in N^{(N)}}$. We let $b_m = T(z^m)$ and we have

$$T\left(\sum_{m\in N^{(N)}} a_m z^m\right) = \sum_m a_m b_m .$$

We identify T with a function, $\tilde{T}$, defined on some subset of E' by letting

$$\tilde{T}(w) = \sum_{m\in N^{(N)}} b_m w^m; \qquad w \in E'.$$

It is easily seen that T uniquely determines $\tilde{T}$. For example (see [8]) $T \in (H(U), \tau_0)'$ if and only if $\tilde{T} \in H(U^M)$ and $T \in (H(U), \tau_\omega)'$ if and only if $\tilde{T} \in H_{HY}(U^M)$. Since the basis is unconditional, it follows that

$$\sum_{m\in N^{(N)}} |a_m b_m| < \infty$$

if $\sum_{m\in N^{(N)}} a_m z^m \in H(U)$ and $\{b_m\}_{m\in N^{(N)}}$ is τ-continuous on $H(U)$.

The converse is true if $\tau = \tau_\delta$ since $(H(U), \tau_\delta)$ is barrelled, i.e. we have the following result.

PROPOSITION 13. *$\{b_m\}_{m\in N^{(N)}} \in (H(U), \tau_\delta)'$ if and only if $\sum_{m\in N^{(N)}} |a_m b_m| < \infty$ for each $\sum_m a_m z^m \in H(U)$.*

PROPOSITION 14. *Let U denote an open polydisc in a fully nuclear space with a basis $E \approx \Lambda(P)$. Then the monomials form an absolute basis for $(H(U), \tau)'_\beta$ in the following cases:*

(1) *$\tau = \tau_0$ or $\tau_{0,b}$,*

(2) *$\tau = \tau_\omega$ or $\tau_{\omega,b}$ if E is Λ-nuclear.*

Proof. For each τ we consider there exists $\tilde{\tau}$ such that the monomials form an absolute basis for the nuclear space $(H(U), \tilde{\tau})$ and τ and $\tilde{\tau}$ have the same bounded sets. Hence if $B = \{\sum_{m\in N^{(N)}} a_m^\lambda z^m\}_{\lambda\in\Lambda}$ is a τ-bounded subset of $H(U)$ and p is a $\tilde{\tau}$ continuous semi-norm,

$$\sum_{m\in N^{(N)}} \sup_\lambda p(a_m^\lambda z^m) < \infty.$$

Thus

$$B' = \left\{\sum_J e^{i\theta_{J,m}} a_m^{\lambda_{J,m}} z^m\right\}_{J,\theta_{J,m},\lambda_{J,m}}$$

is also τ-bounded, where J ranges over the finite subsets of $N^{(N)}$, $\theta_{J,m}$ ranges over $[0, 2\pi]$ and $\lambda_{J,m}$ ranges over Λ. If $\{b_m\}_{m\in N^{(N)}}$ is bounded on the τ-bounded subsets of $H(U)$, then

$$\|\{b_m\}_{m\in N^{(N)}}\|_{B'} = \sum_{m\in N^{(N)}} \sup_{\lambda\in\Lambda} |a_m^\lambda b_m| < \infty.$$

Thus, for any finite subset J of $N^{(N)}$,

$$\|\{b_m\}_{N^{(N)}\setminus J}\|_B = \sup_{\lambda\in\Lambda} \Big|\sum_{m\in N^{(N)}\setminus J} a_m^\lambda b_m\Big| \leqslant \sum_{m\in N^{(N)}\setminus J} \sup_\lambda |a_m^\lambda b_m| \to 0 \quad \text{as} \quad J \to \infty.$$

Hence the monomials form an absolute basis in $(H(U), \tau)'_\beta$. This completes the proof.

COROLLARY 15. *Let U denote an open polydisc in a fully nuclear space with a basis (resp. a reflexive Λ-nuclear space). Then the following are equivalent:*

(a) *τ_0 and $\tau_{0,b}$ (resp. τ_ω and $\tau_{\omega,b}$) are compatible topologies;*

(b) *$(H(U), \tau_{0,b})'_\beta = H(U^M)$ (resp. $(H(U), \tau_{\omega,b})' = H_{HY}(U^M)$);*

(c) *$(H(U), \tau_0)'_\beta$ (resp. $(H(U), \tau_\omega)'_\beta$) is complete or quasi-complete or sequentially complete.*

Proof. The three conditions in (c) are equivalent by Proposition 14 and the result of [15]. By Theorems 18 and 20 of [8] (a) and (b) are equivalent. Since the strong dual of a bornological space is complete (Proposition 6, p. 223 of [14]), (b) and (c) are equivalent by Proposition 14. This completes the proof.

We now look at various properties of the "function" spaces on subsets of E'_β defined by $(H(U), \tau)'$.

DEFINITION 16 (see [2], [3], [17] and [23]). E is a *regular inductive limit of* $(E_\alpha)_{\alpha\in A}$ if $E = \varinjlim_{\alpha\in A} E_\alpha$ and each bounded subset of E is contained and bounded in some E_α.

The following result has been found independently by P. J. Boland.

