for $f \in \Omega_\mu^{n-1}(X; \mathfrak{X})$. Then η is subadditive and $\eta(f) \leqslant 2f_c^*$. Now for $f \in \Omega_\mu^{n-1}(X;\mathfrak{X})$ we choose, as in the proof of Theorem 8, a sequence $\{f_k\}$ of simple functions having support of finit emeasure, such that $\lim_{k\to\infty} |||f-f_k||| = 0$ pointwise and $|||f-f_k||| \leqslant 2|||f|||$ for all $k \geqslant 1$. From assumption $({}^*{}_*{}^*)$, Corollary 2 and Theorem 5, it is seen that $\eta(f_k) = 0$ and $\eta(f) \leqslant \eta(f-f_k) + \eta(f_k) \leqslant 2(f-f_k)_c^*$ for all $k \geqslant 1$. Therefore the same argument as that in the proof of Theorem 8 then gives $\eta(f) = 0$ almost everywhere on X after applying Theorem 7 with $\theta > 1$ and $\lambda > 0$. Hence the theorem follows.

Remark. It is known [5] that Theorems 8,9 and 10 hold without the assumption $({}^*{}_*{}^*)$ in the case of $n = 1$, for $({}^*{}_*{}^*)$ is in fact true for $n = 1$. Particularly, for the operators in the setting of Corollaries 3 and 5, these Theorems 8, 9 and 10 remain also true for functions in the larger class $\Omega_\mu^0(X; \mathfrak{X})$.

Finally, I'd like to raise a problem unanswered at this time. In the setting of Section 5, let $n \geqslant 2$ and let f^* be the ergodic maximal function given by (5.2). The following question is open as yet.

Are there L_∞-bounded sublinear operators $U_1, \ldots, U_n$ of weak type (1,1) such that the product operation $U_n \ldots U_1 f$ dominates the function f^*?

References

[1] M. A. Alcoglu, *A pointwise ergodic theorem for L_p-spaces*, Canad. J. Math. 27 (1975), 1075–1082.

[2] R. V. Chacon, *An ergodic theorem for operators satisfying norm conditions*, J. Math. Mech. 11 (1962), 165–172.

[3] N. Dunford and J. T. Schwartz, *Convergence almost everywhere of operator averages*, J. Rat. Mech. Anal. 5 (1956), 129–178.

[4] N. A. Fava, *Weak type inequalities for product operators*, Studia Math. 62 (1972), 271–288.

[5] T. Yoshimoto, *Vector valued ergodic theorems for operators satisfying norm conditions*, Pacific J. Math. 85 (1979), 485–499.

[6] — *Some inequalities for ergodic power functions*, Acta Math. Acad. Sci. Hungar. 36 (1980), 19–24.

[7] — *Pointwise ergodic theorems and function classes M_p^α*, to appear in Studia Math.

[8] — *Ergodic theorems for operators in L_1+L_∞*, to appear.

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Geodesics on open surfaces containing horns

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Abstract. Symbolic dynamics for the geodesic flow on an open surface containing a hyperbolic horn is constructed in a neighbourhood of the geodesic which escapes in the horn in the future and in the past. It provides a variety of geodesics with different asymptotical behaviour. It is proved that a hyperbolic horn is sharp if and only if the geodesic escaping in this horn form a one-parameter family. A description is given of the set of escaping geodesics for a hyperbolic horn which is not sharp. Also the existence of a continuum of oscillating geodesics is proved under the condition that the topology of the surface is sufficiently rich.

Introduction. Let X be a locally compact, separable topological space and let $\{\varphi_t\}_{t\in\boldsymbol{R}}$ be a continuous flow on X.

DEFINITION 1. We call $x \in X$

(a) *bounded for* $t\to+\infty$ $(t\to-\infty)$ iff there is a compact set K such that $\varphi_t x \in K$ for $t \geqslant 0$ $(t \leqslant 0)$;

(b) *escaping for* $t\to+\infty$ $(t\to-\infty)$ iff for every compact set K there is a $T \in \boldsymbol{R}$ such that $\varphi_t x \notin K$ for $t \geqslant T$ $(t \leqslant T)$;

(c) *oscillating for* $t\to+\infty$ $(t\to-\infty)$ iff (a) and(b) are not satisfied.

Obviously the properties (a), (b) and (c) are the properties of the whole trajectories of the flow. In this paper we study the problem of the existence of oscillating trajectories for geodesic flows on complete open surfaces.

Let M be a 2-dimensional manifold (a surface) with a Riemannian metric of class C^2. We assume that M is complete in this metric, not compact and finitely connected. In view of the last assumption there is a homeomorphism h taking M onto $X\setminus\{x_1, \ldots, x_n\}$ where X is a compact surface and $x_i \in X$, $i = 1, \ldots, n$. The points $\{x_1, \ldots, x_n\}$ are called *points at infinity*.

DEFINITION 2. A *tube* is an open subset $\mathscr{T} \subset M$ homeomorphic to a punctured disk such that $h(T)$ is a punctured neighbourhood of a point at infinity and the closure of $h(\mathscr{T})$ in X contains only one point at infinity.

Consider closed rectifiable Jordan curves in a tube $\mathscr{T}$ which cannot be contracted to a point in $\mathscr{T}$. We denote by $w(\mathscr{T})$ the infimum of their lengths.

DEFINITION 3. A tube $\mathscr{T}$ in M is called a *horn* iff no sequence of curves that realises the infimum $w(\mathscr{T})$ is contained in a compact subset of M. A tube $\mathscr{T}$ is called a *cup* iff it is not a horn.

The concept of a tube and the division of tubes into horns and cups is due to Cohn-Vossen [4].

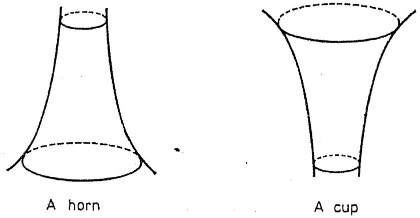

Fig. 1

DEFINITION 4. A horn $\mathscr{T}$ is called *sharp* iff $w(\mathscr{T}) = 0$.

In Chapter I we prove (Theorem 1) that if a surface M contains a horn with some additional property (for instance, the horn is sharp) and M is not homeomorphic to a plane, a cylinder or a projective plane without a point, then for every point $m \in M$ there is a continuum of oscillating geodesics starting at m. The simplest surfaces to which this theorem can be applied are a torus with a hole and a sphere with three holes.

Except for the cases excluded above a fundamental group of an open surface is a free group with more than one generator. In the proof of Theorem 1 we use the correspondence between infinite words of the generators and geodesic rays. The idea goes back to Hadamard [6] and Morse [8].

DEFINITION 5. A horn $\mathscr{T}$ is called *hyperbolic* iff the Gaussian curvature in $\mathscr{T}$ is non-positive.

In Chapter II we study the geometry of hyperbolic horns. We introduce semigeodesical coordinates of class C^1 in a horn under the assumption that there are no focal points in the horn. Such coordinates were constructed in hyperbolic horns by Verner [10]. He considered metrics with bounded curvature in the sense of A. D. Aleksandrov (not necessarily smooth) and he did not study the smoothness of the coordinates obtained. Such coordinates appear also in the work of Eberlein [5] in another context.

Using these coordinates, we prove that a hyperbolic horn is sharp if and only if the geodesic escaping in this horn form a one parameter family (Theorem 3). In Theorem 4 we describe the set of escaping geodesics for a hyperbolic horn which is not sharp.

Assume that there is a hyperbolic horn on M and a geodesic γ which escapes in this horn both in the future and in the past (in the case of a horn which is not sharp we need more: the geodesic is in the boundary of the set of all escaping geodesics and it goes round the horn in the same direction in the future and in the past). In Chapter III (Theorem 2), for some transversal section of the geodesic flow in a small neighbourhood of γ, we construct (under some non-degeneracy conditions) symbolic dynamics with entrances and exits (for a definition see [2]). This theorem gives a variety of geodesics with different geometrical properties. In particular, we obtain a continuum of oscillating geodesics. Also all kinds of asymptotical behaviour in the future and in the past can be freely combined. In the case of a sharp horn we obtain that a geodesic can change an infinite number of times the direction in which it goes round the horn. We have also an infinite family of closed geodesics (all of them hyperbolic).

Theorem 2 can be applied to the geodesic flow on a modular region, a problem studied by Artin [3]. For a large family of geodesics Artin obtained a one-to-one correspondence between geodesics and sequences of positive integers. Theorem 2 gives the same coding but only sequences with elements bigger than a certain N are considered.

The proof of Theorem 2 is obtained by general methods developed by Alekseev [1]. In Chapter III, §1 we formulate the particular theorem we need, and it is a slight generalization of a theorem given by Moser [9].

The symbolic dynamics described in Theorem 2 is analogous to the one constructed by Alekseev in the special case of the three body problem ([1], [9]). We obtain a similar behaviour of trajectories also for some dispersing billiards—a limit case of a horn which is not sharp [11].

In Chapter III, §3 we discuss the situations where Theorem 2 can be applied. In particular, there are no oscillating geodesics on a surface of rotation containing a hyperbolic horn, but after a C^∞ small change of metric (in an arbitratily small domain) Theorem 2 can be applied and we obtain the same picture as for the surface with a large fundamental group.

Part of the results were announced in [12] and the sketches of proofs given there can be helpful to the reader.

I am indebted to Professor V. M. Alekseev, who inspired me to do this work. Also thanks go to Dr. F. Przytycki, who read the manuscript and contributed valuable remarks.

I. Topological theorem

DEFINITION 1. A horn $\mathscr{T}$ is called *regular* iff every tube $\mathscr{T}_1$ such that $\mathscr{T}$ and $\mathscr{T}_1$ are neighbourhoods of the same point at infinity is also a horn.

It is easy to see that if, for a tube $\mathscr{T}$ on $M \neq \mathbf{R}^2$, $w(\mathscr{T}) = 0$ then $\mathscr{T}$ is a regular horn.

THEOREM 1. *If M contains a regular horn and M is not homeomorphic to a plane, a cylinder or a projective plane without a point, then for every point $m \in M$ there is a continuum of oscillating geodesics beginning at m.*

Before proceeding with the proof let us recall the notion of a geodesic domain [4].

DEFINITION 2. A connected subset G of a surface M is called a *geodesic domain* iff for every point $x \in G$ there is a ball $K(x)$ such that the intersection $K_G(x) = K(x) \cap G$ is of one of the following types:

(1) $K_G(x) = K(x)$,

(2) $K_G(x) =$ half of the ball $K(x)$,

(3) $K_G(x) =$ sector of the ball $K(x)$.

Geodesic domain is called *convex* if all interior angles on its boundary are less than π.

We introduce a metric ϱ_G on G by the infimum of the lengths of all rectifiable curves in G connecting the respective points. It is proved in [4] that if G is complete in the metric ϱ_G then in an arbitrary homotopy class of curves connecting two fixed points there is always a curve with minimal length. If, in addition, G is convex then the curve is a smooth geodesic. Note that if G is a closed subset of a complete surface then G is complete in the metric ϱ_G.

The proof of Theorem 1 will be obtained in two steps. We will first prove, for a special geodesic domain, the existence of a continuum of oscillating geodesics beginning at an arbitrarily chosen point. Afterwards we will prove that such a domain can be found on some covering space of the surface M.

Step 1. Let us consider a subset G of the plane C

$$G = \{z \in C \mid 0 < |z| \leqslant 1,\ |z-1/2| \geqslant 1/4\}.$$

We assume that G is a complete geodesic domain for some Riemannian metric defined on a neighbourhood of G in $C \setminus \{0\}$. In particular, $\{|z| = 1\}$ and $\{|z-1/2| = 1/4\}$ are smooth closed geodesics and G is convex.

By a smooth change of variables we can achieve that the curve $\{|z| = 1/2\} \cap G$ will become a geodesic, the shortest one in its homotopy class (homotopy with endpoints on the circle $\{|z-1/2| = 1/4\}$). Indeed, let $c\colon G_2 \to G$ be a 2-sheeted covering of G, where

$$G_2 = \{z \in C \mid 0 < |z| \leqslant 1,\ |z-1/2| \geqslant 1/4,\ |z+1/2| \geqslant 1/4\}.$$

G_2 becomes a complete convex geodesic domain by lifting the metric from G to G_2. We consider the shortest geodesic interval connecting the circles $\{|z-1/2| = 1/4\}$ and $\{|z+1/2| = 1/4\}$ in G_2 and rotate it by the angle π. We obtain two geodesic intervals connecting the circles, which,

being minimal, do not intersect in G_2. Hence they project on a geodesic interval without selfintersections in G. The geodesic interval in G connects different points of the circle $\{|z-1/2| = 1/4\}$ (in view of its minimality it is perpendicular to the circle at both endpoints). Clearly it can be taken onto the arc of the circle $\{|z| = 1/2\}$ by a diffeomorphism of G.

We assume also that the tube $\mathscr{T} = G \cap \{|z| < 1/2\}$ is a horn. Consider

$$\tilde{G} = \{z \in C \mid \operatorname{Im} z \geqslant -1,\ |z-n| \geqslant 1/4 \text{ for } n \in Z\}.$$

Identifying points $z+n$, $n \in Z$ of $\tilde{G}$, we get a domain diffeomorphic to G and a covering $(\tilde{G}, p)$, $p\colon \tilde{G} \to G$. We can assume that an interval $I_0 = \{z \in C \mid 1/4 \leqslant \operatorname{Re} z \leqslant 3/4,\ \operatorname{Im} z = 0\}$ projects onto the arc of the circle $\{|z| = 1/2\}$. $\tilde{G}$ inherits the Riemannian metric from G.

Take $x \in G$. We will prove that there is a continuum of oscillating geodesics in G beginning at x. For simplicity of exposition, we assume that $p^{-1}(x)$ contains the point $z = \frac{1}{2}i$.

Consider curves $z_0(t) = \frac{1}{2}ie^{2\pi it}$, $0 \leqslant t \leqslant 1$ and $z_1(t) = \frac{1}{2}i+t$, $0 \leqslant t \leqslant 1$ in $\tilde{G}$. They project onto closed curves in G with endpoints at x. Denote their homotopy classes in $\pi_1(G, x)$ by a and b, respectively (π_1 is a fundamental group). Let $n_1, n_2, \ldots$ be a sequence of integers $n_k \geqslant 2$, $k = 1, 2, \ldots$ Consider the sequence $c_1, c_2, \ldots$ of elements of $\pi_1(G, x)$:

$$c_1 = ab^{n_1}a$$
$$c_2 = ab^{n_1}ab^{n_2}a$$
$$\dots\dots\dots\dots$$
$$c_k = ab^{n_1}ab^{n_2}a \ldots ab^{n_k}a$$

We choose γ_k to be the shortest geodesic interval in G with x as the beginning and the end, such that its homotopy class is equal to c_k. Let $\tilde{\gamma}_k$ be the lift of γ_k on $\tilde{G}$ with a beginning at $z = \frac{1}{2}i$. $\tilde{\gamma}_k$ goes from $\frac{1}{2}i$ to $\frac{1}{2}i+\sum_{j=1}^{k} n_j$, intersecting the intervals I_0-1, I_0, I_0+n_1-1, I_0+n_1, $I_0+n_1+n_2-1, \ldots$ $\ldots, I_0+\sum_{j=1}^{k} n_j-1$, $I_0+\sum_{j=1}^{k} n_j$, each of them once and no other ones among the intervals I_0+n, $n \in Z$ (Fig. 2). Indeed, this follows from the fact

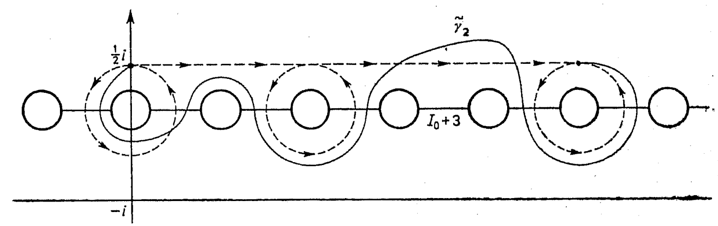

Fig. 2. $n_1 = 2$, $n_2 = 3$

that I_0+n, $n \in \boldsymbol{Z}$ are the shortest geodesic intervals in the respective homotopy classes.

