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Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces*

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Abstract. Using differentiation techniques and methods from model theory, we study uniformly continuous and Lipschitz mappings, the connection between them and their linearization. Applications to the problem of uniform and Lipschitz classification of Banach spaces are given. For example: A Banach space which is Lipschitz homeomorphic to a reflexive Orlicz space $L_M[0, 1]$ is isomorphic to it. Or: If two superreflexive Banach spaces, isomorphic to their respective squares, are uniformly homeomorphic, then they have isomorphic ultrapowers.

Introduction. In this paper we shall study the interrelation between the uniform structure, the Lipschitz structure and the linear-topological structure in Banach spaces. The typical problem in this area can be described as follows:

(P) Let X and Y be Banach spaces and assume that there is a uniformly continuous (or Lipschitz) mapping f from X into Y with some additional property (e.g., homeomorphism, embedding etc.). Does this imply the existence of a linear continuous mapping $F\colon X\to Y$ with the same property (i.e., isomorphism, linear embedding etc.)?

The first results of this kind were obtained by Lindenstrauss in [21]. He proved, for instance, that under some additional assumption the existence of a uniformly continuous projection implies the existence of a linear one. We shall be mainly interested in the uniform and Lipschitz classification, that is in the question whether every two uniformly (or Lipschitz) homeomorphic Banach spaces are isomorphic. In view of the results of Lindenstrauss [21], Bessaga [5], Enflo [12] and the beautiful theorem of Enflo [13] — "a locally convex space uniformly homeomorphic with a Hilbert space is isomorphic to it" — one could expect that the answer to the latter question is positive. Other results supporting this conjecture have been obtained. Namely, in [27], [28] the second named

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author established that reflexivity and RNP are invariant under Lipschitz embeddings. Finally, Ribe [34] used some sophisticated combinatorial argument to prove that uniformly homeomorphic Banach spaces have the same local subspace structure and that superreflexivity and "being an $\mathscr{L}_p$-space for given $p \in (1, \infty)$" are invariant under uniform homeomorphisms [35]. But in 1978 came the surprise — Aharoni and Lindenstrauss [2] produced a striking example of two nonisomorphic Banach spaces (both of them nonreflexive and nonseparable) which were Lipschitz homeomorphic. On the other hand, it is known [29] that a locally convex space uniformly homeomorphic to a Montel–Fréchet space is isomorphic to it. In our paper we shall show that, in spite of the example of Aharoni and Lindenstrauss, under some natural assumption on the Banach spaces X and Y, the existence of a Lipschitz homeomorphism between X and Y implies that X and Y are isomorphic. Combining this with some model-theoretic techniques, we find that under some natural assumption on X and Y the existence of a uniform homeomorphism between X and Y implies that they have isomorphic ultrapowers.

The standard way of solving problem (P) usually consists of two steps (cf. [21], [13], [14], [27], [28], [29], [34], [35], [31]). Namely, step A — using some kind of "compactness" and a diagonal procedure to replace the uniformly continuous mapping by a Lipschitz one (by means of the Corson–Klee Lemma) and step B — linearizing the Lipschitz mapping (using either differentiation or some averaging techniques).

An essentially new feature is the use of ultrapowers and model-theoretic techniques. Methods from nonstandard analysis and model theory have been applied to various linear problems in Banach space theory (cf. works by Dacunha-Castelle, Krivine [9], [10], Henson [17], [18], Henson, Moore [19], Stern [36], [37], and the first named author [15], [16]). It turns out that these methods are also suitable for the investigation of nonlinear questions. In the present context, ultrapowers establish a direct connection between uniform and Lipschitz maps (step A) as well as between the infinite dimensional and local linear structures involved (the outcome of step B). In order to exploit this connection properly, a detailed analysis of the resulting structures (ultrapowers) is required. The machinery for this is supplied by model theory.

In Section 2 we study the differentials of Lipschitz homeomorphisms to prove that in some cases the range of such a differential is complemented (Th. 2.4). This, combined with Pełczyński's Decomposition Method, shows that if X is a superreflexive rearrangement invariant function space on $[0, 1]$ (for ex. $L_p[0, 1]$ for $p \in (1, \infty)$ or a reflexive Orlicz space on $[0, 1]$), then every locally convex space Lipschitz homeomorphic to X is isomorphic to it (Cor. 2.10). Also, we solve the problem of Lipschitz classification within some classes of Banach spaces.

In Section 3 we investigate the w^*-differentials in dual spaces. We prove that in some cases the w^*-differentials of a Lipschitz embedding are isomorphic embeddings a.e. (Th. 3.2). This together with the Loewenheim–Skolem Theorem shows that if a separable Banach space is Lipschitz embeddable in a dual space Y, then it is isomorphic to a subspace of Y (Th. 3.5). Let us mention that, as a byproduct, we also obtain the following result — every dual Banach space with density character greater than $2^{\aleph_0}$ contains a nontrivial complemented subspace (Cor. 3.8).

In the next section we first prove that uniformly homeomorphic spaces have Lipschitz homeomorphic ultrapowers and next use the results from Section 2 and the Keisler–Shelah Isomorphism Theorem to deduce that under some natural assumptions on the Banach spaces X and Y the existence of a uniform homeomorphism between X and Y implies that the ultrapowers of X and Y with respect to some ultrafilter are isomorphic (Th. 4.5). Using a result of Lindenstrauss, we also prove that a Banach space which is uniformly homeomorphic to an $\mathscr{L}_\infty$-space is an $\mathscr{L}_\infty$ space itself (Th. 4.9).

Finally, in Section 5, we show that from the theory developed in this paper one can, in a natural way, obtain both theorems of Ribe [34], [35], whereas the last section contains remarks, comments and problems — the solution of which could be, in our opinion, essential to the progress in the field of uniform and Lipschitz classification of Banach spaces.

Since we are using methods from theories distant from one another (Banach space theory, differentiation and model theory), in order to make the paper more selfcontained we provide the reader with rather extensive preliminaries.

1. Preliminaries. We shall consider Banach spaces over the field of reals only and our terminology and notation will be standard, the same as that used, for example, in [25], [26]. The capital letters X, Y and Z will always denote Banach spaces. We shall mainly deal with nonlinear maps, but when speaking of an isomorphism or an isometry we always mean linear isomorphism or linear isometry. Also "complemented" means "linearly complemented", while the analogous nonlinear situations are marked as "being the range of a uniform (Lipschitz) projection". The notation $X \simeq Y$ means: X and Y are isomorphic.

"Subspace" stands for closed linear subspace and $X \subset Y$ denotes that X is a subspace of Y. With a slight abuse of notation, $X \subset_c Y$ means: X is isomorphic to a complemented subspace of Y.

A mapping f from a Banach space X into a Banach space Y is said to be *uniformly continuous* iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x_1, x_2 \in X$ with $\|x_1 - x_2\| < \delta$ the inequality

$$\|f(x_1) - f(x_2)\| < \varepsilon$$

holds. Such a mapping f is said to be a *uniform embedding* iff it is one-to-one and both f and f^{-1} are uniformly continuous. The mapping f is called a *uniform homeomorphism* if it is a surjective uniform embedding. Finally, if Z is a subspace of X and f is a uniformly continuous mapping of X onto Z such that f restricted to Z is the identity on Z, then f is called a *uniform projection* and Z is said to be the *range of a uniform projection in* X. In the sequel we shall need the following famous result of Lindenstrauss [21]; for another proof of it see [31].

THEOREM 1.1 (On uniform projections). *Let Z be the range of a uniform projection in X. Assume that Z is complemented in its second dual. Then Z is complemented in X.*

A mapping f from a subset A of a Banach space X into a Banach space Y is said to be *Lipschitz* iff there is a constant $K > 0$ such that for all $x_1, x_2 \in A$

$$\|f(x_1)-f(x_2)\| \leqslant K\|x_1-x_2\|.$$

The smallest K satisfying the relation above is called the *Lipschitz constant of* f, in the sequel usually denoted by K_f. A one-to-one mapping f from a subset A of X into Y is said to be a *Lipschitz embedding* iff both f and f^{-1} are Lipschitz. The *Lipschitz embedding constant* is the product of the Lipschitz constants of f and f^{-1}. Obviously a *Lipschitz homeomorphism* means a surjective Lipschitz embedding and the meaning of the *Lipschitz homeomorphism constant* of such a map is also clear. We say that a subset A of X is the *range of a Lipschitz projection in* X iff there is a Lipschitz mapping f of X onto A such that f restricted to A is the identity on A.

A mapping f from X onto Y is said to be *Lipschitz on large distances* iff for every $\delta > 0$ there exists a $K > 0$ such that for all $x_1, x_2 \in X$ satisfying $\|x_1-x_2\| \geqslant \delta$ we have

$$\|f(x_1)-f(x_2)\| \leqslant K\|x_1-x_2\|.$$

One of the crucial tools in the problem of uniform classification of Banach spaces is the following

LEMMA 1.2 (Corson, Klee [8]). *Every uniformly continuous map between Banach spaces is Lipschitz on large distances.*

In the sequel we shall need a generalization of the well-known theorem of Rademacher [33] on differentiation of Lipschitz mappings. Since in infinite dimensional Banach spaces there is no canonical measure, we begin with introducing a notion of zero-sets in an arbitrary separable Banach space. Similar but different notions can be found in [4], [7] and [32].