PROPOSITION 17. *Let U denote an open polydisc in a fully nuclear space with a basis.*

(a) *$(H(U), \tau_0)$ is infrabarrelled if and only if $H(U^M) = \lim\limits_{\overrightarrow{V \supset U^M}} (H^\infty(V), \| \|_V)$ is a regular inductive limit.*

(b) *$(H(U), \tau_0)$ is reflexive if and only if it is complete and $H(U^M) = \lim\limits_{\overrightarrow{V \supset U^M}} (H^\infty(V), \| \|_V)$ is a regular inductive limit.*

(c) *If the τ_ω-bounded subsets of $H(U)$ are equibounded, then the following are equivalent*

(1) *$\tau_\omega = \tau_\delta$ on $H(U)$,*

(2) *τ_ω is infrabarrelled,*

(3) *$H_{HY}(U^M) = \lim\limits_{\overrightarrow{V \supset U^M}} (H_{HY}(V), \tau_0)$ is a regular inductive limit.*

Proof. (a) By Theorem 20 of [8]

$$(H(U), \tau_0)'_\beta = H(U^M) = \lim_{\overrightarrow{V \supset U^M}} (H^\infty(V), \| \|_V).$$

$(H(U), \tau_0)$ is infrabarrelled if and only if the bounded subsets of $(H(U), \tau_0)'$ are contained in the polars of neighbourhoods of zero in $(H(U), \tau_0)$. Since the sets of the form $\{f \in H(V);\ \|f\|_V \leqslant C\}$ form a fundamental system of equicontinuous subsets of $(H(U), \tau_0)'_\beta$ (see the proof of Theorem 20 of [8]) as V ranges over the neighbourhoods of U^M and C ranges over the positive numbers, we have completed the proof.

(b) follows immediately from (a) and the fact that $(H(U), \tau_0)$ is semi-reflexive if and only if it is complete.

(c) If the bounded subsets of $(H(U), \tau_\omega)$ are equibounded, then by Theorem 23 of [8]

$$(H(U), \tau_\omega)'_\beta = H_{HY}(U^M) = \lim_{\overrightarrow{V \supset U^M}} (H_{HY}(V), \tau_0).$$

Moreover, $(H(U), \tau_\omega)$ is complete. By the results of [20] $\tau_{\omega,b}$ is the infrabarrelled topology associated with τ_ω and hence $\tau_\omega = \tau_\delta$ if and only if $(H(U), \tau_\omega)$ is infrabarrelled.

Since the equicontinuous subsets of $(H(U), \tau_\omega)'_\beta$ are exactly the subsets of $H_{HY}(U^M)$ which are defined and τ_0-bounded on some neighbourhood of U^M, this completes the proof.

COROLLARY 18. *Let E denote a fully nuclear space with a basis. Then $(H(U), \tau_0)$ is infrabarrelled for every open polydisc U in E if and only if $(H(E), \tau_0)$ is infrabarrelled.*

Proof. The proof is immediate since U^M is a convex balanced compact subset of E'_β and we may apply Proposition 17 (a).

Remarks. Proposition 17 allows us to relate certain known results hitherto not seen as being related. We confine ourselves to a few examples.

(a) If E is a Fréchet space and K is a compact subset of E, then $\lim\limits_{\overrightarrow{V \supset K}} (H^\infty(V), \| \|_V)$ is a regular inductive limit (see [17] for further details). Hence by Proposition 17 (b), since metrizable spaces are k-spaces, $(H(U), \tau_0)$ is barrelled if U is an open polydisc in a $\mathcal{DFN}$ (strong dual of a Fréchet nuclear space) with a basis. Hence $\tau_0 = \tau_\delta$ on $H(U)$. This result is known and in fact has been extended by other methods to $\mathcal{DFS}$ spaces (strong duals of Fréchet Schwartz spaces) ([1]) and to $\mathcal{DFM}$ spaces (strong duals of Fréchet Montel spaces) ([12]).

(b) By [10], p. 45, if E is a Fréchet space which does not admit a continuous norm, then $\tau_0 \neq \tau_\delta$ on $H(E)$. In particular, $(H(\prod_{n=1}^\infty C), \tau_0) \neq (H(\prod_{n=1}^\infty C), \tau_\delta)$. A result of R. Aron ([23], p. 45–46) states that $\lim\limits_{\substack{\overrightarrow{V \ni 0} \\ V \text{ open}}} (H^\infty(V), \| \|_V)$ is not a regular inductive limit when $E = \sum_{n=1}^\infty C$.

Since $(H(\prod_{n=1}^\infty C), \tau_0)$ is complete, Proposition 17 (b) shows that these two results are equivalent. Proposition 17 implies that if E is a fully nuclear space with a basis and E'_β does not admit a continuous norm, then the germs at 0 in E do not form a regular inductive limit.

(c) Results similar to Proposition 17 can also be proved for entire functions on a fully nuclear space (not necessarily having a basis) by using the results of the final section of [8].

In [8] we identified $(H(U), \tau_0)'$ and $(H(U), \tau_\omega)'$ with germs about U^M. We now give a characterization of $(H(E), \tau_\delta)'$ in terms of germs.

PROPOSITION 19. *Let E denote a reflexive A-nuclear space and suppose $(H(E), \tau_0)$ is complete. Then $\{b_m\}_{m \in N^{(N)}}$ defines an element of $(H(E), \tau_\delta)'$ if and only if for each infinite subset J of $N^{(N)}$ there exists an infinite subset J' of J such that $\sum_{m \in J'} b_m \omega^m$ defines a holomorphic function in some neighbourhood of zero in E'_β.*

Proof. First suppose that $T = \{b_m\}_{m \in N^{(N)}}$ is τ_δ continuous. Let $J \subset N^{(N)}$ be infinite. If $b_m = 0$ for all m belonging to an infinite subset of J, then our condition is trivially satisfied. Hence we may suppose $b_m \neq 0$ for all $m \in J$.

We claim that there exists a neighbourhood of zero in E'_β, V, such that

$$\sum_{m \in J} (1/\|b_m \omega^m\|_V) = \infty.$$

Otherwise let $f_m(z) = z^m/b_m$ all $m \in J$ and let $f = \sum_{m \in J} f_m$. Then if K is a compact subset of E, we have

$$\sum_{m \in J} \|f_m\|_K = \sum_{m \in J} \|z^m\|_K / |b_m| = \sum_{m \in J} (1/\|b_m \omega^m\|_{K^M}) < \infty$$

and

$$\sum_{m\in J} f_m \in H(E).$$

This contradicts the fact that $T(f_m) = 1$ all m.

