Symbolic calculus on weighted group algebras

by

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Abstract. Let $G$ be a locally compact group of polynomial growth and $\omega$ a non-constant polynomial weight on $G$. It is shown that $L^1(G, \omega)$ is a symmetric algebra and that functions with arbitrarily small support operate on hermitian $L^1$-functions in it.

Let $A$ be a semi-simple, commutative Banach $*$-algebra. The Gel’fand transform * maps $A$ onto a subalgebra $\hat{A}$ in $C_0(M)$ of all continuous functions vanishing at infinity on the space $M$ of all maximal ideals of $A$. A function $F: R \to G$ with $F(0) = 0$ operates on an element $f \in A$ if the range of $f$ is real and $F_0f \in \hat{A}$.

In 1959 Helson, Kahane, Katznelson and Rudin [3] showed that if $F$ operates on every $f = f'$ in $L^1(G)$, where $G$ is an infinite locally compact abelian group, then $F$ is real analytic. On the other hand, under some assumptions on a weight $\omega$ on a discrete abelian group $G$ also non-analytic functions operate on $L^1(G, \omega)$ (cf. [2], [9], [7] and [3]).

All these results are obtained by estimating the growth of $\|w(\eta f)\|$ as $n \to \infty$, where $w(f) = \sum_{k=0}^{\infty} \frac{i^k f^k}{k!}$. In particular,

$$\|w(\eta f)\| = O(\eta^n) \quad \text{as} \quad n \to \infty$$

(1)

for a $\gamma$, $0 < \gamma < 1$, guarantees that functions with arbitrarily small supports operate on $f$.

J. Dixmier [1] extended the notion of function operating on an element $f$ in a non-commutative Banach $*$-algebra $B$ by repeating the above definition with $A$ being the maximal commutative Banach subalgebra of $B$ containing $f$. He proved that, if $G$ is a nilpotent group (or, generally, a group of polynomial growth), then any function $F$, where $F(0) = 0$ and $F'$ is differentiable sufficiently many times, operates on every $f$ in the algebra $L^1(G)$, provided that $f$ is an $L^1$-function with compact support. Later A. Hulanicki [4] obtained a result of this type for nilpotent groups of class 2 and hermitian functions in $L^1(G, \omega)\cap L^1(G)$, where $\omega$ is a polynomial (c.f. (2) below) non-constant weight on $G$. 
In this note we investigate the algebra $L^1(G, \omega)$, where $G$ is a locally compact group of polynomial growth and $\omega$ a polynomial weight with $\int \omega(s^p)ds < +\infty$ for some $p > 0$.

Theorem 1 is a generalization of our earlier result [9] and shows that $L^1(G, \omega)$ is symmetric. If $G$ is that of [5], we get an example of a dense symmetric subalgebra in non-symmetric algebra $L^1(G)$.

In Theorem 2 we prove (1) for any hermitian $L^1$-function $f$ in the algebra $L^1(G, \omega)$. The condition $f \subseteq L^1(G)$ is of course automatically satisfied if $G$ is discrete; thus Theorem 3 implies that some non-analytic classes of functions operate on the algebra $L^1(G, \omega)$.

Let $G$ be a locally compact topological group. A measurable and locally bounded (e.g. bounded on compact sets) function $\omega: G \rightarrow [1, \infty)$ is called a weight if $\int \omega(us)ds \leq \omega(u)\omega(s)$ and $\omega(s^{-1}) = \omega(s)$ for every $u, s \in G$. The space $L^1(G, \omega)$ of all complex functions $f$ on $G$ for which $f \cdot \omega$ belongs to $L^1(G)$ is a Banach $\ast$-algebra with the algebra operations inherited from $L^1(G)$ and the norm

$$||f||_\omega = ||f \cdot \omega||_1.$$ 

It is obvious that the sum and the product of two weights is a weight and the same is true for any positive power of a weight. We say that a weight $\omega_1$ is dominated by $\omega_2$ if there is a positive constant $M$ such that $\omega_1 \leq M\omega_2$. Two weights $\omega_1, \omega_2$ are equivalent if $\omega_1$ is dominated by $\omega_2$ and $\omega_2$ is dominated by $\omega_1$. For equivalent weights $\omega_1, \omega_2$ the algebras $L^1(G, \omega_1)$ and $L^1(G, \omega_2)$ are isomorphic. We call $\omega$ a polynomial weight if

$$\omega(su) = \omega(u) + \omega(s)$$

is bounded on $G \times G$.