For a separable Banach space X denote by $\mathscr{A}(X)$ the family of all linearly independent, linearly dense, bounded sequences in X. Let Q be the Cartesian product of a sequence of disjoint copies of the interval

$[0,1]$ endowed with the measure μ being the product of the Lebesgue measures on corresponding intervals. Let X be a separable Banach space. For every $(x_i) \in \mathscr{A}(X)$ and $x_0 \in X$ define $T_{(x_i),x_0}\colon Q \to X$ by the equality

$$T_{(x_i),x_0}\big((t_i)\big) = x_0 + \sum_{i=1}^{\infty} 2^{-i} t_i x_i \quad \text{for} \quad (t_i) \in Q$$

and let $\mu_{(x_i),x_0}$ be the Borel measure on X defined for every Borel subset A of X by

$$\mu_{(x_i),x_0}(A) = \mu\Big(T^{-1}_{(x_i),x_0}\big(A \cap T_{(x_i),x_0}(Q)\big)\Big).$$

A Borel subset A of X is said to be a *zero-set* iff $\mu_{(x_i),x_0}(A) = 0$ for all $(x_i) \in \mathscr{A}(X)$ and $x_0 \in X$. It can easily be seen that a zero-set cannot contain an open set and therefore any complement of a zero-set is a dense subset of X. Also, it is obvious that zero-sets form a countably additive ideal of subsets of X.

Let f be mapping from X into Y. We say that f is *Gateaux differentiable at the point* $x_0 \in X$ iff for every $x \in X$ the limit

$$\lim_{\lambda\to 0} \frac{f(x_0+\lambda x)-f(x_0)}{\lambda} = f'_x(x_0)$$

exists and the mapping $(Df)_{x_0}\colon X \to Y$ defined by $(Df)_{x_0}(x) = f'_x(x_0)$ is linear (in x).

Finally, let us recall that a Banach space X is said to have the *Radon–Nikodým Property* (shortly the RNP) iff every Lipschitz mapping from the interval $[0,1]$ into X is almost everywhere differentiable. Note that this definition is different from the classical one but it is equivalent and suits our purposes best. It easily follows from the definition that the RNP is invariant under isomorphisms and that every subspace of a Banach space with the RNP has the RNP as well. It is known that every reflexive Banach space and every separable dual Banach space has the RNP. For more details on this subject the reader is referred to the monograph [11].

The following result on differentiation of Lipschitz mappings in Banach spaces, due to the second named author [28], proved to be very useful in the study of the problem of uniform and Lipschitz classification. Other, slightly different versions of this result (with different notions of zero-sets) have been obtained independently by Christensen [7] and Aronszajn [4].

THEOREM 1.3 (Infinite Dimensional Rademacher Theorem). *Let f be a Lipschitz mapping from a separable Banach space X into a Banach space Y with the Radon–Nikodým Property. Then*

(i) *f is Gateaux differentiable almost everywhere* (*i.e. the set where f fails to possess a differential is a zero-set in X*).

(ii) *if $(Df)_x$ exists for some $x \in X$, then it is a continuous linear operator from X into Y and its norm is not greater than the Lipschitz constant of f.*

(iii) *if f is a Lipschitz embedding of X into Y, then $(Df)_x$ is an isomorphic embedding of X into Y and the isomorphism constant of $(Df)_x$ (i.e. the product of the norm of $(Df)_x$ and the norm of its inverse) is not greater than the Lipschitz embedding constant of f.*

Our main tool to establish an isomorphism between Banach spaces will be Pełczyński's decomposition method [30], [25]. To describe the different variants of it, we first need some more notation. For a Banach space X we denote by $X \oplus X$ the direct sum, by $l_p(X)$, $1 \leqslant p < \infty$, the space of all sequences $(x_i) \subset X$ satisfying $\|(x_i)\| = \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{1/p} < \infty$. If X is a Banach lattice, $\widetilde{X(l_2)}$ is the space of all sequences $(x_i) \subset X$ satisfying

$$\|(x_i)\| = \sup_n \left\| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\| < \infty,$$

where the expression $\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ is defined by Th.1.d.1 of [26]. $X(l_2)$ is the closure in $\widetilde{X(l_2)}$ of the set of sequences which are eventually zero.

Let X and Y be Banach spaces. We shall say that the pair X, Y *satisfies the Decomposition Scheme* if one of the following conditions (D1)–(D3) is fulfilled:

(D1) $X \cong X \oplus X$ and $Y \cong Y \oplus Y$.

(D2) Either X or Y contains a complemented subspace isomorphic to $l_p(X)$ ($l_p(Y)$, respectively) for some p, $1 \leqslant p < \infty$.

(D3) Either X or Y is a Banach lattice which contains a complemented subspace isomorphic to $X(l_2)$ ($Y(l_2)$, respectively).

The main use of this scheme lies in the following

THEOREM 1.4 (Pełczyński's Decomposition Theorem). *Assume that the pair of Banach spaces X, Y satisfies the Decomposition Scheme. If each of the spaces X, Y is isomorphic to a complemented subspace of the other, then X and Y are isomorphic.*

Checking the case of (D1) is easy, (D2) is due to Pełczyński [30] (cf. also [25], 2.a.3). (D3) comes from Johnson, Maurey, Schechtman, Tzafriri [20], Cor. 9.2 (see also [26], 2.d.5).

Let us now turn to the necessary prerequisites concerning ultrapowers of Banach spaces. For proofs and further details we refer to the survey [15]. Let $\mathcal{U}$ be an ultrafilter on a set I and let $(X_i)_{i \in I}$ be a family of Banach spaces. Consider the space $l_\infty(I, X_i)$ consisting of all families $(x_i)_{i \in I}$ with $x_i \in X_i$ and

$$\|(x_i)\| = \sup_I \|x_i\| < \infty.$$

Let $N_{\mathcal{U}}$ be the closed subspace of all those families (x_i) which satisfy

$\lim_{\mathcal{U}} \|x_i\| = 0$. Then the ultraproduct $(X_i)_{\mathcal{U}}$ is defined to be the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$, equipped with the natural quotient norm. By $(x_i)_{\mathcal{U}}$ we denote the element of the ultraproduct which is determined by the family (x_i). Its norm can be computed as $\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|$.

We shall mainly deal with the ultrapower $(X)_{\mathcal{U}}$ of a space X, i.e. with the ultraproduct $(X_i)_{\mathcal{U}}$ where all spaces X_i are identical with X. There is a canonical isometric embedding J_X of X into $(X)_{\mathcal{U}}$ which maps an element $x \in X$ onto the equivalence class of the constant family $(x)_{i \in I}$, i.e. $J_X x = (x_i)_{\mathcal{U}}$ with $x_i = x$ for all $i \in I$. On the other hand, there is also a canonical map $Q_X\colon (X)_{\mathcal{U}} \to X^{**}$ which assings to each family its w^*-limit along $\mathcal{U}$ in X^{**}, i.e.

$$Q_X (x_i)_{\mathcal{U}} = w^* - \lim_{\mathcal{U}} x_i.$$

Q_X has norm one and satisfies $Q_X J_X = I_X$ (the identity on X). Consequently, if X is reflexive (or a dual space), X can be identified with a norm-one complemented subspace of $(X)_{\mathcal{U}}$. Finally, the dual space $(X)_{\mathcal{U}}^*$ contains an isometric copy of X^*. The isometry, which we will denote by J_{X^*} (if no confusion with the notation above is possible), is defined by

$$\langle (x_i)_{\mathcal{U}}, J_{X^*} x^* \rangle = \lim_{\mathcal{U}} \langle x_i, x^* \rangle \quad \text{for} \quad (x_i)_{\mathcal{U}} \in (X)_{\mathcal{U}}.$$

Returning to the ultrafilter $\mathcal{U}$ itself, we say that $\mathcal{U}$ is *countably incomplete* if there exists a sequence (I_n) of subsets of I with $I_n \in \mathcal{U}$ for $n \in N$ and $\bigcap_{n=1}^{\infty} I_n = \emptyset$. For example, each free (i.e. nontrivial) ultrafilter on N is countably incomplete. Given ultrafilters $\mathcal{U}$ on I and $\mathcal{V}$ on J, we define a new ultrafilter $\mathcal{U} \times \mathcal{V}$ on $I \times J$ in the following way: $A \in \mathcal{U} \times \mathcal{V}$ iff $\{j\colon \{i\colon (i, j) \in A\} \in \mathcal{U}\} \in \mathcal{V}$. This ultrafilter appears when we iterate the ultrapower construction: $((X)_{\mathcal{U}})_{\mathcal{V}}$ is (canonically) isometric to $(X)_{\mathcal{U} \times \mathcal{V}}$.

Finally, we present two basic results from the model theory of Banach spaces. The first one is a variant of the Keisler–Shelah Isomorphism Theorem, [6], Th. 6.1.15. The second one is a combination of the Keisler–Shelah result with the Loewenheim–Skolem Theorem, [6], Th. 3.3.6. Their Banach space versions are due to Henson [17], [18] and Stern [37]. The form of the Loewenheim–Skolem Theorem which we state here is an immediate consequence of the proof of Th. 2.2. in [37].

THEOREM 1.5 (Keisler–Shelah Isomorphism Theorem). *Let X be a Banach space and let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on I and J, respectively. Then there exists an ultrafilter $\mathcal{W}$ on a set K such that the spaces $(X)_{\mathcal{U} \times \mathcal{W}}$ and $(X)_{\mathcal{V} \times \mathcal{W}}$ are isometric.*

THEOREM 1.6 (Loewenheim–Skolem Theorem). *Let X be a Banach space and let $Y \subset X$ be a subspace, $Y \neq \{0\}$. Then there exists a subspace*

Z, $Y \subset Z \subset X$, *with* $\operatorname{dens} Z = \operatorname{dens} Y$, *an ultrafilter* $\mathcal{U}$ *and an isometry* S *from* $(Z)_{\mathcal{U}}$ *onto* $(X)_{\mathcal{U}}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} (Z)_{\mathcal{U}} & \xrightarrow{S} & (X)_{\mathcal{U}} \\ {\scriptstyle J_Z}\uparrow & & \uparrow{\scriptstyle J_X} \\ Z & \xrightarrow[I_Z]{} & X \end{array}$$

where J_X, J_Z *and* I_Z *are the canonical embedding maps.*

As usual, $\operatorname{dens} X$ denotes the density character of X, i.e. the smallest cardinal $\varkappa$ such that X has a dense subset of cardinality $\varkappa$.

