The assertion (3.1) can be formulated without reference to translation structure or to Riesz transforms. For p sufficiently near 1, assertion (3.1) holds on a number of "spaces of homogeneous type" (see [2]). We would merely like to comment that 3.1 holds, in particular, on the unit ball in C^n (for a proof, see [5]) and that the proofs of 2.1 and 2.2 transfer, without significant change, to that context.

Finally, we note that the techniques of this paper can be used to give some sufficient conditions that an operator $f \to \int K(x,t)f(t)\,dt$ map $H^p(\boldsymbol{R}^m)$ to $h^q(\boldsymbol{R}^n)$, $m \neq n$, $q > p$.

References

[1] S. Ross Barker, Math. Scand. 44 (1979), 92–102.
[2] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.
[3] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.
[4] G. B. Folland and E. M. Stein, *Estimates for the $\bar\partial_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. 27 (1974), 429–522.
[5] J. Garnett and R. Latter, *The atomic decomposition for Hardy spaces in several complex variables*, Duke Math. J. 45 (1978), 815–845.
[6] D. Goldberg, *A local version of real Hardy spaces*, ibid. 46 (1979), 27–42.
[7] R. Latter, *The atomic decomposition of Hardy spaces*, Ph.D. Thesis, UCLA, 1977.
[8] — *A characterization of $H^p(\boldsymbol{R}^n)$ in terms of atoms*, Studia Math. 62 (1978), 93–101.
[9] E. M. Stein, *On the theory of harmonic functions of several variables II*, Acta Math. 106 (1961), 137–174.
[10] — *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton 1970.
[11] E. M. Stein and G. Weiss, *On the theory of several variables I, The theory of H^p spaces*, Acta Math. 103 (1960), 25–62.
[12] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge 1959.

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Inequalities for product operators and vector valued ergodic theorems*

by

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Abstract. The basic setting in this consideration is the function class $\Omega_\mu^\alpha(X;\mathfrak{X})$ consisting of all strongly $\mathscr{B}$-measurable $\mathfrak{X}$-valued functions f defined on X such that $[|||f|||/t][\log^+|||f|||/t]^\alpha$ is integrable on the set where $|||f|||>t$ for every $t>0$, where $(X,\mathscr{B},\mu)$ is a σ-finite measure space and $(\mathfrak{X},|||\cdot|||)$ is a Banach space. In this setting some strong type and weak type inequalities (which are indispensable for studying ergodic theorems) for products of L_∞-bounded quasi-linear operators of weak type (1,1) are proved as generalizations of the maximal and the dominated ergodic theorems for the ergodic maximal operators. Moreover, we demonstrate that these results enable us to obtain some vector valued extensions of the ergodic theorems of Dunford–Schwartz type and further generalizations to functions in the class $\Omega_\mu^\alpha(X;\mathfrak{X})$. Local (mean and pointwise) ergodic theorems are also obtained to add to the above results.

1. Introduction. The ergodic theorems (usually called the mean ergodic theorem and the pointwise ergodic theorem) had received a considerably general operator-theoretic treatment in the case where the underlying space is just the Lebesgue space $L_p = L_p(X,\mathscr{B},\mu)$, where $(X,\mathscr{B},\mu)$ is a σ-finite measure space. After the lapse of time, especially, a new approach to the study of pointwise ergodic theorems has been developed in several recent papers. The contrivance to this is the weak type inequality powerful and indispensable for investigating the convergence almost everywhere of operator averages. The first step in this direction was taken by Fava [4] who extended the so-called "non-commuting ergodic theorems" of Dunford and Schwartz for positive operators to functions in a larger class than the space L_p by means of a weak type inequality for products of maximal operators on L_1+L_∞ which is the class of all functions of the form $f = g+h$ with $g \in L_1$ and $h \in L_\infty$. The method of proving the Fava inequality, as his proof shows, calls for the positivity of maximal operators. Recently the author [8] has generalized his inequality to the case of quasi-linear operators without assumption

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of positivity and then extended his generalizations of non-commuting ergodic theorems to the case without assuming the operators involved to be positive (cf. [7]). However, the vector valued ergodic theorems have not been given a comparable general operator-theoretic formulation. Much less is indeed known about the vector valued ergodic theorems of Chacon type. One such result was obtained in the author's paper [5] being appropriated for extending the theorem of Chacon [2]. Now it would be an interesting problem to study the Dunford–Schwartz type apropos of convergence of simultaneous averages of several single operators and of continuous parameter semigroups of operators in the reflexive Banach space valued case. In fact, our principal objectives in this paper will be to demonstrate the weak type and the strong type inequalities for products of quasi-linear operators of weak type (1,1) in the Banach space valued case and to obtain some vector valued extensions of the Dunford–Schwartz theorems. The obtained inequalities are further extensions of those by Fava [4] and the author [8], and the obtained ergodic theorems are further generalizations of those due to Chacon [2] and the author [5], [8].

Now our inequalities are of interest for some reasons. The first is just the fact that they regulate the behaviors of quasi-linear operators including those induced by the Hardy–Littlewood maximal function, the ergodic maximal functions and the ergodic power functions ([6]). Another reason for studying them is that they are natural extensions of the maximal ergodic theorem and the dominated ergodic theorem and that the major role resembles the fact that the maximal-dominated theorems are of essential use in investigating ergodic theorems.

Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space and let $(\mathfrak{X}, |||\cdot|||)$ be a Banach space. We denote by $\Omega_\mu^\alpha(X;\mathfrak{X})$ $(0 \leqslant \alpha < \infty)$ the class of all strongly measurable $\mathfrak{X}$-valued functions f defined on X such that

$$(1.1)\qquad \int_{\{|||f|||>\lambda\}} \frac{|||f(x)|||}{\lambda}\left[\log\frac{|||f(x)|||}{\lambda}\right]^\alpha d\mu(x) \quad \text{for every } \lambda>0.$$

Then $\Omega_\mu^\alpha(X;\mathfrak{X})$, $0 \leqslant \alpha < \infty$ turns out to be a general decreasing sequence of linear spaces. We prove the weak type and the strong type inequalities for quasi-linear operators on $L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$ of weak type (1,1) and for functions in $\Omega_\mu^\alpha(X;\mathfrak{X})$ which permit to obtain some vector valued ergodic theorems of Dunford–Schwartz type for any function in $\Omega_\mu^{n-1}(X;\mathfrak{X})$, where n is the number of operators involved. Furthermore, the analogous extensions for the case of continuous parameter semigroups of linear operators will also be obtained.

2. Quasi-linear operators and function classes $\Omega_\mu^\alpha(X;\mathfrak{X})$. Let $(X, \mathscr{B}, \mu)$ and $(Y, \mathscr{F}, \nu)$ be two σ-finite measure spaces and let $(\mathfrak{X}, |||\cdot|||)$ be a Banach space. Unless stated otherwise from now on, all equalities and inequalities are understood to hold almost everywhere. In what follows it will be convenient to suppress the argument of a function, writing f for $f(x)$. For an operator U mapping strongly $\mathscr{B}$-measurable $\mathfrak{X}$-valued functions defined on X to strongly $\mathscr{F}$-measurable $\mathfrak{X}$-valued functions defined on Y, we say that U is (M)-*quasi-linear* if there is a positive constant M depending only upon U such that

$$|||U(f+g)||| \leqslant M(|||Uf|||+|||Ug|||),$$
$$|||U(cf)||| = |c|\cdot|||Uf|||$$

for any complex number c. In particular, U is said to be *sublinear* when $M=1$. The operator U is *of weak type* (1,1) provided that for every $f\in D(U)$ (the domain of U) and for any $\lambda>0$, there exists a positive constant W independent of f and λ such that

$$\nu\{|||Uf|||>\lambda\} \leqslant \frac{W}{\lambda}\int_X |||f(x)|||\,d\mu(x).$$

Moreover, the least value of such W is called the *weak type* (1,1)-*norm* of U and denoted by $W_{1,1}(U)$. Now, the case that $(X, \mathscr{B}, \mu) \equiv (Y, \mathscr{F}, \nu)$ will be considered throughout the remainder of the present paper. Let $L_p(X;\mathfrak{X}) = L_p(X, \mathscr{B}, \mu; \mathfrak{X})$, $1 \leqslant p \leqslant \infty$, be the usual Banach spaces of strongly $\mathscr{B}$-measurable $\mathfrak{X}$-valued functions defined on X. Let $L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$ denote the class of all functions f of the form $f=g+h$ with $g\in L_1(X;\mathfrak{X})$ and $h\in L_\infty(X;\mathfrak{X})$. An operator U defined on $L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$ is called L_∞-bounded if for all f in $L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$ there exists a constant K independent of f for which $\|Uf\|_\infty \leqslant K\cdot\|f\|_\infty$. Furthermore, the least value of such K is said to be the *strong type* (∞,∞)-*norm* of U and denoted by $K_{\infty,\infty}(U)$. The typical examples of L_∞-bounded quasi-linear operators of weak type (1,1) in ergodic theory setting are the operators induced from the Hardy–Littlewood maximal function, the ergodic maximal functions and the ergodic power functions as mentioned in the introduction. Evidently the class of L_∞-bounded quasi-linear operators of weak type (1,1) is considerably larger than that of maximal operators in the sense of Fava, which are operators U defined on $L_1(X)+L_\infty(X)$ satisfying that

(1) $f\geqslant 0$ implies $Uf\geqslant 0$.
(2) U is sublinear.
(3) $0\leqslant f\leqslant g$ implies $Uf\leqslant Ug$.
(4) $\|Uf\|_\infty \leqslant \|f\|_\infty$.
(5) U is of weak type (1,1).