Let V denote a neighbourhood of zero such that

$$\sum_{m\in J} (1/\|b_m \omega^m\|_V) = \infty.$$

Let $(\delta_n)_n$ denote a sequence of positive real numbers such that $\sum_n (1/\delta_n) < \infty$ and $V' = (1/\delta^2) V = \{(\omega_n/\delta_n^2)_n \in E'_\beta;\ (\omega_n)_n \in V\}$ is a neighbourhood of zero in E'_β. Since $\sum_{m\in J}(1/\delta^m) < \infty$, it follows that $1/\|b_m \omega^m\|_V \geq 1/\delta^m$ for an infinite number of m in J say J'. V' is a neighbourhood of zero and

$$\sum_{m\in J'} \|b_m \omega^m\|_{V'} \leq \sum_{J'} (\delta^m/(\delta^2)^m) = \sum_{m\in J'} (1/\delta^m) < \infty.$$

This completes the proof in one direction. Now suppose $T = \{b_m\}_{m\in N^{(N)}}$ is not τ_δ continuous. Then there exists $f = \sum_{m\in N^{(N)}} a_m z^m \in H(E)$ such that $\sum_{m\in N^{(N)}} |a_m b_m| = \infty$.

Since E is A-nuclear, there exists a sequence of real numbers $(\delta_n)_n$, $\delta_n > 1$ and $\sum_n (1/\delta_n) < \infty$, such that δV is a neighbourhood of zero if and only if V is a neighbourhood of zero.

Since $\sum_m (1/\delta^m) < \infty$, it follows that $|a_m b_m| \geq 1/\delta^m$ for all m belonging to some infinite subset J of $N^{(N)}$. Hence, for any compact subset K of E and $V = K^M$ we have

$$\sum_{m\in J} (1/\|b_m \omega^m\|_{\delta V}) \leq \sum_{m\in J} (1/\delta^m \|b_m \omega^m\|_V)$$
$$\leq \sum_{m\in J} (|a_m|/\|\omega^m\|_V) = \sum_{m\in J} \|a_m z^m\|_K < \infty.$$

Hence $\{b_m\}_{m\in J'}$ is not τ_δ continuous for any infinite subset J' of J and $\sum_{m\in J'} b_m \omega^m$ does not define a holomorphic function in any neighbourhood of zero in E'_β for any infinite subset J' of J. This completes the proof.

Remark. Proposition 19 is equivalent to the following: $\{b_m\}_{m\in N^{(N)}}$ is τ_δ continuous if and only if each infinite subset J of $N^{(N)}$ contains an infinite subset J' such that $\{b_m\}_{m\in J'}$ is τ_0-continuous.

COROLLARY 20. *Let E denote a reflexive A-nuclear space and suppose $(H(E), \tau_0)$ is complete. Then $\{b_m\}_{m\in N^{(N)}}$ defines an element of $(H(E), \tau_\delta)'$ if and only if for each infinite subset J of $N^{(N)}$ there exists an infinite subset J' of J such that $\{b_m\}_{m\in J'}$ defines an element of $(H(E), \tau_\delta)'$.*

We shall use Corollary 20 to give an example of a Fréchet nuclear space E which admits a continuous norm but $(H(E), \tau_0)$ is not bornological. Before proceeding we introduce some notation. Let $E \approx \Lambda(P)$ denote a Fréchet nuclear space with a basis, where $P = (\omega^m)_m$ and $\omega^m = (\omega_n^m)_n$ for all m and n. Let $V_m = \{(z_n)_n \in E;\ \sup_n |z_n \omega_n^m| < 1\}$ and let $\|\ \|_m$ denote the corresponding norm.

Now suppose $(n_i)_{i=1}^\infty$ is a strictly increasing sequence of integers with $n_1 = 1$. Let

$$U_{(n_1)} = \{(z_n)_n \in E;\ \sup_n |z_n \omega_n^1| \leq 1\},$$

$$U_{(n_1,\ldots,n_k)} = \{(z_n)_n \in E;\ |z_n \omega_n^m| \leq 1 \text{ for } n_m \leq n < n_{m+1},\quad m = 1, \ldots, k-1 \text{ and } |z_n \omega_n^k| \leq 1 \text{ for } n \geq n_k\}$$

and

$$U_{(n_i)_{i=1}^\infty} = \{(z_n)_n \in E;\ |z_n \omega_n^m| \leq 1 \text{ for } n_m \leq n < n_{m+1},\ m = 1, 2, \ldots\}.$$

It is immediate that $U_{(n_i)_{i=1}^\infty}$ is a compact subset of E and if V is a neighbourhood of $U_{(n_i)_{i=1}^\infty}$, then there exists a positive integer k such that $V \supset U_{(n_1,\ldots,n_k)}$. Moreover, if K is a compact subset of E, then K is contained in a compact polydisc

$$\tilde{K} = \{(z_n)_n \in E;\ \sup_n |z_n \alpha_n| \leq 1\} \quad \text{and} \quad \alpha_n > 0 \text{ all } n.$$

The sequence $\alpha = (1/\alpha_n)_n \in E$. Let $\delta = \|\alpha\|_1$. Choose $n_2 > 1$ such that $|\omega_n^2/\alpha_n| \leq \delta$ all $n \geq n_2$ and by induction choose $n_{m+1} > n_m$ such that

$$|\omega_n^{m+1}/\alpha_n| < \delta \quad \text{all } n \geq n_{m+1}.$$

Let $n_1 = 1$. If $z = (z_n)_n \in K$ and $n_m \leq n < n_{m+1}$, $m = 1, 2, \ldots$, then $|z_n \omega_n^m| \leq |\omega_n^m|/|\alpha_n| \leq \delta$ and hence $K \subset \delta U_{(n_m)_{m=1}^\infty}$ and we have constructed a fundamental system of compact sets in E.