One can easily verify that the sum of two polynomial weights and any positive power of polynomial weights are polynomial weights. That this is not true for the product we will see in Example 3.

**Example 1.** Let $G_1 \subset G_2 \subset \ldots$ be an increasing sequence of closed subgroups of a locally compact group $G$ such that $\bigcup_{n=1}^{\infty} G_n = G$. Let $\omega_n \leq \omega_{n+1} \leq \ldots$ be an arbitrary increasing sequence of non-negative numbers. Then the function $\omega: G \rightarrow [1, \infty)$ defined by

$$\omega = 1 + \sum_{n=1}^{\infty} \omega_n \omega_{n+1} - \omega_n$$

is a weight on $G$ if it is locally bounded (for example if any compact set of $G$ is fully contained in some $G_n$). It satisfies the equality

$$\omega(su) = \max \{\omega(u), \omega(s)\}, \quad u, s \in G,$$

and therefore it is a polynomial weight.

**Example 2.** Let $G$ be a compactly generated group. Let $K$ be a pre-compact symmetric (e.g. $K^{-1} = K$) open set in $G$ such that $\bigcup_{n=1}^{\infty} K^n = G$.

Define $\sigma_K(s) = \inf \{n: s \in K^n\}$. Then $\sigma_K(s) \leq \sigma_K(s) + \sigma_K(s)$ for all $s, u \in G$, and hence

$$\sigma_K(s) = e^{\sigma_K(s)}, \quad s \in G,$$

$$\sigma_K(s) = 1 + \sigma_K(s), \quad s \in G$$

are weights on $G$. If the group $G$ is non-compact (in this case $\sigma_K$ is unbounded on $G$) then only the second weight is polynomial.

**Proposition 1.** Let $G$ be a compactly generated group. Let $K$ and $\sigma_K$ be as in Example 2. Every weight $\omega$ on $G$ is dominated by $L^p$, for suitable $\delta > 0$. If $\omega$ is a polynomial weight then it is dominated by $\sigma_K^\omega$.

**Proof.** Denote $A = \sup \{\omega(s): s < +\infty\}$. If $s \in K^n - K^{n-1}$, then $s = a_1 s_2 \ldots s_n$ with $a_k \in K, \quad k = 1, 2, \ldots, n$. Hence $\omega(s) \leq \prod_{k=1}^{n} \omega(a_k) \leq A^n = A^{\omega(s)}$ and it suffices to put $\delta = \log A$. If, moreover, $\omega$ is a polynomial weight and $C$ is a bound of (2) then for $s = a_1 s_2 \ldots s_n$ we have $\omega(s) \leq A(2C)^n = A^n \leq A\sigma_K^s(s)$, where $\delta = 1 + \log \frac{\pi}{2}$. For $2^{m-1} < n < 2^m$, then $s = a_1 s_2$, where $a_1, s_2 \in K^n$, thus $\omega(s) \leq \omega(a_1) + \omega(s_2) \leq A\sigma_K^s(s) < A^\sigma_K^s(s)$.

**Example 3.** On the additive group $\mathbb{R}$ the functions $\omega_1, \omega_2$ given by

$$\omega_1([a_1, a_2]) = 1 + |a_1|, \quad t = 1, 2,$$

are polynomial weights, but their product

$$\omega_1 \omega_2$$

is not. Indeed,

$$\omega_1([t_1, t_2]) = \frac{1 + |t_2|}{2}, \quad t \in E$$

is unbounded.

**Lemma 1.** Let $\omega$ be a polynomial weight. There exists a constant $C$ such that for every $f, g$ in $L^1(G, \omega)$ we have

$$||fg||_\omega \leq C(||f||_1||g||_1 + ||f||_1||g||_1).$$

This is a trivial consequence of (2).

**Corollary 1.** Let $\omega$ be a polynomial weight and let $f \in L^1(G, \omega)$. The spectral radius $r_\omega(f)$ and $r(f)$ in algebras $L^1(G, \omega)$ and $L^1(G)$ coincide.

**Proof.** By (5) we have

$$||f^m||_\omega \leq 2C||f||_1||f||_1,$$

$$||f^n||_\omega \leq 2C||f||_1||f||_1 \leq (2C)^1||f||_1||f||_1,$$

and so on. Generally

$$||f^n||_\omega \leq (2C)^n||f||_1||f||_1^{n-1}.$$
Therefore
\[ r_\omega(f) = \lim_{n \to \infty} \| f^n \|_{L^p}^{\frac{1}{n}} \leq \| f \|_1. \]

Substituting \( f^n \) instead of \( f \) in the last inequality, we get

\[ r_\omega(f^n) = \left( r_\omega(f^n) \right)^{\frac{1}{n}} \leq \left( \| f^n \|_{L^p} \right)^{\frac{1}{n}}, \]

and thus \( r_\omega(f) \leq r_\omega(f^n) \). The converse inequality is obvious, and so \( r_\omega(f) = r_\omega(f^n) \) and the corollary follows.