Throughout this paper the notation $E_g(\lambda)$ will be used instead of the set $\{x:\ |||g(x)|||>\lambda\}$.

LEMMA 1. *Let U be an L_∞-bounded (M)-quasi-linear operator of weak type* (1,1) *defined on* $L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$. *Let $W_{1,1}(U)$ be the weak type* (1,1)*-norm of U and let $K_{\infty,\infty}(U)$ be the strong type (∞,∞)-norm of U. Then for every $f\in L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$ and for any $\lambda>0$,*

$$\mu\Big(E_{Uf}\big(M(K_{\infty,\infty}(U)+1)\lambda\big)\Big)\leqslant \frac{W_{1,1}(U)}{\lambda}\int_{E_f(\lambda)}|||f(x)|||\,d\mu(x). \tag{2.1}$$

Proof. It is sufficient to prove the lemma for the case where $|||f|||$ is integrable over the set $E_f(\lambda)$ since if the right hand side of (2.1) is infinite then the lemma holds trivially. Define $f^\lambda(x)=f(x)\chi_{E_f(\lambda)}(x)$ and $f_\lambda(x)=f(x)-f^\lambda(x)$, where χ_E stands for the characteristic function of the set E. Then we have

$$E_{Uf}\big(M(K_{\infty,\infty}(U)+1)\lambda\big)\subset E_{Uf^\lambda}(\lambda)$$

and so

$$\mu\Big(E_{Uf}\big(M(K_{\infty,\infty}(U)+1)\lambda\big)\Big)\leqslant \mu\big(E_{Uf^\lambda}(\lambda)\big)\leqslant \frac{W_{1,1}(U)}{\lambda}\int_X|||f^\lambda(x)|||\,d\mu(x)$$

which, when appealled to the definition of f, gives (2.1) and completes the proof.

COROLLARY 1. *On the hypothesis of Lemma* 1, *let* $f\in L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$.

(1) *For $1<p<\infty$ there is a constant $A>0$ independent of f, such that*

$$\int_X|||Uf(x)|||^p\,d\mu(x)\leqslant A\cdot\int_X|||f(x)|||^p\,d\mu(x).$$

(2) *There are two constants $B>0$ and $C>0$ independent of f, such that*

$$\int_X|||Uf(x)|||\,d\mu(x)\leqslant B\Big[\mu(X)+C\cdot\int_X|||f(x)|||\log^+|||f(x)|||\,d\mu(x)\Big].$$

The proof of Corollary 1 can directly be done by the standard argument (see [5], Theorem 1). Particularly, (1) may be given by using the Marcinkiewicz interpolation theorem.

In the sequel we denote by $\Omega_\mu^\alpha(X;\mathfrak{X})$ $(0\leqslant\alpha<\infty)$ the class of all strongly $\mathscr{B}$-measurable $\mathfrak{X}$-valued functions f defined on X satisfying (1.1). Such classes were considered by Fava in a special case where $\mathfrak{X}$ is the linear space of complex numbers and $\alpha=0,1,2,\ldots$

Let $L(X;\mathfrak{X})[\log^+L(X;\mathfrak{X})]^\alpha$ $(0\leqslant\alpha<\infty)$ denote the class of all strongly $\mathscr{B}$-measurable $\mathfrak{X}$-valued functions f defined on X such that

$$\int_X|||f(x)|||\,[\log^+|||f(x)|||]^\alpha\,d\mu(x)<\infty,$$

where $\log^+u=\log\max(1,u)$ for $u\geqslant 0$.

THEOREM A ([5], [6]). *Let $1\leqslant p<\infty$ and $0\leqslant\alpha,\beta<\infty$.*

(i) *$\Omega_\mu^\alpha(X;\mathfrak{X})$ is a linear space.*

(ii) $L_1(X;\mathfrak{X})\subset\Omega_\mu^0(X;\mathfrak{X})\subset L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$.

(iii) $\Omega_\mu^\beta(X;\mathfrak{X})\subset\Omega_\mu^\alpha(X;\mathfrak{X})$ *for* $\alpha<\beta$.

(iv) $L_p(X;\mathfrak{X})\subset\Omega_\mu^\alpha(X;\mathfrak{X})\subset L(X;\mathfrak{X})[\log^+L(X;\mathfrak{X})]^\alpha\subset L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$ *for* $p>1$.

(v) $\Omega_\mu^\alpha(X;\mathfrak{X})=L(X;\mathfrak{X})[\log^+L(X;\mathfrak{X})]^\alpha$ *if and only if* $\mu(X)<\infty$.

(vi) $\Omega_\mu^\alpha(X;\mathfrak{X})\supset$ Linear span of $\bigcup_{p>1}L_p(X;\mathfrak{X})$.

3. Inequalities for products of quasi-linear operators. Of convergence in some sense of the operator averages, a central role has frequently been played by a weak type estimate (sometimes called the maximal ergodic theorem). With this comprehension we appropriate the present section to demonstrate the weak type and the strong type inequalities for products of quasi-linear operators.

LEMMA 2 ([4], Lemma 3, p. 273). *Let $\varphi(t)$ be a non-decreasing function on the real interval $0\leqslant t<\infty$, such that $\varphi(0)=0$ and $\varphi(t)$ is absolutely continuous on every finite subinterval. Then for any non-negative function f defined on X*

$$\int_E\varphi\{f(x)\}\,d\mu(x)=\int_0^\infty\mu\big(E\cap E_f(t)\big)\varphi'(t)\,dt,\qquad E\in B.$$

THEOREM 1 (Strong type inequality). *Let U_i, $1\leqslant i\leqslant n$, be n L_∞-bounded (M_i)-quasi-linear operators of weak type* (1,1) *defined on* $L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$. *For each i $(1\leqslant i\leqslant n)$ let $W_1(i)=W_{1,1}(U_i)$ be the weak type* (1,1)*-norm of U_i and $K_\infty(i)=K_{\infty,\infty}(U_i)$ the strong type (∞,∞)-norm of U_i. Let θ be any real number with $\theta>1$. Then for every $f\in\Omega_\mu^{k+n}(X;\mathfrak{X})$ $(k=0,1,2,\ldots)$ and for any $\lambda>0$*

$$\int_{E_{U_n\ldots U_1f}(\theta\lambda)}\frac{|||U_n\ldots U_1f(x)|||}{\lambda}\Big[\log\frac{|||U_n\ldots U_1f(x)|||}{\lambda}\Big]^k d\mu(x) \tag{3.1}$$
$$\leqslant A_{n,k}\cdot\int_{E_{B_nf}(\theta\lambda)}\frac{B_n\cdot|||f(x)|||}{\lambda}\Big[\log\frac{B_n\cdot|||f(x)|||}{\lambda}\Big]^{k+n}d\mu(x),$$

where

$$A_{n,k}=\prod_{i=1}^n\Big(\frac{2}{\log\theta}+\frac{1}{k+i}\Big)W_1(n-i+1),$$

$$B_n=\prod_{i=1}^n M_i\big(K_\infty(i)+1\big).$$

Consequently $f\in\Omega_\mu^{k+n}(X;\mathfrak{X})$ *implies* $U_n\ldots U_1f\in\Omega_\mu^k(X;\mathfrak{X})$.