EXAMPLE 21 (see [21], p. 99). If $(\alpha_r)_{r=1}^\infty$ is an increasing sequence of positive real numbers such that $\sum_r q^{\alpha_r} < \infty$ for all q, $0 < q < 1$, then the sequence space $\Lambda(P)$, $P = (\omega_p)_{0<p<p_0}$ and $p_0 < +\infty$, $\omega_p = (p^{\alpha_n})_n$ is a Fréchet nuclear space with a basis and is called a *power series space of finite type*. A simple example is $H(D)$ — the space of holomorphic functions on the open unit disc in C. We note that $\alpha_R \to \infty$ as $R \to \infty$. We now show that $(H(\Lambda(P)), \tau_0)$ is not bornological.

Without loss of generality we may suppose $p_0 = 1$ and since we may choose subsequences whenever necessary in our construction, we may also assume $\alpha_n > \alpha_{n-1} + n$ for all $n \geq 2$.

Let $[x]$ denote the integer part of $x+1$. For any integers $m, n \geq 2$, let

$$(m, n)' = ([m\alpha_{[m\alpha_n]}], 0, \ldots, \underset{\substack{\uparrow \\ [m\alpha_n]\text{position}}}{[m\alpha_n]}, \ldots).$$

Let $W_n = \{(i, j)';\ j = n\}$ and $W = \bigcup_n W_n$. Since $\alpha_n > \alpha_{n-1} + n$, we have $(m, n)' = (m_1, n_1)'$ if and only if $m = m_1$ and $n = n_1$. If $j \in N^{(N)}$, we let

$$b_j = \begin{cases} a_n^{ma[ma_n]} & \text{if} \quad j = (m, n)', \\ 0 & \text{otherwise.} \end{cases}$$

We show that $\{b_j\}_{j\in N^{(N)}}$ is τ_δ continuous but not τ_0 continuous on $H(\Lambda(P))$.

By Propositions 13 and 19 and Corollary 20 it suffices to show that for every infinite subset J of $N^{(N)}$ there exists an infinite subset J' of J such that $\sum_{m\in J'} |a_m b_m| < \infty$ for every $\sum_{m\in N^{(N)}} a_m z^m \in H(E)$. Obviously we need only consider J for which $J\cap W$ is infinite. There are two possibilities and we consider each one in turn.

(1) $J \cap W_n$ is infinite for some integer n. Since $(C, 1, \ldots, 1, 1, \ldots) \in \Lambda(P)$ for every positive number C, it follows that if $\sum_{m\in N^{(N)}} a_m z^m \in H(E)$, then

$$\sum_{m\in N} |a_{(m,n)'}| \cdot C^{[ma[ma_n]]} < \infty.$$

Hence $|a_{(m,n)'}|^{1/[ma[ma_n]]} \to 0$ as $m \to \infty$. Thus given ε, $0 < \varepsilon < 1$, we can choose m_0 such that $|a_{(m,n)'}| \leqslant (\varepsilon/a_n)^{[ma[ma_n]]}$ for all $m \geqslant m_0$. Hence

$$\begin{aligned} \sum_{m\geqslant m_0} |a_{(m,n)'} b_{(m,n)'}| &= \sum_{m\geqslant m_0}^{\infty} |a_{(m,n)'} a_n^{ma[ma_n]}| \\ &\leqslant \sum_{m\geqslant m_0} (\varepsilon/a_n)^{[ma[ma_n]]} a_n^{ma[ma_m]} \\ &\leqslant \sum_{m\geqslant m_0} \varepsilon^{[ma[ma_n]]} \leqslant \sum_{m\geqslant m_0} \varepsilon^m < \infty. \end{aligned}$$

(2) If $J\cap W_n$ is finite for all n, then there exists an increasing sequence of integers $(n_i)_{i=1}^\infty$, and another sequence of integers $(m_i)_{i=1}^\infty$ such that $(m_i, n_i)' \in J\cap W$. Let $(\beta_n)_n$ denote the sequence, where

$$\beta_1 = 1,$$

$$\beta_{[m_i a_{n_i}]} = a_{n_i}^{a[m_i a_{n_i}]/a_{n_i}}$$

and

$$\beta_n = 0 \quad \text{for all other } n.$$

$(\beta_n)_n \in \Lambda(P)$ since

$$\left(a_{n_i}^{a[m_i a_{n_i}]/a_{n_i}}\right)^{1/a[m_i a_{n_i}]} = a_{n_i}^{1/a_{n_i}} \to 1 \quad \text{as} \quad n_i \to \infty$$

and hence

$$\sum_i a_{n_i}^{a[m_i a_{n_i}]/a_{n_i}} \varrho^{a[m_i a_{n_i}]} < \infty \quad \text{for} \quad 0 < \varrho < 1.$$

Hence if $\sum_{m\in N^{(N)}} a_m z^m \in H(\Lambda(P))$, then

$$\sum_i |a_{(m_i,n_i)'} (a_{n_i}^{a[m_i a_{n_i}]/a_{n_i}})^{[m_i a_{n_i}]}| < \infty$$

and

$$\begin{aligned} \sum_i |a_{(m_i,n_i)'} b_{(m_i,n_i)'}| &= \sum_i |a_{(m_i n_i)'} a_{n_i}^{m_i a[m_i a_{n_i}]}| \\ &\leqslant \sum_i |a_{(m_i,n_i)'} (a_{n_i}^{a[m_i a_{n_i}]/a_{n_i}})^{[m_i a_{n_i}]}| < \infty. \end{aligned}$$

Now suppose $\{b_j\}_{j\in N^{(N)}}$ is τ_0 continuous. Then we can find $C > 0$, $0 < \delta < \infty$ and $(n_i)_{i=1}^\infty$, an increasing sequence of positive integers with $n_1 = 1$ such that

$$|b_m| \leqslant C \|z^m\|_{\delta U_{(n_i)_{i=1}^\infty}} \quad \text{for all } m \in N^{(N)}.$$

