The assertion (3.1) can be formulated without reference to translation
structure or to Bessel transforms. For p sufficiently near 3, assertion (3.1)
holds on a number of "spaces of homogeneous type" (see [23]). We would
merely like to comment that 3.1 holds, in particular, on the unit ball
in $C^n$ (for a proof, see [5]) and that the proofs of 2.1 and 2.2 transfer,
without significant change, to that context.

Finally, we note that the techniques of this paper can be used to
give some sufficient conditions that an operator $f\rightarrow \int K(x, t)f(t)dt$ map
$H^q(X, \sigma)$ to $H^s(X, \mu)$, $m \neq s, q > p$.

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Inequalities for product operators
and vector valued ergodic theorems

by

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Abstract. The basic setting in this consideration is the function class $L_p^*(X, \mathcal{F})$
consisting of all strongly $\sigma$-measurable $X$-valued functions $f$ defined on $X$ such that
$||f||_{L_p^*} = (\int ||f(t)||^p d\mu(t))^{1/p}$ is integrable on the set where $||f||_p > t$ for every $t > 0$,
where $(X, \mathcal{F}, \mu)$ is a $\sigma$-finite measure space and $(X, ||-||)$ is a Banach space. In this setting
some strong type and weak type inequalities (which are indispensable for studying
ergodic theorems) for products of $L_{\infty}$-bounded quasi-linear operators of weak type
(1,1) are proved as generalizations of the maximal and the dominated ergodic theo-
remas for the ergodic maximal operators. Moreover, we demonstrate that these results
enable us to obtain some vector valued extensions of the ergodic theorems of Dun-
ford-Schwartz type and further generalizations to functions in the class $L_p^*(X, X)$.
Local (mean and pointwise) ergodic theorems are also obtained to add to the above
results.

1. Introduction. The ergodic theorems (usually called the mean
ergodic theorem and the pointwise ergodic theorem) had received a con-
siderably general operator-theoretic treatment in the case where the
underlying space is just the Lebesgue space $L_p = L_p(X, \sigma, \mu)$, where
$(X, \sigma, \mu)$ is a $\sigma$-finite measure space. After the lapse of time, especially,
a new approach to the study of pointwise ergodic theorems has been devel-
oped in several recent papers. The contravariance to this is the weak type
inequality powerful and indispensable for investigating the convergence
almost everywhere of operator averages. The first step in this direction
was taken by Fava [4] who extended the so-called "non-commuting
ergodic theorems" of Dunford and Schwartz for positive operators to
functions in a larger class than the space $L_p$ by means of a weak type
inequality for products of maximal operators on $L_1 + L_\infty$ which is the
class of all functions of the form $f = g + h$ with $g \in L_1$ and $h \in L_\infty$. The
method of proving the Fava inequality, as his proof shows, calls for the
positivity of maximal operators. Recently the author [5] has generalized
his inequality to the case of quasi-linear operators without assumption

* Some partial results in this paper were presented at the Meeting of Mathematical Society of Japan on October 4, 1979.
of positivity and then extended his generalizations of non-commuting ergodic theorems to the case without assuming the operators involved to be positive (cf. [7]). However, the vector valued ergodic theorems have not been given a comparable general operator-theoretic formulation. Much less is known about the vector valued ergodic theorems of Chacon type. One such result was obtained in the author’s paper [5] being appropriated for extending the theorem of Chacon [2]. Now it would be an interesting problem to study the Dunford–Schwartz type apropos of convergence of simultaneous averages of several single operators and of continuous parameter semigroups of operators in the reflexive Banach space valued case. In fact, the principal objectives in this paper will be to demonstrate the weak type and the strong type inequalities for products of quasi-linear operators of weak type (1,1) in the Banach space valued case and to obtain some vector valued extensions of the Dunford–Schwartz theorems. The obtained inequalities are further extensions of those by Fava [4] and the author [8], and the obtained ergodic theorems are further generalizations of those due to Chacon [2] and the author [5], [8].

Now our inequalities are of interest for some reasons. The first is just the fact that they regulate the behaviors of quasi-linear operators including those induced by the Hardy–Littlewood maximal function, the ergodic maximal functions and the ergodic power functions ([6]). Another reason for studying them is that they are natural extensions of the maximal ergodic theorem and the dominated ergodic theorem and that the major role resembles the fact that the maximal-dominated theorems are of essential use in investigating ergodic theorems.

Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and let \((X, \|\cdot\|)\) be a Banach space. We denote by \(\mathcal{L}^p_r(X; \mathbb{X})\) the class of all strongly measurable \(\mathbb{X}\)-valued functions \(f\) defined on \(X\) such that

\[
\int_{\|f(x)\| > \lambda} \frac{||f(x)||}{\lambda} \left( \log \frac{||f(x)||}{\lambda} \right) \frac{d\mu(x)}{\lambda} \quad \text{for every } \lambda > 0.
\]

Then \(\mathcal{L}^p_r(X; \mathbb{X})\), \(0 \leq p < \infty\) turns out to be a general decreasing sequence of linear spaces. We prove the weak type and the strong type inequalities for quasi-linear operators on \(L_p(X; \mathbb{X})\) of weak type (1,1) and for functions in \(\mathcal{L}^p_r(X; \mathbb{X})\) which permit to obtain some vector valued ergodic theorems of Dunford–Schwartz type for any function in \(\mathcal{L}^p_r(X; \mathbb{X})\), where \(n\) is the number of operators involved. Furthermore, the analogous extensions for the case of continuous parameter semigroups of linear operators will also be obtained.

2. Quasi-linear operators and function classes \(\mathcal{L}^p_r(X; \mathbb{X})\). Let \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{F}, \nu)\) be two \(\sigma\)-finite measure spaces and let \((X, ||\cdot||)\) be a Banach space. Unless stated otherwise from now on, all equalities and inequalities are understood to hold almost everywhere. In what follows it will be convenient to suppress the argument of a function, writing \(f\) for \(f(x)\). For an operator \(U\) mapping strongly \(\mathcal{F}\)-measurable \(\mathbb{X}\)-valued functions defined on \(X\) to strongly \(\mathcal{F}\)-measurable \(\mathbb{X}\)-valued functions defined on \(Y\), we say that \(U\) is \((\mathbb{M}, \mathcal{F})\)-quasi-linear if there is a positive constant \(\mathcal{M}\) depending only on \(U\) such that

\[
\left\| U(f+g) \right\| \leq \mathcal{M} \left( \left\| Uf \right\| + \left\| Ug \right\| \right),
\]

\[
\left\| U(cf) \right\| = |c| \left\| Uf \right\|
\]

for any complex number \(c\). In particular, \(U\) is said to be sublinear when \(\mathcal{M} = 1\). The operator \(U\) is of weak type \((1,1)\) provided that for every \(f \in D(U)\) (the domain of \(U\)) and for any \(\lambda > 0\), there exists a positive constant \(W\) independent of \(f\) and \(\lambda\) such that

\[
\nu(\left\| Uf \right\| > \lambda) \leq \frac{W}{\lambda} \int \left\| f(x) \right\| d\mu(x).
\]

Moreover, the least value of such \(W\) is called the weak type \((1,1)\)-norm of \(U\) and denoted by \(W_{1,1}(U)\). Now, the case that \((X, \mathcal{A}, \mu) \equiv (Y, \mathcal{F}, \nu)\) will be considered throughout the remainder of the present paper. Let \(L_p(X; \mathbb{X}) = L_p(X, \mathcal{F}, \mu; \mathbb{X})\), \(1 \leq p < \infty\), be the usual Banach spaces of strongly \(\mathcal{F}\)-measurable \(\mathbb{X}\)-valued functions defined on \(X\). Let \(L_1(X; \mathbb{X}) + + L_\infty(X; \mathbb{X})\) denote the class of all functions \(f\) of the form \(f = g + h\) with \(g \in L_1(X; \mathbb{X})\) and \(h \in L_\infty(X; \mathbb{X})\). An operator \(U\) defined on \(L_1(X; \mathbb{X}) + + L_\infty(X; \mathbb{X})\) is called \(L_\infty\)-bounded if for all \(f \in L_1(X; \mathbb{X}) + + L_\infty(X; \mathbb{X})\) there exists a constant \(K\) independent of \(f\) for which \(\left\| Uf \right\|_\infty \leq K \left\| f \right\|_\infty\).