Proof. The proof will be achieved by induction on the number of operators. Let $\theta > 1$, $\lambda > 0$ and $f \in \Omega_\mu^{k+1}(X; \mathfrak{X})$, and put $g = f/\lambda$. Then by Theorem A, $g \in \Omega_\mu^{k+1}(X; \mathfrak{X})$ and thus $U_i f$ and $U_i g$ are well defined for each i since $\Omega_\mu^{k+1}(X; \mathfrak{X}) \subset L_1(X; \mathfrak{X}) + L_\infty(X; \mathfrak{X})$. With the function φ defined by $\varphi(u) = u[\log^+ u]$ for $u \geqslant 0$, it follows from Lemma 2 that

$$\int_{E_{U_i f}(\theta\lambda)} \frac{|||U_i f(x)|||}{\lambda}\left[\log\frac{|||U_i f(x)|||}{\lambda}\right]^k d\mu(x) = \int_{E_{U_i g}(\theta)} |||U_i g(x)|||[\log|||U_i g(x)|||]^k d\mu(x) = \int_1^\infty \mu\big(E_{U_i g}(\theta)\cap E_{U_i g}(u)\big)\varphi'(u)\,du.$$

However, since $E_{U_i g}(\theta) \subset E_{U_i g}(u)$ for $u \leqslant \theta$, we have by Lemma 1

$$\int_1^\theta \mu\big(E_{U_i g}(\theta)\cap E_{U_i g}(u)\big)\varphi'(u)\,du \tag{3.2}$$
$$= \int_1^\theta \mu\big(E_{U_i g}(\theta)\big)\varphi'(u)\,du$$
$$\leqslant (\log\theta)^k W_1(i) \int_{E_{M_i(K_\infty(i)+1)g}(\theta)} M_i\big(K_\infty(i)+1\big)|||g(x)|||\,d\mu(x)$$
$$= (\log\theta)^k W_1(i) \int_{E_{M_i(K_\infty(i)+1)f}(\theta\lambda)} \frac{M_i\big(K_\infty(i)+1\big)|||f(x)|||}{\lambda}\,d\mu(x)$$
$$\leqslant \frac{W_1(i)}{\log\theta}\int_{E_{M_i(K_\infty(i)+1)f}(\theta\lambda)} \frac{M_i\big(K_\infty(i)+1\big)|||f(x)|||}{\lambda} \times \left[\log\frac{M_i\big(K_\infty(i)+1\big)|||f(x)|||}{\lambda}\right]^{k+1} d\mu(x),$$

while, also by Lemma 1

$$\int_\theta^\infty \mu\big(E_{U_i g}(\theta)\cap E_{U_i g}(u)\big)\varphi'(u)\,du \tag{3.3}$$
$$= \int_\theta^\infty \mu\big(E_{U_i g}(u)\big)\varphi'(u)\,du$$

$$\leqslant W_1(i)\int_\theta^\infty \frac{1}{u}\varphi'(u)\,du \int_{E_{M_i(K_\infty(i)+1)g}(u)} M_i\big(K_\infty(i)+1\big)|||g(x)|||\,d\mu(x)$$
$$\leqslant W_1(i)\int_{E_{M_i(K_\infty(i)+1)g}(\theta)} M_i\big(K_\infty(i)+1\big)|||g(x)|||\,d\mu(x) \times \int_\theta^{M_i(K_\infty(i)+1)|||g(x)|||} \frac{1}{u}\varphi'(u)\,du$$
$$\leqslant W_1(i)\int_{E_{M_i(K_\infty(i)+1)g}(\theta)} M_i\big(K_\infty(i)+1\big)|||g(x)||| \times \Big\{[\log M_i\big(K_\infty(i)+1\big)|||g(x)|||]^k + \frac{1}{k+1}[\log M_i\big(K_\infty(i)+1\big)|||g(x)|||]^{k+1}\Big\}\,d\mu(x)$$
$$\leqslant \left(\frac{1}{\log\theta}+\frac{1}{k+1}\right)W_1(i)\int_{E_{M_i(K_\infty(i)+1)g}(\theta)} M_i\big(K_\infty(i)+1\big)|||g(x)||| \times [\log M_i\big(K_\infty(i)+1\big)|||g(x)|||]^{k+1}\,d\mu(x)$$
$$= \left(\frac{1}{\log\theta}+\frac{1}{k+1}\right)W_1(i)\int_{E_{M_i(K_\infty(i)+1)f}(\theta\lambda)} \frac{M_i\big(K_\infty(i)+1\big)|||f(x)|||}{\lambda} \times \left[\log\frac{M_i\big(K_\infty(i)+1\big)|||f(x)|||}{\lambda}\right]^{k+1} d\mu(x).$$

Therefore, adding (3.2) and (3.3) and letting

$$A_{1,k} = \left(\frac{2}{\log\theta}+\frac{1}{k+1}\right)W_1(1),$$
$$B_1 = M_1\big(K_\infty(1)+1\big)$$

give

$$\int_{E_{U_1 f}(\theta\lambda)} \frac{|||U_1 f(x)|||}{\lambda}\left[\log\frac{|||U_1 f(x)|||}{\lambda}\right]^k d\mu(x) \leqslant A_{1,k}\cdot\int_{E_{B_1 f}(\theta\lambda)} \frac{B_1\cdot|||f(x)|||}{\lambda}\left[\log\frac{B_1\cdot|||f(x)|||}{\lambda}\right]^{k+1} d\mu(x)$$

which establishes the fact that the theorem is true for the case of $n=1$. We next assume that the theorem has been proved to hold for the case of m quasi-linear operators $U_2, \ldots, U_{m+1}$. Let $f \in \Omega_\mu^{k+m+1}(X;\mathfrak{X})$. The fact just observed then implies $U_1 f \in \Omega_\mu^{k+m}(X;\mathfrak{X})$ and so we see from the induction hypothesis that

$$(3.4)\qquad \int\limits_{E_{U_{m+1}\cdots U_1 f}(\theta\lambda)} \frac{|||U_{m+1}\cdots U_1 f(x)|||}{\lambda}\left[\log\frac{|||U_{m+1}\cdots U_1 f(x)|||}{\lambda}\right]^k d\mu(x)$$

$$\leqslant \tilde{A}_{m,k} \int\limits_{E_{\tilde{B}_m U_1 f}(\theta\lambda)} \frac{\tilde{B}_m\cdot|||U_1 f(x)|||}{\lambda}\left[\log\frac{\tilde{B}_m\cdot|||U_1 f(x)|||}{\lambda}\right]^{k+m} d\mu(x),$$

where

$$\tilde{A}_{m,k} = \prod_{i=1}^{m}\left(\frac{2}{\log\theta}+\frac{1}{k+i}\right)W_1(m+2-i),$$

$$\tilde{B}_m = \prod_{i=2}^{m+1} M_i\big(K_\infty(i)+1\big).$$

Moreover, using the quasi-linearity and the result for $n=1$, the last term of (3.4) is equal to

$$(3.5)\qquad \tilde{A}_{m,k}\int\limits_{E_{U_1(\tilde{B}_m f)}(\theta\lambda)} \frac{|||U_1(\tilde{B}_m f)(x)|||}{\lambda}\left[\log\frac{|||U_1(\tilde{B}_m f)(x)|||}{\lambda}\right]^{k+m} d\mu(x)$$

$$\leqslant A_{m+1,k}\int\limits_{E_{B_{m+1}f}(\theta\lambda)} \frac{B_{m+1}|||f(x)|||}{\lambda}\left[\log\frac{B_{m+1}|||f(x)|||}{\lambda}\right]^{k+m+1} d\mu(x),$$

where

$$A_{m+1,k} = \tilde{A}_{m,k}\left(\frac{2}{\log\theta}+\frac{1}{k+m+1}\right)W_1(1)$$

$$= \prod_{i=1}^{m+1}\left(\frac{2}{\log\theta}+\frac{1}{k+i}\right)W_1(m+2-i),$$

$$B_{m+1} = \tilde{B}_m M_1\big(K_\infty(1)+1\big) = \prod_{i=1}^{m+1} M_i\big(K_\infty(i)+1\big).$$

Consequently, combining (3.4) and (3.5) shows that the theorem holds good for $n=m+1$ and establishes (3.1). Hence the proof is complete.