Consider $\tilde{\gamma}$ — one of the limit geodesic rays for the sequence $\tilde{\gamma}_k$, $k = 1, 2, \ldots$ Obviously $\tilde{\gamma}$ intersects all those intervals I_0+n, $n \in \boldsymbol{Z}$ that are intersected by some $\tilde{\gamma}_k$ and no others. Hence the intervals intersected by $\tilde{\gamma}$ are determined only by the sequence $n_1, n_2, \ldots$ Consequently, for different sequences $n_1, n_2, \ldots$ we get different geodesics rays $\tilde{\gamma}$.

To end step 1 we will show that if the sequence $n_1, n_2, \ldots$ is unbounded then the corresponding geodesic ray $\tilde{\gamma}$ is oscillating. Indeed, $\tilde{\gamma}$ contains the shortest geodesic intervals connecting I_0+n and I_0+n+m with m arbitrarily large (the shortest in the respective homotopy class with fixed endpoints). But for every compact subset $K \subset G$ there is an $m(K)$ such that if $m \geqslant m(K)$ then the projection of the geodesic interval above on G is not contained in K. To prove this recall that $\mathcal{T} = G \cap \{|z| < 1/2\}$ is a horn and so for every compact subset $K \subset G$ we have w_K, $w_K > w(\mathcal{T}) + \varepsilon$ for a certain $\varepsilon > 0$, such that any closed rectifiable Jordan curve in $K \cap \mathcal{T}$ that cannot be contracted to a point has length not less than w_K. Hence it is not difficult to see that any curve in $\tilde{G}$ connecting I_0+n and I_0+n+m and projecting onto $K \cap \mathcal{T}$ has length not less than $(m-1)w_K$. On the other hand, if we do not restrict ourselves to K we can find a curve connecting I_0+n and I_0+n+m with length not bigger than $m\big(w(\mathcal{T})+\varepsilon\big)+$ $+C$ where the constant C does not depend on m. Now we take $m(K)$ so large that

$$m(K)\big(w(\mathcal{T})+\varepsilon\big)+C < \big(m(K)-1\big)w_K,$$

i.e.

$$m(K) > \frac{C+w_K}{w_K - w(\mathcal{T}) - \varepsilon}.$$

The corresponding geodesic interval cannot project onto K in view of its minimality.

Step 2. It is not difficult to find a geodesic domain of the type considered in step 1 for a special M. For instance, let $M \approx X \setminus \{x_1\}$ where X is a torus. Consider l — one of the generators of the fundamental group of X. l can be treated as an element of the fundamental group of M. Take the smooth closed geodesic in M which has the homotopy class l (free homotopy) and is the shortest possible. Cut M along this geodesic. We obtain the geodesic domain we need.

The difficulty of our problem lies in the fact that we want a general construction that would apply to all possible M. Let $m \in M$ and $\mathcal{T} \subset M$ be a regular horn. We will show that if M is not of a topological type excluded in the theorem then there is a geodesic domain like G from step 1 on some covering space of M and it contains a point projecting on m.

Obviously oscillating geodesics in this domain project onto oscillating geodesics on M.

Taking a 2-sheeted covering space, we can restrict ourselves to the case of orientable M. We have $M = X \setminus \{x_1, \ldots, x_n\}$ where X is a compact surface and $\mathcal{T}$ is a punctured neighbourhood of x_1 in X. If X is not homeomorphic to a sphere $\boldsymbol{S}^2$ then we consider the universal covering of X, $(\boldsymbol{R}^2, p)$, $p\colon \boldsymbol{R}^2 \to X$. If, on the contrary, $X = \boldsymbol{S}^2$ then $n \geqslant 3$ and let $(\boldsymbol{R}^2, p)$ denote the universal covering of the cylinder $X \setminus \{x_{n-1}, x_n\}$.

The set $p^{-1}(\{x_1, \ldots, x_n\})$ $(p^{-1}(\{x_1, \ldots, x_{n-2}\})$ in the case $X = \boldsymbol{S}^2)$ is denumerable and discrete. Denote its elements by $y_1, y_2, \ldots$ where $p(y_1) = x_1$. The pair $(Y, p|Y)$, where $Y = \boldsymbol{R}^2 \setminus \{y_1, y_2, \ldots\}$, is a covering of M. Y inherits the Riemannian metric from M. Let $y_0 \in p^{-1}(m)$. We will prove the following alternative:

(1) there is a smooth closed geodesic without selfintersections in Y with points y_0, y_1, y_2, y_3 in its interior;

(2) there is a closed geodesic without selfintersections in Y with endpoints in y_0, with points y_1, y_2, y_3 in its interior and with the interior angle in y_0 not bigger than π.

Consider the family $\mathcal{A}$ of all closed rectifiable curves without selfintersections in Y with endpoints in y_0 and with points y_1, y_2, y_3 (and possibly some other deleted points) in their interiors. By $\tilde{\mathcal{A}} \supset \mathcal{A}$ we denote the family of closed rectifiable curves in Y with endpoints in y_0 homotopic to curves form $\mathcal{A}$ (homotopy in Y with a fixed point in y_0). Let $\inf \tilde{\mathcal{A}}$ be the infimum of the lengths of curves from $\tilde{\mathcal{A}}$. Obviously there is a closed geodesic γ_0 in $\tilde{\mathcal{A}}$ with length equal to $\inf \tilde{\mathcal{A}}$. One can check that in view of its minimality γ_0 belongs to $\mathcal{A}$ (i.e. γ_0 has no selfintersections). Denote the interior of γ_0 in Y by W_0. If the interior angle in y_0 is not bigger than π, then (2) is satisfied. If the angle is bigger than π, then $Y \setminus W_0$ is a complete convex geodesic domain. In $Y \setminus W_0$ we can find a smooth closed geodesic $\tilde{\gamma}_0$ without selfintersections such that its interior in Y, $\tilde{W}_0$, contains γ_0. Hence (1) is satisfied. The construction of $\tilde{\gamma}_0$ is analogous to that of γ_0 and is here omitted.

We proceed differently in cases (1) and (2). Suppose (2) holds. Then $W_0 \cup \gamma_0$ is a complete convex geodesic domain. There is a smooth closed geodesic γ_1 in $W_0 \cup \gamma_0$ without selfintersections the interior of which contains all those deleted points y_i, $i = 1, 2, \ldots$, which were inside γ_0 except for y_1 (i.e. at least points y_2, y_3). To obtain γ_1 we take a geodesic ray γ beginning on γ_0 and escaping in y_1. We cut $W_0 \cup \gamma_0$ along γ and obtain a convex complete geodesic domain with a boundary composed of γ_0 and γ taken twice. Now γ_1 is a smooth closed geodesic without selfintersections in $W_0 \setminus \gamma$ such that its interior contains all the deleted points which are in $W_0 \setminus \gamma$. Denote the interior of γ_1 in Y by W_1. $(W_0 \setminus W_1) \cup \gamma_0$

is a complete convex geodesic domain with all the properties of G from step 1 except for the fact that γ_0 is not generally smooth. But returning to the proof of step 1 we see that this does not really matter.

If (1) holds, we consider the convex geodesic domain $\tilde{W}_0 \cup \tilde{\gamma}_0$. There is a closed geodesic without selfintersections $\tilde{\gamma}_1$, $\tilde{\gamma}_1 \subset \tilde{W}_0$ with endpoints in y_0 such that its interior contains all the deleted points that are inside $\tilde{\gamma}_0$ except y_1. Denote the interior of $\tilde{\gamma}_1$ in Y by $\tilde{W}_1$. If the interior angle in y_0 is not less than π, then $(\tilde{W}_0 \smallsetminus \tilde{W}_1) \cup \tilde{\gamma}_0$ is the geodesic domain we need. If the angle is less than π, then $\tilde{W}_1 \cup \tilde{\gamma}_1$ is a convex geodesic domain. We consider a smooth closed geodesic without selfintersections $\tilde{\tilde{\gamma}}_1$ in $\tilde{W}_1$ such that its interior contains all the deleted points that are inside $\tilde{\gamma}_1$. Let $\tilde{\tilde{W}}_1$ be the interior of $\tilde{\tilde{\gamma}}_1$ in Y. Then $(\tilde{W}_0 \smallsetminus \tilde{\tilde{W}}_1) \cup \tilde{\gamma}_0$ is a complete convex geodesic domain containing y_0 with all the properties of G from step 1. To end the proof we have to show that the tube in the domain we have constructed is a regular horn.

Let $\tilde{\mathcal{T}}$ be a tube on Y with a point at infinity y_1 and such that it projects one-to-one onto the regular horn $\mathcal{T}$ on M. Obviously $\tilde{\mathcal{T}}$ is a horn. We claim it is a regular horn. Suppose, on the contrary, that there is a cup $\tilde{\mathcal{T}}_1 \subset Y$, which is a punctured neighbourhood of y_1. Then (see [4]) there is a smooth closed geodesic γ in Y such that its interior $\tilde{\mathcal{T}}_2$ is a punctured neighbourhood of y_1 and the length of γ is equal to $w(\tilde{\mathcal{T}}_2)$. $\tilde{\mathcal{T}}_2$ is also a cup and it projects one-to-one on M, which contradicts the regularity of the horn $\mathcal{T}$. Indeed, $\pi_1(X, m)$ acts on Y as a group of isometries and if there are points $y, \bar{y} \in \tilde{\mathcal{T}}_2$ such that $p(y) = p(\bar{y})$, then there is an $a \in \pi_1(X, m)$ such that $ay = \bar{y}$. Hence $a\gamma \cap \gamma \neq \emptyset$ and $\boldsymbol{R}^2 \smallsetminus (a\gamma \cup \gamma)$ has more than two connected components. Take the components that contain y_1 and ay_1, respectively. They are different components because otherwise we would have $ay_1 = y_1$ and $\pi_1(X, m)$ acts freely on $\boldsymbol{R}^2$. One of these components has a boundary with length not greater than the length of γ. But since the boundary is not smooth, it contradicts the minimality of γ. Theorem 1 is proved. ■

Consider surfaces of rotation: M_1 with one horn, homeomorphic to a plane and M_2 with two horns, homeomorphic to a cylinder (Fig. 3). These examples show that Theorem 1 is not true for the excluded cases where the fundamental group is trivial or commutative.

In the rest of the paper we will show that if the horns on M_1 and M_2 are hyperbolic then by an arbitrarily small change of the metric (in an arbitrarily small neighbourhood of an arbitrary point) we obtain a continuum of oscillating geodesics for the new metric.

It is worth mentioning that one can prove the existence of a continuum of oscillating geodesics on a surface with a regular horn homeomorphic to a cylinder (a plane) if there is a smooth closed geodesic contractible to a point on that surface (two such geodesics, outside each other in the case of a plane). The proof is the same: we remove the interiors of the geodesics and obtain a complete convex geodesic domain containing a regular horn with the fundamental group sufficiently large.

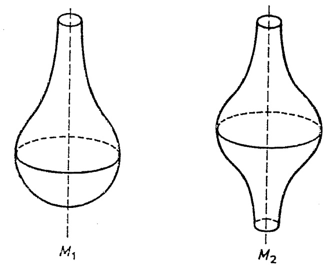

Fig. 3

II. Hyperbolic horns

§ 1. Lemmas about principal solutions and construction of horospheres on a plane without focal points. Let us consider a second order differential equation

$$y'' + k(t)y = 0 \tag{1}$$

where $k(t)$ is a continuous function defined on an interval $I = [a, b)$.

DEFINITION 1. Equation (1) is said to *have no focal points in* I iff for every solution $y(t)$ of (1) such that $y(t_1) = 0$ and $y'(t_2) = 0$, $t_1, t_2 \in I$, we have $y(t) \equiv 0$, $t \in I$.

DEFINITION 2. Equation (1) is said to *have no conjugated points in* I iff for every solution $y(t)$ of (1) such that $y(t_1) = 0$ and $y(t_2) = 0$, $t_1, t_2 \in I$, we have $y(t) \equiv 0$, $t \in I$.

If (1) has no focal points then obviously it has no conjugated points. Also if $k(t) \leqslant 0$, $t \in I$ then (1) has no focal points.

DEFINITION 3. A solution $y(t)$ of (1) is called *principal* iff there is a $T \in I$ such that $y(t) \neq 0$ for $t \in [T, b)$ and $\int_T^b \big(1/y^2(t)\big)\,dt = \infty$.

It is proved in [7] that every equation (1) with no conjugated points (and even under weaker assumptions) has a principal solution and it is determined uniquely up to a multiplicative constant.

We shall assume that if $b < +\infty$ then $k(t)$ is a continuous function on the closed interval $[a, b]$.

LEMMA 1. *If a solution $y(t)$ of* (1) *has the property $y(t) > 0$, $y'(t) \leqslant 0$ for $t \in I$ in case $b = +\infty$ ($y(t) > 0$, $y'(t) \leqslant 0$ for $t \in I$ and $\lim_{t\to b} y(t) = 0$ in case $b < +\infty$) then $y(t)$ is a principal solution of* (1) *and* (1) *has no conjugated points.* ■

LEMMA 2. *If* (1) *has no focal points in I then it has a principal solution $y_0(t)$ with the property $y_0(t) > 0$, $y_0'(t) \leqslant 0$ for $t \in I$ in case $I = [a, +\infty)$ ($y_0(t) > 0$, $y_0'(t) \leqslant 0$ for $t \in I$ and $\lim_{t\to b} y_0(t) = 0$ in case $b < +\infty$).*

Proof. Consider the family of solutions of (1) $y(t; r)$, $r \in (a, b)$ such that $y(a; r) = 1$ and $y(r; r) = 0$. Then for every $r \in (a, b)$ there is a $t \in (a, r)$ such that $y'(t; r) < 0$ and hence $y'(t; r) < 0$ for every $t \in [a, r]$ because (1) has no focal points. We claim that $y'(a; r)$ is strictly increasing as a function of r. Indeed, let us suppose, on the contrary, that $y'(a; r_2) \leqslant y'(a; r_1)$ for some $r_2 > r_1$; then $y(t) = y(t; r_2) - y(t; r_1)$ is a solution of (1) such that $y(a) = 0$, $y'(a) \leqslant 0$ and $y(r_1) > 0$. It follows that $y(t) \equiv 0$ and so $y(t; r_2) \equiv y(t; r_1)$ which is contradictory.