2. Lipschitz homeomorphisms between spaces with RNP. The differential Df of a Lipschitz embedding $f\colon X \to Y$ provides a linear embedding of X into Y. It turns out that under certain assumptions the position of the image $Df(X)$ in Y can be specified:

PROPOSITION 2.1. *Let* f *be a Lipschitz embedding of a Banach space* X *into a Banach space* Y *such that* $f(X)$ *is the range of a Lipschitz projection in* Y. *Assume that* f *is Gateaux differentiable at* $x_0 \in X$ *and that there exists a Lipschitz projection from* X^{**} *onto* X.

Then there exists a Lipschitz projection from Y *onto* $(Df)_{x_0}(X)$. *Moreover, if there is a linear projection from* X^{**} *onto* X, *then* $(Df)_{x_0}(X)$ *is a linearly complemented subspace of* Y.

Proof. By a simple translation argument, without any loss of generality, we may assume that $x_0 = 0$ and $f(0) = 0$. In the sequel, for the sake of simplicity, we shall write Df insteado f $(Df)_0$. Also, we shall identify X with its canonical image in X^{**}.

Let $\pi_{f(X)}$ be a Lipschitz projection from Y onto $f(X)$ and set $g = f^{-1} \circ \circ \pi_{f(X)}$, thus $g \circ f = I_X$ (the identity on X). Since g is Lipschitz with, say, constant K_g, we have, for all $y_1, y_2 \in Y$ and all positive integers n,

(1) $$\|ng(n^{-1}y_1) - ng(n^{-1}y_2)\| \leqslant K_g \|y_1 - y_2\|.$$

Let $\mathcal{U}$ be a free ultrafilter on the set of positive integers. Note that it follows from (1) and the assumption $f(0) = 0$ that for each $y \in Y$ the sequence $\{ng(n^{-1}y)\}$ is norm bounded, and hence w^*-relatively compact in X^{**}. We can therefore define a mapping $\tilde{g}\colon Y \to X^{**}$ by

$$\tilde{g}(y) = w^* - \lim_{\mathcal{U}} ng(n^{-1}y) \quad \text{for} \quad y \in Y.$$

By the w^*-lower semicontinuity of the norm in X^{**}, we get from (1)

$$\|\tilde{g}(y_1) - \tilde{g}(y_2)\| \leqslant K_g \|y_1 - y_2\|;$$

hence $\tilde{g}$ is a Lipschitz mapping. We shall show next that, for all $x \in X$,

(2) $$\tilde{g}\big((Df)x\big) = x.$$

Indeed, denote $(Df)x = z$. This implies by definition

$$\lim_{n\to\infty} \|z - nf(n^{-1}x)\| = 0.$$

Therefore, we have

$$\lim_{\mathcal{U}} \|ng(n^{-1}z) - x\| = \lim_{\mathcal{U}} n \|g(n^{-1}z) - (g\circ f)(n^{-1}x)\| \leqslant K_g \lim_{\mathcal{U}} n \|n^{-1}z - f(n^{-1}x)\| = 0,$$

which proves (2). Now set $Z = Df(X)$ and define a map $h\colon Y \to Z$ by setting $h = Df \circ \pi_X \circ \tilde{g}$, where π_X is a Lipschitz projection from X^{**} onto X. Obviously, as a composition of Lipschitz mappings, h is Lipschitz, and (2) ensures that h is a projection from Y onto Z.

If X is linearly complemented in its second dual, then so is Z, and we can apply Theorem 1.1 on uniform projections to obtain a linear projection from Y onto Z.

Next, we draw some conclusions from Proposition 2.1, combining it with the Infinite Dimensional Rademacher Theorem 1.3.

THEOREM 2.2. *Let* f *be a Lipschitz embedding of a separable Banach space* X *into a reflexive Banach space* Y *such that* $f(X)$ *is the range of a Lipschitz projection in* Y. *Then* X *is isomorphic to a complemented subspace of* Y.

Proof. By Theorem 1.3 there is an $x \in X$ such that f is Gateaux differentiable at x and $(Df)_x\colon X \to Y$ is an isomorphism. Hence X is a reflexive space. Thus the assumptions of Proposition 2.1 are satisfied and we conclude that $(Df)_x(X)$ is complemented in Y.

As an application of the theorem above, we have (compare also Cor. 4.8 below)

COROLLARY 2.3. *If a separable Banach space* X *is Lipschitz embeddable in an* $L_p(\mu)$ *space for some* $p \in (1, \infty)$ *in such a way that its image is the range of a Lipschitz projection in* $L_p(\mu)$, *then* X *is isomorphic to a complemented subspace of* $L_p(\mu)$.

For Lipschitz homeomorphisms we get the following

THEOREM 2.4. *Let* X *be a separable Banach space with the Radon–Nikodým Property, complemented in* X^{**}. *If* X *is Lipschitz homeomorphic to another Banach space* Y, *then* X *is isomorphic to a complemented subspace of* Y.

Proof. Let $f\colon X \to Y$ be a Lipschitz homeomorphism between X and Y. By Theorem 1.3, f^{-1} is a.e. differentiable and every such differential is an isomorphic embedding of Y into X. Thus Y has the RNP. Again, by Theorem 1.3, f is a.e. differentiable and to complete the proof it is enough to apply Prop. 2.1.

In particular, we have

Corollary 2.5. *If a Banach space is Lipschitz homeomorphic to l_1, then it contains an isomorphic copy of l_1 as a complemented subspace.*

The results above imply that for large classes of Banach spaces Lipschitz equivalence implies linear equivalence. The main result of this section is the following

Theorem 2.6. *Let X and Y be separable dual Banach spaces and assume that the pair X, Y satisfies the Decomposition Scheme. If X and Y are Lipschitz homeomorphic, then they are isomorphic.*

Proof. It is easy to see that the assumptions of Theorem 2.4 are satisfied. Thus X is isomorphic to a complemented subspace of Y and Y is isomorphic to a complemented subspace of X. This together with Pełczyński's Decomposition Theorem 1.4 concludes the proof.

A variation of Theorem 2.6 is

Theorem 2.7. *Let a Banach space X be Lipschitz homeomorphic to a separable, reflexive Banach space Y. Assume that the pair X, Y satisfies the Decomposition Scheme. Then X is isomorphic to Y.*

Proof. An easy proof is omitted.

Using the decomposition properties of particular spaces, we obtain from Theorem 2.6 and 2.7

Corollary 2.8. *A dual space Lipschitz homeomorphic to l_1 is isomorphic to l_1.*

Corollary 2.9. *Let $p \in (1, \infty)$ and let X be isomorphic either to $L_p[0,1]$ or to l_p. Then every Banach space Lipschitz homeomorphic to X is isomorphic to X.*

An analogous result for an essentially wider class of Banach spaces can be obtained in the same way by relying on (D3) instead of (D2). It was shown in [20] (cf. also [26], Prop. 2.d.4 and 2.d.5) that superreflexive rearrangement invariant function spaces on $[0,1]$ (for the definition see [26], Ch.2.a) satisfy (D3) of the Decomposition Scheme. Consequently we have

Corollary 2.10. *Let Y be a superreflexive rearrangement invariant function space on $[0,1]$. If X is Lipschitz homeomorphic to Y, then X is isomorphic to Y. In particular, this statement holds when $Y = L_M[0,1]$, where M is a reflexive Orlicz function.*

Concluding this topic, let us mention that Theorem 2.6 in particular accomplishes the Lipschitz classifiation of the following classes of Banach spaces:

Separable dual Banach spaces which are isomorphic to their squares, including:

all spaces with a boundedly complete symmetric basis, and

all reflexive rearrangement invariant function spaces on $[0,1]$ or $[0,\infty)$.

The rest of this section is devoted to the question what can be said if we have no additional information about X (such as X being complemented in X^{**}) but instead some more about Y. It turns out that we can still assert the existence of a Lipschitz projection (instead of a linear one).

We need the following

Lemma 2.11. *Let X be a Banach space, and Y a dual Banach space. Then each Lipschitz map $f\colon X\to Y$ can be extended to a Lipschitz map $\tilde{f}\colon X^{**}\to Y$.*

Proof. We shall apply the principle of local reflexivity [24]. It suits our purposes to use the ultrapower version of it ([15], Prop. 6.7): There exists an ultrafilter $\mathcal{U}$ on a set I and an isometric embedding $J\colon X^{**}\to(X)_{\mathcal{U}}$ such that J restricted to X is the canonical embedding of X into $(X)_{\mathcal{U}}$. We define a map $F\colon (X)_{\mathcal{U}}\to Y$ by

$$F\big((x_i)_{\mathcal{U}}\big) = w^*-\lim_{\mathcal{U}} f(x_i).$$

Since f is Lipschitz, the family $\big(f(x_i)\big)_{i\in I}$ is uniformly bounded and $\lim_{\mathcal{U}}\|x_{1,i}-x_{2,i}\| = 0$ implies $\lim_{\mathcal{U}}\|f(x_{1,i})-f(x_{2,i})\| = 0$. Thus, F is well-defined. F is a Lipschitz map since

$$\begin{aligned}\big\|F\big((x_{1,i})_{\mathcal{U}}\big)-F\big((x_{2,i})_{\mathcal{U}}\big)\big\| &= \big\|w^*-\lim_{\mathcal{U}}\big(f(x_{1,i})-f(x_{2,i})\big)\big\|\\ &\leqslant \lim_{\mathcal{U}}\|f(x_{1,i})-f(x_{2,i})\| \leqslant K_f\lim_{\mathcal{U}}\|x_{1,i}-x_{2,i}\|\\ &= K_f\|(x_{1,i})_{\mathcal{U}}-(x_{2,i})_{\mathcal{U}}\|.\end{aligned}$$

Obviously, we have for $x\in X$

$$F(Jx) = w^*-\lim_{\mathcal{U}} f(x) = f(x),$$

and we can therefore define the desired extension as $\tilde{f} = F\circ J$.

The next proposition is a variation on the theme of Proposition 2.1.