Furthermore, the least value of such \(K\) is said to be the strong type \((\infty, \infty)\)-norm of \(U\) and denoted by \(K_{\infty,\infty}(U)\). The typical examples of \(L_\infty\)-bounded quasi-linear operators of weak type \((1,1)\) in ergodic theory setting are the operators induced from the Hardy–Littlewood maximal function, the ergodic maximal functions and the ergodic power functions as mentioned in the introduction. Evidently the class of \(L_\infty\)-bounded quasi-linear operators of weak type \((1,1)\) is considerably larger than that of maximal operators in the sense of Fava, which are operators \(U\) defined on \(L_1(X) + + L_\infty(X)\) satisfying that

(1) \(f \geq 0\) implies \(Uf \geq 0\).

(2) \(U\) is sublinear.

(3) \(0 \leq f \leq g\) implies \(Uf \leq Ug\).

(4) \(\left\| Uf \right\|_\infty \leq \left\| f \right\|_\infty\).

(5) \(U\) is of weak type \((1,1)\).

Throughout this paper the notation \(E_p(\lambda)\) will be used instead of the set \(\{ x : \|f(x)\| > \lambda \} \).
LEMMA 1. Let $U$ be an $L_a$-bounded $(M)$-quasi-linear operator of weak type $(1,1)$ defined on $L_a(X;\mathbb{X})+L_a(X;\mathbb{X})$. Let $W_{a,1}(U)$ be the weak type $(1,1)$-norm of $U$ and let $K_{a,m}(U)$ be the strong type $(\infty,\infty)$-norm of $U$. Then for every $f \in L_a(X;\mathbb{X})+L_a(X;\mathbb{X})$, and for any $\lambda > 0$, 
\begin{equation}
E_{\mathcal{G}}[M(K_{a,m}(U)+\lambda)] \leq \frac{W_{a,1}(U)}{\lambda} \int \frac{||f||}{\mathcal{A}_d}(x) \delta \mu(x).
\end{equation}

Proof. It is sufficient to prove the lemma for the case where $||f||$ is integrable over the set $\mathcal{E}_d(\lambda)$ since the right hand side of (2.1) is infinite then the lemma holds trivially. Define $f^*(x) = f(x)\mathcal{A}_d(\lambda)$ and $f_0(x) = f(x)-f^*(x)$, where $L_a$ stands for the characteristic function of the set $A$. Then we have 
\begin{equation}
E_{\mathcal{G}}[M(K_{a,m}(U)+\lambda)] = E_{\mathcal{G}}[\lambda] \leq \frac{W_{a,1}(U)}{\lambda} \int ||f^*(x)|| \delta \mu(x)
\end{equation}
and so 
\begin{equation}
\mathcal{E}_d[\mathcal{G}(M(K_{a,m}(U)+\lambda))] \leq \frac{W_{a,1}(U)}{\lambda} \int ||f^*(x)|| \delta \mu(x)
\end{equation}
which, when applied to the definition of $f^*$ gives (2.1) and completes the proof.

Corollary 1. Under the hypothesis of Lemma 1, let $f \in L_a(X;\mathbb{X})+L_a(X;\mathbb{X})$. 
(1) For $1 < p < \infty$ there is a constant $A > 0$ independent of $f$, such that 
\begin{equation}
\mathcal{E}_d[\mathcal{G}(M(K_{a,m}(U)+\lambda))] \leq A \int ||f|| \delta \mu(x).
\end{equation}

(2) There are two constants $B > 0$ and $C > 0$ independent of $f$, such that 
\begin{equation}
\mathcal{E}_d[\mathcal{G}(M(K_{a,m}(U)+\lambda))] \leq B \int ||f|| \delta \mu(x).
\end{equation}

The proof of Corollary 1 can be directly done by the standard argument (see [5]). Theorem 1. Particularly, (1) may be given by using the Marcinkiewicz interpolation theorem.

In the sequel we denote by $\mathcal{G}^*(X;\mathbb{X})$ the class of all strongly $\alpha$-measurable $\mathbb{X}$-valued functions $f$ defined on $X$ satisfying (1.1). Such classes were considered by Fava in a special case where $\mathbb{X}$ is the linear space of complex numbers and $\alpha = 0, 1, 2, \ldots$

Let $L(X;\mathbb{X})[\log^\alpha L(X;\mathbb{X})]$, $0 < \alpha < \infty$, denote the class of all strongly $\alpha$-measurable $\mathbb{X}$-valued functions $f$ defined on $X$ such that 
\begin{equation}
\mathcal{E}_d[\mathcal{G}(M(K_{a,m}(U)+\lambda))] \leq A \int ||f|| \delta \mu(x),
\end{equation}
where $\log^\alpha s = \log \max(1, s^\alpha)$ for $s \geq 1$.

THEOREM \[5]\, [6]. Let $1 < p < \infty$ and $0 < \alpha, \beta < \infty$. 
(i) $\mathcal{G}^*(X;\mathbb{X})$ is a linear space. 
(ii) $L(X;\mathbb{X}) \subseteq \mathcal{G}^*(X;\mathbb{X}) \subseteq L(X;\mathbb{X}) + L_a(X;\mathbb{X})$. 
(iii) $\mathcal{G}^*(X;\mathbb{X})$ for $0 < \alpha < \infty$. 
(iv) $L(X;\mathbb{X}) \subseteq \mathcal{G}^*(X;\mathbb{X}) \subseteq L(X;\mathbb{X})[\log^\alpha L(X;\mathbb{X})]^\alpha \subseteq L(X;\mathbb{X}) + L_a(X;\mathbb{X})$ for $p > 1$. 
(v) $\mathcal{G}^*(X;\mathbb{X})$ and $L(X;\mathbb{X})[\log^\alpha L(X;\mathbb{X})]^\alpha$ if and only if $\mu(X) < \infty$. 
(vi) $\mathcal{G}^*(X;\mathbb{X})$ is the linear span of $\bigcup \mathcal{G}^*(X;\mathbb{X})$.

3. Inequalities for products of quasi-linear operators. Of convergence in some sense of the operator averages, a central role has frequently been played by a weak type estimate (sometimes called the maximal ergodic theorem). With this comprehension we appropriate the present section to demonstrate the weak type and the strong type inequalities for products of quasi-linear operators.

LEMMA 2 ([4], [3]). Let $\psi(t)$ be a non-decreasing function on the real interval $0 < t < \infty$, such that $\psi(0) = 0$ and $\psi(t)$ is absolutely continuous on every finite subinterval. Then for any non-negative function $f$ defined on $X$ 
\begin{equation}
\int \psi(f(x)) \delta \mu(x) = \int \frac{1}{\mu(E \cap \mathcal{E}_d(t))} \psi(t) \delta t, \quad E \in \mathcal{B}.
\end{equation}

THEOREM 1 (Strong type inequality). Let $U_i, 1 \leq i \leq n$, be an $L_a$-bounded $(M_i)$-quasi-linear operators of weak type $(1,1)$ defined on $L_a(X;\mathbb{X}) + L_a(X;\mathbb{X})$. For each $i (1 \leq i \leq n)$ let $W_{a,k}(U_i) = W_{a,k}(U_i)$ be the weak type $(1,1)$-norm of $U_i$. Let $\mu$ be an arbitrary measure on $\theta > 1$. Then for every $f \in \mathcal{G}^*(X;\mathbb{X})$ ($k = 0, 1, 2, \ldots$) and for any $\lambda > 0$ 
\begin{equation}
\mathcal{E}_d[\mathcal{G}(M(K_{a,m}(U)+\lambda))] \leq A \int \frac{||U_1 \ldots U_n f||}{\lambda} \left[ \log \frac{||U_1 \ldots U_n f||}{\lambda} \right] \delta \mu(x),
\end{equation}
where 
\begin{equation}
A_{n,\lambda} = \prod_{n=1}^N \left( \frac{2}{\log \delta} + \frac{1}{k+1} \right) \delta \mu(x), \quad B_{n} = \prod_{n=1}^N M_i(K_{a,m}(U_i) + 1).
\end{equation}

Consequently $f \in \mathcal{G}^*(X;\mathbb{X})$ implies $U_1 \ldots U_n f \in \mathcal{G}^*(X;\mathbb{X})$. 