Consider a solution $y_0(t)$ such that $y_0(a) = 1$ and $y'(a) = \lim_{r\to b} y'(a; r)$. In view of continuous dependence on initial conditions we have $y_0(t) \geqslant 0$ and $y_0'(t) \leqslant 0$ for $t \in I$ and in case $b < +\infty$ also $\lim_{t\to b} y_0(t) = 0$. Now from Lemma 1 it follows that $y_0(t)$ is a principal solution. ■

LEMMA 3. *If* (1) *has no focal points in $I = [a, +\infty)$ and $y(t)$ is a nonconstant solution of* (1) *such that $y(a) \geqslant 0$, $y'(a) \geqslant 0$, then $y(t) > 0$ for $t > a$ and $\lim_{t\to\infty} y(t) = +\infty$. Moreover, if $y(a) = 0$ and $y'(a) > 0$ then $y(t)$ is increasing.*

Proof. Suppose $y(t) = 0$ for a certain $t \geqslant a$; then $y' > 0$ because (1) has no focal points. This proves that $y(t) > 0$ for $t > a$ and that the second part of the lemma holds.

In view of Lemma 2, $y(t)$ is not a principal solution and so $\int_t^\infty (1/y^2(s))ds < \infty$. If $y(a) = 0$ then it follows that $\lim_{t\to\infty} y(t) = \infty$. If $y(a) > 0$ then

$$y_1(t) = y(t) + cy(t)\int_t^\infty \frac{ds}{y^2(s)}$$

is a solution of (1). Take $c < 0$ such that $y_1(a) = 0$; then

$$y_1'(a) = y'(a) + cy'(a)\int_a^\infty \frac{ds}{y^2(s)} - c\frac{1}{y(a)} = -c\frac{1}{y(a)} > 0$$

and so $y_1(t)$ is increasing and $\lim_{t\to\infty} y_1(t) = \infty$. But $y_1(t) < y(t)$ for all t. ■

LEMMA 4 (Continuity of principal solutions). *Let $k_1(t), k_2(t), \ldots$ be continuous functions defined on $[a, b_i]$ and such that the equations*

$$(2_i) \qquad y'' + k_i(t)y = 0$$

have no focal points in $I_i = [a, b_i)$ for $i = 1, 2, \ldots$ and $b_i \to \infty$, $k_i(t) \to k_\infty(t)$ as $i \to \infty$, uniformly on closed intervals.

Denote by y_{0i} the principal solution of (2_i) such that $y_{0i}(a) = 1$. Then the limits $\lim_{i\to\infty} y_{0i}(t) = y_\infty(t)$ and $\lim_{t\to\infty} y_{0i}'(t) = y_\infty'(t)$ exist, convergence is uniform on closed intervals and $y_\infty(t)$ is a principal solution of (2_∞).

Proof. In view of Lemma 1 and Lemma 2 the proof of conclusion 6.6, Chapter XI, [7] can be repeated. ■

Lemma 4 will be crucial in the proof of the following theorem, which can be viewed as a theorem on the existence and smoothness of horospheres on a plane without focal points.

Consider a 2-manifold M homeomorphic to $\boldsymbol{R}^2$ with a Riemannian metric of class C^2 without focal points (i.e. the Jacobi equation along any geodesic has no focal points). Suppose M is complete in this metric. Let $\gamma_0(r)$, $r \geqslant 0$ be a geodesic ray in M (γ_0: $[0, \infty) \to M$) and $\delta(s)$, $|s| \leqslant c$ a geodesic interval such that $\gamma_0(0) = \delta(0)$ and δ is transversal to γ_0 in $\delta(0)$. For every $r > 0$ we connect $\gamma_0(r)$ with points of the geodesic interval δ by geodesic intervals. We denote by $\beta_r(s)$ the angle between the velocity vector of δ and these geodesic intervals (Fig. 4).

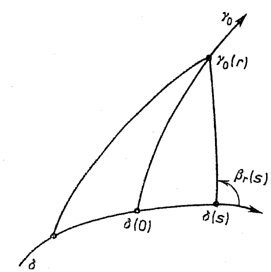

Fig. 4

THEOREM 1. (a) *The functions $\beta_r(s)$ converge uniformly together with their first derivatives to a function $\beta(s)$ of class C^1 as $r \to \infty$. Moreover,*

$$(3) \qquad \frac{d\beta}{ds} = -u_t'(0; s)\sin\beta(s)$$

where $u(t; s)$ is a principal solution of the equation $u_t'' = -k(t; s)u$ such that $u(0; s) = 1$ and $u_t'(t; s) \leqslant 0$ for all $t \geqslant 0$. $k(t; s)$ denotes the Gaussian *curvature along the geodesic γ_s starting from $\delta(s)$ at the angle $\beta(s)$ with the velocity vector of δ.*

(b) *If we apply the construction above to γ_{s_0} and δ (instead of γ_0 and δ), we obtain by* (a) *another function $\tilde\beta(s)$; then $\beta(s)=\tilde\beta(s)$ for every $|s|\leqslant c$.*

Proof. (a) We shall consider graphs of the functions $\beta_r(s)$ in the plane (s,β). By $\tau_{(s,r)}\colon [0,b(s,r)]\to M$ we denote the geodesic interval connecting $\delta(s)$ and $\gamma_0(r)$.

Taking into account the geometrical sense of the Jacobi equation, we see that $\beta_r(s)$ is a C^1 function such that

$$\frac{d\beta_r}{ds} = -u_t'(0;s,r)\sin\beta_r(s)$$

where $u(0;s,r)=1$ and $u(t;s,r)$ is a principal solution of the equation $u_t''=-k(t;s,r)u$, $0\leqslant t<b(s,r)$; $k(t;s,r)$ is the Gaussian curvature along $\tau_{(s,r)}$. It is not difficult to see that

$$0\geqslant u_t'(0;s,r)\geqslant \text{const} \quad \text{for} \quad r\geqslant r_0>0 \text{ and } |s|\leqslant c.$$

Hence all β_r, $r\geqslant r_0$ are Lipschitz functions with Lipschitz constants bounded uniformly in r. We have

$$\beta_r(s)\leqslant \beta_{r_1}(s) \quad \text{for} \quad s\leqslant 0,\ r_1\geqslant r$$

and

$$\beta_{r_1}(s)\leqslant \beta_r(s) \quad \text{for} \quad s\geqslant 0,\ r_1\geqslant r.$$

Hence $\beta_r(s)$ have a pointwise limit $\beta(s)$ as $r\to\infty$. We conclude that β_r converge uniformly to β as $r\to\infty$.

In view of the continuity of principal solutions the functions $d\beta_r/ds$ have a pointwise limit (3). Moreover by the Lipschitz condition for β_r and the continuity of principal solutions the functions $d\beta_r/ds$ are continuous uniformly in r. (To prove this one should check that, for every s, $|s|\leqslant c$, $d\beta_r/ds$ are continuous at s uniformly in $r\geqslant r_0$ and the case $s=0$ has to be treated separately.) And so $d\beta_r/ds$, $r\geqslant r_0$ form a compact family of functions in the space of continuous functions with the topology of uniform convergence. Hence $d\beta_r/ds$ are uniformly convergent to their pointwise limit as $r\to\infty$.

(b) Let $s_0>0$. Simple geometrical considerations show that $\tilde\beta(s)=\beta(s)$ for $s<0$ and $s>s_0$. Take $0<s_1<s_0$; then $\tilde\beta(s_1)\leqslant\beta(s_1)$ (Fig. 5). If $\tilde\beta(s_1)<\beta(s_1)$, we obtain two geodesic rays starting at $\delta(s_1)$ and contained in the domain covered by geodesic rays γ_s, $0\leqslant s\leqslant s_0$. The distance between $\gamma_0(r)$ and $\gamma_{s_0}(r)$ is bounded in r. On the other hand, on a plane without focal points two geodesics starting from one point must depart unboundedly (this follows essentially from Lemma 3, the details will be repeated in

Lemma 5 in §2). Thus the resulting contradiction shows that $\tilde\beta(s_1)=\beta(s_1)$. Theorem 1 is proved. ■

By Theorem 1 for a given geodesic ray γ_0 we obtain a family of geodesic rays γ_s; they are called asymptotic for γ_0 because the distance between $\gamma_0(r)$ and $\gamma_s(r)$ is bounded in r. The existence and smoothness (C^1) of horospheres (limit circles) follows from Theorem 1.

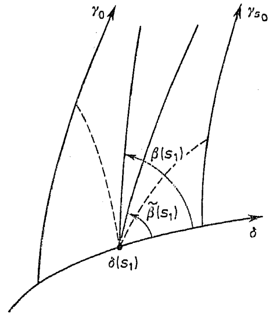

Fig. 5

§2. Construction of semigeodesical coordinates in a horn without focal points. We shall consider a horn $\mathcal{T}$ on some complete surface M with a Riemannian metric of class C^2. We assume that there are no focal points in $\mathcal{T}$ (i.e. that the Jacobi equation along any geodesic interval contained in $\mathcal{T}$ has no focal points).

Let $x_n\in\mathcal{T}$, $n=0,1,\ldots$ and $x_n\to\infty$. If x_0 and x_n are far enough from the boundary of $\mathcal{T}$ in M then the shortest geodesic interval connecting x_0 and x_n is contained in $\mathcal{T}$; it is so because $\mathcal{T}$ is a horn. We choose x_n, $n=0,1,\ldots$ to have this property. Let γ_0 be a limit geodesic ray for the sequence of the shortest geodesic intervals connecting x_0 and x_n, $n=1,2,\ldots$ γ_0 is an escaping geodesic ray contained in $\mathcal{T}$ and has the property of being the shortest connection between any two of its points. (It will turn out that because there are no focal points in $\mathcal{T}$ the ray γ_0 is uniquely determined by x_0.)

Let δ be a geodesic interval in $\mathcal{T}$ with endpoints in x_0 which goes once around $\mathcal{T}$. There will be such an interval in $\mathcal{T}$ provided x_0 is far enough from the boundary of $\mathcal{T}$.

Consider a tube $\mathcal{T}_1\subset\mathcal{T}$ such that δ is the boundary of $\mathcal{T}_1$. We cut $\mathcal{T}_1$ along γ_0 and obtain an unbounded geodesical triangle Δ with sides γ_0, δ and γ_0 (more strictly, Δ is a geodesic triangle on a two-sheeted covering space).

Let $s \in [0, d]$ denote the length parameter on δ and $r \in [0, +\infty)$ the length parameter on γ_0. We fix $\bar{s} \in [0, d]$ and connect $\delta(\bar{s})$ and $\gamma_0(r)$ in Δ by a geodesic interval. This can be done in two ways by taking $\gamma_0(r)$ on the left or on the right side of Δ. Denote by $\beta_L(r)$ and $\beta_R(r)$ the angles between δ and the corresponding geodesic intervals (Fig. 6).

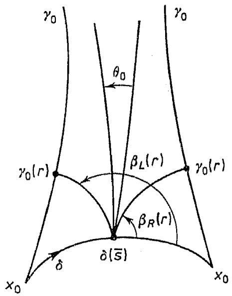

Fig. 6

LEMMA 5. *$\beta_R(r)$ increases and $\beta_L(r)$ decreases as r increases and*

$$\lim_{r\to\infty}\beta_L(r) = \lim_{r\to\infty}\beta_R(r).$$

Proof. Two geodesics in Δ can intersect only once because there are no focal points in $\mathcal{T}$. It follows that $\beta_L(r)$ and $\beta_R(r)$ are monotonous functions of r as stated in the lemma.

Suppose that $\lim\limits_{r\to\infty}\beta_L(r)-\beta_R(r) = \theta_0 > 0$. Between the limit geodesics we introduce geodesic polar coordinates (ϱ, θ) with a pole in $\delta(\bar{s})$ $(\varrho > 0$, $\theta \in [0, \theta_0])$. The Riemannian metric in these coordinates has the form $d^2\varrho + Y^2(\varrho, \theta)d^2\theta$ where $Y(\varrho, \theta)$ is the solution of the equation $Y''_\varrho = -k(\varrho, \theta)Y$ with initial values $Y(0, \theta) = 0$, $Y'_\varrho(0, \theta) = 1$ ($k(\varrho, \theta)$ is the Gaussian curvature).

By Lemma 3, $Y(\varrho, \theta)$ increases as ϱ increases and $\lim\limits_{\varrho\to\infty} Y(\varrho, \theta) = +\infty$. It follows that an arbitrary rectifiable curve $(\varrho(\theta), \theta)$, $0 \leqslant \theta \leqslant \theta_0$ which is contained in the domain $\varrho \geqslant \varrho_0$ has arbitrarily great lenght, provided ϱ_0 is sufficiently large. Indeed,

$$\int_0^{\theta_0}\sqrt{\left(\frac{d\varrho}{d\theta}\right)^2 + Y^2}\, d\theta \geqslant \int_0^{\theta_0} Y(\varrho_0, \theta)\,d\theta \to \infty \quad \text{as} \quad \varrho_0 \to \infty.$$

But $\mathcal{T}$ is a horn and hence there is a sequence of curves, divergent to infinity, connecting the sides of Δ and with bounded lenghts. The contradiction shows that $\theta_0 = 0$. The lemma is proved. ∎

From Lemma 5 we obtain that for every $s \in [0, d]$ there is a unique geodesic ray γ_s starting in $\delta(s)$ and contained in Δ (in other words, not intersecting γ_0). We denote by $\beta(s)$ the angle at $\delta(s)$ between δ and γ_s. It follows from Theorem 1 that $\beta(s)$ is of class C^1.

We introduce in Δ coordinates (r, s) such that s is the length parameter on δ and r the length parameter on γ_s. The coordinates (r, s) are of class C^1. By elementary differential geometry the metric tensor in these coordinates has the form

$$d^2r + 2\cos\beta(s)\,dr\,ds + \big(\cos^2\beta(s) + u^2(r, s)\sin^2\beta(s)\big)d^2s$$

where $u(r, s)$ is the principal solution along γ_s such that $u(0, s) = 1$. Consider the change of variables $r = r' + c(s')$, $s = s'$ where $dc/ds' = -\cos\beta(s')$ and $c(0)$ is so large that $c(s') \geqslant 0$ for $s' \in [0, d]$. The metric tensor in these coordinates has the form

$$d^2r' + u^2\big(r' + c(s'), s'\big)\sin^2\beta(s')\,d^2s'.$$

LEMMA 6. $c(0) = c(d)$ *(i.e.* $\int_0^d \cos\beta(s)\,ds = 0$).