In particular, for each integer n we would have

$$a_n^{ma[ma_n]} \leqslant C\cdot \delta^{[ma[ma_n]]+[ma_n]} (1+1/a_n)^{a[ma_n][ma_n]}$$

for all m sufficiently large (since $\left((1+(1/a_n))^{-a_m}\right)_{m=1}^\infty$ forms a fundamental system of weights as n ranges over the positive integers). Since for each fixed n

$$ma_{[ma_n]}/[ma_{[ma_n]}] \to 1 \quad \text{as} \quad m\to\infty,$$

$$[ma_{[ma_n]}] \to \infty \quad \text{as} \quad m \to \infty$$

and

$$[ma_n]/[ma_{[ma_n]}] \to 0 \quad \text{as} \quad m\to\infty$$

and we have $(1+1/a_n)^{a_n} \leqslant 3$ for all $n \geqslant 3$, this would imply $a_n \leqslant \delta 3$ for all $n \geqslant 3$ and hence $\delta = \infty$. This shows that $\{b_j\}_{j\in N^{(N)}}$ is not τ_0 continuous on $H(\Lambda(P))$ and completes the proof.

We now show that $\tau_0 = \tau_\delta$ on $H(U)$ when U is an open polydisc in a certain class of Fréchet nuclear spaces with a basis.

DEFINITION 22. E is a *B-nuclear space* if E is a Fréchet nuclear space with a basis and E is isomorphic to $\Lambda(P)$, where P has the following properties:

(1) $P = (\omega^m)_{m=1}^\infty$, $\omega^m = (\omega_n^m)_{n=1}^\infty$ and $\omega_n^m > 0$ for all m and n (hence E admits a continuous norm),

(2) if $\beta_n^m = \omega_n^{m+1}/\omega_n^m$, then $\left(\omega_n^m(\beta_n^m)^p\right)_{n=1}^\infty \in [P]$ *for every integer* p. (We may, and shall assume $\beta_n^m > 1$ for all m and n.)

EXAMPLE 23. (a) If $\omega_n^m = (\omega_n)^m$ for all m and n, where $\omega_n > 1$ and $\sum_n (1/\omega_n) < \infty$, then $\Lambda(P)$ is a nuclear Fréchet space with a basis.

$$\omega_n^{m+1}/\omega_n^m = \beta_n^m = (\omega_n)^{m+1}/(\omega_n)^m = \omega_n$$

and so

$$\omega_n^m(\beta_n^m)^p = (\omega_n)^m\cdot(\omega_n)^p = (\omega_n)^{m+p}$$

and thus

$$\left(\omega^m(\beta_n^m)^p\right)_n \in P.$$

Hence $\Lambda(P)$ is a B-nuclear space.

(b) If we let $\omega_n = (n+1)^2$ all n, then $\omega_n^m = (n+1)^{2m}$ and we obtain S, the space of rapidly decreasing sequences, which is isomorphic to $\mathscr{S}$, the space of rapidly decreasing functions.

(c) If $(a_n)_n$ is an increasing sequence of positive real numbers such that $\sum_n q^{a_n} < \infty$ for some q, $0 < q < 1$, we let $\omega_n = 2^{a_n}$ for all n.

We have $\omega_n^m = (2^m)^{a_n}$, $\beta_n^m = \omega_n^{m+1}/\omega_n^m = 2^{a_n}$ and $\omega_n^m(\beta_n^m)^p = (2^{m+p})^{a_n} = \omega_n^{m+p}$. Hence, the nuclear power series spaces of infinite type are B-nuclear. In particular, $H(C)$ is B-nuclear (see [21], p. 99 for further details).

(d) Other B-nuclear spaces with weights of even more rapid growth than those of the power series spaces can easily be constructed, e.g. let $\omega_n^m = 2^{n^m}$ for all m and n. Then $\beta_n^m = 2^{n^{m+1}}/2^{n^m} = 2^{n^m}\cdot(n-1)$ and

$$\omega_n^m(\beta_n^m)^p = 2^{n^m}\left(2^{n^m}\cdot(n-1)p\right) \leqslant C\cdot 2^{n^{m+2}}$$

for a given fixed p and some $C > 0$ and all n sufficiently large. Hence $\Lambda(P)$ is a B-nuclear space and it is easily seen that it is not a power series sequence space.

(e) $H(D)$, D is the open unit disc, is a reflexive A-nuclear space but it is not a B-nuclear space. The usual sequence space representation of $H(D)$ does not satisfy property (2) of Definition 22. The fact that it is not B-nuclear follows from Example 21 and the next proposition.

PROPOSITION 24. *If U is an open polydisc in a B-nuclear space, then $(H(U), \tau_0)$ is a bornological (and hence a reflexive nuclear) space.*

Proof. By Corollary 18, it suffices to consider $U = E$ and since $\tau_0 = \tau_\omega$ on $H(E)$, ([8], Proposition 22), we need only show that any Banach valued linear functional T on $H(E)$ which is bounded on the τ_0-bounded subsets of $H(E)$ is ported by a compact subset of E.

Now suppose for some non-negative integer k there exists $\delta > 0$, $(n_1, \ldots, n_k)$ an increasing sequence of integers, $n_1 = 1$, and $C(k) > 0$ such that

$$\|T(z^m)\| \leqslant C(k)\,\|z^m\|_{\delta U_{(n_1,\ldots,n_k)}} \quad \text{for all } m \in N^{(N)}.$$

Then we claim if $\delta' > \delta$, there exists a positive integer j and $C(k+1) > 0$ ($C(k+1)$ will depend on j and δ') such that

$$\|T(z^m)\| \leqslant C(k+1)\,\|z^m\|_{\delta' U_{(n_1,\ldots,n_k,j+n_k)}} \quad \text{for all } m \in N^{(N)}.$$

If not, then we can choose for each positive integer j, $m_j \in N^{(N)}$ such that

$$\|T(z^{m_j})\| \geqslant j\,\|z^{m_j}\|_{\delta' U_{(n_1,\ldots,n_k,j+n_k)}}.$$

We first show that $|m_j| \to \infty$ as $j \to \infty$. If not, there exists a subsequence, $(j_n)_n$, such that $|m_{j_n}| = \alpha$ for all n. Since $U_{(n_1,\ldots,n_k,j+n_k)} \supset V_{k+2}$ for all j, it follows that

$$\{z^{m_{j_n}}/\|z^{m_{j_n}}\|_{\delta' U_{(n_1,\ldots,n_k,j_n+n_k)}}\}_{j=1}^{\infty}$$

is uniformly bounded on a fixed neighbourhood of zero and hence is τ_0-bounded (note that we use the existence of a continuous norm here).