Inequalities for product operators
Proof. The proof will be achieved by induction on the number of operators. Let \( \theta > 1, \lambda > 0 \) and \( f \in L_{p}^{\lambda}(X; E) \), and put \( g = f/\lambda \). Then by Theorem A, \( g \in L_{p}^{\lambda+1}(X; E) \) and thus \( U_{g}f \) and \( U_{g}E_{\sigma} \) are well defined for each \( i \) since \( L_{p}^{\lambda+1}(X; E) \subset L_{p}^{\lambda}(X; E) \). With the function \( \varphi \) defined by \( \varphi(u) = u^{[\log u]} \) for \( u > 0 \), it follows from Lemma 2 that

\[
\int_{E_{\varphi}(\theta)} \left[ \log \left( \frac{1}{\lambda} \right) \right]^{b} \frac{d\mu(\varphi)}{\lambda} = \int_{E_{\varphi}(\theta)} \left( \log \left( \frac{1}{\lambda} \right) \right)^{b} d\mu(\varphi)
\]

However, since \( E_{\varphi}(\theta) \subset E_{\varphi}(u) \) for \( u < \theta \), we have by Lemma 1

\[
(3.5) \quad \int_{E_{\varphi}(\theta)} \mu(E_{\varphi}(\theta) \cap E_{\varphi}(u)) \varphi'(u) d\mu(u)
\]

\[
= \int_{E_{\varphi}(\theta)} \mu(E_{\varphi}(\theta)) \varphi'(u) d\mu(u)
\]

\[
\leq \int_{E_{\varphi}(\theta)} \mu(E_{\varphi}(\theta) \cap E_{\varphi}(u)) \varphi'(u) d\mu(u)
\]

\[
= \int_{E_{\varphi}(\theta)} \left( \log \left( \frac{1}{\lambda} \right) \right)^{b} d\mu(\varphi)
\]

\[
= \frac{1}{\lambda^{b+1}} \int_{E_{\varphi}(\theta)} \left( \log \left( \frac{1}{\lambda} \right) \right)^{b+1} \frac{d\mu(\varphi)}{\lambda}
\]

\[
\leq W_{1}(i) \frac{1}{\log \theta} \int_{E_{\varphi}(\theta)} \frac{M_{\left( K_{\varphi}(i) + 1 \right)}}{\lambda} \frac{1}{\lambda} \left( \log \left( \frac{1}{\lambda} \right) \right)^{b+1} \frac{d\mu(\varphi)}{\lambda}
\]

while, also by Lemma 1

\[
(3.3) \quad \int_{E_{\varphi}(\theta)} \mu(E_{\varphi}(\theta) \cap E_{\varphi}(u)) \varphi'(u) d\mu(u)
\]

\[
= \int_{E_{\varphi}(\theta)} \mu(E_{\varphi}(\theta)) \varphi'(u) d\mu(u)
\]

\[
\leq W_{1}(i) \frac{1}{\log \theta} \int_{E_{\varphi}(\theta)} \frac{M_{\left( K_{\varphi}(i) + 1 \right)}}{\lambda} \frac{1}{\lambda} \left( \log \left( \frac{1}{\lambda} \right) \right)^{b+1} \frac{d\mu(\varphi)}{\lambda}
\]

Therefore, adding (3.2) and (3.3) and letting

\[
A_{1} = \left( \frac{1}{\log \theta} + \frac{1}{k+1} \right) W_{1}(i),
\]

\[
B_{1} = M_{\left( K_{\varphi}(i) + 1 \right)}
\]

give

\[
\int_{E_{\varphi}(\theta)} \frac{d\mu(\varphi)}{\lambda} \left( \log \left( \frac{1}{\lambda} \right) \right)^{b+1} \frac{d\mu(\varphi)}{\lambda}
\]

\[
\leq A_{1} \cdot \int_{E_{\varphi}(\theta)} \frac{B_{1}}{\lambda} \left( \log \left( \frac{1}{\lambda} \right) \right)^{b+1} \frac{d\mu(\varphi)}{\lambda}
\]
which establishes the fact that the theorem is true for the case of \( n = 1 \). We next assume that the theorem has been proved to hold for the case of \( m \) quasi-linear operators \( U_1, \ldots, U_m \). Let \( f \in L_\alpha^{\gamma+1}(X; \mathcal{B}) \). The fact just observed then implies \( U_{j f} \in L_\alpha^{\gamma+1}(X; \mathcal{B}) \) and so we see from the induction hypothesis that

\[
\mathcal{A}_{m, \lambda} \int_{E_{m+1}} \left[ \log \frac{||U_{m+1} \cdots U_{j f}||}{\lambda} \right] d\mu(x) \\
\leq \mathcal{A}_{m, \lambda} \int_{E_{m+1}} \left[ \log \frac{\mathcal{B}_{m+1} \cdots \mathcal{B}_{j f}||}{\lambda} \right] d\mu(x),
\]

where

\[
\mathcal{A}_{m, \lambda} = \prod_{i=1}^{m} \left( \log \frac{\theta}{\lambda} + \frac{1}{k+1} \right) W_i(m+2-i),
\]

\[
\mathcal{B}_{m} = \prod_{i=1}^{m} M_i(K_{\alpha}(i) + 1).
\]

Moreover, using the quasi-linearity and the result for \( n = 1 \), the last term of (3.4) is equal to

\[
\mathcal{A}_{m+1, \lambda} \int_{E_{m+1}} \left[ \log \frac{||U_{m+1} \cdots U_{j f}||}{\lambda} \right] d\mu(x)
\leq \mathcal{A}_{m+1, \lambda} \int_{E_{m+1}} \left[ \log \frac{\mathcal{B}_{m+1} \cdots \mathcal{B}_{j f}||}{\lambda} \right] d\mu(x),
\]

where

\[
\mathcal{A}_{m+1, \lambda} = \mathcal{A}_{m, \lambda} \left( \frac{2}{\log \theta} + \frac{1}{k+1} \right) W_i(1)
\]

\[
= \prod_{i=1}^{m+1} \left( \frac{2}{\log \theta} + \frac{1}{k+1} \right) W_i(m+2-i),
\]

\[
\mathcal{B}_{m+1} = \mathcal{B}_m M_i(K_{\alpha}(i) + 1) = \prod_{i=1}^{m+1} M_i(K_{\alpha}(i) + 1).
\]

Consequently, combining (3.4) and (3.5) shows that the theorem holds good for \( n = m+1 \) and establishes (3.1). Hence the proof is complete.

**Theorem 2.** (Weak type inequality). Let \( U_i, 1 \leq i \leq n \), be \( L_\alpha \)-bounded \((M_i)\)-quasi-linear operators of weak type \((1,1)\), defined on \( L_1(X; \mathcal{B}) + L_\alpha(X; \mathcal{B}) \). For each \( i \) \( (1 \leq i \leq n) \) let \( W_i = W_i(U_i) \) be the weak type \((1,1)\)-norm of \( U_i \) and \( K_{\alpha}(i) = K_{\alpha, \lambda}(U_i) \) the strong type \((\infty, \infty)\)-norm of \( U_i \). Let \( \theta \) be any real number with \( \theta > 1 \). Then for every \( f \in L_\alpha^{\gamma+1}(X; \mathcal{B}) \) \( (\theta, \lambda > 0, 1 \leq i \leq n) \) and for any \( \lambda > 0 \)

\[
\mu(E_{m+1} \cup J_{m+1} ||f||^\lambda d\mu(x)) \\
\leq \mathcal{A}_{n, \lambda} \mathcal{B}_{m+1} \int_{E_{m+1}} \left[ \log \frac{\mathcal{B}_{m+1} \cdots \mathcal{B}_{j f}||}{\lambda} \right] d\mu(x),
\]

where \( \mathcal{A}_{n, \lambda} \) and \( \mathcal{B}_{m+1} \) are as given in Theorem 1.