Proof. Suppose on the contrary that for instance $\int_0^d -\cos\beta(s)\,ds = a > 0$. For $x > a$ consider a curve in Δ: $r' = (x-a)t$, $s' = d\cdot t$, $0 \leqslant t \leqslant 1$. In $\mathcal{T}$ the curve connects the points $\gamma_0\big(c(0)\big)$ and $\gamma_0\big(c(0)+x\big)$ (Fig. 7). Its length is equal to

$$\int_0^1 \sqrt{(x-a)^2 + d^2\cdot u^2\sin^2\beta(s)}\,dt \leqslant \sqrt{(x-a)^2 + d^2}.$$

For sufficiently large x this length is strictly smaller than x, which contradicts the fact that γ_0 is the shortest curve connecting $\gamma_0\big(c(0)\big)$ and $\gamma_0\big(c(0)+x\big)$. ∎

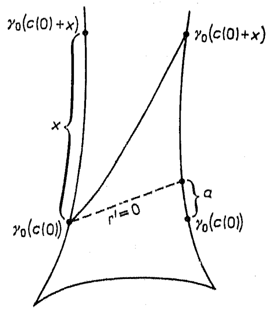

Fig. 7

THEOREM 2. *If $\mathcal{T}$ is a horn without focal points then there is a tube $\mathcal{T}_1 \subset \mathcal{T}$ such that in* cl $\mathcal{T}_1$ *we can introduce coordinates (r, s) of class C^1, $r \geq 0$, $s \bmod d$, such that*

(a) *the curves $s = \text{const}$ are the only escaping geodesics in $\mathcal{T}_1$ with the property of being the shortest connection between any two of their points;*

(b) *the curve $r = 0$ with s as a parameter is of class C^2;*

(c) *the Riemannian metric in the coordinates (r, s) is of the form*

$$d^2r + Y^2(r, s)d^2s$$

where $Y(r, s)$ is a solution of the equation $Y_r'' = -k(r, s)Y$ such that $Y(0, s) = 1$ and $Y_r'(r, s) \leq 0$ ($Y(r, s)$ is a principal solution); $k(r, s)$ denotes the Gaussian curvature;

(d) *for an arbitrary geodesic in $\mathcal{T}_1$ let $a(s)$ denote the angle between the vector of velocity of the geodesic and the coordinate line $s = \text{const}$; then $a(s)$ is of class C^1 and*

$$\frac{da}{ds} = -Y_r'.$$

Proof. In view of Lemma 6 the line $r' = 0$ is closed in $\mathcal{T}$. From this and from the form of the metric tensor we obtain that the geodesics $s' = \text{const}$ are the only escaping geodesics in $\mathcal{T}$ which have the property of being the shortest connection between any two of their points.

The line $r' = 0$ is of class C^1 with respect to parameter s' and its unit normal vector is also a C^1 function of s'. Hence the line $r' = 0$ is of class C^2 with respect to the length parameter. Smoothness is perhaps violated at the point of intersection with γ_0. But we can start all our construction not from γ_0 but from any other geodesic $s' = \text{const}$. And so the line $r' = 0$ must be smooth everywhere.

Now we obtain the coordinates (r, s) from the coordinates (r', s') by replacing the coordinate s' by the lenght parameter s on the line $r' = 0$. Then obviously (a), (b) and (c) are true.

(d) follows from Theorem 1. Indeed, the angle $\beta(s)$ from that theorem is our $a(s)$ and if we take into account the difference between the parameter s from Theorem 1 and our parameter s, we obtain equation (d). ■

From now on we shall constantly use the coordinates (r, s).

§ 3. Geodesics in a hyperbolic horn. In the rest of the paper we consider only hyperbolic horns, i.e. horns with a nonpositive Gaussian curvature. Theorem 2 has important consequences concerning the behaviour of geodesics in a hyperbolic horn.

THEOREM 3. *Suppose $\mathcal{T}$ is a hyperbolic horn. $\mathcal{T}$ is a sharp horn if and only if the curves $s = \text{const}$ are the only escaping geodesics in $\mathcal{T}$.*

Proof. Suppose $(r(\tau), s(\tau))$, $\tau \geq 0$ is an escaping geodesic in $\mathcal{T}_1$ and $s(\tau) \neq \text{const}$, for instance $ds/d\tau > 0$. Then $a(s(\tau))$ increases and $a(s(\tau)) < \pi/2$ for $\tau \geq 0$. Consider a real function $f(r) = \int_0^d Y(r, s)ds$. f is positive, decreasing and convex ($f'' \geq 0$). $\mathcal{T}$ is a sharp horn if and only if $\lim_{r \to +\infty} f(r) = f(\infty) = 0$.

Let $r_n < r_{n+1}$, $n = 0, 1, \ldots$ denote the coordinate r of the points of intersection of our escaping geodesic and the curve $s = 0$, and let a_n denote the corresponding angles at those points. We have

$$\frac{dr}{d\tau} = \cos a \quad \text{and} \quad \frac{ds}{d\tau} = \frac{\sin a}{Y},$$

and so

$$\frac{dr}{ds} = Y \operatorname{ctg} a.$$

Further,

$$\Delta r_n = r_{n+1} - r_n = \int_0^d \frac{dr}{ds}\, ds = \int_0^d Y \operatorname{ctg} a\, ds$$
$$\leq \operatorname{ctg} a_n \int_0^d Y ds \leq \operatorname{ctg} a_n f(r_n),$$

$$\Delta a_n = a_{n+1} - a_n = \int_0^d \frac{da}{ds}\, ds$$
$$= \int_0^d -Y_r' ds \geq -f'(r_{n+1}).$$

So we have

$$\sum_{n=1}^{\infty} -f'(r_n) \leq \sum_{n=0}^{\infty} \Delta a_i = a_\infty - a_0 < \pi/2.$$

On the other hand, $f(r_{n+1}) \geq f(r_n) + f'(r_n)\Delta r_n$ and

$$\frac{f(r_{n+1})}{f(r_n)} \geq 1 + f'(r_n)\frac{\Delta r_n}{f(r_n)}$$
$$\geq 1 + f'(r_n)\operatorname{ctg} a_n \geq 1 + f'(r_n)\operatorname{ctg} a_0$$

where $1 + f'(r_n)\operatorname{ctg} a_0 > 0$ for sufficiently large n. Now we get

$$\frac{f(\infty)}{f(r_{n_0})} = \prod_{n=n_0}^{\infty} \frac{f(r_{n+1})}{f(r_n)} \geq \prod_{n=n_0}^{\infty} (1 + f'(r_n)\operatorname{ctg} a_0)$$

and, once the infinite product converges, we obtain $f(\infty) > 0$; so T is not a sharp horn.

To prove the reverse part of the theorem suppose that $f(\infty) > 0$. Take $\alpha < \pi/2$. We will show that there is an escaping geodesic, different from $s =$ const and such that all angles α_n are smaller than α. Consider a geodesic starting in $(0, 0)$ at the angle α_0 (with the line $s = 0$) so small that at the first point of intersection with the line $s = 0$, $\alpha_1 < \alpha/2$ and r_1 is so large that

$$-f'(r_1) + \frac{f(r_1) - f(\infty)}{f(\infty)\operatorname{ctg}\alpha} < \alpha/2.$$

We claim that such a geodesic is escaping. Indeed, let us assume by induction that the angles α_i, $i = 0, \ldots, n$ are less than α. Then

$$\Delta r_i > f(\infty)\operatorname{ctg}\alpha, \quad i = 0, \ldots, n-1,$$

and

$$\begin{aligned}\sum_{i=1}^{n} -f'(r_i) &= -f'(r_1) + \sum_{i=2}^{n} \frac{-f'(r_i)\Delta r_{i-1}}{\Delta r_{i-1}} \\ &< -f'(r_1) + \frac{1}{f(\infty)\operatorname{ctg}\alpha} \sum_{i=2}^{n} -f'(r_i)\Delta r_{i-1} \\ &\leqslant -f'(r_1) + \frac{1}{f(\infty)\operatorname{ctg}\alpha} \int_{r_1}^{\infty} -f'(r)\,dr \\ &= -f'(r_1) + \frac{f(r_1) - f(\infty)}{f(\infty)\operatorname{ctg}\alpha}.\end{aligned}$$

We have

$$\alpha_{n+1} = \alpha_1 + \sum_{i=1}^{n} \Delta\alpha_i \leqslant \alpha_1 + \sum_{i=1}^{n} -f'(r_i) < \alpha/2 + \alpha/2 = \alpha.$$

Hence by induction the geodesic is escaping. ■

PROPOSITION 1. *$\mathscr{T}$ is a sharp horn if and only if $\int_0^\infty rk(r, s)\,dr = \infty$ for almost all s.*

Proof. $\mathscr{T}$ is a sharp horn if and only if $\lim_{r\to\infty} \int_0^d Y(r, s)\,ds = 0$. But $Y(r, s)$ is decreasing as a function of r and so $\mathscr{T}$ is a sharp horn if and only if $\lim_{r\to\infty} Y(r, s) = 0$ for almost all s. Now the proposition follows from the properties of principal solutions [7]. For the sake of completeness we give a straightforward proof.

Suppose $Y(r) > c > 0$ for $r \geqslant 0$. Then

$$\begin{aligned}\int_0^\infty -rk(r)\,dr &= \int_0^\infty \frac{Y''}{Y} r\,dr \leqslant \frac{1}{c}\int_0^\infty Y'' r\,dr \\ &= \frac{1}{c}\Big(Y'(r)\cdot r\Big|_0^\infty - \int_0^\infty Y'\,dr\Big) \leqslant \frac{1-c}{c}.\end{aligned}$$

The last inequality follows from the fact that $Y' \leqslant 0$.

Conversely, we see that $\int_0^\infty Y'' r\,dr \leqslant 1 - Y(\infty)$ and so $\lim_{r\to\infty} Y'(r)\cdot r$ exists. If $\lim_{r\to\infty} Y'(r)\cdot r = -a$, $a > 0$ then, for sufficiently big r $Y'(r) < -a/2r$. We obtain $Y(\infty) - Y(r) = -\infty$, which contradicts the fact that $Y > 0$. Hence $\lim_{r\to\infty} Y'(r)\cdot r = 0$. If $Y(\infty) = 0$, we have for every $t > 0$

$$\begin{aligned}\int_t^\infty -k(r) r\,dr &= \int_t^\infty \frac{Y''}{Y} r\,dr \geqslant \frac{1}{Y(t)} \int_t^\infty Y'' r\,dr \\ &= \frac{1}{Y(t)}\Big(Y'(r)\cdot r\Big|_t^\infty - \int_t^\infty Y'(r)\,dr\Big) = \frac{-Y'(t)\cdot t}{Y(t)} + 1 \geqslant 1.\end{aligned}$$

The last inequality shows that $\int_0^\infty rk(r)\,dr = \infty$. ■

Consider the set $\mathscr{R}$ of unit tangent vectors with carriers at the points of the curve $r = 0$. We introduce coordinates (s, φ) in $\mathscr{R}$: $s \,(\mathrm{mod}\, \boldsymbol{d})$ is the length parameter on the curve $r = 0$ and $\varphi \,(\mathrm{mod}\, 2\pi)$ is the angle between the tangent vector and the line $s =$ const. It follows from Theorem 2(b) that $\mathscr{R}$ with coordinates (s, φ) is a closed submanifold of T_1M of class C^1 (T_1M is the unit tangent bundle of M).

Let $\gamma(t; s, \varphi)$, $t \in \boldsymbol{R}$ denote a geodesic in M starting $(t = 0)$ at $(0, s)$ on the curve $r = 0$ with an initial velocity vector equal to $(s, \varphi) \in \mathscr{R}$. By $\mathscr{U}^+$ we denote the set of $(s, \varphi) \in \mathscr{R}$ such that $\gamma(t; s, \varphi)$, $t \geqslant 0$ is contained in $\mathscr{T}_1$. Similarly, $\mathscr{U}^-$ is the set of $(s, \varphi) \in \mathscr{R}$ such that $\gamma(t; s, \varphi)$, $t \leqslant 0$ is contained in $\mathscr{T}_1$. Obviously $\mathscr{U}^- = \{(s, \varphi) \mid (s, \varphi - \pi) \in \mathscr{U}^+\}$.

According to Theorem 2 (d) for $\gamma(t; s, \varphi)$, $(s, \varphi) \in \mathscr{U}^+$ the angle between $d\gamma/dt$ and the coordinate lines $s =$ const is not decreasing if $\varphi \leqslant 0$ and not increasing if $\varphi \geqslant 0$ as t increases. We denote by $\alpha^+(s, \varphi)$ the limit value of this angle as $t \to +\infty$.

For a sharp horn Theorem 3 gives

$$\mathscr{U}^+ = \{(s, \varphi) \mid \varphi = 0\}, \qquad \mathscr{U}^- = \{(s, \varphi) \mid \varphi = \pi\}.$$

$\mathscr{U}^+$ for a horn which is not sharp is described in the following theorem.

THEOREM 4. *If $\mathcal{T}$ is a hyperbolic horn which is not sharp then there is a family of functions* $f_\alpha(s)$, $s(\operatorname{mod} d)$, $-\pi/2 \leqslant \alpha \leqslant \pi/2$, *of class* C^1 *such that*

(a) f_α *and* df_α/ds, $-\pi/2 \leqslant \alpha \leqslant \pi/2$ *are continuous families of functions in the topology of uniform convergence;*

(b) $-\pi/2 < f_{-\pi/2} < 0 < f_{\pi/2} < \pi/2$ *and* $\mathcal{U}^+ = \{(s, \varphi) | f_{-\pi/2}(s) \leqslant \varphi \leqslant f_{\pi/2}(s)\}$;

(c) $\{(s, \varphi) |\ \alpha^+(s, \varphi) = \alpha\} = \{(s, \varphi) |\ \varphi = f_\alpha(s)\}$.

Proof. Consider intervals $I_s = \{(s, \varphi) |\ -\pi/2 \leqslant \varphi \leqslant \pi/2\} \subset \mathcal{R}$. Obviously $\gamma(t; s, \pm\pi/2)$ goes out of $\mathcal{T}_1$ or coincides with the curve $r = 0$. The last case is impossible because then $Y_r' = 0$ on the curve $r = 0$ (in view of Theorem 2 (d)), and this contradicts the fact that $\mathcal{T}_1$ is a horn. Geodesics $\gamma(t; s, \varphi)$ with φ close to $-\pi/2$ or $\pi/2$ also leave $\mathcal{T}_1$. Hence in view of Theorem 3 there are $f_{-\pi/2}(s)$ and $f_{\pi/2}(s)$ such that $-\pi/2 < f_{-\pi/2}(s) < 0 < f_{\pi/2}(s) < \pi/2$ and

$$\mathcal{U}^+ \cap I_s = \{(s, \varphi) |\ f_{-\pi/2}(s) \leqslant \varphi \leqslant f_{\pi/2}(s)\}.$$

By an argument similar to that used at the end of the proof of Theorem 3 one can check that $\alpha^+\big(s, f_{\pm\pi/2}(s)\big) = \pm\pi/2$.

By Theorem 2 (d) α^+ is continuous and increasing as a function of φ on $I_s \cap \mathcal{U}^+$. Consequently, α^+ takes on all values from $-\pi/2$ to $\pi/2$ and each of them once as φ changes from $f_{-\pi/2}(s)$ to $f_{\pi/2}(s)$ (s is fixed). It follows that (c) defines uniquely a family of functions f_α, $-\pi/2 \leqslant \alpha \leqslant \pi/2$.