Proposition 2.12. *Let f be a Lipschitz embedding of a Banach space X into a Banach space Y such that $f(X)$ is the range of a Lipschitz projection in Y. Assume that f is Gateaux differentiable at $x_0\in X$ and that Y is complemented in Y^{**}. Then $(Df)_{x_0}(X)$ is the range of a Lipschitz projection in Y.*

Proof. According to Proposition 2.1, it suffices to show that X is the range of a Lipschitz projection in X^{**}. Let $\tilde{f}\colon X^{**}\to Y^{**}$ be an extension of f (regarded as a map from X into Y^{**}) to X^{**}, which exists according to Lemma 2.11. Furthermore, let π_Y be a projection from Y^{**} onto

Y, and $\pi_{f(X)}$ a Lipschitz projection from Y onto $f(X)$. It is now clear that the mapping $\pi_X\colon X^{**}\to X$ defined by $\pi_X = f^{-1}\circ\pi_{f(X)}\circ\pi_Y\circ\tilde{f}$ is the desired projection.

Proposition 2.12 yields immediately

THEOREM 2.13. *Let a Banach space X be Lipschitz embeddable in a separable dual Banach space Y in such a way that its image is the range of a Lipschitz projection in Y. Then X is isomorphic to a subspace of Y which is the range of a Lipschitz projection in Y.*

As a particular case of Theorem 2.13 we get (compare also Cor. 2.5)

COROLLARY 2.14. *If a Banach space is Lipschitz homeomorphic to l_1, then it is isomorphic to a subspace of l_1 which is the range of a Lipschitz projection in l_1.*

3. Lipschitz embeddings into dual Banach spaces. It follows from Th. 1.3 that if a Banach space is Lipschitz embeddable in a separable dual space Y, then it is isomorphic to a subspace of Y. This section is mainly devoted to the proof that, in fact, the assumption of the separability of Y can be omitted.

Let $Y = Z^*$ and $f\colon X\to Y$ be a Lipschitz mapping. Then, we say that f is *w^*-differentiable at $x_0\in X$* iff for every $x\in X$ the limit

(3) $$w^*-\lim_{\lambda\to 0}\frac{f(x_0+\lambda x)-f(x_0)}{\lambda} = f_x^*(x_0)$$

exists (in the w^*-topology of Y) and the mapping $(D^*f)_{x_0}(x) = f_x^*(x_0)$ is linear (in x). The mapping $(D^*f)_{x_0}$ is said to be the *w^*-differential of f at x_0* (compare [4]).

In the sequel by a *Lebesgue measure on an n-dimensional Banach space X* we shall mean any measure on X which is an image of the Lebesgue measure on $\mathbf{R}^n$ via an isomorphism between X and $\mathbf{R}^n$. Since all Lebesgue measures on X are equal up to a constant, the class of Lebesgue zero-sets in a finite-dimensional Banach space is well-defined.

LEMMA 3.1 (w^*-Rademacher Theorem). *Let X be a finite-dimensional Banach space, Z a separable Banach space, and $f\colon X\to Z^*$ a Lipschitz mapping. Then*

(i) *$(D^*f)_x$ exists for almost all $x\in X$,*

(ii) *if $\|f(x_1)-f(x_2)\|\leqslant K\|x_1-x_2\|$ for $x_1, x_2\in X$, then $\|(D^*f)_x\|\leqslant K$, whenever $(D^*f)_x$ exists,*

(iii) *if, in addition, $k\|x_1-x_2\|\leqslant\|f(x_1)-f(x_2)\|$ for $x_1, x_2\in X$, then, for almost every $x\in X$, $(D^*f)_x$ is an isomorphic embedding of X into Z^* with $\|(D^*f)_x^{-1}\|\leqslant k^{-1}$.*

Proof. During the proof we shall skip all (standard) arguments showing that all sets and functions involved are measurable.

(i): Set $\varphi_n(x) = \langle z_n, f(x)\rangle$ for all $n\in\mathbf{N}$ and $x\in X$, where (z_n) is an arbitrary fixed sequence of points dense in the unit sphere of Z. By Rademacher's Theorem, all φ_n's are almost everywhere differentiable. Thus

$$W = \{x\in X\colon (D\varphi_n)_x \text{ exists for every } n\in\mathbf{N}\}$$

is of full measure. Fix $x\in W$. For every $a\in X$, $n\in\mathbf{N}$

$$(D\varphi_n)_x(a) = \lim_{\lambda\to 0}\left\langle z_n, \frac{f(x+\lambda a)-f(x)}{\lambda}\right\rangle.$$

Since the z_n's are dense in the unit sphere of Z, the routine computations show that $\lambda^{-1}(f(x+\lambda a)-f(x))$ is a w^*-Cauchy system when $\lambda\to 0$. Thus, for every $a\in X$, the limit (3) exists. Its linearity easily follows from the linearity of $(D\varphi_n)_x$'s. Hence $(D^*f)_x$ exists for all $x\in W$.

(ii):

$$\begin{aligned}\|(D^*f)_x(a)\| &= \sup_{n\in\mathbf{N}}|\langle z_n, (D^*f)_x(a)\rangle|\\ &= \sup_{n\in\mathbf{N}}\left|\lim_{\lambda\to 0}\left\langle z_n, \frac{f(x+\lambda a)-f(x)}{\lambda}\right\rangle\right|\\ &\leqslant \sup_{n\in\mathbf{N}}\|z_n\|\cdot K\cdot\|a\| = K\|a\|.\end{aligned}$$

(iii): Without any loss of generality assume $k=1$. First, we claim that, for every fixed $a\in X$ with $\|a\|=1$, the set

$$M_a = \{x\in W\colon \|(D^*f)_x(a)\|\geqslant 1\}$$

is of full measure. Indeed, assume the contrary. Then there is an $a\in X$ with $\|a\|=1$ and $m\in\mathbf{N}$ such that the set

$$N = \{x\in W\colon \|(D^*f)_x(a)\| < 1-1/m\}$$

is of positive measure. By Fubini's Theorem, there is a line L of the form $L = \{x_0+ta\colon t\in\mathbf{R}\}$ in X with the property that the one-dimensional Lebesgue measure of $L\cap N$ is positive. Let x be a Lebesgue point of the set $L\cap N$ on the line L. Then there is an $\varepsilon>0$ such that the Lebesgue measure of the set

$$A = \{t\in(0,\varepsilon)\colon x+ta\in L\cap N\}$$

is greater than $\varepsilon(1-1/(2Km))$. Set $B = (0,\varepsilon)\setminus A$. Since

$$\|f(x+\varepsilon a)-f(x)\|\geqslant\varepsilon\|a\| = \varepsilon,$$

one can find $z\in Z$ with $\|z\|=1$ such that $\varphi(\varepsilon)-\varphi(0) > \varepsilon(1-1/(2m))$, where

$$\varphi(t) = \langle z, f(x+ta)\rangle \quad\text{for}\quad t\in[0,\varepsilon].$$

Clearly, φ is a Lipschitz function with the Lipschitz constant not greater than K and therefore $\varphi'(t)$ exists a.e. in $(0, \varepsilon)$ and $|\varphi'(t)| \leqslant K$ for $t \in (0, \varepsilon)$. On the other hand, $\varphi'(t) = \langle z, (D^*f)_{x+ta}(a)\rangle$ for $t \in A$, and we have

$$\begin{aligned}\varepsilon\big(1-1/(2m)\big) < \varphi(\varepsilon)-\varphi(0) &= \int_{(0,\varepsilon)} \varphi'(t)\,dt \\ &\leqslant \int_A \langle z, (D^*f)_{x+ta}(a)\rangle dt + \int_B K\,dt \\ &\leqslant \int_A \|(D^*f)_{x+ta}(a)\|\,dt + K\varepsilon/(2Km) \\ &\leqslant \varepsilon(1-1/m) + \varepsilon/(2m) = \varepsilon\big(1-1/(2m)\big),\end{aligned}$$

a contradiction, which completes the proof of the claim.

Now, let $\mathscr{A}$ be a countable dense subset of the unit sphere in X and let $M = \bigcap_{a \in \mathscr{A}} M_a$. Then for every $x \in M$

(4) $$\|(D^*f)_x(a)\| \geqslant \|a\| \quad \text{for every } a \in \mathscr{A},$$

and M is of full measure. (iii) easily follows from (4) and the continuity of $(D^*f)_x$.

Using the same arguments as in the proof of the Infinite-Dimensional Rademacher Theorem in [28], one can deduce from the lemma above its infinite-dimensional version. So we omit its proof. (For statement (i) see also [4]).

THEOREM 3.2 (Infinite-Dimensional w^*-Rademacher Theorem). *Let X and Z be separable Banach spaces and let $f\colon X \to Z^*$ be a Lipschitz mapping. Then*

(i) *$(D^*f)_x$ exists for almost all $x \in X$,*

(ii) *if $(D^*f)_x$ exists, then $\|(D^*f)_x\| \leqslant K_f$, where K_f is the Lipschitz constant of f,*

(iii) *if f is a Lipschitz embedding, then for almost all $x \in X$, $(D^*f)_x$ is an isomorphic embedding of X into Z^* with $\|(D^*f)_x^{-1}\| \leqslant K_{f^{-1}}$.*

Theorem 3.2 yields directly

COROLLARY 3.3. *If a separable Banach space X is Lipschitz embeddable into the dual Z^* of a separable Banach space Z, then X is isomorphic to a subspace of Z^*.*

The following proposition, which will enable us to pass from separable Z in Corollary 3.3 to the general case, is an extension of a result of Lindenstrauss [22] on projections in nonseparable reflexive Banach spaces (cf. also [3], [17]).