Remark. Inequality (3.6) looks like the natural weak type analogue of inequality (3.1) at the first glance. Indeed, (3.6) can immediately be derived from (3.1) for functions in \( L_\alpha^{\gamma+1}(X; \mathcal{B}) \). However, (3.1) is no guarantee of (3.6) for functions in \( L_\alpha^{\gamma+1}(X; \mathcal{B}) \) but not in \( L_\alpha^{\gamma+1}(X; \mathcal{B}) \).

**Proof.** The proof will be by induction on \( n \) (the number of operators). Let \( \theta > 1 \), \( \lambda > 0 \) and \( f \in L_\alpha^{\gamma+1}(X; \mathcal{B}) \). Then \( U_{j f} \) is well defined for each \( i \), so that by Lemma 1 we have

\[
\mu(E_{m+1} \cup J_{m+1} ||f||^\lambda d\mu(x)) \\
\leq \mathcal{A}_{n, \lambda} \mathcal{B}_{m+1} \int_{E_{m+1}} \left[ \log \frac{\mathcal{B}_{m+1} \cdots \mathcal{B}_{j f}||}{\lambda} \right] d\mu(x),
\]

where

\[
\mathcal{A}_{n, \lambda} = \mathcal{A}_{m, \lambda} \left( \frac{2}{\log \theta} + \frac{1}{k+1} \right) W_i(1)
\]

\[
= \prod_{i=1}^{m+1} \left( \frac{2}{\log \theta} + \frac{1}{k+1} \right) W_i(m+2-i),
\]

\[
\mathcal{B}_{m+1} = \mathcal{B}_m M_i(K_{\alpha}(i) + 1) = \prod_{i=1}^{m+1} M_i(K_{\alpha}(i) + 1).
\]

Then taking

\[
\mathcal{C}_{n, \lambda} = \frac{W_i(1)}{\log \theta} \mathcal{A}_{m, \lambda} \mathcal{B}_{m+1} \int_{E_{m+1}} \left[ \log \frac{\mathcal{B}_{m+1} \cdots \mathcal{B}_{j f}||}{\lambda} \right] d\mu(x),
\]

and inserting the constant \( \mathcal{C}_{n, \lambda} \) into (3.7) yield

\[
\mu(E_{m+1} \cup J_{m+1} ||f||^\lambda d\mu(x)) \leq \mathcal{C}_{n, \lambda} \mathcal{B}_{m+1} \int_{E_{m+1}} \left[ \log \frac{\mathcal{B}_{m+1} \cdots \mathcal{B}_{j f}||}{\lambda} \right] d\mu(x).
\]
which shows that the theorem holds for \( n = 1 \). To apply induction we suppose that the theorem has already been established for the case of \( m \) quasi-linear operators \( U_{n1}, \ldots, U_{mn} \), and let \( f \in L_n^{p+1}(\mathbb{X}) \). Since \( U_{nf} \in L_n^{p+1}(\mathbb{X}) \) on account of Theorem 1, it follows by the induction hypothesis that

\begin{equation}
\mu(E_{U_{n+1} \cdots U_{nf}}(\mathbb{X})) \\
\leq C_{m,b} \int_{E_{U_{n+1} \cdots U_{nf}}(\mathbb{X})} \frac{B_m ||f(x)||}{\lambda} \left[ \log \frac{B_m ||f(x)||}{\lambda} \right]^{b+1} d\mu(x),
\end{equation}

where the constant \( B_m \) is as given in the proof of Theorem 1. Again, by Theorem 1 and the quasi-linearity of the right hand side of (3.8) is equal to

\begin{equation}
C_{m,b} \int_{E_{U_{n+1} \cdots U_{nf}}(\mathbb{X})} \frac{||U_1(B_m f)||}{\lambda} \left[ \log \frac{||U_1(B_m f)||}{\lambda} \right]^{b+1} d\mu(x)
\end{equation}

where the constant \( B_{m+1} \) given in Theorem 1, where \( C_{m+1,b} = C_{m,b} \cdot A_{m+1,b} \). This observation, when combined with (3.8), proves that the theorem is true for the case of \( n = m + 1 \) and concludes the proof of the theorem.

**Corollary 2.** Let \( U_{1}, M_{i}, W_{i}(\mathbb{X}), K_{n}(\mathbb{X}), 1 \leq i \leq m, \) be as in Theorem 1.

1. If \( 1 < p < \infty \) then for every \( f \in L_1(\mathbb{X}; \mathbb{X}) + L_\infty(\mathbb{X}; \mathbb{X}) \)

\begin{equation}
\int_{\mathbb{X}} ||U_{n} \cdots U_{nf}(x)||^{p} d\mu(x) \leq D_{n,p} \int_{\mathbb{X}} ||f(x)||^{p} d\mu(x),
\end{equation}

where

\begin{equation}
D_{n,p} = \left( \frac{p}{p-1} \right) \sum_{i=1}^{m} \left[ M_i(K_{n}(\mathbb{X})) + 1 \right]^{p} W_i(\mathbb{X}).
\end{equation}

2. Let \( \theta > 1 \). Then for every \( f \in L_1(\mathbb{X}; \mathbb{X}) + L_\infty(\mathbb{X}; \mathbb{X}) \) and for each \( h (h = 0, 1, 2, \ldots) \) holds

\begin{equation}
\int_{\mathbb{X}} ||U_{n} \cdots U_{nf}(x)|| d\mu(x) \\
\leq \theta \left\{ \mu(\mathbb{X}) + \frac{C_{m,b}}{n+k} \int_{\mathbb{X}} B_n ||f(x)|| \left[ \log \frac{B_n ||f(x)||}{u} \right]^{b+1-k} d\mu(x) \right\}
\end{equation}

with the constants \( B_n \) and \( C_{m,b} \) given in Theorems 1 and 2.

**Proof.** It suffices to show (3.9) for functions in \( L_1(\mathbb{X}; \mathbb{X}) \), for if the right hand side of (3.9) is infinite then it holds trivially. So, if \( f \in L_1(\mathbb{X}; \mathbb{X}) \) one gets by Lemma 1

\begin{equation}
\int_{\mathbb{X}} ||U_{nf}(x)||^{p} d\mu(x) \\
= p \left[ M_i(K_{n}(\mathbb{X})) + 1 \right]^{p} \int_{\mathbb{X}} \left[ \log \frac{B_n ||f(x)||}{u} \right]^{b+1-k} d\mu(x)
\end{equation}

implying \( U_{nf} \in L_1(\mathbb{X}; \mathbb{X}) \). Hereby the desired conclusion may be understood to hold after using an induction argument.

As for (2), by the same reasoning as (1), we may consider only the case where \( \mu(\mathbb{X}) < \infty \) and \( f \in L_1(\mathbb{X}; \mathbb{X}) \) \( \log^+ L_1(\mathbb{X}; \mathbb{X}) \) \( p+1 \). Let \( \theta > 1 \) and \( f \in L_1^{p+1}(\mathbb{X}; \mathbb{X}) \). It follows that

\begin{equation}
\int_{\mathbb{X}} ||U_{n} \cdots U_{nf}(x)|| d\mu(x) = \int_{\mathbb{X}} \mu(E_{U_{n} \cdots U_{nf}}(\mathbb{X})) d\mu(x)
\end{equation}

Using Theorem 2, the estimate of the second term of the last part of (3.11) is as follows:

\begin{equation}
\int_{\mathbb{X}} \mu(E_{U_{n} \cdots U_{nf}}(\mathbb{X})) d\mu(x)
\end{equation}

\begin{equation}
\leq C_{n,k} \int_{\mathbb{X}} \left[ \sum_{k=0}^{m} \frac{B_n ||f(x)||}{u} \left[ \log \frac{B_n ||f(x)||}{u} \right]^{b+1-k} d\mu(x) \right]
\end{equation}

\begin{equation}
\leq C_{n,k} \int_{\mathbb{X}} B_n ||f(x)|| d\mu(x) \times
\end{equation}

\begin{equation}
\frac{\max(1, B_n ||f(x)||)}{n+k} \int_{\mathbb{X}} \left[ \log \frac{B_n ||f(x)||}{u} \right]^{b+1-k} d\mu(x)
\end{equation}

where

\begin{equation}
\int_{\mathbb{X}} \mu(E_{U_{n} \cdots U_{nf}}(\mathbb{X})) d\mu(x)
\end{equation}

\begin{equation}
= \frac{C_{n,k}}{n+k} \int_{\mathbb{X}} B_n ||f(x)|| \left[ \log \frac{B_n ||f(x)||}{u} \right]^{b+1-k} d\mu(x).
\end{equation}
Recall that $L^p_{	ext{loc}}(X; \mathcal{X}) = L^p(X; \mathcal{X})[\log^+ L(X; \mathcal{X})]^{\text{loc}}$ by Theorem A. Then, combining (3.11) and (3.12) gives (3.10) and completes the proof.