**Corollary 2.** Let \( \omega \) be a polynomial weight. Every continuous homomorphism of \( L^1(G, \omega) \) into a commutative \( C^* \) algebra can be extended to the whole \( L^p(G) \).

**Proof.** Let \( A \) be a commutative \( C^* \) algebra and let \( \psi : L^1(G, \omega) \to A \). For \( f \in L^1(G, \omega) \) we have

\[ \| \psi(f) \| = \lim_{n \to \infty} \| \psi(f^n) \|^{\frac{1}{n}} \leq r_\omega(f) \leq \| f \|_1. \]

So \( \psi \) is continuous in the \( L^P(G) \) norm.

**Remark 1.** The results of Corollaries 1 and 2 are not true for every weight \( \omega \). This is shown in the following example.

**Example 4.** Let \( G \) be the additive group of real numbers and \( \omega(t) = e^{-t} \). The mapping \( f \to f(t) = \int_0^\infty f(t) e^{-it} dt \) is a continuous linear and multiplicative functional on \( L^1(G, \omega) \) for any complex number \( a \) from the set

\[ D = \{ x \in G : -1 \leq \text{Re} x \leq 1 \}. \]

Therefore

\[ r_\omega(f) > \sup_{x \in D} |f(x)|. \]

In particular, for the characteristic function of the interval \((0, 1)\) we have \( \| f \|_1 = \| f^0 \|_1 = 1 \); thus \( r_\omega(f) = 1 \), but

\[ f(t) = \int_0^1 e^{-it} dt = e^{-1}, \]

and so \( r_\omega(f) \geq e^{-1} > r_\omega(f) \).

**Lemma 2.** Let \( \omega \) be a polynomial weight on \( G \) such that \( \omega^{-1} \) belongs to \( L^p(G) \) for a \( 0 < p < \infty \). Suppose \( f \in L^1(G, \omega) \cap L^p(G) \). Then there exists a constant \( A \), depending only on \( \omega \) and such that

\[ \| f \|_1 \leq A \| f^p \|_{L^p}^{\frac{1}{n}} \| f^0 \|_{L^0}^{\frac{1}{n}}. \]

**Proof.** Let \( a \) be an arbitrary positive number and let

\[ U_a = \{ x \in G : \omega(x) \leq a \}. \]

Then, since \( 1 \leq a^p \omega^{-p}(x) \) for \( x \in U_a \), we have

\[ \| f \|_1 = \int_{U_a} 1 \, d\omega \leq a^p \int_{U_a} \omega^{-p}(x) \, d\omega \leq B a^p \]

(here \( |U_a| \) denotes the Haar measure of the set \( U_a \)). Thus

\[ \| f \|_1 \leq \| f \|_{L^1} + \| f \|_{L^p} + \| f \|_{L^p} \leq \| f \|_{L^p} \leq \| f \|_{L^p} \leq B \| f \|_{L^p} + \| f \|_{L^p}. \]

For \( n = (\| f \|_{L^P(G)}^{1+1/n})^{1/n} \) both factors of the right side of the last inequality are equal and so

\[ \| f \|_1 \leq 2 (\| f \|_{L^P(G)}^{1+1/n})^{1/n} \| f \|_{L^P(G)} \]

This lemma has many consequences, so at first let us consider when the assumption on \( \omega \) may be realized.

**Proposition 2.** Let \( G \) be a compactly generated locally compact group. There exists a polynomial weight \( \omega \) on \( G \) such that \( \omega^{-1} \) belongs to \( L^p(G) \) (or \( L^P(G) \)) if and only if \( G \) is of polynomial growth, e.g. for any compact set \( K \subset G \) there exists a polynomial \( P \) such that \( |K^n| \leq P(n) \), \( n = 1, 2, \ldots \)

**Proof.** Suppose \( \omega^{-1} \) is integrable on \( G \). Let \( K \) be any compact set in \( G \). It is contained in a compact set \( K_0 \) such that \( K_0 \cap K = K \). Then by Proposition 1, \( \omega \leq C \omega_{K_0} \), and thus

\[ |K^n| \leq C (1+n)^n \leq \int \omega_{K_0}^n \leq C \int_{K_0} \omega_{K_0} \leq C_1. \]

Therefore \( |K^n| \leq C_1 (1+n)^n \), and \( G \) is of polynomial growth.