THEOREM 2. (Weak type inequality). *Let U_i, $1\leqslant i\leqslant n$, be n L_∞-bounded (M_i)-quasi-linear operators of weak type $(1,1)$ defined on $L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$. For each i $(1\leqslant i\leqslant n)$ let $W_1(i)=W_{1,1}(U_i)$ be the weak type $(1,1)$-norm of U_i and $K_\infty(i)=K_{\infty,\infty}(U_i)$ the strong type (∞,∞)-norm of U_i. Let θ be any real number with $\theta>1$. Then for every $f\in\Omega_\mu^{k+n-1}(X;\mathfrak{X})$ $(k=0,1,2,\ldots)$ and for any $\lambda>0$*

$$(3.6)\qquad \mu\big(E_{U_n\cdots U_1 f}(\theta\lambda)\big) \leqslant C_{n,k}\int\limits_{E_{B_n f}(\theta\lambda)} \frac{B_n|||f(x)|||}{\lambda}\left[\log\frac{B_n|||f(x)|||}{\lambda}\right]^{k+n-1} d\mu(x),$$

$$C_{n,k} = \begin{cases} \dfrac{W_1(1)}{\theta(\log\theta)^k} & \text{for } n=1,\\ C_{n-1,k}\cdot A_{1,k} & \text{for } n\geqslant 2,\end{cases}$$

where B_n and $A_{1,k}$ are as given in Theorem 1.

Remark. Inequality (3.6) looks like the natural weak type analogue of inequality (3.1) at the first glance. Indeed, (3.6) can immediately be derived from (3.1) for functions in $\Omega_\mu^{k+n}(X;\mathfrak{X})$. However, (3.1) is no guarantee of (3.6) for functions in $\Omega_\mu^{k+n-1}(X;\mathfrak{X})$ but not in $\Omega_\mu^{k+n}(X;\mathfrak{X})$.

Proof. The proof will be by induction on n (the number of operators). Let $\theta>1$, $\lambda>0$ and $f\in\Omega_\mu^k(X;\mathfrak{X})$. Then $U_i f$ is well defined for each i, so that by Lemma 1 we have

$$(3.7)\qquad \mu\big(E_{U_i f}(\theta\lambda)\big)\leqslant\frac{W_1(i)}{\theta}\int\limits_{E_{M_i(K_\infty(i)+1)f}(\theta\lambda)} \frac{M_i\big(K_\infty(i)+1\big)|||f(x)|||}{\lambda}\,d\mu(x)$$

$$\leqslant\frac{W_1(i)}{\theta(\log\theta)^k}\int\limits_{E_{M_i(K_\infty(i)+1)f}(\theta\lambda)} \frac{M_i\big(K(i)+1\big)|||f(x)|||}{\lambda}\times$$

$$\times\left[\log\frac{M_i\big(K_\infty(i)+1\big)|||f(x)|||}{\lambda}\right]^k d\mu(x).$$

Then taking

$$C_{1,k} = \frac{W_1(1)}{\theta(\log\theta)^k}$$

and inserting the constant $C_{1,k}$ into (3.7) yield

$$\mu\big(E_{U_1 f}(\theta\lambda)\big)\leqslant C_{1,k}\cdot\int\limits_{E_{B_1 f}(\theta\lambda)} \frac{B_1|||f(x)|||}{\lambda}\left[\log\frac{B_1|||f(x)|||}{\lambda}\right]^k d\mu(x)$$

which shows that the theorem holds for $n=1$. To apply induction we suppose that the theorem has already been established for the case of m quasi-linear operators $U_2, \ldots, U_{m+1}$ and let $f \in \Omega_\mu^{k+m}(X; \mathfrak{X})$. Since $U_1 f \in \Omega_\mu^{k+m-1}(X; \mathfrak{X})$ on account of Theorem 1, it follows by the induction hypothesis that

$$(3.8)\qquad \mu\big(E_{U_{m+1}\ldots U_1 f}(\theta\lambda)\big) \leqslant C_{m,k} \int_{E_{\tilde{B}_m U_1 f}(\theta\lambda)} \frac{\tilde{B}_m |||U_1 f(x)|||}{\lambda}\left[\log \frac{\tilde{B}_m |||U_1 f(x)|||}{\lambda}\right]^{k+m-1} d\mu(x),$$

where the constant $\tilde{B}_m$ is as given in the proof of Theorem 1. Again, by Theorem 1 and the quasi-linearity the right hand side of (3.8) is equal to

$$C_{m,k} \int_{E_{U_1(\tilde{B}_m f)}(\theta\lambda)} \frac{|||U_1(\tilde{B}_m f)(x)|||}{\lambda}\left[\log \frac{|||U_1(\tilde{B}_m f)(x)|||}{\lambda}\right]^{k+m-1} d\mu(x)$$

$$\leqslant C_{m+1,k} \int_{E_{B_{m+1}f}(\theta\lambda)} \frac{B_{m+1}|||f(x)|||}{\lambda}\left[\log \frac{B_{m+1}|||f(x)|||}{\lambda}\right]^{k+m} d\mu(x)$$

with the constant B_{m+1} given in Theorem 1, where $C_{m+1,k} = C_{m,k}\cdot A_{1,k}$. This observation, when combined with (3.8), proves that the theorem is true for the case of $n=m+1$ and concludes the proof of the theorem.

Corollary 2. *Let U_i, M_i, $W_1(i)$, $K_\infty(i)$, $1 \leqslant i \leqslant n$, be as in Theorem 1.*

(1) *If $1 < p < \infty$ then for every $f \in L_1(X; \mathfrak{X}) + L_\infty(X; \mathfrak{X})$*

$$(3.9)\qquad \int_X |||U_n \ldots U_1 f(x)|||^p \, d\mu(x) \leqslant D_{n,p} \int_X |||f(x)|||^p \, d\mu(x),$$

where

$$D_{n,p} = \left(\frac{p}{p-1}\right)^n \prod_{i=1}^n \big[M_i\big(K_\infty(i)+1\big)\big]^p W_1(i).$$

(2) *Let $\theta > 1$. Then for every $f \in L_1(X; \mathfrak{X}) + L_\infty(X; \mathfrak{X})$ and for each k $(k = 0, 1, 2, \ldots)$ there holds*

$$(3.10)\qquad \int_X |||U_n \ldots U_1 f(x)|||\, d\mu(x) \leqslant \theta\cdot\Big\{\mu(X) + \frac{C_{n,k}}{n+k}\int_X B_n|||f(x)|||\big[\log^+ B_n|||f(x)|||\big]^{n+k} d\mu(x)\Big\}$$

with the constants B_n and $C_{n,k}$ given in Theorems 1 and 2.

Proof. It suffices to show (3.9) for functions in $L_p(X; \mathfrak{X})$, for if the right hand side of (3.9) is infinite then it holds trivially. So, if $f \in L_p(X; \mathfrak{X})$ one gets by Lemma 1

$$\int_X |||U_i f(x)|||^p \, d\mu(x) = p\big[M_i\big(K_\infty(i)+1\big)\big]^p \int_0^\infty u^{p-1}\,\mu\Big(E_{U_i f}\big(M_i\big(K_\infty(i)+1\big)u\big)\Big)\,du$$

$$\leqslant p\big[M_i\big(K_\infty(i)+1\big)\big]^p W_1(i) \int_0^\infty u^{p-2}\,du \int_{E_f(u)} |||f(x)|||\, d\mu(x)$$

$$= \left(\frac{p}{p-1}\right)\big[M_i\big(K_\infty(i)+1\big)\big]^p W_1(i) \int_X |||f(x)|||^p \, d\mu(x)$$

implying $U_i f \in L_p(X; \mathfrak{X})$. Hereby the desired conclusion may be understood to hold after using an induction argument.

As for (2), by the same reasoning as (1), we may consider only the case where $\mu(X) < \infty$ and $f \in L(X; \mathfrak{X})[\log^+ L(X; \mathfrak{X})]^{n+k}$. Let $\theta > 1$ and $f \in \Omega_\mu^{n+k}(X; \mathfrak{X})$. It follows that

$$(3.11)\qquad \int_X |||U_n \ldots U_1 f(x)|||\, d\mu(x) = \int_0^\infty \mu\big(E_{U_n\ldots U_1 f}(u)\big)\,du \leqslant \theta\Big[\mu(X) + \int_1^\infty \mu\big(E_{U_n\ldots U_1 f}(\theta u)\big)\,du\Big].$$

Using Theorem 2, the estimate of the second term of the last part of (3.11) is as follows:

$$(3.12)\qquad \int_1^\infty \mu\big(E_{U_n\ldots U_1 f}(\theta u)\big)\,du \leqslant C_{n,k}\int_1^\infty du \int_{E_{B_n f}(\theta u)} \frac{B_n|||f(x)|||}{u}\left[\log \frac{B_n|||f(x)|||}{u}\right]^{n+k-1} d\mu(x)$$

$$\leqslant C_{n,k}\int_X B_n|||f(x)|||\,d\mu(x) \times \int_1^{\max(1,\, B_n|||f(x)|||)} \frac{1}{u}\left[\log^+ \frac{B_n|||f(x)|||}{u}\right]^{n+k} du$$

$$= \frac{C_{n,k}}{n+k}\int_X B_n|||f(x)|||\,\big[\log^+ B_n|||f(x)|||\big]^{n+k} d\mu(x).$$

Recall that $\Omega_\mu^{n+k}(X;\mathfrak{X}) = L(X;\mathfrak{X})[\log^+ L(X;\mathfrak{X})]^{n+k}$ by Theorem A. Then, combining (3.11) and (3.12) gives (3.10) and completes the proof.