4. Vector valued ergodic theorems for $L_p(X; \mathcal{X})$. Let $(\mathcal{T}(t); t \geq 0)$ be a strongly continuous semigroup of bounded linear operators on $L_p(X; \mathcal{X})$ with $1 \leq p \leq \infty$. For each $f \in L_p(X; \mathcal{X})$ there exists a scalar representation $\xi(t, \cdot)$ of $\mathcal{T}(t)f$ (which is often denoted by $\mathcal{T}(t)f(x)$) such that

1. $\xi(t, x)$ is measurable in $(t, x)$ on the product space $[0, \infty) \times X$;
2. $\xi(t, x)$ is uniquely determined up to a set of points $(t, x)$ whose product measure is zero;
3. $\xi(t, \cdot) = \mathcal{T}(t)f(\cdot)$ in $L_p(X; \mathcal{X})$ for almost every $t \geq 0$;
4. there exists a null set $N(f)$ which may depend on $f$ but which is independent of $t$ and is such that for any $x \in X - N(f)$, $\xi(t, x)$ is integrable over every finite $t$-interval and

$$\int_0^\infty \xi(t, x) \, dt = \int_0^\infty \mathcal{T}(t)f(x) \, dt$$

in $L_p(X; \mathcal{X})$, $0 \leq a < \infty$.

With this function $\xi(t, x)$ we define

$$\int_0^\infty \mathcal{T}(t)f(x) \, dt = \int_0^\infty \xi(t, x) \, dt$$

which will be continuous in $a$ for almost every $x$. In this section $X$ is assumed to be reflexive. In fact this assumption is requisite for proving theorems about convergence of operator averages.

Theorem 3. Let $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ be (not necessarily commuting) linear operators on $L_p(X; \mathcal{X})$ with $||\mathcal{T}_j|| \leq 1$ and sup $\{||\mathcal{T}_j||: k \geq 0 \} < K$, $1 \leq j \leq n$, for some constant $K \geq 1$.

1. For every $f \in L_p(X; \mathcal{X})$ with $1 < p < \infty$, the multiple sequence

$$V(h_1, \ldots, h_n)f = \frac{1}{k_1 \cdots k_n} \sum_{k_1=1}^{k_1} \cdots \sum_{k_n=1}^{k_n} \mathcal{T}_{n1} \cdots \mathcal{T}_{1f}$$

is convergent in the norm of $L_p(X; \mathcal{X})$ as $h_1, \ldots, h_n \to \infty$ independently.

2. If for every $f \in L_p(X; \mathcal{X})$ with $1 < p < \infty$, one can show that

$$\sup_{k_1, \ldots, k_n} \left|\left| V(h_1, \ldots, h_n)f(x) \right|\right| < \infty$$

for almost all points $x$ in $X$, then the multiple sequence $V(h_1, \ldots, h_n)f$ converges strongly in $X$ almost everywhere on $X$ as $h_1, \ldots, h_n \to \infty$ independently.

3. If $T_1, \ldots, T_n$ are commutative and condition (1) is fulfilled for all $f \in L_p(X; \mathcal{X})$ then the strong limit $\lim_{n \to \infty} V(h_1, \ldots, h_n)f(x)$ exists for almost all $x$ in $X$.

where

$$V(h_1, \ldots, h_n)f(x) = \frac{1}{k_1 \cdots k_n} \sum_{k_1=1}^{k_1} \cdots \sum_{k_n=1}^{k_n} \mathcal{T}_{n1} \cdots \mathcal{T}_{1f}$$

for every $f \in L_p(X; \mathcal{X})$.

Proof. Let us put

$$V(T_j, k_j)f = \frac{1}{k_j} \sum_{k=1}^{k_j-1} \mathcal{T}_j f$$

then $V(h_1, \ldots, h_n)f = V(T_{n1}, k_n) \cdots V(T_1, k_1)f$ and it follows from the Helly-convexity theorem that

$$\sup\{||V(T_j, k_j)||_p: k_j \geq 1 \} < K, \quad 1 \leq j \leq n,$$

$$\sup\{||V(h_1, \ldots, h_n)||_p: h_1, \ldots, h_n \geq 1 \} < K^n.$$

According to [5], Theorems 1 and 3, there are projection operators $E_j$, $1 \leq j \leq n$, such that

$$\lim_{k_{n-1} \to \infty} ||V(T_j, k_j)f - E_j f||_p = 0$$

when $1 < p < \infty$.

Then the induction argument (similar to the Dunford–Schwarz’s) yields that

$$\lim_{k_{n-1} \to \infty} ||V(h_1, \ldots, h_n)f - E_n \cdots E_1 f||_p = 0$$

as $h_1, \ldots, h_n \to \infty$ independently, which proves (1). Now it is known ([5], Lemma 3) that (2) is true for the case of $n = 1$. To establish (2) for the general case we shall therefore apply induction and assume that it has been proved for the case of $m$ operators $T_1, \ldots, T_{m-1}$. Since $L_p(X; \mathcal{X})$ is reflexive for $1 < p < \infty$ and $V(T_1, k_1)f = V(T_2, k_2)f$, we can use the Kakutani–Yosida mean ergodic theorem to say that the linear manifold $\mathcal{M}_k$, generated by the functions of the form $f = g + (h - T_1)g$ with $g \in L_p(X; \mathcal{X})$, $T_1g = g$, $h \in L_p(X; \mathcal{X})$, is dense in $L_p(X; \mathcal{X})$. For such a function $f$ we have

$$||V(h_1, \ldots, h_m)f(x) - V(k_1, \ldots, h_m+1)g(x)|| \leq \frac{2m+1}{k_1} ||g||_p$$

tending to zero as $k_1 \to \infty$ for almost all $x$, which, together with the induction hypothesis, implies the almost everywhere convergence of the multiple sequence $V(h_1, \ldots, h_n)f$ for every $f \in \mathcal{M}_n$ dense in $L_p(X; \mathcal{X})$. We may now deduce (2) from the principle of Banach (convergence theorem) by considering (1). A propos of (3) it is sufficient to note that (3)
holds for functions in $L_p(X; \mathcal{X}) \cap L_1(X; \mathcal{X})$ ($p > 1$) dense in $L_1(X; \mathcal{X})$ as a special case of (2), for nothing remains but to use $(*)$ and the Banach convergence theorem.

Now it is very important to indicate the class of operators for which condition $(*)$ appearing in Theorem 3 is satisfied for all functions in $L_p(X; \mathcal{X})$ with $1 \leq p < \infty$. In connection with this question, $(*)$ is known to hold for a general semigroup of Dunford-Schwartz type operators ([3], [5]). By the way we note that in the special case where $\mathcal{X}$ is the linear space of complex numbers, this condition is also known to hold for several semigroups of Dunford-Schwartz operators ([3], [4], [3]) and of positive $L_p$ ($p > 1$) and $L_\infty$-contractions ([1], [7]). Here we indicate two special cases in which $\mathcal{X}$ is a reflexive Banach space.