Consider a geodesic ray $\gamma(t; s, \varphi)$, $t \geqslant 0$, $(s, \varphi) \in \mathcal{U}^+$; for its asymptotic geodesic rays α^+ has the same value. Then from Theorem 1 it follows that f_α is of class C^1 and (using Lemma 8)

$$(*) \qquad \frac{df_\alpha}{ds} = u_t'\big(0; s, f_\alpha(s)\big)\cos f_\alpha(s) - Y_r'(0, s)$$

where $u(t; s, \varphi)$ is a principal solution of the Jacobi equation along $\gamma(t; s, \varphi)$ such that $u(0; s, \varphi) = 1$. The last formula shows that df_α/ds are bounded uniformly in s and α. It follows that f_α, $-\pi/2 \leqslant \alpha \leqslant \pi/2$ is a compact family of functions and hence it is continuous in the topology of uniform convergence. Then once again, by formula $(*)$, df_α/ds, $-\pi/2 \leqslant \alpha \leqslant \pi/2$, is also a continuous family. ■

Note that, by Theorem 4, α^+ is a continuous function on $\mathcal{U}^+$.

PROPOSITION 2. *Denote by* $k(t; s, \varphi)$ *the Gaussian curvature along a geodesic* $\gamma(t; s, \varphi)$. *Then for almost all* s

$$\int_0^\infty t k\big(t; s, f_{\pm\pi/2}(s)\big)\,dt = \infty.$$

Proof. From what was said in the proof of Proposition 1 it follows that we have to show that, for almost all s, $u\big(t; s, f_{\pm\pi/2}(s)\big) \to 0$ as $t \to \infty$ where $u(t; s, \varphi)$ is a principal solution of the Jacobi equation along $\gamma(t; s, \varphi)$ such that $u(0; s, \varphi) = 1$.

The lengths of the closed curves $r = \text{const}$ in $\mathcal{T}_1$ are bounded. Consider the geodesics $\gamma\big(t; s, f_{\pi/2}(s)\big)$ for all s. Denote by $\alpha_r(s)$ the angle between $d\gamma/dt$ and the curve $s = \text{const}$ at the point of intersection with the curve $r = \text{const}$. Then the length of the curve $r = r_0$ is equal to

$$\int_0^d u\big(t(s, r_0); s, f_{\pi/2}(s)\big)\frac{\cos f_{\pi/2}(s)}{\cos \alpha_{r_0}(s)}\,ds$$

where $t(s, r_0)$ is the length of the geodesic $\gamma\big(t; s, f_{\pi/2}(s)\big)$ between the curves $r = 0$ and $r = r_0$. We have $t(s, r_0) > r_0$ and $\alpha_{r_0}(s) \to \pi/2$ as $r_0 \to \infty$.

Suppose $u\big(t; s, f_{\pi/2}(s)\big) > c > 0$ for all t for a set of s of positive measure. Then the lengths of the curves $r = r_0$ grow unboundedly as $r_0 \to \infty$. The resulting contradiction ends the proof. ■

Note that Proposition 1 is the limit case of Proposition 2.

Let

$$\mathcal{R}_1^+ = \{(s, \varphi) |\ -\pi/2 < \varphi < f_{-\pi/2}(s)\}, \qquad \mathcal{R}_2^+ = \{(s, \varphi) |\ f_{\pi/2}(s) < \varphi < \pi/2\}.$$

and $\mathcal{R}^+ = \mathcal{R}_1^+ \cup \mathcal{R}_2^+$. Similarly

$$\mathcal{R}_1^- = \{(s, \varphi) |\ (s, \varphi - \pi) \in \mathcal{R}_2^+\}, \qquad \mathcal{R}_2^- = \{(s, \varphi) |\ (s, \varphi - \pi) \in \mathcal{R}_1^+\}$$

and $\mathcal{R}^- = \mathcal{R}_1^- \cup \mathcal{R}_2^-$.

The geodesic flow in $T_1 M$ defines the Poincaré map $\Phi\colon \mathcal{R}^+ \to \mathcal{R}^-$. Φ is a diffeomorphism and $\Phi(\mathcal{R}_i^+) = \mathcal{R}_i^-$, $i = 1, 2$. By Theorem 4 the sets $\Pi_i^\pm = \mathcal{U}^\pm \cap \operatorname{cl} \mathcal{R}_i^\pm$, $i = 1, 2$, are smooth curves. In the case of a sharp horn we put $f_{-\pi/2}(s) = f_{\pi/2}(s) \equiv 0$ in the definitions of $\mathcal{R}_i^\pm$, $i = 1, 2$. In this case $\Pi_1^\pm = \Pi_2^\pm = \mathcal{U}^\pm$. Our aim is to investigate Φ in a small neighbourhood of Π_i^+, $i = 1, 2$.

LEMMA 7. *Let s be fixed and* $(s_1, \varphi_1) = \Phi(s, \varphi)$.

If $\varphi \nearrow f_{-\pi/2}(s)$ *then* $\varphi_1 - f_{\pi/2}(s_1) \to \pi$ *and* $s_1 \to -\infty$.

If $\varphi \searrow f_{\pi/2}(s)$ *then* $\varphi_1 - f_{-\pi/2}(s_1) \to \pi$ *and* $s_1 \to +\infty$.

Proof. We will prove for instance the second part of the lemma. Consider the covering space of $\mathcal{T}_1$ with coordinates (r, s), $r \geqslant 0$, $s \in \mathbf{R}$ (a projection of this covering space on $\mathcal{T}_1$ is obtained by taking $s(\operatorname{mod} d)$). The geodesics $\gamma(t; s, \varphi)$, $t \geqslant 0$, $f_{\pi/2}(s) < \varphi < \pi/2$ lifted onto the covering space and beginning at $(0, s)$ intersect only at this point. Hence s_1 increases as $\varphi \searrow f_{\pi/2}(s)$. For a horn which is not sharp $s_1 \to \infty$ because the geodesic $\gamma(t; s, \varphi)$ converges to an escaping geodesic when $\varphi \searrow f_{\pi/2}(s)$.

From the fact that Φ is a diffeomorphism of $\mathcal{R}_2^+$ on $\mathcal{R}_2^-$ it follows that $\varphi_1 - f_{-\pi/2}(s_1) \to \pi$.

For a sharp horn $\varphi_1 \to \pi$. We use Theorem 2 (d) and obtain

$$\varphi_1 - \varphi = \int_s^{s_1} -Y_r' \, ds.$$

Suppose s_1 is bounded. Then the integral on the right tends to zero and the left side tends to π. So also in this case $s_1 \to \infty$. ∎

Now we shall study $d\Phi$ in a small neighbourhood of Π_i^+ $i = 1, 2$. We begin with a description of Fermi coordinates in a tangent space of T_1M. Let $x \in M$, $v \in (T_1M)_x$. By $v^\perp \in (T_1M)_x$ we denote a vector orthogonal to v such that the pair $(v^\perp, v)$ is positively oriented. We introduce linear coordinates (Fermi coordinates) $(y, \tau, \varkappa)$ in $\big(T(T_1M)\big)_{(x,v)}$ in the following way. We represent a vector in $\big(T(T_1M)\big)_{(x,v)}$ by a parametrized curve in T_1M: $\big(x(\sigma), v(\sigma)\big)$, $|\sigma| < \varepsilon$, $x(0) = x$, $v(0) = v$. Then (y, τ) are coordinates of the vector $(dx/d\sigma)|_0$ in the basis $(v^\perp, v)$.

Let $D_\sigma v(\sigma)$ denote a covariant derivative of $v(\sigma)$ along $x(\sigma)$ with respect to σ (i.e. we translate the vectors $v(\sigma)$ parallely along the curve $x(\sigma)$, $|\sigma| < \varepsilon$ to a common tangent space $(TM)_{x(\sigma)}$ and then differentiate them with respect to the parameter σ). If $(dx/d\sigma)|_{\sigma=0} \neq 0$ then $D_\sigma v(\sigma)|_{\sigma=0} = \nabla_{dx/d\sigma} v(\sigma)|_{\sigma=0}$. $D_\sigma v(\sigma)|_{\sigma=0}$ is always orthogonal to v and $\varkappa$ is defined by the equality $D_\sigma v(\sigma)|_{\sigma=0} = \varkappa v^\perp$. The Jacobi equations in the coordinates $(y, \tau, \varkappa)$ take the form

$$\begin{aligned} y' &= \varkappa \\ \tau' &= 0 \\ \varkappa' &= -k(t)y \end{aligned}$$

where $k(t)$ is a Gaussian curvature along the respective geodesic.

Let $\omega: \mathscr{R} \to T_1M$ be a natural inclusion. Consider the coordinates (ξ^s, ξ^φ) in $T\mathscr{R}$ with respect to the basis $(\partial/\partial s, \partial/\partial\varphi)$. Fix $(\bar{s}, \bar{\varphi}) \in \mathscr{R}$.

LEMMA 8. *The differential* $d\omega$: $(T\mathscr{R})_{(\bar{s},\bar{\varphi})} \to \big(T(T_1M)\big)_{\omega(\bar{s},\bar{\varphi})}$ *has in the coordinates* (ξ^s, ξ^φ) *and* $(y, \tau, \varkappa)$ *the form*

$$\begin{aligned} y &= \cos\bar{\varphi}\,\xi^s \\ \tau &= \sin\bar{\varphi}\,\xi^s \\ \varkappa &= Y_r'(0, \bar{s})\,\xi^s + \xi^\varphi. \end{aligned}$$

Proof. Let us represent a vector $(\xi^s, \xi^\varphi) \in (T\mathscr{R})_{(\bar{s},\bar{\varphi})}$, by a curve $\big(s(\sigma), \varphi(\sigma)\big)$, $|\sigma| < \varepsilon$, $s(0) = \bar{s}$, $\varphi(0) = \bar{\varphi}$, that is

$$\left.\frac{ds}{d\sigma}\right|_0 = \xi^s, \qquad \left.\frac{d\varphi}{d\sigma}\right|_0 = \xi^\varphi.$$

We have

$$\omega\big(s(\sigma), \varphi(\sigma)\big) = \left(\big(0, s(\sigma)\big), \cos\varphi(\sigma)\frac{\partial}{\partial r} + \sin\varphi(\sigma)\frac{\partial}{\partial s}\right).$$

Hence (Fig. 8) $y = \cos\bar{\varphi}\,\xi^s$, $\tau = \sin\bar{\varphi}\,\xi^s$. Suppose $\xi^s \neq 0$. Then

$$(1) \quad D_\sigma\left(\cos\varphi(\sigma)\frac{\partial}{\partial r} + \sin\varphi(\sigma)\frac{\partial}{\partial s}\right)\Bigg|_0 = -\sin\bar{\varphi}\,\xi^\varphi\frac{\partial}{\partial r} + \cos\bar{\varphi}\,\xi^\varphi\frac{\partial}{\partial s} + \\ + \cos\bar{\varphi}\,\xi^s \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial r}\Bigg|_{\bar{s}} + \sin\bar{\varphi}\,\xi^s\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}\Bigg|_{\bar{s}}.$$

From the construction of the coordinates (r, s) it follows (one can also use Theorem 2 (d)) that

$$(2) \qquad \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial r}\Bigg|_{\bar{s}} = Y_r'(0, \bar{s})\frac{\partial}{\partial s}.$$

(Note that we cannot use Christoffel symbols in computing covariant derivatives because the coordinates (r, s) are only of class C^1; nevertheless, formal computation gives the right answer.)

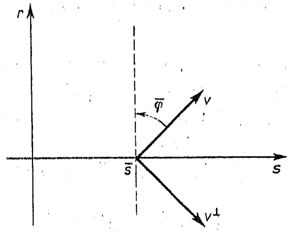

Fig. 8

Parallel translation along a fixed curve is an isometry of tangent spaces. So, in view of $\partial/\partial r$ and $\partial/\partial s$ being orthogonal, we obtain that $\nabla_{\partial/\partial s}(\partial/\partial s)|_{\bar{s}}$ has equal length and is orthogonal to $\nabla_{\partial/\partial s}(\partial/\partial r)|_{\bar{s}}$. Hence

$$(3) \qquad \nabla_{\partial/\partial s}\frac{\partial}{\partial s}\Bigg|_{\bar{s}} = -Y_r'(0, \bar{s})\frac{\partial}{\partial r}.$$

Combining (1), (2) and (3), we obtain

$$D_\sigma\left(\cos\varphi(\sigma)\frac{\partial}{\partial r} + \sin\varphi(\sigma)\frac{\partial}{\partial s}\right)\Bigg|_0 = \big(Y_r'(0, \bar{s})\,\xi^s + \xi^\varphi\big)\left(-\sin\bar{\varphi}\frac{\partial}{\partial r} + \cos\bar{\varphi}\frac{\partial}{\partial s}\right).$$

Hence $\varkappa = Y_r'(0, \bar{s})\,\xi^s + \xi^\varphi$. ∎

In a neighbourhood of the curves Π_i^+, $i = 1, 2$ we introduce coordinates (s, ψ) by the formula

$$\psi = \varphi - f_{\pm\pi/2}(s)$$

where we take "+" in a neighbourhood of Π_2^+ and "−" in a neighbourhood of Π_1^+. In neighbourhoods of Π_i^-, $i = 1, 2$ we define (s, ψ) by the same formula but taking "+" in a neighbourhood of Π_1^- and "−" in a neighbourhood of Π_2^-. In the limit case of a sharp horn the coordinates (s, ψ) and (s, φ) coincide.

If (η^s, η^ψ) are coordinates in a tangent space of $\mathscr{R}$ in the basis $(\partial/\partial s, \partial/\partial\psi)$ then they are connected with the coordinates (ξ^s, ξ^φ) by the formulae

$$\eta^s = \xi^s, \qquad \eta^\psi = \xi^\varphi - \frac{df_\pm}{ds}\xi^s. \tag{4}$$

From Theorem 1 and Lemma 8 it follows (see the proof of Theorem 4) that

$$\frac{df_{\pm\pi/2}}{ds} = u_t'\big(0; s, f_{\pm\pi/2}(s)\big)\cos f_{\pm\pi/2}(s) - Y_r'(0, s)$$

where $u(t; s, \varphi)$ has the same sense as in the proof of Theorem 4. For simplicity we put $u_i(t; s) = u\big(t; s, f_{(-1)^i\pi/2}(s)\big)$, $i = 1, 2$.