PROPOSITION 3.4. *Let X be a Banach space and Y a subspace of X. Then there exists a subspace Z of X containing Y such that $\operatorname{dens} Z = \operatorname{dens} Y$ and there is an isometric embedding $T\colon Z^* \to X^*$ with the property that $\langle z, Tz^*\rangle = \langle z, z^*\rangle$ for every $z \in Z$ and $z^* \in Z^*$. In particular, $T(Z^*)$ is norm-one complemented in X^*.*

Proof. Take a subspace Z of X containing Y, an ultrafilter $\mathscr{U}$ and an isometry $S\colon (Z)_{\mathscr{U}} \to (X)_{\mathscr{U}}$ which satisfy the conclusion of the Loewenheim–Skolem Theorem 1.6. The embedding $T\colon Z^* \to X^*$ is defined by $T = J_X^*(S^{-1})^*J_{Z^*}$, where J_X^* and J_{Z^*} are the canonical embeddings of X into $(X)_{\mathscr{U}}$ and Z^* into $(Z)_{\mathscr{U}}^*$, respectively. Then, for $z \in Z$ and $z^* \in Z^*$, we have

$$\begin{aligned}\langle z, Tz^*\rangle &= \langle z, J_X^*(S^{-1})^*J_{Z^*}z^*\rangle = \langle S^{-1}J_Xz, J_{Z^*}z^*\rangle \\ &= \langle J_Zz, J_{Z^*}z^*\rangle = \lim_{\mathscr{U}}\langle z, z^*\rangle = \langle z, z^*\rangle.\end{aligned}$$

This completes the proof.

Now we are ready to prove

THEOREM 3.5. *If a separable Banach space is Lipschitz embeddable into a dual space Y, then it is isomorphic to a subspace of Y.*

Proof. Let $Y = Z^*$ and let $f\colon X \to Z^*$ be a Lipschitz embedding of a separable Banach space X into Z^*. Choose $Z_0 \subset Z$ so that Z_0 is separable and norming for the closed linear span of $f(X)$. By Proposition 3.4, there exists a separable space Z_1 such that $Z_0 \subset Z_1 \subset Z$ and Z_1^* embeds isometrically into Z^*. Let $Q\colon Z^* \to Z_1^*$ be the restriction map. Then, since Z_1 is norming for $f(X)$, the map $Q \circ f$ is a Lipschitz embedding of X into Z_1^*. But Z_1 is separable, and so, by Corollary 3.3, X is isomorphic to a subspace of Z_1^*. But Z_1^*, in turn, is isomorphic to a subspace of $Z^* = Y$, which completes the proof.

Note that it follows directly from the estimates in Theorem 3.2 and the proof above that the isomorphism constant of the embedding of X into Y does not exceed the Lipschitz embedding constant of f.

Let us also mention that Theorem 3.5 is no longer true if Y is an arbitrary Banach space. Indeed, by a result of Aharoni [1] each separable Banach space is Lipschitz embeddable into c_0 while obviously not always linearly embeddable in c_0.

COROLLARY 3.6. *If X and Y are separable Lipschitz homeomorphic Banach spaces, then X embeds isomorphically into Y^{**} and Y into X^{**}.*

COROLLARY 3.7. *Lipschitz homeomorphic dual spaces have the same separable linear dimension (i.e. each separable subspace of one of them embeds isomorphically into the other).*

Although it is off the major theme of the present paper, it seems worthwhile to mention the following observation, which is a consequence of Proposition 3.4 and concerns the problem of existence of nontrivial complemented subspaces in arbitrary Banach spaces (cf. [23], p. 166).

COROLLARY 3.8. *Let X be a dual Banach space with $\operatorname{dens} X > 2^{\aleph_0}$ Then X contains a nontrivial complemented subspace. More precisely, there exists an infinite-dimensional subspace $Y \subset X$ with $\operatorname{dens} Y \leqslant 2^{\aleph_0}$ and a projection of norm one from X onto Y.*

Proof. Apply Proposition 3.4 to the predual of X and note that the dual of a separable space has density character not greater than $2^{\aleph_0}$.

4. Uniform homeomorphisms and isomorphism of ultrapowers. We shall now apply the results of the previous sections to the study of uniform equivalence of Banach spaces. The connection between uniform and Lipschitz mappings is supplied by ultrapowers, as the following proposition shows.

PROPOSITION 4.1. *Let X and Y be Banach spaces and let $\mathscr{U}$ be a countably incomplete ultrafilter. Then the following holds:*

(i) *If there is a uniform embedding f of X into Y such that $f(X)$ is the range of a uniform projection in Y, then there exists a Lipschitz embedding F of $(X)_\mathscr{U}$ into $(Y)_\mathscr{U}$ such that there is a Lipschitz projection from $(Y)_\mathscr{U}$ onto $F\big((X)\big)_\mathscr{U}$.*

(ii) *If X and Y are uniformly homeomorphic, then $(X)_\mathscr{U}$ and $(Y)_\mathscr{U}$ are Lipschitz homeomorphic.*

Proof. (i): Let $\pi_{f(X)}$ be a uniform projection from Y onto $f(X)$ and set $g = f^{-1}\circ\pi_{f(X)}$. We may assume without loss of generality that $f(0) = 0$. Define $f_n(x) = n^{-1}f(nx)$ for each positive integer n and $x \in X$. We claim that the family f_n is uniformly equicontinuous (compare [29], Lemma 2). Given $\varepsilon > 0$, choose $\delta > 0$ such that $\|x_1-x_2\| < \delta$ implies $\|f(x_1)-f(x_2)\| < \varepsilon$. We shall show that, for all n,

$$\|f_n(x_1)-f_n(x_2)\| < \varepsilon \quad \text{if} \quad \|x_1-x_2\| < \delta. \tag{5}$$

Indeed,

$$\begin{aligned}\|f_n(x_1)-f_n(x_2)\| &= n^{-1}\|f(nx_1)-f(nx_2)\| \\ &\leqslant n^{-1}\sum_{k=0}^{n-1}\big\|f\big((n-k)x_1+kx_2\big)-f\big((n-k-1)x_1+(k+1)x_2\big)\big\| \\ &< n^{-1}n\varepsilon = \varepsilon,\end{aligned}$$

which proves (5). Applying Lemma 1.2 for $\delta = 1$ gives a constant K such that $\|f(x_1)-f(x_2)\| \leqslant K\|x_1-x_2\|$ whenever $\|x_1-x_2\| \geqslant 1$. For the f_n's we get

$$\|f_n(x_1)-f_n(x_2)\| \leqslant K\|x_1-x_2\| \tag{6}$$

whenever $\|x_1-x_2\| \geqslant n^{-1}$. Next, we define the mapping $F\colon (X)_\mathscr{U}\to(Y)_\mathscr{U}$. Since $\mathscr{U}$ is countably incomplete, there is a family $(I_n)_{n\in N}$ with $I_n \subset I$, $I_n \in \mathscr{U}$ for all n, and $\bigcap_{n=1}^{\infty} I_n = \emptyset$. We may assume that $I_1 = I$ and that (I_n) is decreasing. For $i \in I_n\setminus I_{n+1}$, we put $f_i = f_n$. Now, given $(x_i)_\mathscr{U} \in (X)_\mathscr{U}$, we define

$$F\big((x_i)_\mathscr{U}\big) = \big(f_i(x_i)\big)_\mathscr{U};$$

thus F is the ultraproduct $(f_i)_\mathscr{U}$ of the family of mappings $(f_i)_{i\in I}$. By (5), (6) and the assumption $f(0) = 0$, the family $\big(f_i(x_i)\big)_{i\in I}$ is bounded whenever $(x_i)_{i\in I}$ is. Furthermore (5) guarantees that $\lim_\mathscr{U}\|x_{i,1}-x_{i,2}\| = 0$ implies $\lim_\mathscr{U}\|f_i(x_{i,1})-f_i(x_{i,2})\| = 0$. Consequently, F is well-defined. We claim that F is Lipschitz. Namely, given $(x_{i,1})_\mathscr{U}, (x_{i,2})_\mathscr{U} \in (X)_\mathscr{U}$, $(x_{i,1})_\mathscr{U} \neq (x_{i,2})_\mathscr{U}$, there is a positive integer n such that $\|(x_{i,1})_\mathscr{U}-(x_{i,2})_\mathscr{U}\| > n^{-1}$. By definition, this means that there exists a $D \in \mathscr{U}$ with $\|x_{i,1}-x_{i,2}\| > n^{-1}$ for all $i \in D$. The set $D\cap I_n$ also belongs to the ultrafilter $\mathscr{U}$. By (6) and the definition of the f_i's we have, for all $i \in D\cap I_n$,

$$\|f_i(x_{i,1})-f_i(x_{i,2})\| \leqslant K\|x_{i,1}-x_{i,2}\|.$$

Passing to the limit along the ultrafilter $\mathscr{U}$, we get

$$\big\|\big(f_i(x_{i,1})\big)_\mathscr{U}-\big(f_i(x_{i,2})\big)_\mathscr{U}\big\| \leqslant K\|(x_{i,1})_\mathscr{U}-(x_{i,2})_\mathscr{U}\|,$$

which proves the claim.

Similarly, we can define $g_n(y) = n^{-1}g(ny)$ and $G = (g_i)_\mathscr{U}$. The argument above shows as well that G is Lipschitz. On the other hand, since $g_n\circ f_n = I_X$, we have $G\circ F = (g_i)_\mathscr{U}\circ(f_i)_\mathscr{U} = I_{(X)_\mathscr{U}}$. This concludes the proof of (i).

Part (ii) follows in the same way, with $g = f^{-1}$ if we observe that in this case also $(f_i)_\mathscr{U}\circ(g_i)_\mathscr{U} = I_{(Y)_\mathscr{U}}$.

In the sequel we shall need the following consequence of Proposition 4.1, essentially due to Ribe [34].

COROLLARY 4.2. *If a Banach space X embeds uniformly into a superreflexive space Y such that the image of X is the range of a uniform projection in Y, then X is superreflexive as well.*

Proof. Let $\mathscr{U}$ be a countably incomplete ultrafilter. By Proposition 4.1, $(X)_\mathscr{U}$ is Lipschitz embeddable into $(Y)_\mathscr{U}$. Since Y is superreflexive, $(Y)_\mathscr{U}$ is also superreflexive (cf. [15], Ch. 6). By Theorem 1.3, each separable subspace of X (regarded as a subspace of $(X)_\mathscr{U}$) embeds isomorphically into $(Y)_\mathscr{U}$, and hence is superreflexive. Therefore X itself is superreflexive.