**Corollary 3.** Let $S_{ij}$, $1 \leq j \leq n$, be operators in the $B$-space $B(\mathcal{X})$ of bounded linear operators on $\mathcal{X}$ with $\sup \{0 \leq S_k^+ \leq K, 1 \leq k \leq n\}$ for some constant $K \geq 1$. Then the statements (1), (2) and (3) of Theorem 3 hold for $S_{11}, \ldots, S_{nn}$ and for all $f \in L_p(X; \mathcal{X})$, $1 \leq p < \infty$.

**Proof.** Define $(T_j f)(x) = S_{ij}(f(x))$, $1 \leq j \leq n$. It follows that

$$\sup_{k_1, \ldots, k_n \geq 1} \|V(k_1, \ldots, k_n)\|_d \leq K^{n-1} \|f\|_d$$

almost everywhere on $X$. Then the desired conclusion follows at once from Theorem 3.

**Corollary 4.** Let $\varphi_j$, $1 \leq j \leq n$, be measure preserving transformations on $\mathcal{X}$. Then all the conclusions corresponding to (1)-(3) of Theorem 3 remain true.

**Proof.** For $f \in L_p(X; \mathcal{X})$ with $1 \leq p < \infty$, define $(T_j f)(x) = \varphi_j f(x)$, $1 \leq j \leq n$. Then there holds

$$\sup_{k_1, \ldots, k_n \geq 1} \|V(k_1, \ldots, k_n)\|_d \leq U_{ij} \ldots U_{nn} \|f\|_d$$

almost everywhere on $X$, where $U_{ij}$, $1 \leq j \leq n$, are $L_\infty$-contracting sublinear operators of weak type $(1,1)$ which are given by

$$\{U_{ij}\} d = \frac{1}{k} \sum_{k \geq 1} \|f\|_d (\varphi_k^j) d \leq K^{n-1} \|f\|_d$$

Therefore, the corollary may be deduced from Theorem 3.

The continuous analogue of Theorem 3 is also true and may be stated as follows.

**Theorem 4.** Let $\{T_j(t); t \geq 0\}$, $1 \leq j \leq n$, be strongly continuous semigroups of linear operators on $L_1(X; \mathcal{X})$ with $\|T_j(t)\|_d \leq 1$ and suppose $\{\|T_j(t)\|_d; t \geq 0\} \leq K$, $1 \leq j \leq n$, for some constant $K \geq 1$.

(1) For every $f \in L_1(X; \mathcal{X})$ with $1 < p < \infty$, the functions

$$A(a_1, \ldots, a_n) f = \frac{1}{a_1 \cdots a_n} \int_0^1 \ldots \int_0^1 T_{a_1}(t) \ldots T_{a_n}(t) f dt_1 \ldots dt_n$$

are convergent in the norm of $L_p(X; \mathcal{X})$ as $a_1, \ldots, a_n \to \infty$ independently. (2) If for every $f \in L_p(X; \mathcal{X})$ with $1 < p < \infty$, one can show that

$$\sup_{a_1, \ldots, a_n \to \infty} \|A(a_1, \ldots, a_n) f(a)\|_d < \infty$$

for almost all $a$ in $\mathcal{X}$, then $A(a_1, \ldots, a_n) f$ converges strongly in almost everywhere on $X$ as $a_1, \ldots, a_n \to \infty$ independently.

(3) If $\{T_1(t); t \geq 0\}, \ldots, \{T_n(t); t \geq 0\}$ are commutative and condition $(**)$ is fulfilled for every $f \in L_1(X; \mathcal{X})$ then the strong limit $\lim A(a) f$ exists for almost all $a$ in $\mathcal{X}$, where

$$A(a) = \frac{1}{a^n} \int_0^1 \ldots \int_0^1 T_{a_1}(t) \ldots T_{a_n}(t) f dt_1 \ldots dt_n$$

We omit the proof of this theorem since the same argument as that in Theorem 3 applies to the continuous-parameter case.

**Corollary 5.** Let $\{S_j(t); t \geq 0\}$, $1 \leq j \leq n$, be strongly continuous semigroups of operators in $B(\mathcal{X})$ with $\sup \{0 \leq S_j(t) \leq K, 1 \leq j \leq n\}$ for some constant $K \geq 1$. Then the statements (1)-(3) of Theorem 4 hold for these semigroups and for all $f \in L_p(X; \mathcal{X})$, $1 \leq p < \infty$.

**Corollary 6.** Let $\{\varphi_j(t); t \geq 0\}$, $1 \leq j \leq n$, be measurable semigroups on $X$. Then all the conclusions corresponding to (1)-(3) of Theorem 4 remain true for these semigroups.

**Theorem 5.** Let $\{T_j(t); t \geq 0\}$, $1 \leq j \leq n$, be strongly continuous (and not necessarily commuting) semigroups of linear operators on $L_p(X; \mathcal{X})$ with $\|T_j(t)\|_d \leq 1$ and suppose $\{\|T_j(t)\|_d; t \geq 0\} \leq K$, $1 \leq j \leq n$, for some constant $K \geq 1$. Then for every $f \in L_p(X; \mathcal{X})$, $1 < p < \infty$,

$$\lim_{a_1, \ldots, a_n \to \infty} \|A(a_1, \ldots, a_n) f - f\|_p = 0.$$

**Proof.** Let $1 < p < \infty$ and define for $f \in L_p(X; \mathcal{X})$,

$$A_j(a) f = \frac{1}{a} \int_0^1 T_j(t) f dt, \quad a > 0, 1 \leq j \leq n.$$
Then it follows that
\[
A(a_1, \ldots, a_n)f = A_{a_1}(a_n) \cdots \cdots A_{a_1}(a_1)f,
\]
\[
\|A_{a_j}(a_j)\|_p \leq K, \quad 1 \leq j \leq n,
\]
\[
\|A(a_1, \ldots, a_n)\|_p \leq K^n, \quad a_1 > 0, \ldots, a_n > 0.
\]
Now we have by Theorems 1 and 4 of [5]
\[
\lim_{a \to 0^+} \frac{1}{a} \int_{\mathbb{R}} T_a(t) f dt = 0, \quad f \in L_p(X; \mathbb{X}),
\]
and, moreover, (4.2) is also valid for \(p > 1\), since there holds
\[
\left\| \frac{1}{a} \int_{\mathbb{R}} T_a(t) f dt - f \right\|_p \leq \frac{1}{p} \int_{\mathbb{R}} \|T_a(t) f - f\|_p dt,
\]
which tends to zero as \(a \to 0^+\) because of the strong continuity of \(T_a(t)\) in \(L_p(X; \mathbb{X})\). The theorem is therefore true for the case of \(n = 1\). In order to show (4.1) for \(n \geq 2\) we apply induction and assume that it has already been established for the case of \(m\) semigroups \(T_i(t), \ldots, T_{m+1}(t)\). Then for every \(f \in L_p(X; \mathbb{X})\), \(1 < p < \infty\), we have
\[
\left\| \sum_{i=1}^{m+1} A_{a_{m+1}}(a_{m+1}) \cdots \cdots A_{a_1}(a_1) f - f \right\|_p
\]
\[
\leq \left\| \sum_{i=1}^{m+1} A_{a_{m+1}}(a_{m+1}) \cdots \cdots A_{a_1}(a_1) f - f \right\|_p
\]
\[
+ \left\| \sum_{i=1}^{m+1} A_{a_{m+1}}(a_{m+1}) \cdots \cdots A_{a_1}(a_1) f - f \right\|_p
\]
which approaches zero as \(a_1, \ldots, a_{m+1} \to 0^+\) independently by the fact observed for \(n = 1\) and the induction hypothesis. Consequently the theorem follows by the what we have observed above.