4. Vector valued ergodic theorems for $L_p(X;\mathfrak{X})$. Let $\{T(t):\ t\geqslant 0\}$ be a strongly continuous semigroup of bounded linear operators on $L_p(X;\mathfrak{X})$ with $1\leqslant p\leqslant\infty$. For each $f\in L_p(X;\mathfrak{X})$ there exists a scalar representation $\xi(t,x)$ of $T(t)f$ (which is often denoted by $T(t)f(x)$) such that

(1) $\xi(t,x)$ is measurable in (t,x) on the product space $[0,\infty)\times X$;

(2) $\xi(t,x)$ is uniquely determined up to a set of points (t,x) whose product measure is zero;

(3) $\xi(t,\cdot) = T(t)f(\cdot)$ in $L_p(X;\mathfrak{X})$ for almost every $t\geqslant 0$;

(4) there exists a null set $N(f)$ which may depend on f but which is independent of t and is such that for any $x\in X-N(f)$, $\xi(\cdot,x)$ is integrable over every finite t-interval and

$$\int_0^a \xi(t,\cdot)\,dt = \int_0^a T(t)f(\cdot)\,dt \quad \text{in} \quad L_p(X;\mathfrak{X}),\quad 0\leqslant a<\infty.$$

With this function $\xi(t,x)$ we define

$$\int_0^a T(t)f(x)\,dt = \int_0^a \xi(t,x)\,dt,\quad 0\leqslant a<\infty,$$

which will be continuous in a for almost every x. In this section $\mathfrak{X}$ is assumed to be reflexive. In fact this assumption is requisite for proving theorems about convergence of operator averages.

THEOREM 3. *Let T_j, $1\leqslant j\leqslant n$, be (not necessarily commuting) linear operators on $L_1(X;\mathfrak{X})$ with $\|T_j\|_1\leqslant 1$ and $\sup\{\|T_j^k\|_\infty:\ k\geqslant 0\}\leqslant K$, $1\leqslant j\leqslant n$, for some constant $K\geqslant 1$.*

(1) *For every $f\in L_p(X;\mathfrak{X})$ with $1<p<\infty$, the multiple sequence*

$$V(k_1,\ldots,k_n)f = \frac{1}{k_1\ldots k_n}\sum_{i_n=0}^{k_n-1}\cdots\sum_{i_1=0}^{k_1-1} T_n^{i_n}\ldots T_1^{i_1}f$$

is convergent in the norm of $L_p(X;\mathfrak{X})$ as $k_1,\ldots,k_n\to\infty$ independently.

(2) *If for every $f\in L_p(X;\mathfrak{X})$ with $1<p<\infty$, one can show that*

$$(*)\qquad \sup_{k_1,\ldots,k_n>0} |||V(k_1,\ldots,k_n)f(x)||| < \infty$$

for almost all points x in X, then the multiple sequence $V(k_1,\ldots,k_n)f$ converges strongly in $\mathfrak{X}$ almost everywhere on X as $k_1,\ldots,k_n\to\infty$ independently.

(3) *If $T_1,\ldots,T_n$ are commutative and condition $(*)$ is fulfilled for all $f\in L_1(X;\mathfrak{X})$ then the strong limit $\lim_{k\to\infty} V(k)f(x)$ exists for almost all x in X, where*

$$V(k)f(x) = \frac{1}{k^n}\sum_{i_1=0}^{k-1}\cdots\sum_{i_n=0}^{k-1} T_n^{i_n}\ldots T_1^{i_1}f(x),\quad f\in L_1(X;\mathfrak{X}).$$

Proof. Let us put

$$V(T_j,k_j)f = \frac{1}{k_j}\sum_{i_j=0}^{k_j-1} T_j^{i_j}f,\quad 1\leqslant j\leqslant n,$$

then $V(k_1,\ldots,k_n)f = V(T_n,k_n)\ldots V(T_1,k_1)f$ and it follows from the Riesz convexity theorem that

$$\sup\{\|V(T_j,k_j)\|_p:\ k_j\geqslant 1\}\leqslant K,\quad 1\leqslant j\leqslant n,$$

$$\sup\{\|V(k_1,\ldots,k_n)\|_p:\ k_1,\ldots,k_n\geqslant 1\}\leqslant K^n.$$

According to [5], Theorems 1 and 3, there are projection operators E_j, $1\leqslant j\leqslant n$, such that

$$\lim_{k\to\infty}\|V(T_j,k)f-E_jf\|_p = 0,\quad 1\leqslant j\leqslant n,\quad f\in L_p(X;\mathfrak{X}).$$

Then the induction argument (similar to the Dunford–Schwartz's) yields that

$$\lim\|V(k_1,\ldots,k_n)f-E_n\ldots E_1f\|_p = 0,\quad f\in L_p(X;\mathfrak{X})$$

as $k_1,\ldots,k_n\to\infty$ independently, which proves (1). Now it is known ([5], Lemma 3) that (2) is true for the case of $n=1$. To establish (2) for the general case we shall therefore apply induction and assume that it has been proved for the case of m operators $T_2,\ldots,T_{m+1}$. Since $L_p(X;\mathfrak{X})$ is reflexive for $1<p<\infty$ and $\lim_{k\to\infty} V(T_1,k)f$ exists strongly in $L_p(X;\mathfrak{X})$, we can use the Kakutani–Yosida mean ergodic theorem to say that the linear manifold $\mathfrak{M}_1$ generated by functions of the form $f = g+(h-T_1h)$ with $g\in L_p(X;\mathfrak{X})$, $T_1g=g$, $h\in L_p(X;\mathfrak{X})\cap L_\infty(X;\mathfrak{X})$, is dense in $L_p(X;\mathfrak{X})$. For such a function f we have

$$|||V(k_1,\ldots,k_{m+1})f(x)-V(k_2,\ldots,k_{m+1})g(x)|||\leqslant\frac{2K^{m+1}}{k_1}\|h\|_\infty$$

tending to zero as $k_1\to\infty$ for almost all x, which, together with the induction hypothesis, implies the almost everywhere convergence of the multiple sequence $V(k_1,\ldots,k_{m+1})f$ for every $f\in\mathfrak{M}_1$ dense in $L_p(X;\mathfrak{X})$. We may now deduce (2) from the principle of Banach (convergence theorem) by considering $(*)$. Apropos of (3) it is sufficient to note that (3) holds for functions in $L_p(X;\mathfrak{X})\cap L_1(X;\mathfrak{X})$ $(p>1)$ dense in $L_1(X;\mathfrak{X})$ as a special case of (2), for nothing remains but to use (*) and the Banach convergence theorem.

Now it is very important to indicate the class of operators for which condition (*) appearing in Theorem 3 is satisfied for all functions in $L_p(X;\mathfrak{X})$ with $1\leqslant p<\infty$. In connection with this question, (*) is known to hold for a general semigroup of Dunford–Schwartz type operators ([2], [5]). By the way we note that in the special case where $\mathfrak{X}$ is the linear space of complex numbers, this condition is also known to hold for several semigroups of Dunford–Schwartz operators ([3], [4], [8]) and of positive L_p- $(p>1)$ and L_∞-contractions ([1], [7]). Here we indicate two special cases in which $\mathfrak{X}$ is a reflexive Banach space.