The following result reflects Theorem 2.2.

PROPOSITION 4.3. *Let X be a Banach space which is uniformly embeddable into a superreflexive space Y in such a way that its image is the range of a uniform projection in Y. Then there exists an ultrafilter $\mathscr{U}$ such that X is isomorphic to a complemented subspace of $(Y)_\mathscr{U}$.*

Proof. By the Loewenheim–Skolem Theorem 1.6, there exists a separable subspace Z of X and an ultrafilter $\mathscr{U}$ such that $(Z)_\mathscr{U}$ and $(X)_\mathscr{U}$ are isometric. We may assume that $\mathscr{U}$ is countably incomplete (if this is not the case, replace $\mathscr{U}$ by $\mathscr{U}\times\mathscr{V}$, where $\mathscr{V}$ is any non-trivial ultrafilter on N). From Proposition 4.1 we infer that there exist Lipschitz mappings $F\colon (Z)_\mathscr{U}\to(Y)_\mathscr{U}$ and $G\colon (Y)_\mathscr{U}\to(Z)_\mathscr{U}$ with $G\circ F = I_{(Z)_\mathscr{U}}$. We define a mapping $f\colon Z\to(Y)_{\mathscr{U}}$ by $f=F\circ J_Z$, where J_Z is the canonical embedding of Z into $(Z)_{\mathscr{U}}$. Furthermore, by Corollary 4.2, X and hence Z are superreflexive. Therefore, we can define $g\colon (Y)_{\mathscr{U}}\to Z$ by $g=Q_Z\circ G$, where $Q_Z\colon (Z)_{\mathscr{U}}\to Z$ is the canonical projection of $(Z)_{\mathscr{U}}$ onto Z. Then $g\circ f=I_Z$ and $(Y)_{\mathscr{U}}$ is reflexive—so we are able to apply Theorem 2.2. Thus we conclude that Z is isomorphic to a complemented subspace of $(Y)_{\mathscr{U}}$. Taking ultrapowers, we get $(Z)_{\mathscr{U}}\subset_c (Y)_{\mathscr{U}\times\mathscr{U}}$, and hence $(X)_{\mathscr{U}}\subset_c (Y)_{\mathscr{U}\times\mathscr{U}}$. Since X is reflexive, it is complemented in $(X)_{\mathscr{U}}$ and we conclude finally that X is isomorphic to a complemented subspace of $(Y)_{\mathscr{U}\times\mathscr{U}}$, which completes the proof.

PROPOSITION 4.4. *Let X and Y be uniformly homeomorphic Banach spaces and assume that Y is superreflexive. Then there exists an ultrafilter $\mathscr{U}$ such that $(X)_{\mathscr{U}}$ is isomorphic to a complemented subspace of $(Y)_{\mathscr{U}}$ and $(Y)_{\mathscr{U}}$ is isomorphic to a complemented subspace of $(X)_{\mathscr{U}}$.*

Proof. By Proposition 4.3 and Corollary 4.2, there exist ultrafilters $\mathscr{U}_1$ and $\mathscr{U}_2$ such that

(7) $$X\subset_c (Y)_{\mathscr{U}_1}$$

and

(8) $$Y\subset_c (X)_{\mathscr{U}_2}.$$

The Keisler–Shelah Isomorphism Theorem 1.5 asserts that there is an ultrafilter $\mathscr{U}_3$ such that $(X)_{\mathscr{U}_3}$ and $(X)_{\mathscr{U}_2\times\mathscr{U}_3}$ are isometric. Taking powers with respect to $\mathscr{U}_3$ in (7) and (8), we get

(9) $$(X)_{\mathscr{U}_3}\subset_c (Y)_{\mathscr{U}_1\times\mathscr{U}_3}$$

and

(10) $$(Y)_{\mathscr{U}_3}\subset_c (X)_{\mathscr{U}_3}.$$

Applying Theorem 1.5 once more, we find an ultrafilter $\mathscr{U}_4$ such that $(Y)_{\mathscr{U}_1\times\mathscr{U}_3\times\mathscr{U}_4}$ is isometric to $(Y)_{\mathscr{U}_3\times\mathscr{U}_4}$. From (9) and (10) we derive

$$(X)_{\mathscr{U}_3\times\mathscr{U}_4}\subset_c (Y)_{\mathscr{U}_3\times\mathscr{U}_4}$$

and

$$(Y)_{\mathscr{U}_3\times\mathscr{U}_4}\subset_c (X)_{\mathscr{U}_3\times\mathscr{U}_4}.$$

We set $\mathscr{U}=\mathscr{U}_3\times\mathscr{U}_4$, which accomplishes the proof.

The following theorem is the main result of this section. It shows that in many situations uniform equivalence of Banach spaces implies linear equivalence of closely related structures—the ultrapowers of the original spaces.

THEOREM 4.5. *Let X and Y be Banach spaces. Assume that Y is superreflexive and that the pair X, Y satisfies the Decomposition Scheme. If X and Y are uniformly homeomorphic, then they have isomorphic ultrapowers, i.e. there exists an ultrafilter $\mathscr{U}$ such that $(X)_{\mathscr{U}}$ and $(Y)_{\mathscr{U}}$ are isomorphic.*

Proof. In view of Proposition 4.4 and Pełczyński's Decomposition Theorem 1.4 it remains to check that the ultrapowers $(X)_{\mathscr{U}}$ and $(Y)_{\mathscr{U}}$ also satisfy the Decomposition Scheme. In the case of (D1) this is obvious: $X\simeq X\oplus X$ implies $(X)_{\mathscr{U}}\cong (X)_{\mathscr{U}}\oplus(X)_{\mathscr{U}}$ (and, of course, the same holds for Y). We shall verify (D3) only, because the proof for (D2) is technically quite similar, only easier. So assume that Y is a Banach lattice and that $Y(l_2)\subset_c Y$. Then $\big(Y(l_2)\big)_{\mathscr{U}}\subset_c (Y)_{\mathscr{U}}$. The ultrapower $(Y)_{\mathscr{U}}$ is a lattice as well, and we claim that

$$(Y)_{\mathscr{U}}(l_2)\subset_c \big(Y(l_2)\big)_{\mathscr{U}}.$$

To prove the claim, we first define a map J_0 from the dense subset of $(Y)_{\mathscr{U}}(l_2)$, consisting of the sequences with only finitely many non-zero coordinates, into $\big(Y(l_2)\big)_{\mathscr{U}}$ by setting

$$J_0\big((y_{i,1})_{\mathscr{U}},\ldots,(y_{i,n})_{\mathscr{U}},0,0,\ldots\big)=\big((y_{i,1},\ldots,y_{i,n},0,0,\ldots)\big)_{\mathscr{U}}.$$

J_0 is an isometry. This is a consequence of the identity

(11) $$\Big(\sum_{k=1}^{n}|(y_{i,k})_{\mathscr{U}}|^2\Big)^{1/2}=\bigg(\Big(\sum_{k=1}^{n}|y_{i,k}|^2\Big)^{1/2}\bigg)_{\mathscr{U}}.$$

The verification of (11) can be carried out in a standard way: Approximate the function $\big(\sum_{k=1}^{n}|t_k|^2\big)^{1/2}\colon \boldsymbol{R}^n\to\boldsymbol{R}$ by functions which are obtained from $f_k(t_1,\ldots,t_n)=t_k$, $k=1,\ldots,n$, by applying linear operations finitely many times or taking finite suprema (i.e. functions from $\mathscr{H}_n$ in the notation of [26], Ch.I.d). Then (11) follows from Th.1.d.1 of [26] and the fact that in ultrapowers of Banach lattices $\sup\big((x_i)_{\mathscr{U}},(y_i)_{\mathscr{U}}\big)=\big(\sup(x_i,y_i)\big)_{\mathscr{U}}$.

We extend J_0 to an isometry J from $(Y)_{\mathscr{U}}(l_2)$ into $\big(Y(l_2)\big)_{\mathscr{U}}$. In order to define a projection onto the range of J, we denote by P_n the projection onto the nth coordinate in $Y(l_2)$, and set for $(v_i)_{\mathscr{U}}\in\big(Y(l_2)\big)$

$$Q\big((v_i)_{\mathscr{U}}\big)=\big((P_1v_i)_{\mathscr{U}},(P_2v_i)_{\mathscr{U}},\ldots\big).$$

Again, by (11), Q is a linear map from $\big(Y(l_2)\big)_{\mathscr{U}}$ into $\overline{(Y)_{\mathscr{U}}(l_2)}$ of norm 1. Since Y is superreflexive, $(Y)_{\mathscr{U}}$ is reflexive and hence $\overline{(Y)_{\mathscr{U}}(l_2)}=(Y)_{\mathscr{U}}(l_2)$, (see [26], p. 47). It remains to define $P=JQ$, which yields the desired projection and completes the proof.

A direct consequence of the theorem above is

COROLLARY 4.6. *Let Y be a superreflexive rearrangement invariant function space on $[0,1]$. If X is uniformly homeomorphic to Y, then X and Y have isomorphic ultrapowers.*

An analogous statement also holds for $\mathscr{L}_p$ spaces, $p\in(1,\infty)$. Namely, we have

COROLLARY 4.7. *Let $p\in(1,\infty)$. If X is uniformly homeomorphic to an $\mathscr{L}_p$ space Y, then X and Y have isomorphic ultrapowers.*

Proof. According to a result of Stern ([37], the proof of Th. 4.4) there is a countably incomplete ultrafilter $\mathscr{U}$ such that $(Y)_{\mathscr{U}} \cong (l_p)_{\mathscr{U}}$. By the proof of Theorem 4.5 of the present paper (or by Lemma 4.1 (i) of [37]), $(l_p)_{\mathscr{U}}$ satisfies (D2) of the Decomposition Scheme. Therefore $(Y)_{\mathscr{U}}$ satisfies (D2), as well. On the other hand, Proposition 4.1 shows that $(X)_{\mathscr{U}}$ and $(Y)_{\mathscr{U}}$ are Lipschitz homeomorphic. Since Y is superreflexive, it suffices to apply Theorem 4.5 to the pair $(X)_{\mathscr{U}}, (Y)_{\mathscr{U}}$ to obtain the desired result.