Since

$$\|T(z^{m_{j_n}})\|/\|z^{m_{j_n}}\|_{\delta' U_{(n_1,\ldots,n_k,j_n+n_k)}} \geqslant j_n,$$

this contradicts the fact that T is bounded on the τ_0 bounded subsets of $H(E)$. Thus we will assume that $|m_j| \to \infty$ as $j \to \infty$. For each j we let $m_j = (R_j, S_j)$, where R_j are the first $j+n_k-1$ coordinates and S_j are the remaining coordinates. We have $z^{m_j} = z^{R_j} z^{S_j}$.

We now consider two possibilities:

$$(*) \qquad \lim_{j\to\infty} |(\beta_n^k)^{S_j}|^{1/|m_j|} = 1,$$

$$(**) \qquad \limsup_{j\to\infty} |(\beta_n^k)^{S_j}|^{1/|m_j|} = \omega > 1.$$

$(\beta_n^k)^{S_j}$ is the value of the monomial z^{S_j} at the point $(\beta_n^k)_n$. Since $\beta_n^k > 1$ all n and k, it follows that $(*)$ and $(**)$ cover all possibilities.

We first suppose that $(*)$ is satisfied. Then

$$\begin{aligned}
\|T(z^{m_j})\|/\|z^{m_j}\|_{U_{(n_1,\ldots,n_k)}} &\geqslant j(\delta')^{|m_j|}\left(\|z^{m_j}\|_{U_{(n_1,\ldots,n_k,j+n_k)}}/\|z^{m_j}\|_{U_{(n_1,\ldots,n_k)}}\right)\\
&= j(\delta')^{|m_j|}\left(\|z^{S_j}\|_{U_{(n_1,\ldots,n_k,j+n_k)}}/\|z^{S_j}\|_{U_{(n_1,\ldots,n_k)}}\right)\\
&= j(\delta')^{|m_j|}\left(\|z^{S_j}\|_{V_{k+1}}/\|z^{S_j}\|_{V_k}\right)\\
&= j(\delta')^{|m_j|}\left(1/(\omega_n^{k+1})^{S_j}\right)(\omega_n^k)^{S_j} = j(\delta')^{|m_j|}/(\beta_n^k)^{S_j}.
\end{aligned}$$

Hence

$$\liminf_{j\to\infty}\left(\|T(z^{m_j})\|/\|z^{m_j}\|_{U_{(n_1,\ldots,n_k)}}\right)^{1/|m_j|} \geqslant \liminf_{j\to\infty} j^{1/|m_j|}\cdot\delta'/\lim_{j\to\infty}|(\beta_n^k)^{S_j}|^{1/|m_j|} \geqslant \delta'.$$

Since

$$\|T(z^m)\|/\|z^m\|_{U_{(n_1,\ldots,n_k)}} \leqslant C(k)\,\delta^{|m|} \quad \text{for all } m \in N^{(N)},$$

we must have

$$\limsup_{j\to\infty}\left(\|T(z^{m_j})\|/\|z^{m_j}\|_{U_{(n_1,\ldots,n_k)}}\right)^{1/|m_j|} \leqslant \delta.$$

This is a contradiction and so $(*)$ cannot hold.

We now suppose $(**)$ holds. In fact, we may suppose without loss of generality that $\lim_{j\to\infty}[(\beta_n^k)^{S_j}]^{1/|m_j|} = \omega > 1$. Let

$$f(z) = \sum_{j=1}^{\infty}\left(z^{m_j}/\|z^{m_j}\|_{U_{(n_1,\ldots,n_k,j+n_k)}}\right).$$

Since each monomial is continuous and E is a Fréchet space, f will be an entire function if it converges at all points of E. Let $\alpha = (\alpha_n)_n \in E$. We have $(0, 0, \ldots, \alpha_n, 0, \ldots) \in V_{k+1}$ for all n sufficiently large, say, $n > l$. For each j let $m_j^1, m_j^2, \ldots, m_j^l$ be the first l coordinates of $m_j \in N^{(N)}$, and let $C_i = \|(0, \ldots, \alpha_i, 0, \ldots)\|_{k+1}$ for $i = 1, \ldots, l$. We now have

$$|\alpha^{m_j}/\|z^{m_j}\|_{U_{(n_1,\ldots,n_k,j+n_k)}}| \leqslant C_1^{m_j^1} C_2^{m_j^2} \ldots C_j^{m_j^l} F(\alpha, m_j),$$

where $F(\alpha, m_j) = |\alpha^{S_j}|/\|z^{S_j}\|_{U(n_1,\ldots,n_k,j+n_k)}$ (the terms between l and $j+n_k$ are all less than one, in fact all the terms greater than l are less than one but we need a sharper estimate).