Now, using Lemma 8, (4) and Jacobi equations, we can describe the differential $d\Phi$ in the following way: Let $(s_0, \psi_0) \in \mathscr{R}_i^+$ $(s_1, \psi_1) = \Phi(s_0, \psi_0) \in \mathscr{R}_i^-$, $d\Phi_{(s_0,\psi_0)}(\eta^s, \eta^\psi) = (\eta_1^s, \eta_1^\psi)$. Given (η^s, η^ψ) we put

$$\begin{aligned} y(0) &= \cos\big(\psi_0 + f_{(-1)^i\pi/2}(s_0)\big)\eta^s, \\ y'(0) &= u_i'(0; s_0)\cos f_{(-1)^i\pi/2}(s_0)\eta^s + \eta^\psi. \end{aligned} \tag{5}$$

Then we solve the equation

$$y'' = -k\big(t; s_0, \psi_0 + f_{(-1)^i\pi/2}(s_0)\big)y$$

with initial values (5). In the end we find (η_1^s, η_1^ψ) by the formulae

$$\begin{aligned} \eta_1^s &= \frac{1}{\cos\big(\psi_1 + f_{(-1)^j\pi/2}(s_1)\big)}\,y(t_1), \\ \eta_1^\psi &= -\frac{\cos f_{(-1)^j\pi/2}(s_1)}{\cos\big(\psi_1 + f_{(-1)^j\pi/2}(s_1)\big)}\,u_j'(0; s_1)y(t_1) + y'(t_1) \end{aligned} \tag{6}$$

where t_1 is the length of the geodesic $\gamma\big(t; s_0, \psi_0 + f_{(-1)^i\pi/2}(s_0)\big)$ between the points $(0, s_0)$ and $(0, s_1)$, and $(i, j) = (1, 2)$ or $(2, 1)$ ($k(t; s, \varphi)$ is the Gaussian curvature along $\gamma(t; s, \varphi)$).

We shall use the description of $d\Phi$ above to prove the following theorem, crucial for the construction of symbolic dynamics in Chapter III.

THEOREM 5. *Let $\bar{s}_0$ and $\bar{s}_1$ be fixed and let $(s_0, \psi_0) \in \mathscr{R}_i^+$, $i = 1$ or 2,*

$$\Phi(s_0, \psi_0) = (s_1, \psi_1), \qquad d\Phi_{(s_0,\psi_0)}(\eta^s, \eta^\psi) = (\eta_1^s, \eta_1^\psi);$$

(a) *for every $\mu > 0$ there is a $\zeta > 0$ such that if*

$$|\psi_0| \leqslant \zeta, \quad |s_0 - \bar{s}_0| \leqslant \zeta, \quad |\psi_1 - \pi| \leqslant \zeta \quad \text{and} \quad |s_1 - \bar{s}_1| \leqslant \zeta$$

then from $|\eta^\psi| \geqslant \mu|\eta^s|$ it follows that $|\eta_1^\psi| \leqslant \mu|\eta_1^s|$;

(b) *if $u_i(t; \bar{s}_0) \to 0$ as $t \to \infty+$ (i.e. $\int_0^\infty tk\big(t; \bar{s}_0, f_{(-1)^i\pi/2}(\bar{s}_0)\big)dt = \infty$) then for every $\mu > 0$ and $\lambda > 1$ there is a $\zeta > 0$ such that if $|\psi_0| \leqslant \zeta$, $|s_0 - \bar{s}_0| \leqslant \zeta$ then from $|\eta^\psi| \geqslant \mu|\eta^s|$ it follows that $|\eta_1^s| \geqslant \lambda|\eta^\psi|$.*

The same theorem can be formulated for $d\Phi^{-1}$ and it does not need a separate proof because Φ^{-1} can be expressed by Φ in the following way: if $\Phi^{-1}(s_1, \psi_1) = (s_0, \psi_0)$ then $\Phi(s_1, \psi_1 - \pi) = (s_0, \psi_0 + \pi)$. In the proof of the theorem we shall use the following lemma.

LEMMA 9. *Let $k_0(t)$ be a continuous, nonpositive function for $t \geqslant 0$ and let $y_0(t)$ be a principal solution of the equation*

$$y'' = -k_0(t)y; \tag{7_0}$$

then for every $\varepsilon > 0$ there are $\delta > 0$ and $T > 0$ such that if $k_1(t)$ is a continuous nonpositive function for $0 \leqslant t \leqslant T$ and $|k_0(t) - k_1(t)| < \delta$ for $0 \leqslant t \leqslant T$ then from $|y_0'(0)/y_0(0) - y_1'(0)/y_1(0)| \geqslant \varepsilon$ it follows that $y_1'(T)/y_1(T) > 0$ where $y_1(t)$, $0 \leqslant t \leqslant T$ is an arbitrary solution of equation (7_1).

Proof. If for an arbitrary solution $y(t)$ of equation (7_0) (with $k_0(t) \leqslant 0$ for $t \geqslant 0$) $y(t_0) \neq 0$ and $y'(t_0)/y(t_0) > 0$ for a certain $t_0 \geqslant 0$ then $y(t) \neq 0$ and $y'(t)/y(t) > 0$ for $t \geqslant t_0$.

Put $\chi(t) = y'(t)/y(t)$ on an interval where $y(t) \neq 0$. Then $\chi(t)$ satisfies the Riccati equation

$$\chi' = -k_0(t) - \chi^2. \tag{8_0}$$

The solution $\chi_0(t) = y_0'(t)/y_0(t)$ of (8_0) corresponding to the principal solution of (7_0) is the only nonpositive solution of (8_0) defined for all $t \geqslant 0$ (i.e. every other solution $\chi(t)$ of (8_0) such that $\chi(0) \leqslant 0$ either becomes positive or goes to $-\infty$ in a finite time).

Let $k_1(t) \geqslant k(t)$, $t \geqslant 0$ and $\chi_1(t)$ denote a solution of equation (8_1) such that $\chi_1(0) \leqslant \chi(0) \leqslant 0$. Then $\chi_1(t) \leqslant \chi(t)$ for all $t \geqslant 0$ for which $\chi_1(t)$ is defined and $\chi(t) \leqslant 0$.

Let $y_{\nu 0}(t)$ be a principal solution of equation (7_ν) where $k_\nu(t) = \min\big(0, k_0(t) + \nu\big)$ for $|\nu| < 1$. From the comparison theorem ([7], Corollary 6.5) it follows that

$$\frac{y_{\nu 0}'(t)}{y_{\nu 0}(t)} \leqslant \frac{y_{\nu_1 0}'(t)}{y_{\nu_1 0}(t)} \quad \text{if} \quad \nu \leqslant \nu_1.$$

Take $\varepsilon > 0$. From the continuity of principal solutions (Lemma 4) it follows that there is a $0 < \bar{\nu} < 1$ such that

$$\frac{y_0'(0)}{y_0(0)} - \frac{\varepsilon}{2} \leqslant \frac{y_{-\bar{\nu}0}'(0)}{y_{-\bar{\nu}0}(0)} \quad \text{and} \quad \frac{y_{\bar{\nu}0}'(0)}{y_{\bar{\nu}0}(0)} \leqslant \frac{y_0'(0)}{y_0(0)} + \frac{\varepsilon}{2}.$$

Further, from the properties of a principal solution we obtain T such that for solutions $y_{\bar{\nu}1}(t)$ of $(7_{\bar{\nu}})$ and $y_{-\bar{\nu}1}(t)$ of $(7_{-\bar{\nu}})$ defined by the initial conditions

$$\frac{y_{\bar{\nu}1}'(0)}{y_{\bar{\nu}1}(0)} = \frac{y_0'(0)}{y_0(0)} + \varepsilon \quad \text{and} \quad \frac{y_{-\bar{\nu}1}'(0)}{y_{-\bar{\nu}1}(0)} = \frac{y_0'(0)}{y_0(0)} - \varepsilon$$

we have

$$\frac{y_{\bar{\nu}1}'(T)}{y_{\bar{\nu}1}(T)} > 0 \quad \text{and} \quad \frac{y_{-\bar{\nu}1}'(T)}{y_{-\bar{\nu}1}(T)} > 0.$$

Put $\delta = \bar{\nu}$. From what was said in the beginning of the proof about solutions of the Riccati equation it follows that T and δ satisfy the lemma. ■

Proof of Theorem 5. Put $i = 1$.

(a) If $|\eta_1^\psi| > \mu|\eta_1^s|$ then, using (6), we obtain

$$\left|u_2'(0;\bar{s}_1) - \frac{-y'(t_1)}{y(t_1)}\right|$$
$$= \left|u_2'(0;\bar{s}_1) + \frac{\cos f_{\pi/2}(s_1)}{\cos(\psi_1 + f_{\pi/2}(s_1))} u_2'(0;s_1) + \frac{1}{\cos(\psi_1 + f_{\pi/2}(s_1))} \frac{\eta_1^\psi}{\eta_1^s}\right|$$
$$\geqslant \left|\frac{\eta_1^\psi}{\eta_1^s}\right| - \left|u_2'(0;\bar{s}_1) + \frac{\cos f_{\pi/2}(s_1)}{\cos(\psi_1 + f_{\pi/2}(s_1))} u_2'(0;s_1)\right| \geqslant \mu/2$$

provided $|s_1 - \bar{s}_1|$ and $|\psi_1 - \pi|$ are sufficiently small. We apply Lemma 9 to the function $k(t; \bar{s}_1, f_{\pi/2}(\bar{s}_1))$ and $\varepsilon = \mu/2$ and obtain δ and T. $y(t_1 - t)$ is a solution of the equation $y'' = -k(t; s_1, \psi_1 - \pi + f_{\pi/2}(s_1))y$. If $|s_1 - \bar{s}_1|$ and $|\psi_1 - \pi|$ are sufficiently small then

$$|k(t; \bar{s}_1, f_{\pi/2}(\bar{s}_1)) - k(t; s_1, \psi_1 - \pi + f_{\pi/2} s_1)| < \delta \quad \text{for} \quad 0 \leqslant t \leqslant T$$

and Lemma 9 yields $-y'(t_1 - T)/y(t_1 - T) > 0$. Consequently, for sufficiently small ζ if $y'(t_1 - T)/y(t_1 - T) > 0$ then $|\eta_1^\psi| \leqslant \mu|\eta_1^s|$. Further, using (5), we have

$$\left|u_1'(0;\bar{s}_0) - \frac{y'(0)}{y(0)}\right|$$
$$= \left|u_1'(0;\bar{s}_0) - \frac{\cos f_{-\pi/2}(s_0)}{\cos(f_{-\pi/2}(s_0) + \psi_0)} u_1'(0;s_0) - \frac{1}{\cos(f_{-\pi/2} s_0 + \psi_0)} \frac{\eta^\psi}{\eta^s}\right|$$
$$\geqslant \left|\frac{\eta^\psi}{\eta^s}\right| - \left|u_1'(0;\bar{s}_0) - \frac{\cos f_{-\pi/2}(s_0)}{\cos(f_{-\pi/2}(s_0) + \psi_0)} u_1'(0;s_0)\right| \geqslant \mu/2$$

provided $|s_0 - \bar{s}_0|$ and $|\psi_0|$ are sufficiently small. We apply Lemma 9 once again to the function $k(t; \bar{s}_0, f_{-\pi/2}(\bar{s}_0))$ and $\varepsilon = \mu/2$ and obtain $\bar{\delta}$ and $\bar{T}$. If ζ is sufficiently small then

$$|k(t; \bar{s}_0, f_{-\pi/2}(\bar{s}_0)) - k(t; s_0, \psi_0 + f_{-\pi/2}(s_0))| < \bar{\delta} \quad \text{for} \quad 0 \leqslant t \leqslant \bar{T}.$$

In consequence $y'(\bar{T})/y(\bar{T}) > 0$. But then also $y'(t_1 - T)/y(t_1 - T) > 0$ provided $t_1 - T > \bar{T}$, which can be achieved by taking ζ sufficiently small. (a) is proved.

(b) Let $|\eta^\psi| \geqslant \mu|\eta^s|$. Denote by $y_1(t)$ the solution of the equation $y'' = -k(t; s_0, \psi_0 + f_{-\pi/2}(s_0))y$ with initial values corresponding by (5) to the vector $(0, \eta^\psi)$; similarly $y_2(t)$ is the solution corresponding to the vector $(\eta^s, 0)$. Hence $y(t) = y_1(t) + y_2(t)$. We can assume that $\eta^\psi > 0$. We have

(9) $$y_1'(t) \geqslant \eta^\psi \quad \text{and} \quad y_1(t) \geqslant \eta^\psi t$$

and consequently

(10) $$y_1'(t) \geqslant \eta^\psi\left(1 + \int_0^t -\tau k(\tau; s_0, \psi_0 + f_{-\pi/2}(s_0))\,d\tau\right).$$

In the rest of the proof we will show that the growth of $y_1(t)$ is much greater than that of $y_2(t)$. We have

$$\frac{y_2'(0)}{y_2(0)} = \frac{\cos f_{-\pi/2}(s_0)}{\cos(\psi_0 + f_{-\pi/2}(s_0))} u_1'(0; s_0),$$

and so from the continuity of principal solutions we obtain that for every $T > 0$ and $\varepsilon > 0$ there is a ζ such that if $|\psi_0| < \zeta$ and $|s_0 - \bar{s}_0| < \zeta$ then

$$\left|\frac{y_2'(t)}{y_2(t)} - \frac{u_1'(t;\bar{s}_0)}{u_1(t;\bar{s}_0)}\right| < \varepsilon \quad \text{for} \quad 0 \leqslant t \leqslant T.$$

Consequently if ε was taken sufficiently small then

$$\frac{y_2'(t)}{y_2(t)} < 0 \quad \text{for} \quad 0 \leqslant t \leqslant T.$$

If the last inequality holds for all $0 \leqslant t \leqslant t_1$ then (b) is satisfied. Indeed, in such a case $|y_2(t_1)| < |\eta^s|$ and in view of (9) we obtain

$$|\eta_1^s| \geqslant |y(t_1)| = |y_1(t_1) + y_2(t_1)| \geqslant |\eta^\psi| t_1 - |\eta^s| \geqslant |\eta^\psi|(t_1 - 1/\mu)$$

and, for sufficiently small ζ, $t_1 - 1/\mu > \lambda$.

Therefore it suffices to consider two cases:

I *case*. $y_2'(T_1) = 0$ for a certain T_1, $T < T_1 \leqslant t_1$.

II *case*. $y_2(T_1) = 0$ for a certain T_1, $T < T_1 \leqslant t_1$.

I *case.* Clearly $|y_2(T_1)| \leqslant |y_2(0)|$. Using (10), we obtain

$$y(T_1) = y_1(T_1)+y_2(T_1) \geqslant y_1(T_1) - |y_2(0)| \geqslant \eta^v T_1 - |\eta^s| \geqslant \eta^v(T_1 - 1/\mu).$$

Hence $y(T_1) > 0$ provided T was taken sufficiently large. We have also $y'(T_1) = y_1'(T_1) > 0$ and so

$$|\eta_1^s| \geqslant y(t_1) \geqslant y(T_1) \geqslant \eta^v(T_1 - 1/\mu).$$

Consequently $|\eta_1^s| \geqslant \lambda|\eta^v|$ provided $T_1 - 1/\mu > \lambda$, which holds if T was taken sufficiently large.