A consequence of Corollary 4.6 is the fact (due to Ribe [35]) that for $p \in (1, \infty)$ the class of $\mathscr{L}_p$ spaces is stable under uniform homeomorphism. Using Proposition 4.3, we get a slightly stronger result (which is implicitly contained in the original proof of Ribe as well).

Corollary 4.8. *Let $p \in (1, \infty)$ and let X be a Banach space which is uniformly embeddable into an $\mathscr{L}_p$ space Y in such a way that the image of X is the range of a uniform projection in Y. Then X either is an $\mathscr{L}_p$ space or is isomorphic to a Hilbert space.*

Proof. By Proposition 4.3, X is isomorphic to a complemented subspace of some ultrapower $(Y)_{\mathscr{U}}$. But $(Y)_{\mathscr{U}}$ is an $\mathscr{L}_p$ space, which implies the desired result (see [24]).

Concluding this section, we shall show that an analogous result holds for $\mathscr{L}_\infty$ spaces. We derive this from a result of Lindenstrauss, using the techniques developed above.

Theorem 4.9. *If a Banach space X is uniformly embeddable into an $\mathscr{L}_\infty$ space Y in such a way that its image is the range of a uniform projection in Y, then X is an $\mathscr{L}_\infty$ space. In particular, the class of $\mathscr{L}_\infty$ spaces is stable under uniform equivalence.*

Proof. In [21], Cor. 4 of Th. 3, Lindenstrauss established the following: Let Z be a Banach space complemented in Z^{**}, which is uniformly embedded into a space $l_\infty(\Gamma)$, for some set Γ, in such a way that the image of Z is the range of a uniform projection from $l_\infty(\Gamma)$. Then Z is an injective space.

Now, let X and Y be as in the hypothesis of the theorem. By the ultrapower version of local reflexivity ([15], Prop. 6.7), there exists a (countably incomplete) ultrafilter $\mathscr{U}$ such that X^{**} is isometric to a complemented subspace of $(X)_{\mathscr{U}}$. On the other hand, according to Proposition 4.1, $(X)_{\mathscr{U}}$ is Lipschitz homeomorphic to a subset of $(Y)_{\mathscr{U}}$ which is the range of a Lipschitz projection in $(Y)_{\mathscr{U}}$. This means that there exist Lipschitz maps $f\colon X^{**} \to (Y)_{\mathscr{U}}$ and $g\colon (Y)_{\mathscr{U}} \to X^{**}$ with $g \circ f = I_{X^{**}}$. The ultrapower $(Y)_{\mathscr{U}}$ is an $\mathscr{L}_\infty$ space, and therefore its second dual $(Y)_{\mathscr{U}}^{**}$ is an injective space, [24]. By Lemma 2.11, g extends to a Lipschitz mapping $\tilde{g}\colon (Y)_{\mathscr{U}}^{**} \to X^{**}$. This shows that X^{**} is Lipschitz embedded into $(Y)_{\mathscr{U}}^{**}$, so that the image admits a Lipschitz projection from $(Y)_{\mathscr{U}}^{**}$. Since $(Y)_{\mathscr{U}}^{**}$, as an injective space, is isomorphic to a complemented subspace of some $l_\infty(\Gamma)$ and X^{**} is complemented in its second dual, we can now apply the result of Lindenstrauss quoted above. Hence X^{**} is an injective space and, consequently, X is an $\mathscr{L}_\infty$ space, [24].

5. The local structure of uniformly homeomorphic Banach spaces. Ribe [34], [35] has established general connections between uniform equivalence and the local structure of the spaces involved (which he used to deduce Corollary 4.8 of the previous section). The proof required a detailed combinatorial analysis of so-called finite point meshes. The aim of this section is to give short proofs of these results based on the theory developed in the previous chapters, which means using essentially only differentiability and ultrapower techniques.

To state the first result, recall that for Banach spaces E and F with $\dim E = \dim F < \infty$ the *Banach–Mazur distance* is defined as $d(E, F) = \inf\{\|T\|\|T^{-1}\|\colon T$ is an isomorphism between E and $F\}$.

Theorem 5.1 (Ribe [34]). *Let X and Y be uniformly homeomorphic Banach spaces. Then there exists a constant $C \geqslant 1$, such that for each finite-dimensional subspace $E \subset X$ there exists a subspace $F \subset Y$ of the same dimension as E with $d(E, F) \leqslant C$.*

Proof. By Proposition 4.1, there is an ultrafilter $\mathscr{U}$ such that $(X)_{\mathscr{U}}$ and $(Y)_{\mathscr{U}}$ are Lipschitz homeomorphic. Let K denote the corresponding Lipschitz homeomorphism constant. Given a finite-dimensional $E \subset X$, E can be considered as a subspace of $(X)_{\mathscr{U}}$; hence it is Lipschitz embedded into $(Y)_{\mathscr{U}}$. Corollary 3.6 and the remark after Theorem 3.5 show that there is a subspace $G \subset (Y)_{\mathscr{U}}^{**}$ such that $d(E, G) \leqslant K$. Applying the principle of local reflexivity, [24], and using the local structure of ultrapowers, [15], Prop. 6.1, we find for each $\varepsilon > 0$ a subspace $F \subset Y$ with $d(F, G) \leqslant 1+\varepsilon$. Hence $d(E, F) \leqslant K(1+\varepsilon)$ and setting $C = K(1+\varepsilon)$ for some $\varepsilon > 0$ completes the proof.

For superreflexive spaces, the following stronger result holds.

Theorem 5.2 (Ribe [35]). *Assume that the Banach spaces X and Y are uniformly homeomorphic and that Y is superreflexive. Then there exists a constant $C \geqslant 1$ such that for each finite-dimensional subspace $E \subset X$ and each positive integer n there is a linear embedding $S\colon E \to Y$ with the following property:*

Whenever F is an n-dimensional subspace of Y, there is a linear mapping $T\colon S(E)+F \to X$ such that TS is the identity on E and $\|S\|\|T\| \leqslant C$.

Proof. It follows from Proposition 4.3 that $X \subset_c (Y)_{\mathscr{U}}$ for some ultrafilter $\mathscr{U}$ on a set I. Let $J\colon X \to (Y)_{\mathscr{U}}$ be an embedding and let $P\colon (Y)_{\mathscr{U}} \to J(X)$ be a projection. Fix any $\varepsilon > 0$ and put $C = \|J\|\|J^{-1}P\| + \varepsilon$. Let $E \subset X$ be a finite-dimensional subspace and fix $n \in \boldsymbol{N}$.

The operator J, restriced to E, induces a family of mappings $S_i\colon E\to Y$, $i\in I$, in the following way: Choose any basis $\{x_k\}_{k=1}^m$ of E, put $y_k = Jx_k \in (Y)_{\mathscr{U}}$ and take representations $y_k = (y_{i,k})_{\mathscr{U}}$. Define S_i, for $i\in I$, by

$$S_i x_k = y_{i,k} \quad \text{for} \quad 1\leqslant k\leqslant m$$

and extend by linearity to the linear span of x_k's. We claim that among the S_i's there is at least one which satisfies the conclusion of the theorem (with $S = S_i$, the fixed n and the constant C above). Assume the claim is false. This means that:

(*) For each $i\in I$ we can find an n-dimensional subspace $F_i\subset Y$ such that, whenever $T\colon S_i(E)+F_i\to X$ is a linear operator with $TS_i = I_E$, then $\|S_i\|\,\|T\| > C$. Denote $G_i = S_i(E)+F_i$. The ultraproduct $(G_i)_{\mathscr{U}}$ can be identified with an r-dimensional subspace G of $(Y)_{\mathscr{U}}$, where $r\leqslant m+n$. We shall show that the operator $J^{-1}P$, restricted to G, induces mappings $T_i\colon G_i\to X$, for $i\in I$, which will, at least for some $i\in I$, contradict (*).

To this end, note that the family $\{y_k\}_{k=1}^m$ belongs to G and forms a basis sequence. Hence it can be extended to a basis $\{y_k\}_{k=1}^r$ of G. Choose representations of the remaining $y_k = (y_{i,k})_{\mathscr{U}}$, for $m<k\leqslant r$ with $y_{i,k}\in G_i$ for $i\in I$ and $m<k\leqslant r$. It is easily checked (cf. e.g. the proof of Prop. 6.1 in [15]) that there is a set $D_1\in\mathscr{U}$ such that, for $i\in D_1$, the family $\{y_{i,k}\}_{k=1}^r$ is a basis of G_i.

For $i\in D_1$ we can now define

$$T_i y_{i,k} = J^{-1}Py_k \quad \text{for} \quad 1\leqslant k\leqslant r.$$

Extend T_i by linearity to the whole of G_i and set $T_i = 0$ for $i\in I\setminus D_1$. By the above definitions, $T_iS_i = I_E$ for $i\in D_1$. It is easy to check (compare again Prop. 6.1 in [15]) that

$$\lim_{\mathscr{U}}\|S_i\|\leqslant\|J\|$$

and

$$\lim_{\mathscr{U}}\|T_i\|\leqslant\|J^{-1}P\|.$$

Hence there is a set $D_2\in\mathscr{U}$ such that $\|S_i\|\,\|T_i\|\leqslant\|J\|\,\|J^{-1}P\|+\varepsilon = C$ for $i\in D_2$. Any T_i with $i\in D_1\cap D_2$ will contradict (*). Since $D_1\cap D_2\neq\emptyset$, this concludes the proof.