**Theorem 6.** Let \(X\) be the linear space of complex numbers. Let \(\{T_i(t)\}_{t \geq 0}\) be strongly continuous, (and not necessarily commuting) semigroups of linear operators on \(L_p(X; \mathbb{X})\) such that \(T_i(t)\) is \(L_p\)-continuous and \(L_p\)-integrable if \(T_i(t)\) is strongly continuous in the norm of \(L_p(X; \mathbb{X})\) when restricted to \(L_p(X; \mathbb{X})\) and strongly integrable on every finite \(t\)-interval when restricted to \(L_p(X; \mathbb{X})\), respectively. Let \(\{T_i(t)\}_{t \geq 0}\) be an \(L_p\)-continuous and \(L_p\)-integrable semigroup of bounded linear operators on \(L_p(X; \mathbb{X})\). For \(f = g + h\), we define
\[
\left( \int_0^\infty T(t) f dt \right) g = \left( \int_0^\infty T(t) g dt \right) + \left( \int_0^\infty T(t) h dt \right),
\]
where the signs preceding the integrals indicate the norms with respect to which each integral is defined. It is easy to verify that the definition (5.1) is consistent. Using scalar representations \(T(t)g(x)\) and \(T(t)h(x)\) of \(T(t)g\) and \(T(t)h\), respectively, we have a scalar representation \(T(t)f(x)\) of \(T(t)f\) by setting \(T(t)f(x) = T(t)g(x) + T(t)h(x)\). As a matter of course, the integral
\[
\int_0^\infty T(t)f(x) dt = \int_0^\infty T(t)g(x) dt + \int_0^\infty T(t)h(x) dt,
\]
as a function of \(a\), is a scalar representation of \(\int T(t) f dt\). In the sequel, \(X\) will be assumed to be reflexive. Let \(T_i, 1 \leq i \leq n\), be linear operators almost everywhere on \(X\). Furthermore by Corollary 2, \(U_{a_1} \cdots U_{a_n} f \in L_p(X)\) for every \(f \in L_p(X)\). Therefore, the theorem follows from Theorem 5 and the Lebesgue dominated convergence theorem.

It is an interesting problem to study the pointwise local ergodic theorem corresponding to Theorem 5 (cf. [8]). In connection with this we state the following two special cases.

**Corollary 7.** On the hypothesis of Corollary 5, let \(f \in L_p(X; \mathbb{X})\), \(1 < p < \infty\). Then the equation
\[
\lim_{\alpha \to 0^+} \frac{1}{\alpha} \int_{\mathbb{R}} S_\alpha(t_0) \cdots \cdots S_\alpha(t_0) f(x) dt_1 \cdots \cdots dt_n = f(x)
\]
holds strongly in \(X\) for almost all \(x \in X\).

**Corollary 8.** Under the hypothesis of Corollary 6, let \(f \in L_p(X; \mathbb{X})\), \(1 < p < \infty\). Then the equation
\[
\lim_{\alpha \to 0^+} \frac{1}{\alpha} \int_{\mathbb{R}} \cdots \cdots \int_{\mathbb{R}} f(\varphi_\alpha(t_0) \cdots \cdots \varphi_\alpha(t_0) x) dt_1 \cdots \cdots dt_n = f(x)
\]
holds strongly in \(X\) for almost all \(x \in X\).

**5. More about vector valued ergodic theorems.** Given a semigroup \(\{T(t)\}_{t \geq 0}\) of linear operators on \(L_p(X; \mathbb{X})\) we shall say that it is \(L_p\)-continuous and \(L_p\)-integrable if \(T(t)\) is strongly continuous in the norm of \(L_p(X; \mathbb{X})\) when restricted to \(L_p(X; \mathbb{X})\) and strongly integrable on every finite \(t\)-interval when restricted to \(L_p(X; \mathbb{X})\), respectively. Let \(\{T(t)\}_{t \geq 0}\) be an \(L_p\)-continuous and \(L_p\)-integrable semigroup of bounded linear operators on \(L_p(X; \mathbb{X})\). For \(f = g + h\) with \(g \in L_p(X; \mathbb{X})\) and \(h \in L_p(X; \mathbb{X})\), we define
\[
\int_0^\infty T(t) f dt = \int_0^\infty T(t) g dt + \int_0^\infty T(t) h dt,
\]
where the signs preceding the integrals indicate the norms with respect to which each integral is defined. It is easy to verify that the definition (5.1) is consistent. Using scalar representations \(T(t)g(x)\) and \(T(t)h(x)\) of \(T(t)g\) and \(T(t)h\), respectively, we have a scalar representation \(T(t)f(x)\) of \(T(t)f\) by setting \(T(t)f(x) = T(t)g(x) + T(t)h(x)\). As a matter of course, the integral
\[
\int_0^\infty T(t)f(x) dt = \int_0^\infty T(t)g(x) dt + \int_0^\infty T(t)h(x) dt,
\]
as a function of \(a\), is a scalar representation of \(\int T(t) f dt\). In the sequel, \(X\) will be assumed to be reflexive. Let \(T_i, 1 \leq i \leq n\), be linear operators
on $L_I(X; \mathbb{X}) + L_w(X; \mathbb{X})$ with $\|T_k\|_{L_I} \leq 1$ and $\sup \|T_k\|_{L_w} : k \geq 0 \leq K$, $1 \leq j \leq n$, for some constant $K > 1$. Let $(T_j(t) ; t \geq 0)$, $1 \leq j \leq n$, be $L_r$-continuous and $L_s$-integrable semigroups of linear operators on $L_I(X; \mathbb{X}) + L_w(X; \mathbb{X})$ with $\|T_j(1)\|_{L_I} \leq 1$ and $\sup \|T_j(1)\|_{L_w} : t \geq 0 \leq K$, $1 \leq j \leq n$, for some constant $K > 1$. For $f \in L_I(X; \mathbb{X}) + L_w(X; \mathbb{X})$, put
\[ f^*(x) = \sup_{v \in X} \|v(f(x))||, \]
\[ f^*_v(x) = \sup_{a \in A} \|\Lambda(a) f(x)||. \]

In what follows we write $f^*(x)$ for $f^*_v(x)$ in the discrete time case and for $f^*_v(x)$ in the continuous time case. We set up the following assumption.

(\ast) There are $L_s$-bounded sublinear operators $U_1, \ldots, U_n$ of weak type $(1,1)$ such that for each $f \in L_I(X; \mathbb{X}) + L_w(X; \mathbb{X})$
\[ f^*(x) \leq U_1 f(x), \ldots, U_n f(x), \quad \|f(x)|| = \|f(x)||. \]

In fact, we have some examples of operators materializing the assumption (\ast) as mentioned in the preceding section.

1. The operators in the setting of Corollary 3 realize assumption (\ast).
2. The semigroups in the setting of Corollary 5 realize assumption (\ast).
3. The measure preserving transformations in the setting of Corollary 4 realize assumption (\ast).
4. The semiflows in the setting of Corollary 6 realize assumption (\ast).
5. The semigroups in the setting of Theorem 6 realize assumption (\ast).
6. The operators in the discrete setting of Theorem 6 realize assumption (\ast).

Now, as a direct consequence of Theorems 1 and 2, we have the following theorem which offers a powerful tool to the study of ergodic theorems.

**Theorem 7.** Under assumption (\ast), let $\theta > 0$ and $\lambda > 0$.

1. For every $f \in L_\mu^b(X; \mathbb{X})$ (k = 0, 1, 2, \ldots) there holds the strong type inequality (which implies $f \in L_\mu^b(X; \mathbb{X})$ whenever $f \in L_\mu^b(X; \mathbb{X})$)
\[ \mu \left( \frac{f^*(x)}{\lambda} \right) \leq A_{b,k} \int \frac{B_n ||f(x)||}{\lambda} \frac{1}{\mu} \, dm(x), \]
where $A_{b,k} = \int \frac{B_n ||f(x)||}{\lambda} \frac{1}{\mu} \, dm(x)$.

2. For every $f \in L_\mu^{b+1}(X; \mathbb{X})$ (k = 0, 1, 2, \ldots) there holds the weak type inequality
\[ \mu(f^* > \lambda) \leq C_{b,k} \int \frac{B_n ||f(x)||}{\lambda} \frac{1}{\mu} \, dm(x). \]

Here the constants $A_{b,k}$, $B_n$ and $C_{b,k}$ are the same as in Theorems 1 and 2.