II *case.* We have

$$|y_2'(T_1)| \leqslant |y_2'(0)| \leqslant -u_1'(0; s_0)|\eta^s| \leqslant \eta^v|u_1'(0; s_0)|/\mu.$$

So, using (10), we obtain

$$y'(T_1) = y_1'(T_1)+y_2'(T_1) \geqslant y_1'(T_1) - |y_2'(T_1)| \geqslant \eta^v\Big(1+\int_0^{T_1} -\tau k\big(\tau; s_0, \psi_0+f_{-\pi/2}(s_0)\big)d\tau - |u_1'(0; s_0)|/\mu\Big).$$

Consequently $y'(T_1) > 0$ provided T_1 is sufficiently large and ζ sufficiently small. Moreover, if $T_1 \geqslant \lambda$ then

$$|\eta_1^s| \geqslant y(t_1) = y_1(t_1)+y_2(t_1) \geqslant y_1(T_1)+y_2(T_1) = y_1(T_1) \geqslant \eta^v T_1 \geqslant \lambda\eta^v.$$

Theorem 5 is proved. ∎

III. Symbolic dynamics

§1. Abstract theorem. Let A denote a denumerable or finite set. A will be referred to as an alphabet. Let $\mathscr{A}$ and Ω be two topological compact spaces. We will call $\mathscr{A}$ the set of entrances and Ω the set of exits. We assume that $\mathscr{A} \cup A$ and $\Omega \cup A$ are compact topological spaces and the topologies restricted to A coincide with discrete topology.

By S we denote the set of sequences $\{a_n\}_{n=n_1}^{n_2}$ where $-\infty \leqslant n_1 \leqslant -1 < 0 \leqslant n_2 \leqslant +\infty$ and $a_n \in A$ for $n_1 < n < n_2$, $a_{n_1} \in \mathscr{A}$ if $n_1 > -\infty$, $a_{n_2} \in \Omega$ if $n_2 < +\infty$. So S contains infinite sequences of elements of A and also sequences with a beginning—an element of $\mathscr{A}$, or an end—an element of Ω.

We introduce a topology in S by describing neighbourhood bases. For infinite sequence $\{\tilde a_n\}_{n=-\infty}^{+\infty}$ we take as neighbourhoods the sets

$$O_N = \big\{\{a_n\}_{n=n_1}^{n_2} \in S \mid n_1 \leqslant -N,\ n_2 \geqslant N,\ a_n = \tilde a_n \text{ for } |n| < N\big\}, \quad N = 1, 2, \ldots$$

For a sequence $\{\tilde a_n\}_{n=-\infty}^{\tilde n_2}$ a neighbourhood is given by the set of $\{a_n\}_{n=n_1}^{n_2}$ with $n_1 \leqslant -N$, $n_2 \geqslant \tilde n_2$ and $a_n = \tilde a_n$ for $-N < n < \tilde n_2$, and $a_{\tilde n_2}$ belongs to a fixed neighbourhood of $\tilde a_{\tilde n_2}$ in $A \cup \Omega$. Similarly we introduce the neighbourhoods for sequences infinite on the right and for finite sequences.

We obtain a topology in S such that $\{a_n^{(k)}\}_{n=n_1^{(k)}}^{n_2^{(k)}} \to \{a_n\}_{n=n_1}^{n_2}$ as $k \to +\infty$ if and only if $\overline{\lim}_{k\to\infty} n_1^{(k)} \leqslant n_1$, $\underline{\lim}_{k\to\infty} n_2^{(k)} \geqslant n_2$ and $\lim_{k\to\infty} a_n^{(k)} = a_n$ in A for $n_1 < n < n_2$, $\lim_{k\to\infty} a_{n_1}^{(k)} = a_{n_1}$ in $A \cup \mathscr{A}$ if $n_1 > -\infty$, $\lim_{k\to\infty} a_{n_2}^{(k)} = a_{n_2}$ in $A \cup \Omega$ if $n_2 < +\infty$. S is a compact space in this topology ([2]).

The shift transformation τ is defined on $\Delta^+ = \big\{\{a_n\}_{n_1}^{n_2} \mid n_2 \geqslant 1\big\}$ by

$$\tau(\{a_n\}_{n=n_1}^{n_2}) = \{a_{n+1}\}_{n=n_1-1}^{n_2-1}.$$

$\tau(\Delta^+) = \Delta^-$ consists of $\{a_n\}_{n=n_1}^{n_2}$ with $n_1 \leqslant -2$. Now our goal is to formulate sufficient conditions for a local diffeomorphism to have $\tau\colon \Delta^+ \to \Delta^-$ as a subsystem.

Consider a square in $\boldsymbol{R}^2$ $Q = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$ and fix a number μ, $0 < \mu < 1$. We call a curve $y = u(x)$ in Q: $0 \leqslant u(x) \leqslant 1$ for $0 \leqslant x \leqslant 1$ a horizontal curve if $|u(x_1) - u(x_2)| \leqslant \mu|x_1 - x_2|$ for $0 \leqslant x_1 \leqslant x_2 \leqslant 1$. If $u_1(x)$ and $u_2(x)$ define two horizontal curves and $u_1(x) < u_2(x)$, $0 \leqslant x \leqslant 1$, then we call the set $U = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ u_1(x) \leqslant y \leqslant u_2(x)\}$ a horizontal strip. Similarly, by exchanging the role of x and y, we define vertical curves and vertical strips.

Let $\{U_a, a \in A\}$ and $\{V_a, a \in A\}$ be two families of disjoint horizontal, respectively vertical strips. Further, $\{u_\alpha, \alpha \in \mathscr{A}\}$ and $\{v_\omega, \omega \in \Omega\}$ are families of disjoint horizontal and vertical curves in $Q \setminus \bigcup_{a\in A} U_a$ and $Q \setminus \bigcup_{a\in A} V_a$, respectively.

We assume that the family $\{u_\alpha, \alpha \in \mathscr{A}\}$ and also the union of this family and the family of horizontal boundaries of U_a, $a \in A$ are closed (and hence compact) in the space of all horizontal curves with the topology of uniform convergence. Identifying two horizontal boundaries of the same horizontal strip, we obtain the topology of a compact space on $A \cup \mathscr{A}$. We require that the topology restricted to A should be discrete (in other words the horizontal boundaries of U_a, $a \in A$ form a discrete set in the space of all horizontal curves).

We constrain $\{V_a, a \in A\}$ and $\{v_\omega, \omega \in \Omega\}$ by similar assumptions and obtain a topology of a compact space on $A \cup \Omega$ such that Ω is a compact and A a discrete subspace.

Consider a diffeomorphism of class C^1 $\Phi\colon \bigcup_{a\in A} U_a \to \bigcup_{a\in A} V_a$ satisfying the following conditions:

(1) $\Phi(U_a) = V_a$, $a \in A$ and Φ maps the vertical and the horizontal boundaries of U_a, respectively onto the vertical and the horizontal boundaries of V_a.

(2) $d\Phi$ takes the boundle of sectors $\Sigma^+ = \{(\xi^x, \xi^y) \mid |\xi^x| \leqslant \mu |\xi^y|\}$ in the tangent bundle TQ into itself ((ξ^x, ξ^y) are natural coordinates in a tangent space of Q). $d\Phi^{-1}$ takes the bundle of sectors $\Sigma^- = \{(\xi^x, \xi^y) \mid |\xi^y| \leqslant \mu |\xi^x|\}$ into itself.

(3) There is a $\lambda > 1$ such that for $(\xi_1^x, \xi_1^y) = d\Phi(\xi_0^x, \xi_0^y)$ if $(\xi_0^x, \xi_0^y) \in \Sigma^+$ then $|\xi_1^y| \geqslant \lambda |\xi_0^y|$ and if $(\xi_1^x, \xi_1^y) \in \Sigma^-$ then $|\xi_0^x| \geqslant \lambda |\xi_1^x|$.

THEOREM 1. *If, for* $\Phi\colon \bigcup_{a\in A} U_a \to \bigcup_{a\in A} V_a$, *conditions* (1), (2) *and* (3) *are satisfied, then there exists a homeomorphism* $h\colon S \to Q$ *such that the diagram*

$$\begin{array}{ccc} \Delta^+ & \xrightarrow{\ \tau\ } & \Delta^- \\ \Big\downarrow h & & \Big\downarrow h \\ \bigcup_{a\in A} U_a & \xrightarrow{\ \Phi\ } & \bigcup_{a\in A} V_a \end{array}$$

is commutative and

$$h(\{a_n\}_{n=n_1}^{n_2}) \in U_{a_0} \quad \text{if} \quad n_2 \geqslant 1, \qquad h(\{a_n\}_{n=n_1}^{n_2}) \in u_{a_0} \quad \text{if} \quad n_2 = 0,$$

$$h(\{a_n\}_{n=n_1}^{n_2}) \in V_{a_{-1}} \quad \text{if} \quad n_1 \leqslant -2, \qquad h(\{a_n\}_{n=n_1}^{n_2}) \in v_{a_{-1}} \quad \text{if} \quad n_1 = -1.$$

The proof of this theorem can be obtained by general methods developed by Alekseev [1]. The theorem is a slight generalization of a theorem given by Moser [9]. So we give only a sketch of the proof. For details the reader is referred to [9].

The sketch of the proof. Let $\{a_n\}_{n=n_1}^{n_2} \in S$. Consider the sets

$$U_{a_0, a_1, \ldots, a_k} = \{p \in Q \mid \Phi^i p \in U_{a_i},\ i = 0, \ldots, k\}, \qquad k < n_2.$$

By (1) and (2) they are horizontal strips. If $n_2 = +\infty$ we get an infinite sequence of horizontal strips $U_{a_0} \supset U_{a_0, a_1} \supset \ldots$ and their intersection $\bigcap_{k=0}^{\infty} U_{a_0, a_1, \ldots, a_k}$ is a horizontal curve because by (3) the width of the strips exponentially decreases.

If $n_2 < +\infty$ we consider a horizontal curve $\bigcap_{k=0}^{n_2-1} U_{a_0, a_1, \ldots, a_k} \cap \Phi^{-n_2} u_{a_{n_2}}$.

In both cases the resulting horizontal curve depends continuously (in the respective topologies) on the sequence $\{a_n\}_{n=n_1}^{n_2} \in S$.

Similarly, we consider the sets

$$V_{a_{-1}, \ldots, a_{-k}} = \{p \in Q \mid \Phi^{i+1} p \in V_{a_i},\ i = -1, \ldots, -k\}, \qquad -k > n_1$$

and get a vertical curve $\bigcap_{k=1}^{\infty} V_{a_{-1}, \ldots, a_{-k}}$ if $n_1 = -\infty$ and $\bigcap_{k=1}^{-n_1-1} V_{a_{-1}, \ldots, a_{-k}} \cap \Phi^{-n_1-1} v_{a_{n_1}}$ if $n_1 > -\infty$.

We define $h(\{a_n\}_{n=n_1}^{n_2})$ as the intersection of the resulting horizontal and vertical curves. The point of intersection depends continuously on the curves and hence h is continuous. h is also injective and so it is a homeomorphism for which the statements of the theorem are satisfied. ■

§ 2. Application to geodesic flows on open surfaces with a hyperbolic horn.

We will use the notation and results of §3 Chapter II. Suppose there is a geodesic γ which intersects Π_i^- and Π_i^+, $i = 1$ or 2 (in this case we consider γ as a curve in $T_1 M$). Note that by reversing if necessary the time parameter we can always achieve that γ intersect Π_2^- and Π_2^+. The coincidence of indexes will play an important role in the sequel.

So, as the geodesics which intersect $\Pi_i^\pm$, $i = 1, 2$ are transversal to $\mathscr{R}$, we obtain the Poincaré map Ψ taking a neighbourhood of $\gamma \cap \Pi_2^-$ in $\mathscr{R}$ onto a neighbourhood of $\gamma \cap \Pi_2^+$ in $\mathscr{R}$. In the coordinates (s, ψ) we have $\Psi(\bar{s}_1, \pi) = (\bar{s}_0, 0)$ for some $\bar{s}_0$ and $\bar{s}_1$.

We require that the following conditions hold:

(A) (transversality condition). $\Psi(\Pi_2^-)$ intersects Π_2^+ transversally at the point $(\bar{s}_0, 0)$.

(B) $u_2(t; \bar{s}_0) \to 0$ and $u_1(t; \bar{s}_1) \to 0$ as $t \to \infty$.

Remember that in the limit case of a sharp horn $\Pi_1^\pm = \Pi_2^\pm$ and $u_1(t; s) = u_2(t; s) = Y(t, s)$. For an integer $N > 0$ and real $\delta > 0$ we consider an alphabet $A = \{N, N+1, \ldots\}$, the set of entrances $\mathscr{A} = [0, \delta]$ and the set of exits $\Omega = [0, \delta]$. The topology in $A \cup \mathscr{A} = A \cup \Omega$ is defined by the map onto $\{-1/N, -1/(N+1), \ldots\} \cup [0, \delta] \subset \mathbf{R}$. We denote by $S(N, \delta)$ the corresponding space of sequences.

For an integer $N > 0$ we consider also an alphabet $A = \{\pm N, \pm(N+1), \ldots\}$ and the sets of entrances and exists $\mathscr{A} = \Omega = \{\infty\}$. The topology in $A \cup \mathscr{A} = A \cup \Omega$ is defined by the map onto $\{\pm 1/N, \pm 1/(N+1), \ldots\} \cup \{0\} \subset \mathbf{R}$. $S(N)$ denotes the corresponding space of sequences. Let

$$\mathscr{V}_\zeta = \{(s, \psi) \mid |s - \bar{s}_0| \leqslant \zeta,\ |\psi| \leqslant \zeta\} \quad \text{for} \quad \zeta > 0.$$

THEOREM 2. *If for a geodesic* γ *on a surface with a horn which is not sharp (with a sharp horn) conditions* (A) *and* (B) *are satisfied, then there* *are* $N > 0$, $\delta > 0$, $\zeta > 0$ ($N > 0$, $\zeta > 0$) *and a homeomorphism* $h\colon S(N, \delta) \to \mathscr{V}_\zeta$ ($h\colon S(N) \to \mathscr{V}_\zeta$) *such that the diagram*

$$\begin{array}{ccc} \Delta^+ & \xrightarrow{\ \tau\ } & \Delta^- \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathscr{V}_\zeta & \xrightarrow{\ \Psi\circ\Phi\ } & \mathscr{V}_\zeta \end{array}$$

is commutative. (More exactly, $\Psi\circ\Phi$ *takes* $h(\Delta^+)$ *onto* $h(\Delta^-)$ *and* $\Psi\circ\Phi\circ h = h\circ\tau$ *on* Δ^+.)

Moreover, for $(s_0, \psi_0) = h(\{a_n\}_{n=n_1}^{n_2})$ *if* $n_2 = 0$ *then* $(s_0, \psi_0) \in U^+$ *and*

$$a^+(s_0, \psi_0) = \pi/2 - a_0 \qquad (\psi_0 = 0)$$

and if $n_2 \geqslant 1$ *then for* $(s_1, \psi_1) = \Phi(s_0, \psi_0)$ *we have*

$$E\big((s_1 - s_0)/d\big) = a_0 \qquad \big(E(|s_1 - s_0|/d) = \big(\operatorname{sgn}(s_1 - s_0)\big)a_0\big).$$

Analogously for $(s_1, \psi_1) = (\Psi^{-1}\circ h)(\{a_n\}_{n=n_1}^{n_2})$ *if* $n_1 = -1$ *then* $(s_1, \psi_1) \in U^-$ *and*

$$a^+(s_1, \psi_1 - \pi) = -\pi/2 + a_{-1} \qquad (\psi_1 = \pi).$$

This theorem gives a large family of geodesics with different geometrical properties. If a sequence $\{a_n\}_{n=-\infty}^{\infty}$ is bounded on the right, the corresponding geodesic is bounded in the future and when it is unbounded the geodesic is oscillating. In particular, we obtain a continuum of oscillating geodesics. Periodic sequences correspond to closed geodesics. Finite sequences correspond to escaping geodesics. All kinds of behaviour in the past and in the future can be freely combined. For a sharp horn we find that a geodesic can arbitrarily change the direction in which it goes round the horn.