It seems worthwhile to summarize the essential direction of the argument above. What we actually did in order to prove Theorem 5.2 was "localizing" Theorem 2.2, i.e. passing to ultrapowers, applying (via Proposition 4.3) the infinite-dimensional Theorem 2.2 to them and finally transferring the outcome back to the original spaces, where it takes local shape. Similarly, the proof of Theorem 5.1 consists in fact of the localization of Corollary 3.6. We refer to [15], Ch. 9 and 10, for further applications of this technique.

6. Comments and Problems. After the striking example given by Aharoni and Lindenstrauss [2], the problem remains open whether each two reflexive or separable uniformly (or Lipschitz) homeomorphic Banach spaces are isomorphic. Also, one might ask generally what *can* be said about the linear structure of uniformly (Lipschitz) homeomorphic Banach spaces. In the sequel we shall point out some partial problems, which arise in connection with the contents of the present paper. We also add some comments concerning the results of the particular sections.

Section 2: The only space for which the problem of Lipschitz classification had been solved before is the Hilbert space [13]. Since the differentiation technique applies, by definition, to separable Banach spaces with the RNP, the only "classical" separable Banach space with the RNP for which we are now unable to solve the problem is l_1. In view of the results in Section 2, to solve the problem of Lipschitz classification for l_1 it suffices to give a positive answer to any of the following two problems:

PROBLEM 6.1. Is every Banach space, Lipschitz homeomorphic to a separable dual space, isomorphic to a dual space?

PROBLEM 6.2. Is every Lipschitz complemented subspace of l_1 linearly complemented in it?

We do not know what is essential in the Aharoni–Lindenstrauss example [2]. Nonreflexivity? Nonseparability? Or both? However, if we take into account Theorem 2.7, there seems to be a good chance for a positive answer to

PROBLEM 6.3. Is every Banach space, Lipschitz homeomorphic to a separable reflexive Banach space X, isomorphic to X?

In the context of Pełczyński's Decomposition Method, the following problem is important.

PROBLEM 6.4. Let X be a separable reflexive Banach space isomorphic to its Cartesian square. Is every Banach space, Lipschitz homeomorphic to X, isomorphic to its square as well?

A positive answer to Problem 6.4 would imply that a Banach space Lipschitz homeomorphic to a separable reflexive Banach space X with a symmetric basis is isomorphic to X.

The results of Section 2 admit some extensions. Since the differentiation argument is of local character, all the corresponding results of this section remain valid (after obvious modifications) if we replace Lipschitz homeomorphisms (or embeddings) of spaces by Lipschitz homeomorphisms (resp. embeddings) of their open subset. In particular, we have the following version of Theorem 2.6:

THEOREM 6.5. *Let X and Y be separable dual Banach spaces such that the pair X, Y satisfies the Decomposition Scheme. If an open subset* *of X is Lipschitz homeomorphic to an open subset of Y, then X is isomorphic to Y.*

Finally, we want to mention that all Lipschitz, isomorphism or embedding constants for the mappings obtained can be computed directly from their constructions. For example, in Proposition 2.1 the Lipschitz constant of the projection onto $Df(X)$ is less than or equal to $K_f K_{f^{-1}} K_{\pi_{f(X)}}$. This remark concerns also the subsequent chapters. E.g., in the homeomorphism case of Proposition 4.1, the constant of the produced Lipschitz homeomorphism between $(X)_{\mathscr{U}}$ and $(Y)_{\mathscr{U}}$ is equal to $C_f C_{f^{-1}}$, where C_f denotes the "best Lipschitz constant for large distances" of f, i.e. $C_f = \inf\{C\colon$ there exists an $r > 0$ such that $\|f(x)-f(y)\| \leqslant C\|x-y\|$ for $\|x-y\| \geqslant r\}$.

Section 4: A uniform classification of Banach spaces seems to be far from a solution. The following problem, which is suggested by the results of Section 4, might be more accessible:

PROBLEM 6.6. Do uniformly homeomorphic Banach spaces have isomorphic ultrapowers (i.e. if X and Y are uniformly homeomorphic, does there exist an ultrafilter $\mathscr{U}$ such that $(X)_{\mathscr{U}}$ and $(Y)_{\mathscr{U}}$ are isomorphic)?

We feel that an important special case is the following

PROBLEM 6.7. Do superreflexive uniformly homeomorphic Banach spaces have isomorphic ultrapowers?

In connection with Problem 6.6 one might wonder whether the two spaces exhibited by Aharoni and Lindenstrauss [2] have isomorphic ultrapowers. As the following proposition shows, they do (thus supporting the hope for a positive solution of Problem 6.6).

PROPOSITION 6.8. *Let Γ be a set of cardinality $2^{\aleph_0}$, let $X = \overline{\operatorname{span}}\{c_0(\boldsymbol{N}) \cup \cup (\chi_{N_\gamma})_{\gamma\in\Gamma}\} \subset l_\infty$, where $(N_\gamma)_{\gamma\in\Gamma}$ is a family of almost disjoint subsets of $\boldsymbol{N}$, and let $Y = c_0(\Gamma)$. Then X and Y have isomorphic ultrapowers.*

Proof (Sketch). Let us say that a Banach space Z, $Z \subset X$ is an *elementary subspace of X* if Z satisfies the conclusion of the Loewenheim–Skolem Theorem 1.6 (i.e. if there is an ultrafilter $\mathscr{U}$ and an isometry S such that the diagram in Th. 1.6 commutes). Using Theorem 1.6 and the structure of X, we can find an alternating chain of separable subspaces of X,

$$c_0(\boldsymbol{N}) = X_0 \subset Z_0 \subset X_1 \subset Z_1 \ldots,$$

such that each Z_n is an elementary subspace of X while each X_n is of the form

$$\overline{\operatorname{span}}\{c_0(\boldsymbol{N}) \cup (\chi_{N_\gamma})_{\gamma\in\tilde\Gamma}\},$$

with $\tilde\Gamma$ countable (depending on n). Obviously $\overline{\bigcup_n X_n}$ is then of the same form, and it is not difficult to see that such a space is isomorphic to $c_0(\boldsymbol{N})$.

On the other hand, the Elementary Chain Theorem of model theory ([6], Th. 3.1.13) implies that $\overline{\bigcup_n X_n} = \overline{\bigcup_n Z_n}$ is an elementary subspace of X. Consequently X and $c_0(\boldsymbol{N})$ have isomorphic ultrapowers. The same approach shows that $c_0(\boldsymbol{N})$ and $c_0(\Gamma)$ have isomorphic ultrapowers (cf. also [17]). Finally, we apply iterated powering as in the proof of Proposition 4.4.

The two problems above are closely related to those concerning Lipschitz classification. Indeed, a positive solution of Problem 6.3 implies a positive solution of Problem 6.7. Similarly, a positive answer to the general separable case of Lipschitz classification (mentioned at the beginning of this section) would give a positive answer to Problem 6.6. To see this, use an elementary chain argument as above together with the following observation: Given a Lipschitz homeomorphism $f\colon X \to Y$ and separable subspaces $X_0 \subset X$, $Y_0 \subset Y$, there exist separable subspaces X_1, Y_1 with $X_0 \subset X_1 \subset X$, $Y_0 \subset Y_1 \subset Y$ and $f(X_1) = Y_1$.

Although there is no converse implication, we are convinced that a positive result concerning say, Problem 6.7, might also give some clue for Problem 6.3.

Another related question concerns the classes $\mathscr{L}_p$. As follows from Ribe's result, Corollary 4.8, and Theorem 4.9, these classes are stable under uniform homeomorphisms for $1 < p \leqslant \infty$. There remains

PROBLEM 6.9. *If a Banach space X is uniformly homeomorphic to an $\mathscr{L}_1$ space, is X itself an $\mathscr{L}_1$ space?*

Obviously, an affirmative answer to Problem 6.6 would solve Problem 6.9 as well.

Let us finally mention that there are separable Banach spaces, namely the spaces l_1 and $L_1[0,1]$, which have isomorphic ultrapowers, [37], but are not uniformly homeomorphic [14]. Similarly, the spaces l_p and $L_p[0,1]$ have isomorphic ultrapowers. Therefore, neither Theorem 4.5 nor any other local approach can distiguish them with respect to uniform equivalence. This leaves open the following problem (cf. [23], Problem 3).

PROBLEM 6.10. *Let $1 < p < \infty$, $p \neq 2$. Are the spaces l_p and $L_p[0,1]$ uniformly homeomorphic?*

Section 5: Continuing the idea of the preceding remark, let us note the following: In the proof of Ribe's two theorems, 5.1 and 5.2, we did not exploit the whole uniform homeomorphism but rather its local action only. In fact, the proofs show that both theorems hold true if we replace the assumption "X is uniformly homeomorphic to Y" in the first case by "X is Lipschitz embeddable into some ultrapower of Y" and in the second case by "X is Lipschitz homeomorphic to a Lipschitz complemented subset of some ultrapower of Y". Therefore, it is not difficult to see that the following "local Lipschitz" variant of Theorem 5.1 also holds:

THEOREM 6.11. *If there is a $K \geqslant 1$ such that each finite subset A of X satisfying $\|x_1 - x_2\| \geqslant 1$ for all $x_1 \neq x_2$; $x_1, x_2 \in A$, embeds into Y with Lipschitz embedding constant not greater than K, then each finite-dimensional subspace of X is isomorphically embeddable into Y with isomorphic embedding constant not greater than $K+\varepsilon$ for arbitrary $\varepsilon > 0$.*

A similar but more involved version of Theorem 5.2 can also be given. Both versions can be obtained also by modifying Ribe's original proofs. We want to stress the following fact: In view of these versions of Ribe's theorems it is clear that both of them belong to step B, as described in the introduction, and are of local character. It seems that the best hope for progress in the problem of uniform classification of Banach spaces lies now in achieving some improvements in step A, based on replacing the local use of the Corson–Klee Lemma by some global argument which will preserve the information that we begin with a surjective homeomorphism. So far we have no idea how to make it.

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