**Theorem 8.** Under assumption (\ast), let $f \in L_\mu^{b+1}(X; \mathbb{X})$. Then the functions $V_{(a_1, \ldots, a_n)} f$ converge strongly in $X$ almost everywhere on $X$ as $b \to \infty$ independently.

**Proof.** We denote by $D_s$ the set of all $n$-tuples $(a_1, \ldots, a_n)$ of positive integers $k_i$ with $k_i \geq 1$, $1 \leq i \leq n$, and let $Q(n; a) = V_{(a_1, \ldots, a_n)}$ for $a = (a_1, \ldots, a_n) \in D_s$. Define for $f \in L_\mu^{b+1}(X; \mathbb{X})$
\[ \omega(f)(x) = \lim_{\lambda \to \infty} \sup_{a \in A} \frac{||Q(n; a)f(x) - Q(n; a)f(x)||}{\lambda}, \]
then $\omega(f)$ is clearly subadditive and $\omega(f) \in L_\mu^{b+1}$. Let us choose a sequence $(f_k)$ of simple functions having support of finite measure, such that $f_k \to f$ strongly in $X$ pointwise and $||f - f_k|| \leq 2 ||f||$ for all $k \geq 1$. From assumption (\ast), Corollary 2 and Theorem 3 it follows that $\omega(f_k) = 0$, $\lambda \geq 1$, and so $\omega(f) \leq \omega(f - f_k) + \omega(f_k) \leq 2 ||f - f_k||$ for all $k \geq 1$. Thus by Theorem 7, for a fixed $\theta > 1$ and any $\lambda > 0$, we have
\[ \mu(\omega(f) > \lambda) \leq C_{b,k} \int \frac{B_n ||f - f_k||}{\lambda} \frac{1}{\mu} \, dm(x), \]
\[ \leq C_{b,k} \int \frac{B_n ||f - f_k||}{\lambda} \frac{1}{\mu} \, dm(x), \]
tending to zero as $b \to \infty$ by virtue of the Lebesgue dominated convergence theorem. Hence $\mu(\omega(f) > \lambda) = 0$ for any $\lambda > 0$. This implies $\omega(f) = 0$ almost everywhere on $X$ and completes the proof.

**Theorem 9.** Under assumption (\ast), let $f \in L_\mu^{b+1}(X; \mathbb{X})$. Then the functions $A_{(a_1, \ldots, a_n)} f$ are convergent strongly in $X$ almost everywhere on $X$ as $a_1, \ldots, a_n \to \infty$ independently.

The proof of Theorem 9 follows exactly the same line as that of Theorem 8 and we omit the details.

**Theorem 10.** Under assumption (\ast), let $f \in L_\mu^{b+1}(X; \mathbb{X})$. Then the equation
\[ \lim_{s \to \infty} A_{(a_1, \ldots, a_n)} f = f \]
holds strongly in $X$ almost everywhere on $X$.

**Proof.** For $\delta > 0$ let $G_\delta$ denote the set of all $n$-tuples $(a_1, \ldots, a_n)$ of real numbers $\alpha_i$ with $1/\alpha_i \geq \delta$, $1 \leq i \leq n$, and put $R(\alpha_1, \ldots, \alpha_n) = A_{(a_1, \ldots, a_n)}$ discarding $a = (a_1, \ldots, a_n) \in G_\delta$. We define an operator $\eta$ by
\[ \eta(f)(x) = \lim \sup_{\lambda \to \infty} \frac{||R(\alpha_1, \ldots, \alpha_n)f(x) - R(\alpha_1, \ldots, \alpha_n)f(x)||}{\lambda}. \]
for } f \in L^p_{\text{loc}}(X; \mathbb{X}). \text{ Then } \eta \text{ is subadditive and } \eta(f) \leq 2 \eta^*_. \text{ Now for } f \in L^p_{\text{loc}}(X; \mathbb{X}) \text{ we choose, as in the proof of Theorem 8, a sequence } (f_k) \text{ of simple functions having support of finite measure, such that } \lim_{k \to \infty} ||f - f_k|| = 0 \text{ pointwise and } ||f - f_k|| \leq 2||f|| \text{ for all } k \geq 1. \text{ From assumption } (*)^*, \text{ Corollary 2 and Theorem 5, it is seen that } \eta(f_k) = 0 \text{ and } \eta(f) \leq \eta(f - f_k) + \eta(f_k) \leq 2||f - f_k||^* \text{ for all } k \geq 1. \text{ Therefore the same argument as that in the proof of Theorem 8 then gives } \eta(f) = 0 \text{ almost everywhere on } X \text{ after applying Theorem 7 with } \theta > 1 \text{ and } \lambda > 0. \text{ Hence the theorem follows.}

Remark. It is known [5] that Theorems 8, 9 and 10 hold without the assumption (*) in the case of } n = 1, \text{ for } (*)^* \text{ is in fact true for } n = 1. \text{ Particularly, for the operators in the setting of Corollaries 3 and 5, these Theorems 8, 9 and 10 remain also true for functions in the larger class } L^p(X; \mathbb{X}).

Finally, I'd like to raise a problem unanswered at this time. In the setting of Section 5, let } n \geq 2 \text{ and let } f^* \text{ be the ergodic maximal function given by } (3.2). \text{ The following question is open as yet.}

Are there } L^p_{\text{loc}} \text{-bounded sublinear operators } U_1, \ldots, U_n \text{ of weak type } (1,1) \text{ such that the product operation } U_n \ldots U_1 f \text{ dominates the function } f^*?\text{ }

References

[7] — Pointwise ergodic theorems and function classes } \mathbb{X}_w, \text{ to appear in Studia Math.}
[8] — Ergodic theorems for operators in } L^p + L^q, \text{ to appear.}

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Geodesics on open surfaces containing horns

by

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Abstract. Symbolic dynamics for the geodesic flow on an open surface containing a hyperbolic horn is constructed in a neighbourhood of the geodesic which escapes in the horn in the future and in the past. It provides a variety of geodesics with different asymptotic behaviour. It is proved that a hyperbolic horn is sharp if and only if the geodesic escaping in this horn form a one-parameter family. A description is given of the set of escaping geodesics for a hyperbolic horn which is not sharp. Also the existence of a continuum of oscillating geodesics is proved under the condition that the topology of the surface is sufficiently rich.

Introduction. Let } X \text{ be a locally compact, separable topological space and let } (\sigma_t)_{t \in \mathbb{R}} \text{ be a continuous flow on } X.

Definition 1. We call } \sigma \in \mathcal{X} \text{ (a) bounded for } t \to +\infty \text{ (or } t \to -\infty) \text{ iff there is a compact set } K \text{ such that } \sigma_t \Subset K \text{ for } t \geq 0 \text{ (or } t \leq 0);\text{ (b) escaping for } t \to +\infty \text{ (or } t \to -\infty) \text{ iff for every compact set } K \text{ there is a } T \in \mathbb{R} \text{ such that } \sigma_t \not\Subset K \text{ for } t \geq T \text{ (or } t \leq T);\text{ (c) oscillating for } t \to +\infty \text{ (or } t \to -\infty) \text{ iff (a) and (b) are not satisfied.}

Obviously the properties (a), (b) and (c) are the properties of the whole trajectories of the flow. In this paper we study the problem of the existence of oscillating trajectories for geodesic flows on open surfaces.

Let } M \text{ be a 2-dimensional manifold (a surface) with a Riemannian metric of class } C^2. \text{ We assume that } M \text{ is complete in this metric, not compact and finitely connected. In view of the last assumption there is a homeomorphism } \gamma \text{ taking } M \text{ onto } X \text{ in the compact surface and } x_i \in \mathcal{X}, i = 1, \ldots, n \text{ are called } points \text{ at infinity. Consider closed rectifiable Jordan curves in a tube } \mathcal{F} \text{ which cannot be contracted to a point in } \mathcal{F}. \text{ We denote by } \psi(\mathcal{F}) \text{ the inimity of their lengths.}