Now given any positive integer p

$$(\omega_n^k(\beta_n^k)^p)_n \in [P].$$

Hence

$$\sum_{n=1}^{\infty} |\alpha_n \omega_n^k(\beta_n^k)^p| < \infty \quad \text{and} \quad |\alpha_n \omega_n^k(\beta_n^k)^p| \leqslant 1 \quad \text{for all } n \geqslant l_1 > l.$$

Hence if $n_k+j > l_1$ (in particular for all j sufficiently large), we have

$$|F(\alpha, m_j)| \leqslant (1/\omega_n^k(\beta_n^k)^p)^{S_j}(\omega_n^{k+1})^{S_j} = (1/(\beta_n^k)^{p-1})^{S_j} = (1/(\beta_n^k)^{S_j})^{p-1}.$$

Hence

$$\limsup_{j\to\infty}(|\alpha^{m_j}|/\|z^{m_j}\|_{U(n_1,\ldots,n_k,j+n_k)})^{1/|m_j|} \leqslant C_1^{\gamma_1}C_2^{\gamma_2}\ldots C_l^{\gamma_l}\cdot(\lim_{j\to\infty}|1/(\beta_n^k)^{S_j}|^{1/|m_j|})^{p-1},$$

where $\gamma_i = \limsup_{j\to\infty}(|m_j^i|/|m_j|)$ if $C_i \geqslant 1$ and $1 \leqslant i \leqslant l$ and $\gamma_i = 0$ if $C_i < 1$, $1 \leqslant i \leqslant l$. Since $m_j^i \leqslant |m_j|$ for all i and j; we have $0 \leqslant \gamma_i \leqslant 1$ for all i. Hence

$$\limsup_{j\to\infty}(|\alpha^{m_j}|/\|z^{m_j}\|_{U(n_1,\ldots,n_k,j+n_k)})^{1/|m_j|} \leqslant C_1^{\gamma_1}\ldots C_l^{\gamma_l}/\omega^{p-1}.$$

Since $\omega > 1$, this limit is zero and hence $f \in H(E)$. Hence

$$\tilde{f} = \sum_{j=1}^{\infty}(1/(\delta'))^{|m_j|}\cdot(z^{m_j}/\|z^{m_j}\|_{U(n_1,\ldots,n_k,j+n_k)}) \in H(E)$$

and

$$\{z^{m_j}/\|z^{m_j}\|_{\delta' U(n_1,\ldots,n_k,j+n_k)}\}_{j=1}^{\infty}$$

is a τ_0-bounded subset of $H(E)$. This contradicts the fact that

$$\|T(z^{m_j})\|/\|z^{m_j}\|_{\delta' U(n_1,\ldots,n_k,j+n_k)} \geqslant j$$

and hence we have proved our claim.

Since T is τ_δ continuous and $(nU_{(1)})_n$ is an open cover of E, there exists $C(1) > 0$ and δ_1 a positive integer such that

$$\|T(f)\| \leqslant C(1)\|f\|_{\delta_1 U_{(1)}} \quad \text{for all } f \in H(E).$$

In particular,

$$\|T(z^m)\| \leqslant C(1)\|z^m\|_{\delta_1 U_{(1)}} \quad \text{for all } m \in N^{(N)}.$$

Let $(\delta_n)_{n=2}^{\infty}$ denote a sequence of positive numbers, $\delta_n > 1$, such that $\prod_{n=1}^{\infty}\delta_n = \theta < \infty$. By the above, we can choose an increasing sequence of positive integers, $(n_k)_{k=1}$ and $(C(k))_{k=1}$ a sequence of positive numbers such that

$$\|T(z^m)\| \leqslant C(k)\|z^m\|_{\delta_1\ldots\delta_k U(n_1,\ldots,n_k)}$$

for all $m \in N^{(N)}$ and all k.

Let $K = U_{(n_i)_{i=1}^{\infty}}$. K is a compact subset of E and if V is any neighbourhood of θK, there exists a positive integer k such that

$$\theta U_{(n_1,\ldots,n_k)} \subset V.$$

Hence for all $m \in N^{(N)}$

$$\|T(z^m)\| \leqslant C(k)\|z^m\|_{\delta_1\ldots\delta_k U(n_1,\ldots,n_k)} \leqslant C(k)\|z^m\|_{\theta U(n_1,\ldots,n_k)} \leqslant C(k)\|z^m\|_V.$$

Hence for any neighbourhood V of 0 in E there exists $C(V) > 0$ such that

$$\|T(z^m)\| \leqslant C(V)\|z^m\|_{\theta K+V} \quad \text{for all } m \in N^{(N)}.$$

Let $\varepsilon = (\varepsilon_n)_{n=1}^{\infty}$ denote a sequence of positive numbers such that $\sum_n(1/\varepsilon_n) < \infty$ and εK is a relatively compact subset of E. If V is any neighbourhood of zero in E, we can find a neighbourhood of zero V' and $(\varepsilon_n')_n$ a sequence of positive real numbers such that $\varepsilon' V' \subset V$ and $\sum_n(1/\varepsilon_n') < \infty$ (see [8]). Let $\varepsilon_n'' = \inf(\varepsilon_n, \varepsilon_n')$; then if $f(z) = \sum_{m\in N^{(N)}} a_m z^m \in H(E)$, we have

$$\begin{aligned}\|T(f)\| &\leqslant \sum_{m\in N^{(N)}}\|T(a_m z^m)\| \quad \text{(Proposition 4)}\\ &\leqslant C(V')\cdot\sum_{m\in N^{(N)}}\|a_m z^m\|_{\theta K+V'}\\ &\leqslant C(V')\cdot\sum_{m\in N^{(N)}}(1/(\varepsilon'')^m)\|f\|_{\varepsilon''\theta K+\varepsilon'' V'}\\ &\leqslant C(V')\cdot\Big(\sum_{m\in N^{(N)}}(1/(\varepsilon'')^m)\Big)\cdot\|f\|_{\varepsilon\theta K+V}.\end{aligned}$$

Since $\varepsilon\theta K$ is compact, this shows that T is ported by a compact set and hence is τ_ω and τ_0 continuous, ([8], Proposition 22). This completes the proof.

COROLLARY 25. *If $\Lambda(P)$ is a nuclear power series space, then the following are equivalent:*

(a) *$\Lambda(P)$ is of infinite type,*

(b) *$\Lambda(P)$ is a B-nuclear space,*

(c) *$(H(\Lambda(P)), \tau_0)$ is a bornological space.*

Proof. (a)⇒(b) by Example 23 (c); (b)⇒(c) by Proposition 24; (c)⇒(a) by Example 21.