Conversely, if for a compact set \( K \) in \( G \) such that \( K^{-1} = K \), \( \bigcup_{n=1}^\infty K^n = G \) we have \( |K^n| \leq C n^s \), \( n = 1, 2, \ldots \), then \( \omega \omega^{-k-2} \) is integrable on \( G \).

Now we are ready to prove a result which gives a generalization of [9], Theorem 6.

**Theorem 1.** Let \( G \) be a locally compact group. If \( \omega \) is a polynomial weight on \( G \) such that \( \omega^{-1} \) belongs to \( L^p(G) \) for a \( 0 < p < \infty \), then the algebra \( L^p(G, \omega) \) is symmetric.

**Proof.** It suffices to prove \( r_\omega(f) \leq \lambda(f) \) for any hermitian \( f \in L^1(G, \omega) \), where \( \lambda(f) \) denotes the norm of the operator \( T_f \) acting on \( L^p(G) \) by convolution.

Let \( A \) be a function on \( G \) such that \( A \in \mathcal{S}(G, \omega) \). Then each function \( f^n A \), \( n = 1, 2, \ldots \), satisfies the assumption of Lemma 2, and so

\[ \| f^n A \|_1 \leq A \| f^n A \|_{L^P(G)}^{\frac{1}{n}} \| f^n A \|_{L^0}^{\frac{1}{n}}. \]
But \( \|f^n \cdot h\|_\omega \leq \|f\|_{\omega^n} \cdot \|h\|_\omega \) and \( \|f^m \cdot h\|_\omega \leq \|f\|_{\omega^m} \cdot \|h\|_\omega \); thus

\[
\lim_{n \to \infty} \|f^n \cdot h\|_\omega = [\lambda(f)]^n \cdot \|h\|_\omega.
\]

By [9] Lemma 4, for a given \( f \) there exists a continuous function \( \delta \) with compact support such that \( r(f) \leq \lim_{n \to \infty} \|f^n \cdot h\|_\omega \). Therefore by Corollary 1

\[
r(f) \leq [\lambda(f)]^{n+1} \cdot \|r(f)\|^{\omega(n+1)},
\]

and so \( r(f) \leq \lambda(f) \).

**Remark 2.** It is shown in [5] that there exists a countable locally finite group \( G \), e.g., every finitely generated subgroup of \( G \) is finite and such that \( \mathcal{V}(G) \) is not symmetric. It is easy to construct a weight \( \omega \) on \( G \) as in Example 1 such that \( \omega^{-1} \) is integrable. Then by Theorem 1 the algebra \( \mathcal{L}(\mathcal{M}(G)) \) is a dense symmetric subalgebra in the non-symmetric algebra \( \mathcal{L}(\mathcal{M}(G)) \).

**Lemma 3.** Let \( 1 < \omega < 2 \). If \( a_1, a_2, \ldots \) is a sequence of non-negative real numbers such that

(a) \( a_{m+n} \leq a_m a_n \), \( m, n = 1, 2, \ldots \),

(b) \( a_{2n} \leq 2a_n \), \( n = 1, 2, \ldots \),

then \( a_{\alpha} = o(e^{\gamma}) \) for every \( \gamma > \log_2 \lambda \).

**Proof.** From (b) we have \( a_{2^n} \leq 2a_{n+1} a_{n-1} \leq 2a_{n} \), \( k = 0, 1, 2, \ldots \), where \( \beta = (1-\lambda)^{-2} + \log_2 a \). Now an arbitrary natural number \( n \) is of the form \( n = a_2 b_2 + a_1 b_1 + \cdots + a_0 b_0 \), where \( a_0 \) is 0 or 1, \( a_m = 1 \) and \( \log_2 m < \log_2 n \). Therefore by (a)

\[
a_{\alpha} \leq \prod_{k=1}^{m} a_{\beta+k} \leq \prod_{k=1}^{m} 2a_k \leq e_\alpha m \leq e_\alpha \log_2 n,
\]

where \( \gamma = \log_2 \lambda \). It is obvious that the left side of the last inequality is \( o(e^{\gamma}) \) for every \( \gamma > \gamma \).