COROLLARY 3. *Let S_j, $1\leqslant j\leqslant n$, be operators in the B-space $B(\mathfrak{X})$ of bounded linear operators on $\mathfrak{X}$ with $\sup\{|||S_j^k|||:\ k\geqslant 0\}\leqslant K$, $1\leqslant j\leqslant n$, for some constant $K\geqslant 1$. Then the statements* (1), (2) *and* (3) *of Theorem* 3 *hold for $S_1,\ldots,S_n$ and for all $f\in L_p(X;\mathfrak{X})$, $1\leqslant p<\infty$.*

Proof. Define $(T_jf)(x)=S_j\big(f(x)\big)$, $1\leqslant j\leqslant n$. It follows that

$$\sup_{k_1,\ldots,k_n\geqslant 1}|||V(k_1,\ldots,k_n)f|||\leqslant K^n|||f|||$$

almost everywhere on X. Then the desired conclusion follows at once from Theorem 3.

COROLLARY 4. *Let φ_j, $1\leqslant j\leqslant n$, be measure preserving transformations on X. Then all the conclusions corresponding to* (1)–(3) *of Theorem* 3 *remain true for $\varphi_1,\ldots,\varphi_n$.*

Proof. For $f\in L_p(X;\mathfrak{X})$ with $1\leqslant p<\infty$, define $(T_jf)(x)=f(\varphi_jx)$, $1\leqslant j\leqslant n$. Then there holds

$$\sup_{k_1,\ldots,k_n\geqslant 1}|||V(k_1,\ldots,k_n)f|||\leqslant U_n\ldots U_1|||f|||$$

almost everywhere on X, where U_j, $1\leqslant j\leqslant n$, are L_∞-contracting sublinear operators of weak type (1,1) which are given by

$$(U_j|||f|||)(x)=\sup_{k>0}\frac{1}{k}\sum_{i=0}^{k-1}|||f|||(\varphi_j^ix),\qquad 1\leqslant j\leqslant n,$$

$$|||f|||(x)\equiv|||f(x)|||.$$

Therefore, the corollary may be deduced from Theorem 3.

The continuous analogue of Theorem 3 is also true and may be stated as follows.

THEOREM 4. *Let $\{T_j(t):\ t\geqslant 0\}$, $1\leqslant j\leqslant n$, be strongly continuous semigroups of linear operators on $L_1(X;\mathfrak{X})$ with $\|T_j(t)\|_1\leqslant 1$ and $\sup\{\|T_j(t)\|_\infty:\ t\geqslant 0\}\leqslant K$, $1\leqslant j\leqslant n$, for some constant $K\geqslant 1$.*

(1) *For every $f\in L_p(X;\mathfrak{X})$ with $1<p<\infty$, the functions*

$$A(\alpha_1,\ldots,\alpha_n)f=\frac{1}{\alpha_1\ldots\alpha_n}\int_0^{\alpha_1}\ldots\int_0^{\alpha_n}T_n(t_n)\ldots T_1(t_1)f\,dt_1\ldots dt_n$$

are convergent in the norm of $L_p(X;\mathfrak{X})$ as $\alpha_1,\ldots,\alpha_n\to\infty$ independently.

(2) *If for every $f\in L_p(X;\mathfrak{X})$ with $1<p<\infty$, one can show that*

$$(**)\qquad \sup_{\alpha_1,\ldots,\alpha_n>0}|||A(\alpha_1,\ldots,\alpha_n)f(x)|||<\infty$$

for almost all x in X, then $A(\alpha_1,\ldots,\alpha_n)f$ converges strongly in almost everywhere on X as $\alpha_1\to\infty,\ldots,\alpha_n\to\infty$ independently.

(3) *If $\{T_1(t):\ t\geqslant 0\},\ldots,\{T_n(t):\ t\geqslant 0\}$ are commutative and condition* (**) *is fulfilled for every $f\in L_1(X;\mathfrak{X})$ then the strong limit $\lim\limits_{\alpha\to\infty}A(\alpha)f(x)$ exists for almost all x in X, where*

$$A(\alpha)f=\frac{1}{\alpha^n}\int_0^{\alpha}\ldots\int_0^{\alpha}T_n(t_n)\ldots T_1(t_1)f\,dt_1\ldots dt_n.$$

We omit the proof of this theorem since the same argument as that in Theorem 3 applies to the continuous-parameter case.

COROLLARY 5. *Let $\{S_j(t):\ t\geqslant 0\}$, $1\leqslant j\leqslant n$, be strongly continuous semigroups of operators in $B(\mathfrak{X})$ with $\sup\{|||S_j(t)|||:\ t\geqslant 0\}\leqslant K$, $1\leqslant j\leqslant n$, for some constant $K\geqslant 1$. Then the statments* (1)–(3) *of Theorem* 4 *hold for these semigroups and for all $f\in L_p(X;\mathfrak{X})$, $1\leqslant p<\infty$.*

COROLLARY 6. *Let $\{\varphi_j(t):\ t\geqslant 0\}$, $1\leqslant j\leqslant n$, be measurable semiflows on X. Then all the conclusions corresponding to* (1)–(3) *of Theorem* 4 *remain true for these semiflows.*

THEOREM 5. *Let $\{T_j(t):\ t\geqslant 0\}$, $1\leqslant j\leqslant n$, be strongly continuous (and not necessarily commuting) semigroups of linear operators on $L_1(X;\mathfrak{X})$ with $\|T_j(t)\|_1\leqslant 1$ and $\sup\{\|T_j(t)\|_\infty:\ t\geqslant 0\}\leqslant K$, $1\leqslant j\leqslant n$, for some constant $K\geqslant 1$. Then for every $f\in L_p(X;\mathfrak{X})$ with $1\leqslant p<\infty$.*

$$(4.1)\qquad \lim_{\alpha_1,\ldots,\alpha_n\to 0+}\|A(\alpha_1,\ldots,\alpha_n)f-f\|_p=0.$$

Proof. Let $1<p<\infty$ and define for $f\in L_p(X;\mathfrak{X})$

$$A_j(\alpha)f=\frac{1}{\alpha}\int_0^{\alpha}T_j(t)f\,dt,\qquad \alpha>0,\ 1\leqslant j\leqslant n.$$

Then it follows that

$$A(a_1, \ldots, a_n)f = A_n(a_n) \ldots A_1(a_1)f,$$

$$\|A_j(a_j)\|_p \leqslant K, \quad 1 \leqslant j \leqslant n,$$

$$\|A(a_1, \ldots, a_n)\|_p \leqslant K^n, \quad a_1 > 0, \ldots, a_n > 0.$$

Now we have by Theorems 1 and 4 of [5]

$$\lim_{a\to 0+} \|A_1(a)f-f\|_p = 0, \quad f \in L_p(X; \mathfrak{X}), \tag{4.2}$$

and, moreover, (4.2) is also valid for $p \geqslant 1$, since there holds

$$\left\| \frac{1}{a}\int_0^a T_1(t)f\,dt - f\right\|_1 \leqslant \frac{1}{a}\int_0^a \|T_1(t)f-f\|_1\,dt,$$

which tends to zero as $a \to 0+$ because of the strong continuity of $T_1(t)$ in $L_1(X; \mathfrak{X})$. The theorem is therefore true for the case of $n = 1$. In order to show (4.1) for $n \geqslant 2$ we apply induction and assume that it has already been established for the case of m semigroups $T_2(t), \ldots, T_{m+1}(t)$. Then for every $f \in L_p(X; \mathfrak{X})$, $1 \leqslant p < \infty$, we have

$$\begin{aligned}\|A_{m+1}(a_{m+1}) \ldots A_1(a_1)f-f\|_p &\leqslant \|A_{m+1}(a_{m+1}) \ldots A_2(a_2)[A_1(a_1)f-f]\|_p + \\ &\qquad + \|A_{m+1}(a_{m+1}) \ldots A_2(a_2)f-f\|_p \\ &\leqslant K^m\cdot \|A_1(a_1)f-f\|_p + \|A_{m+1}(a_{m+1}) \ldots A_2(a_2)f-f\|_p\end{aligned}$$

which approaches zero as $a_1, \ldots, a_{m+1} \to 0+$ independently by the fact observed for $n = 1$ and the induction hypothesis. Consequently the theorem follows by what we have observed above.

THEOREM 6. *Let $\mathfrak{X}$ be the linear space of complex numbers. Let $\{T_j(t)$: $t \geqslant 0\}$, $1 \leqslant j \leqslant n$, be strongly continuous (and not necessarily commuting) semigroups of linear operators on $L_1(X) = L_1(X; \mathfrak{X})$ with $\|T_j(t)\|_1 \leqslant 1$ and $\sup\{\|T_j(t)\|_\infty$: $t \geqslant 0\} \leqslant K$, $1 \leqslant j \leqslant n$, for some constant $K \geqslant 1$. Then for every $f \in L_p(X)$ with $1 < p < \infty$ the equation*

$$\lim_{a_1, \ldots, a_n \to 0+} A(a_1, \ldots, a_n)f(x) = f(x)$$

holds for almost all x in X.