We give the proof of Theorem 2 only in the case of a horn which is not sharp. The proof in the case of a sharp horn is almost the same.

Proof. We will show that Theorem 1 can be applied. For simplicity of exposition we assume that $\Psi(\Pi_2^-)$ and Π_2^+ intersect at the point $(\bar{s}_0, 0)$ perpendicularly. The role of the rectangle Q will be played by $\mathscr{V}_\zeta$ with a sufficiently small ζ.

For $\mu > 0$ we introduce the following sectors in the linear space (η^s, η^ψ):

$$\Sigma_1(\mu) = \{(\eta^s, \eta^\psi) \mid\ |\eta^\psi| \leqslant \mu|\eta^s|\},$$
$$\Sigma_1'(\mu) = \{(\eta^s, \eta^\psi) \mid\ |\eta^\psi| \geqslant \mu|\eta^s|\},$$
$$\Sigma_2(\mu) = \{(\eta^s, \eta^\psi) \mid\ |\eta^s| \leqslant \mu|\eta^\psi|\}.$$

If μ, $0 < \mu < 1$ and $\zeta > 0$ are sufficiently small, then for $(s_0, \psi_0) \in \mathscr{V}_\zeta$ we have

(1) $$d\Psi^{-1}_{(s_0, \psi_0)}\Sigma_1(u) \subset \Sigma_1'(\mu).$$

Taking perhaps a smaller ζ and for sufficiently small μ_1, $\mu_1 < \mu$ we have for $(s_1, \psi_1) \in \Psi^{-1}(\mathscr{V}_\zeta)$

(2) $$d\Psi_{(s_1, \psi_1)}\Sigma_1(\mu_1) \subset \Sigma_2(\mu).$$

By Theorem 5 (a) and analogous theorem for Φ^{-1}, if ζ is sufficiently small, $(s_0, \psi_0) \in \mathscr{V}_\zeta$ and $\Phi(s_0, \psi_0) \in \Psi^{-1}(\mathscr{V}_\zeta)$, then we have

(3) $$d\Phi_{(s_0, \psi_0)}\Sigma_1'(\mu_1) \subset \Sigma_1(\mu_1)$$

and

(4) $$d\Phi^{-1}_{(s_1, \psi_1)}\Sigma_1'(\mu) \subset \Sigma_1(\mu), \qquad (s_1, \psi_1) = \Phi(s_0, \psi_0).$$

Combining (2) and (3), we obtain that for a sufficiently small ζ and $(s_0, \psi_0) \in \mathscr{V}_\zeta \cap \Phi^{-1}\circ\Psi^{-1}(\mathscr{V}_\zeta)$

(5) $$d(\Psi\circ\Phi)_{(s_0, \psi_0)}\Sigma_2(\mu) \subset \Sigma_2(\mu).$$

Similarly, from (1) and (4) it follows that for $(s_0, \psi_0) \in \mathscr{V}_\zeta \cap \Psi\circ\Phi(\mathscr{V}_\zeta)$

(6) $$d(\Phi^{-1}\circ\Psi^{-1})_{(s_0, \psi_0)}\Sigma_1(\mu) \subset \Sigma_1(\mu).$$

By Lemma 7, Chapter II and (6), connected components of $\Phi(\mathscr{V}_\zeta) \cap \cap \Psi^{-1}(\mathscr{V}_\zeta)$ are, except for a finite number of them, curvilinear quadrilaterals $P_1, P_2, \ldots$ such that $\Phi^{-1}(P_i) = U_i$, $i = 1, 2, \ldots$ are horizontal strips in the rectangle $\mathscr{V}_\zeta$ (Fig. 9). Similarly, by (5), $\Psi(P_i) = V_i$, $i = 1, 2, \ldots$ are vertical strips in the rectangle $\mathscr{V}_\zeta$.

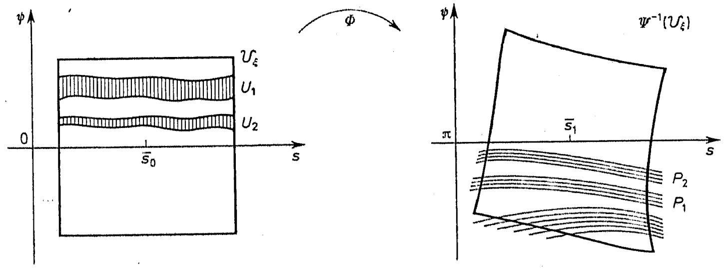

Fig. 9

Moreover, for every $i = 1, 2, \ldots$ there is an integer n_i, $n_{i+1} = n_i + 1$ such that if $(s_0, \psi_0) \in U_i$ and $(s_1, \psi_1) = \Phi(s_0, \psi_0)$ then

$$E\big((s_1 - s_0)/d\big) = n_i.$$

Roughly speaking, the geodesic $\gamma(t; s_0, \psi_0)$ between the points $(0, s_0)$ and $(0, s_1)$ makes approximately n_i loops round the horn. (If, for a fixed $|\bar{s}_1-\bar{s}_0|$, ζ is not sufficiently small then we have only $|E((s_1-s_0)/d)-n_i| \leqslant 1$.)

We change the indices of U_i and V_i in such a way that $n_i = i$ and $i = N, N+1, \ldots$ for a certain $N > 0$. By Theorem 4 (a) the intersection of the curve

$$\psi = f_{\pi/2-\alpha}(s) - f_{\pi/2}(s)$$

with $\mathscr{V}_\zeta$ is a horizontal curve for $0 \leqslant \alpha \leqslant \delta$ and a sufficiently small δ. We denote the curve by u_α.

Correspondingly, the intersection of Ψ-image of the curve

$$\psi = f_{-\pi/2+\omega}(s) - f_{-\pi/2}(s)$$

with $\mathscr{V}_\zeta$ is a vertical curve for $0 \leqslant \omega \leqslant \delta$ and a sufficiently small δ (in this case we use also (2)). We denote the curve by v_ω.

Now it suffices to show that the assumptions of Theorem 1 for $\Psi\circ\Phi$ are satisfied. (1) is a straightforward consequence of the construction of the strips U_i and V_i.

(2) is satisfied by (5) and (6).

Take $\lambda > 1$. Condition (B) allows us to apply Theorem 5 (b) and an analogous theorem for Φ^{-1}. Put

$$\nu = \max(\|d\Psi\|_{\Psi^{-1}V_\zeta}, \|d\Psi^{-1}\|_{\mathscr{V}_\zeta})$$

where

$$\|d\Psi\|_{\Psi^{-1}\mathscr{V}_\zeta} = \sup_{x\in\Psi^{-1}V_\zeta} \|d\Psi_x\|,$$

$$\|d\Psi^{-1}\|_{\mathscr{V}_\zeta} = \sup_{x\in V_\zeta} \|d\Psi_x^{-1}\|$$

and $\|\cdot\|$ is defined by the norm $|(\eta^s, \eta^\psi)| = |\eta^s| + |\eta^\psi|$. Let $\zeta > 0$ be as in Theorem 5 (b); then for

$$(s_0, \psi_0) \in \mathscr{V}_\zeta, \quad \Phi(s_0, \psi_0) = (s_1, \psi_1) \in \Psi^{-1}(\mathscr{V}_\zeta),$$

$$(\eta^s, \eta^\psi) \in \Sigma_2(\mu), \quad (\eta_1^s, \eta_1^\psi) = d\Phi_{(s_0,\psi_0)}(\eta^s, \eta^\psi),$$

$$(\eta_2^s, \eta_2^\psi) = d\Psi_{(s_1,\psi_1)}(\eta_1^s, \eta_1^\psi)$$

we have

$$|\eta_1^s| \geqslant \lambda|\eta^\psi| \quad \text{and} \quad |\eta_2^s| + |\eta_2^\psi| \geqslant (1/\nu)(|\eta_1^s| + |\eta_1^\psi|).$$

Further, (5) means that $|\eta_2^s| \leqslant \mu|\eta_2^\psi|$. Combining these inequalities we get

$$(1+\mu)|\eta_2^\psi| \geqslant |\eta_2^s| + |\eta_2^\psi| \geqslant (1/\nu)(|\eta_1^s| + |\eta_1^\psi|) \geqslant (1/\nu)|\eta_1^s| \geqslant (\lambda/\nu)|\eta^\psi|,$$

and hence

(7) $$|\eta_2^\psi| \geqslant (\lambda/\nu(1+\mu))|\eta^\psi|.$$

Now let

$$(\eta^s, \eta^\psi) \in \Sigma_1(\mu), \quad (\eta_1^s, \eta_1^\psi) = d\Psi^{-1}_{(s_0,\psi_0)}(\eta^s, \eta^\psi),$$

$$(\eta_2^s, \eta_2^\psi) = d\Phi^{-1}_{(s_1,\psi_1)}(\eta_1^s, \eta_1^\psi)$$

where $(s_0, \psi_0) \in \mathscr{V}_\zeta$ and $(s_1, \psi_1) = \Psi^{-1}(s_0, \psi_0) \in \Phi(\mathscr{V}_\zeta)$. Similarly, we have

$$|\eta_2^s| \geqslant \lambda|\eta_1^\psi|, \quad |\eta_1^s| + |\eta_1^\psi| \geqslant (1/\nu)(|\eta^s| + |\eta^\psi|),$$

and

$$|\eta_1^\psi| \geqslant \mu|\eta_1^s|.$$

Combining these inequalities, we get

$$(1+1/\mu)|\eta_2^s| \geqslant \lambda(1+1/\mu)|\eta_1^\psi| \geqslant \lambda(|\eta_1^s| + |\eta_1^\psi|)$$
$$\geqslant (\lambda/\mu)(|\eta^s| + |\eta^\psi|) \geqslant (\lambda/\nu)||\eta^s|$$

and hence

(8) $$|\eta_2^s| \geqslant (\lambda\mu/\nu(1+\mu))|\eta^s|.$$

Assumption (3) of Theorem 1 follows from (7) and (8) provided λ is chosen so large that

$$\lambda\mu/\nu(1+\mu) > 1.$$

The theorem is proved. ∎

§ 3. Final remarks. Consider a surface of rotation homeomorphic to a plane and containing a hyperbolic horn (the surface M_1 on Fig. 3). The Clairaut integral shows that all geodesics starting on Π_2^- eventually arrive on Π_2^+ or all of them are asymptotic to some closed geodesic. The latter case is degenerated in the sense that it can be destroyed by a C^∞ small change of the metric in a compact region (in the class of surfaces of rotation). In the first case the transversality condition (A) is obviously not satisfied for any of the escaping geodesics.

For a surface of rotation homeomorphic to a cylinder (the surface M_2 on Fig. 3) and containing two hyperbolic horns $\mathscr{T}$ and $\mathscr{T}_1$, if $w(\mathscr{T}) > w(\mathscr{T}_1)$ then the geodesics starting on Π_2^- arrive eventually on Π_2^+ or (similarly to the case of one hyperbolic horn) this can by achieved by an arbitrarily small change of the metric.

Assume that there is a geodesic γ intersecting Π_2^- and Π_2^+ for some hyperbolic horn but condition (A) is not satisfied. We take an arbitrarily small neighbourhood of a point on the geodesic γ and change the metric in it in such a way that it continues to be a geodesic in the new metric and the Gaussian curvature on γ increases (decreases) at some points and does not decrease (increase) at the remaining points. From the theory of linear differential equation it follows that condition (A) holds for γ in the new metric at least if the change of the Gaussian curvature is not to large. The change of the metric can be made C^∞ small.

Now, if (A) holds but not (B), then in view of Proposition 2 (Proposition 1 in the limit case of a sharp horn) after a C^∞ small and local change of the metric (A) and (B) will be satisfied for another geodesic close to γ.

Consequently by a small change of the metric we obtain a continuum of oscillating geodesics on the surfaces of rotation from Fig. 3.

Symbolic dynamics similar to that in Theorem 2 can be constructed for a geodesic flow on a surface with several hyperbolic horns which are "connected" by geodesics which escape in the future and in the past and for which conditions analogous to (A) and (B) are satisfied. Now, a geodesic makes a prescribed number of loops round a horn, then goes out of the horn closely to the "connecting" geodesic, makes the prescribed number of loops round another horn and so on. For different graphs of "connections" we obtain as symbolic spaces a variety of topological Markov chains with entrances and exits (for a definition see [2]). Moreover, if some power of a Markov matrix has 1 on its diagonal, we find a posteriori a geodesic that escapes in the future and in the past in one and the same horn.

For two tubes on a complete surface there is always a geodesic that escapes in the past in the one tube and in the future in another tube [4].

The following problem arises:

PROBLEM. Suppose M is a complete surface homeomorphic to a plane and containing a sharp hyperbolic horn. Is there always a geodesic on M that escapes both in the future and in the past? If not, is the existence of such a geodesic in some sense typical?

Added in proof. Recently Bangert solved the problem in the affirmative (there is always such a geodesic, and even without selfintersections): V. Bangert, *On the existence of escaping geodesics*, Comment. Math. Helvetici 56 (1981), 59–65.

References

[1] [V. M. Alekseev] В. М. Алексеев, *Квазислучайные динамические системы I, II, III*, Мат. сб. 76, 118 (1968), 72–134; 77 (119) (1968), 545–601; 78 (120) (1969), 3–50.

[2] — *Символическая динамика*, Одинадцатая летняя математическая школа, Изд-о Института Математики, Kiev 1976.

[3] E. Artin, *Ein mechanisches System mit quasiergodischen Bahnen*, Abh. Math. Sem. Univ. Hamburg 3 (1924), 170–175.

[4] S. Cohn-Vossen, *Kürzeste Wege und Totalkrümmung auf Flächen*, Compositio Math. 2 (1935), 69–133.

[5] P. Eberlein, *Geodesics and ends in certain surfaces without conjugated points*, Mem. Amer. Math. Soc. 13 (1978), 199.

[6] J. Hadamard, *Les surfaces à curbures opposés et leurs lignes géodesiques*, J. de Math. (5) 4 (1898), 27–73.

[7] P. Hartman, *Ordinary Differential Equations*, J. Wiley&Sons, 1964.

[8] H. M. Morse, *A one-to-one representation of geodesics on a surface of negative curvature*, Amer. J. Math. 43 (1921), 33–51.

[9] J. Moser, *Stable and Random Motions in Dynamical Systems*, Princeton Univ. Press, 1973.

[10] [A. L. Verner] А. Л. Вернер, *Полугеодезическая координатная сеть на трубках неположительной кривизны*, Труды Матем. Ин-та АН СССР 76 (1965), 130–140.

[11] [M. Wojtkowski] М. Войтковский, *Символическая динамика в одной бильярдной задаче*, Успехи Мат. Наук 3 (1977), 150–151.

[12] — *Oscillating geodesics on two-dimensional manifolds*, Asterisque 51 (1978), 443–456.

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