**Lemma 4.** Let \( \omega \) be as in Theorem 1. For \( f \in L^1(G, \omega) \cap L^2(G) \), let us define

\[
u(f) = \sum_{\gamma \in \mathbb{G}} |\gamma|^\omega f_{\gamma}.
\]

If \( f \) is hermitian and belongs to \( L^2(G, \omega) \cap L^2(G) \), then

(7) \( \|f\|_{\omega} = [\lambda(f)]^{\omega} \cdot \omega(a^{\omega^2}) \),

(8) \( \|u(\omega f)\|_{\omega} = o(a^{\omega^2}) \)

for every \( \gamma > \log_2((2p+2)/(p+2)) \).

**Proof of (7).** Ignoring the trivial case of \( f = 0 \), we may assume \( \lambda(f) = 1 \). We will show that the sequence \( a_n = D_{(\omega f)} a_n \) \( n = 1, 2, \ldots \) satisfies conditions (a) and (b) of Lemma 3 for \( \lambda = (2p+2)/(p+2) \) and a suitable constant \( D \geq 1 \).

We have \( \|\omega f\|_{\omega} = \|f\|_{\omega} \) and thus condition (a) is satisfied. Now by (5) and (6)

\[
\|\omega f\|_{\omega} \leq 20 \|f\|_{\omega} / 2 \|\omega f\|_{\omega} \leq 20A \|f\|_{\omega} \|\omega f\|_{\omega} \|f\|_{\omega}.
\]

Since \( \lambda(f) = 1 \), we have \( \|\omega f\|_{\omega} \leq \|f\|_{\omega} \) for each \( n \); thus choosing \( D \) such that \( D^{-1} \geq 20A \|f\|_{\omega} \|f\|_{\omega} \) we get (b).

**Proof of (8).** First observe that

(9) \( \|u(\omega f)\|_{\omega} \leq 2 \|f\|_{\omega} \).

Indeed, let \( \nu \) be the analytic function defined by

\[
v(\psi) = e^{\sigma -1} \zeta = \sum_{\gamma \in \mathbb{G}} e^{\sigma -1} \gamma \zeta, \quad \zeta \in \mathcal{O}.
\]

Then we have \( u(f) = \nu(T_f) \), where \( \nu(T_f) \) is the operator on \( L^1(G) \) obtained by the action of \( \nu \) on \( T_f \). Therefore

\[
\|u(f)\|_{\omega} = \|\nu(T_f)\|_{\omega} \leq \|\nu(T_f)_{\omega} \|_{\omega} \leq \sup_{\nu \in \mathcal{O}} \|\nu\|_{\omega} \|f\|_{\omega} \leq \|f\|_{\omega}.
\]

To prove (8) we will apply Lemma 3 with the sequence \( a_n = D_{(\omega f)} a_n \) for a suitable constant \( D \geq 1 \). We have

\[
\|u(\omega f)\|_{\omega} + 1 = \|u(\omega f)\|_{\omega} + u(\omega f) + u(\omega f) + u(\omega f)\|_{\omega} + 1 \leq \|u(\omega f)\|_{\omega} + 1;
\]

thus (a) is satisfied. To prove (b), let us consider two cases: First, let \( \|u(\omega f)\|_{\omega} \leq 1 \). Then

\[
\|u(2\omega f)\|_{\omega} + 1 = \|u(\omega f)\|_{\omega} + 1 \leq 4 \|u(\omega f)\|_{\omega} + 1 \leq 4 \|u(\omega f)\|_{\omega} + 1 \leq 4 \|u(\omega f)\|_{\omega} + 1
\]

and so we have (b) for \( D^{-1} \geq 4 \). Second, let \( \|u(\omega f)\|_{\omega} \geq 1 \). Then by (5), (6), and (9)

\[
\|u(2\omega f)\|_{\omega} + 1 = \|u(\omega f)\|_{\omega} + 2 \|u(\omega f)\|_{\omega} + 1 \leq \|u(\omega f)\|_{\omega} + 2 \|u(\omega f)\|_{\omega} + 1 \leq \|u(\omega f)\|_{\omega} + 3 \|u(\omega f)\|_{\omega},
\]

and it suffices to put \( D^{-1} \geq 20A \|f\|_{\omega} \|f\|_{\omega} + 3 \|u(\omega f)\|_{\omega} \).

**Theorem 2.** Let \( G \) be a locally compact group. If \( \omega \) is a polynomial weight on \( G \) such that \( \omega^{-1} \) belongs to \( \mathcal{L}(\mathcal{M}(G)) \) for a \( 0 < p < \infty \), then functions
with arbitrarily small support operate on each hermitian $L'$-function in the algebra $L'(G, \omega)$.

Proof. This is an immediate consequence of (8).

References


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