Proof. Observe that there exist L_∞-bounded sublinear operators $U_1, \ldots, U_n$ of weak type (1,1) such that for every $f \in L_p(X)$, $1 < p < \infty$,

$$\sup_{a_1, \ldots, a_n > 0} |A(a_1, \ldots, a_n)f| \leqslant U_n \ldots U_1|f|$$

almost everywhere on X. Furthermore by Corollary 2, $U_n \ldots U_1|f| \in L_p(X)$ for every $f \in L_p(X)$. Therefore, the theorem follows from Theorem 5 and the Lebesgue dominated convergence theorem.

It is an interesting problem to study the pointwise local ergodic theorem corresponding to Theorem 5 (cf. [8]). In connection with this we state the following two special cases.

COROLLARY 7. *On the hypothesis of Corollary 5, let $f \in L_p(X; \mathfrak{X})$, $1 \leqslant p < \infty$. Then the equation*

$$\lim_{a_1, \ldots, a_n \to 0+} \frac{1}{a_1 \ldots a_n} \int_0^{a_1} \ldots \int_0^{a_n} S_n(t_n) \ldots S_1(t_1)f(x)\,dt_1 \ldots dt_n = f(x)$$

holds strongly in $\mathfrak{X}$ for almost all x in X.

COROLLARY 8. *Under the hypothesis of Corollary 6, let $f \in L_p(X; \mathfrak{X})$, $1 < p < \infty$. Then the equation*

$$\lim_{a_1, \ldots, a_n \to 0+} \frac{1}{a_1 \ldots a_n} \int_0^{a_1} \ldots \int_0^{a_n} f\big(\varphi_n(t_n) \ldots \varphi_1(t_1)x\big)\,dt_1 \ldots dt_n = f(x)$$

holds strongly in $\mathfrak{X}$ for almost all x in X.

5. More about vector valued ergodic theorems. Given a semigroup $\{T(t)$: $t \geqslant 0\}$ of linear operators on $L_1(X; \mathfrak{X}) + L_\infty(X; \mathfrak{X})$ we shall say that it is *L_1-continuous* and *L_∞-integrable* if $T(t)$ is strongly continuous in the norm of $L_1(X; \mathfrak{X})$ when restricted to $L_1(X; \mathfrak{X})$ and strongly integrable on every finite t-interval when restricted to $L_\infty(X; \mathfrak{X})$, respectively. Let $\{T(t)$: $t \geqslant 0\}$ be an L_1-continuous and L_∞-integrable semigroup of bounded linear operators on $L_1(X; \mathfrak{X}) + L_\infty(X; \mathfrak{X})$. For $f = g+h$ with $g \in L_1(X; \mathfrak{X})$ and $h \in L_\infty(X; \mathfrak{X})$, we define

$$\int_0^a T(t)f\,dt = (L_1)\int_0^a T(t)g\,dt + (L_\infty)\int_0^a T(t)h\,dt, \tag{5.1}$$

where the signs preceding the integrals indicate the norms with respect to which each integral is defined. Then it is easy to verify that the definition (5.1) is consistent. Using scalar representations $T(t)g(x)$ and $T(t)h(x)$ of $T(t)g$ and $T(t)h$, respectively, we have a scalar representation $T(t)f(x)$ of $T(t)f$ by setting $T(t)f(x) = T(t)g(x) + T(t)h(x)$. As a matter of course, the integral

$$\int_0^a T(t)f(x)\,dt = \int_0^a T(t)g(x)\,dt + \int_0^a T(t)h(x)\,dt,$$

as a function of x, is a scalar representation of $\int_0^a T(t)f\,dt$. In the sequel, $\mathfrak{X}$ will be assumed to be reflexive. Let T_j, $1 \leqslant j \leqslant n$, be linear operators on $L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$ with $\|T_j\|_1 \leqslant 1$ and $\sup\{\|T_j^k\|_\infty: k \geqslant 0\} \leqslant K$, $1 \leqslant j \leqslant n$, for some constant $K \geqslant 1$. Let $\{T_j(t): t \geqslant 0\}$, $1 \leqslant j \leqslant n$, be L_1-continuous and L_∞-integrable semigroups of linear operators on $L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$ with $\|T_j(t)\|_1 \leqslant 1$ and $\sup\{\|T_j(t)\|_\infty: t \geqslant 0\} \leqslant K$, $1 \leqslant j \leqslant n$, for some constant $K \geqslant 1$. For $f \in L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$, put

$$f_d^*(x) = \sup_{k_1,\ldots,k_n>0} |||V(k_1,\ldots,k_n)f(x)|||,$$

(5.2)

$$f_c^*(x) = \sup_{a_1,\ldots,a_n>0} |||A(a_1,\ldots,a_n)f(x)|||.$$

In what follows we write $f^*(x)$ for $f_d^*(x)$ in the discrete time case and for $f_c^*(x)$ in the continuous time case. We set up the following assumption.

$(^*_*)$ There are L_∞-bounded sublinear operators $U_1,\ldots,U_n$ of weak type (1,1) such that for every $f \in L_1(X;\mathfrak{X})+L_\infty(X;\mathfrak{X})$

$$f^*(x) \leqslant U_n \ldots U_1|||f|||(x), \qquad |||f|||(x) = |||f(x)|||.$$

In fact, we have some examples of operators materializing the assumption $(^*_*)$ as mentioned in the preceding section.

1. The operators in the setting of Corollary 3 realize assumption $(^*_*)$.

2. The semigroups in the setting of Corollary 5 realize assumption $(^*_*)$.

3. The measure preserving transformations in the setting of Corollary 4 realize assumption $(^*_*)$.

4. The semiflows in the setting of Corollary 6 realize assumption $(^*_*)$.

5. The semigroups in the setting of Theorem 6 realize assumption $(^*_*)$.

6. The operators in the discrete setting of Theorem 6 realize assumption$(^*_*)$.

Now, as a direct consequence of Theorems 1 and 2, we have the following theorem which offers a powerful tool to the study of ergodic theorems.

THEOREM 7. *Under assumption $(^*_*)$, let $\theta > 1$ and $\lambda > 0$.*

(1) *For every $f \in \Omega_\mu^{k+n}(X;\mathfrak{X})$ $(k = 0,1,2,\ldots)$ there holds the strong type inequality (which implies $f^* \in \Omega_\mu^k(X;\mathfrak{X})$ whenever $f \in \Omega_\mu^{k+n}(X;\mathfrak{X})$)*

$$\int_{\{f^*>\theta\lambda\}} \frac{f^*(x)}{\lambda}\left[\log\frac{f^*(x)}{\lambda}\right]^k d\mu(x) \leqslant A_{n,k}\int_{E_{B_n f}(\theta\lambda)} \frac{B_n|||f(x)|||}{\lambda}\left[\log\frac{B_n|||f(x)|||}{\lambda}\right]^{k+n} d\mu(x).$$

(2) *For every $f \in \Omega_\mu^{k+n-1}(X;\mathfrak{X})$ $(k = 0,1,2,\ldots)$ there holds the weak type inequality*

$$\mu\{f^* > \theta\lambda\} \leqslant C_{n,k}\int_{E_{B_n f}(\theta\lambda)} \frac{B_n|||f(x)|||}{\lambda}\left[\log\frac{B_n|||f(x)|||}{\lambda}\right]^{k+n-1} d\mu(x).$$

Here the constants $A_{n,k}$, B_n and $C_{n,k}$ are the same as in Theorems 1 and 2.