COROLLARY 26. *If E is the strong dual of a B-nuclear space, then the germs at 0 in E form a regular inductive limit* (i.e. $\lim_{\substack{V\ni 0\\ V \text{ open}}}(H^\infty(V), \|\ \|_V)$ *is a regular inductive limit*).

COROLLARY 27. *If E is an A-nuclear space and a B-nuclear space, then $(H(E), \tau_0)$ is a reflexive A-nuclear space.*

Proposition 28. *If E and F are both A and B nuclear spaces, then*

$$H'(E\times F) = H'(E)\overline{\otimes} H'(F)$$

where $H(E)$, $H(F)$ and $H(E\times F)$ are given the compact open topology, $H'(E)$, $H'(F)$ and $H'(E\times F)$ are given the strong dual topology and $\overline{\otimes}$ denotes the completed inductive tensor product.

Proof. By Corollary 2.8 of [22]

$$H(E\times F) = H(E)\hat{\otimes} H(F).$$

By the Corollary p. 91 of [13], since $H'(E\times F)$ is complete, we have $H'(E\times F) = \big(H(E)\hat{\otimes} H(F)\big)' = H'(E)\overline{\otimes} H'(F)$. This completes the proof.

Remark. In attempting to prove Proposition 24 for the special case $E = \mathcal{S}$ we come across the following fact which may be of some interest to someone—at any rate it shows a rather curious relationship between $H(C)$ and $\mathcal{S}$. Let S denote the space of rapidly decreasing sequences: $\mathcal{S} \approx S$;

(a) the weights on S' are given by $(1/g(n))_n$, where $g(z) = \sum_n a_n z^n \in H(C)$ and $a_n > 0$ for all n,

(b) $(x_n)_n \in S$ if and only if there exists $g \in H(C)$, $g(z) = \sum_n a_n z^n \in H(C)$ and $a_n > 0$ all n such that

$$x_n/g(n) \to 0 \quad \text{as} \quad n\to\infty.$$

Open problems.

(a) If E is a fully nuclear space with a basis is E an A-nuclear space? This is similar to the following problem: If E is a reflexive nuclear space with a Schauder basis, then is E'_β nuclear if and only if E is A-nuclear?

(b) If E is a Fréchet nuclear space with a basis and $(H(E), \tau_0)$ is infrabarrelled (or equivalently reflexive) is E a B-nuclear space?

Added in proof (14/10/80). Since this paper was written, the following results have been obtained. We refer to S. Dineen, *Complex Analysis on Locally Convex Spaces* (book to appear in the North Holland Mathematical Studies Series) for further details.

(a) Every Fréchet nuclear space with a basis is an A-nuclear space.

(b) A Fréchet nuclear space with a basis is a B nuclear space if and only if it is a DN space (see D. Vogt, *Subspaces and quotients of* (s), Proc. First Paderborn Conf. on Functional Analysis, Ed. K.-D. Bierstedt and K. Fuchssteiner, North Holland, Math. Studies, 27 (1977), 167–188) and hence if and only if it is a closed subspace of S.

(c) The converse to Proposition 24 holds. This answers positively the open problem (b) posed at the end of the paper. Hence we have the following result:

If E is a Fréchet nuclear space with a basis, then $(H(E), \tau_0)$ is reflexive if and only if E is a B nuclear space.

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A generalization of the Banach–Stone theorem

by

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Abstract. We investigate the geometric properties of a Banach space X^* which give the following implication: There is an isomorphism between $C(S, X)$ and $C(S', X)$ with a small bound iff S and S' are homeomorphic.

Let X be a Banach space and let $C(S, X)$ $(C(S))$ denote the space of X-valued (scalar-valued) continuous functions on a compact Hausdorff space S provided with supremum norm. The classical Banach–Stone theorem states that the existence of an isometric isomorphism from $C(S)$ onto $C(S')$ implies that S and S' are homeomorphic. Cambern [3] proved that this property is stable: If Ψ is an isomorphism of $C(S)$ onto $C(S')$ such that $\|\Psi\|\,\|\Psi^{-1}\| < 2$ then S and S' are homeomorphic.

Several authors have considered a vector-valued generalization of the Banach–Stone theorem. Behrends [2] proved that if the centralizers of X and Y are one-dimensional then the existence of an isometric isomorphism between $C(S, X)$ and $C(S', Y)$ implies that S and S' are homeomorphic. (For the definition and properties of the centralizer see [1], [2]). Cambern [4] proved that if X is a finite-dimensional Hilbert space and if Ψ is an isomorphism of $C(S, X)$ onto $C(S', X)$ such that $\|\Psi\|\,\|\Psi^{-1}\| < \sqrt{2}$ then S and S' are homeomorphic. In this paper we investigate the relation between geometric properties of X^* and the stability of X-valued Banach–Stone theorem.

THEOREM 1. *Let S and S' be compact Hausdorff spaces and let X be a complex (real) Banach space. If there is an isomorphism Ψ of a complex (real) Banach space $C(S, X)$ onto $C(S', X)$ with $\|\Psi\|\,\|\Psi^{-1}\| \leqslant k$ and if*

$$\sup\{\|x_1^* - x_2^*\|:\ \|(x_1^* + x_2^*)/2\| = 1,\ \|x_1^*\| \leqslant k,\ \|x_2^*\| \leqslant k\} = a < 4/3,$$

then S and S' are homeomorphic.

We divide the proof of the theorem into a number of lemmas. Let K^* denote the set $\{x^* \in X^*:\ \|x^*\| = 1\}$ provided with the weak * topology. A Banach space $C(S, X)$ can be identified in a natural way with a subspace of $C(S \times K^*)$: $\Phi(f)(s, x^*) := x^*(f(s))$. Hence any $F \in (C(S, X))^*$ gives