THEOREM 8. *Under assumption $(^*_*)$, let $f \in \Omega_\mu^{n-1}(X;\mathfrak{X})$. Then the functions $V(k_1,\ldots,k_n)f$ converge strongly in $\mathfrak{X}$ almost everywhere on X as $k_1,\ldots,k_n \to \infty$ independently.*

Proof. We denote by D_r the set of all n-tuples $a = (k_1,\ldots,k_n)$ of positive integers k_i with $k_i \geqslant r$, $1 \leqslant i \leqslant n$, and let $Q(n;a) = V(k_1,\ldots,k_n)$ for $a = (k_1,\ldots,k_n) \in D_r$. Define for $f \in \Omega_\mu^{n-1}(X;\mathfrak{X})$

$$\omega(f)(x) = \lim_{r\to\infty}\sup_{a,b\in D_r} |||Q(n;a)f(x)-Q(n;b)f(x)|||,$$

then ω is clearly subadditive and $\omega(f) \leqslant 2f_d^*$. Let us choose a sequence $\{f_k\}$ of simple functions having support of finite measure, such that $\lim_{k\to\infty} f_k = f$ strongly in $\mathfrak{X}$ pointwise and $|||f-f_k||| \leqslant 2|||f|||$ for all $k \geqslant 1$. From assumption $(^*_*)$, Corollary 2 and Theorem 3 it follows that $\omega(f_k) = 0$, $1 \leqslant k$, and so $\omega(f) \leqslant \omega(f-f_k)+\omega(f_k) \leqslant 2(f-f_k)_d^*$ for all $k \geqslant 1$. Thus by Theorem 7, for a fixed $\theta > 1$ and any $\lambda > 0$, we have

$$\begin{aligned}\mu\{\omega(f) > 2\theta\lambda\} &\leqslant \mu\{(f-f_k)_d^* > \theta\lambda\}\\ &\leqslant C_{n,0}\cdot\int_{E_{B_n(f-f_k)}(\theta\lambda)} \frac{B_n|||f-f_k|||}{\lambda}\left[\log\frac{B_n|||f-f_k|||}{\lambda}\right]^{n-1} d\mu\\ &\leqslant C_{n,0}\cdot\int_{E_{2B_n f}(\theta\lambda)} \frac{B_n|||f-f_k|||}{\lambda}\left[\log\frac{B_n|||f-f_k|||}{\lambda}\right]^{n-1} d\mu\end{aligned}$$

tending to zero as $k\to\infty$ by virtue of the Lebesgue dominated convergence theorem. Hence $\mu\{\omega(f) > 2\theta\lambda\} = 0$ for any $\lambda > 0$. This implies $\omega(f) = 0$ almost everywhere on X and completes the proof.

The continuous analogue of Theorem 8 is stated as follows.

THEOREM 9. *Under assumption $(^*_*)$, let $f \in \Omega_\mu^{n-1}(X;\mathfrak{X})$. Then the functions $A(a_1,\ldots,a_n)f$ are convergent strongly in $\mathfrak{X}$ almost everywhere on X as $a_1,\ldots,a_n \to \infty$ independently.*

The proof of Theorem 9 follows exactly the same line as that of Theorem 8 and we omit the details.

THEOREM 10. *Under assumption $(^*_*)$, let $f \in \Omega_\mu^{n-1}(X;\mathfrak{X})$. Then the equation*

$$\lim_{a_1,\ldots,a_n\to 0+} A(a_1,\ldots,a_n)f = f$$

holds strongly in $\mathfrak{X}$ almost everywhere on X.

Proof. For $\delta > 0$ let G_δ denote the set of all n-tuples $a = (a_1,\ldots,a_n)$ of real numbers a_i with $1/a_i \geqslant \delta$, $1 \leqslant i \leqslant n$, and put $R(n;a) = A(a_1,\ldots,a_n)$ for $a = (a_1,\ldots,a_n) \in G_\delta$. We define an operator η by

$$\eta(f)(x) = \lim_{\delta\to\infty}\sup_{a,\beta\in G_\delta} |||R(n;a)f(x)-R(n;\beta)f(x)|||$$

for $f \in \Omega_\mu^{n-1}(X; \mathfrak{X})$. Then η is subadditive and $\eta(f) \leqslant 2f_c^*$. Now for $f \in \Omega_\mu^{n-1}(X;\mathfrak{X})$ we choose, as in the proof of Theorem 8, a sequence $\{f_k\}$ of simple functions having support of finit emeasure, such that $\lim_{k\to\infty} |||f-f_k||| = 0$ pointwise and $|||f-f_k||| \leqslant 2|||f|||$ for all $k \geqslant 1$. From assumption $(^*{}_*{}^*)$, Corollary 2 and Theorem 5, it is seen that $\eta(f_k) = 0$ and $\eta(f) \leqslant \eta(f-f_k) + \eta(f_k) \leqslant 2(f-f_k)_c^*$ for all $k \geqslant 1$. Therefore the same argument as that in the proof of Theorem 8 then gives $\eta(f) = 0$ almost everywhere on X after applying Theorem 7 with $\theta > 1$ and $\lambda > 0$. Hence the theorem follows.

Remark. It is known [5] that Theorems 8,9 and 10 hold without the assumption $(^*{}_*{}^*)$ in the case of $n = 1$, for $(^*{}_*{}^*)$ is in fact true for $n = 1$. Particularly, for the operators in the setting of Corollaries 3 and 5, these Theorems 8, 9 and 10 remain also true for functions in the larger class $\Omega_\mu^0(X; \mathfrak{X})$.

Finally, I'd like to raise a problem unanswered at this time. In the setting of Section 5, let $n \geqslant 2$ and let f^* be the ergodic maximal function given by (5.2). The following question is open as yet.

Are there L_∞-bounded sublinear operators $U_1, \ldots, U_n$ of weak type (1,1) such that the product operation $U_n \ldots U_1 f$ dominates the function f^*?

References

[1] M. A. Alcoglu, *A pointwise ergodic theorem for L_p-spaces*, Canad. J. Math. 27 (1975), 1075–1082.

[2] R. V. Chacon, *An ergodic theorem for operators satisfying norm conditions*, J. Math. Mech. 11 (1962), 165–172.

[3] N. Dunford and J. T. Schwartz, *Convergence almost everywhere of operator averages*, J. Rat. Mech. Anal. 5 (1956), 129–178.

[4] N. A. Fava, *Weak type inequalities for product operators*, Studia Math. 62 (1972), 271–288.

[5] T. Yoshimoto, *Vector valued ergodic theorems for operators satisfying norm conditions*, Pacific J. Math. 85 (1979), 485–499.

[6] — *Some inequalities for ergodic power functions*, Acta Math. Acad. Sci. Hungar. 36 (1980), 19–24.

[7] — *Pointwise ergodic theorems and function classes M_p^α*, to appear in Studia Math.

[8] — *Ergodic theorems for operators in L_1+L_∞*, to appear.

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Geodesics on open surfaces containing horns

by

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Abstract. Symbolic dynamics for the geodesic flow on an open surface containing a hyperbolic horn is constructed in a neighbourhood of the geodesic which escapes in the horn in the future and in the past. It provides a variety of geodesics with different asymptotical behaviour. It is proved that a hyperbolic horn is sharp if and only if the geodesic escaping in this horn form a one-parameter family. A description is given of the set of escaping geodesics for a hyperbolic horn which is not sharp. Also the existence of a continuum of oscillating geodesics is proved under the condition that the topology of the surface is sufficiently rich.

Introduction. Let X be a locally compact, separable topological space and let $\{\varphi_t\}_{t\in\boldsymbol{R}}$ be a continuous flow on X.

DEFINITION 1. We call $x \in X$

(a) *bounded for* $t\to+\infty$ $(t\to-\infty)$ iff there is a compact set K such that $\varphi_t x \in K$ for $t \geqslant 0$ $(t \leqslant 0)$;

(b) *escaping for* $t\to+\infty$ $(t\to-\infty)$ iff for every compact set K there is a $T \in \boldsymbol{R}$ such that $\varphi_t x \notin K$ for $t \geqslant T$ $(t \leqslant T)$;

(c) *oscillating for* $t\to+\infty$ $(t\to-\infty)$ iff (a) and(b) are not satisfied.

Obviously the properties (a), (b) and (c) are the properties of the whole trajectories of the flow. In this paper we study the problem of the existence of oscillating trajectories for geodesic flows on complete open surfaces.

Let M be a 2-dimensional manifold (a surface) with a Riemannian metric of class C^2. We assume that M is complete in this metric, not compact and finitely connected. In view of the last assumption there is a homeomorphism h taking M onto $X\setminus\{x_1, \ldots, x_n\}$ where X is a compact surface and $x_i \in X$, $i = 1, \ldots, n$. The points $\{x_1, \ldots, x_n\}$ are called *points at infinity*.

DEFINITION 2. A *tube* is an open subset $\mathscr{T} \subset M$ homeomorphic to a punctured disk such that $h(T)$ is a punctured neighbourhood of a point at infinity and the closure of $h(\mathscr{T})$ in X contains only one point at infinity.

Consider closed rectifiable Jordan curves in a tube $\mathscr{T}$ which cannot be contracted to a point in $\mathscr{T}$. We denote by $w(\mathscr{T})$ the infimum of their lengths.