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W^*-algebras on Banach spaces

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Abstract. The linear span of hermitian operators on any complex Banach space X with a hyperorthogonal basis is a type I W^*-algebra. Moreover, the group of all linear isometries on X is a Banach–Lie subgroup of $\mathrm{GL}(X)$. A slight generalization of the first statement is also proved.

Introduction. Let X be a complex Banach space and denote by $\mathscr{L}(X)$ the algebra of all linear bounded operators on X. An operator $H \in \mathscr{L}(X)$ is called *hermitian* if $\|\exp(itH)\| = 1$ for all real t, or equivalently, if $\|I+itH\| \leqslant 1+o(t)$ for $t \in \boldsymbol{R}$, $t \to 0$. Hermitian operators on Banach spaces (and, more generally, hermitian elements of Banach algebras) were defined in hope that Hilbert space (resp. C^*-algebra) techniques could be applied at least in some cases, and this article is based on the same motivation. Denote by $\mathscr{H}(X)$ the real Banach space of all hermitian operators on X. The elements of $i\mathscr{H}(X)$ are called *skew-hermitian* operators. If $H \in \mathscr{H}(X)$ implies $H^2 \in \mathscr{H}(X)$, then $\mathscr{H}(X)+i\mathscr{H}(X)$ is an algebra. In this case we can define an involution on $\mathscr{H}(X)+i\mathscr{H}(X)$ by $(H+iK)^* = H-iK$ $(H, K \in \mathscr{H}(X))$ and in this way $\mathscr{H}(X)+i\mathscr{H}(X)$ becomes a C^*-algebra (Vidav–Palmer; see [1]).

But even if $\mathscr{H}(X)+i\mathscr{H}(X)$ is a C^*-algebra, it need not be a W^*-algebra.

EXAMPLE. Let $X = C[0,1]$. Then $T \in \mathscr{H}(X)$ if and only if there exists $h_T \in X$ with $\bar{h}_T = h_T$ such that $Tf = h_T f$ for all $f \in X$ ([2], p. 91). Clearly $T \mapsto h_T$ defines a *-isomorphism between the C^*-algebras $\mathscr{H}(X)+i\mathscr{H}(X)$ and the C^*-algebra $C[0,1]$. But $C[0,1]$ is not a W^*-algebra.

Our purpose is to present a class of complex Banach spaces for which $\mathscr{H}(X)+i\mathscr{H}(X)$ is a W^*-algebra. This class includes all spaces with a hyperorthogonal basis (i.e., a Schauder basis such that the norm of any vector depends only on the absolute values of its coefficients relative to this basis).

Finite dimensional complex Banach spaces with a hyperorthogonal basis were studied independently by Schneider and Turner [9] and Vidav [10]. Fleming and Jamison pursued this work to obtain nice results on the structure of such spaces in the infinite dimensional case, too ([4],

[5]). A certain generalization was published by Kalton and Wood [7]. Actually we will only make use of the fact that, for a finite dimensional complex Banach space X with a hyperorthogonal basis, $\mathscr{H}(X)+i\mathscr{H}(X)$ is an algebra. (The properties of infinite dimensional spaces with such basis will come out as a side product.) An important tool will be the characterization of W^*-algebras due to Kadison [6]. As a consequence we will be able to prove that for every complex Banach space X with a hyperorthogonal basis the group $\mathscr{G}$ of all isometries in $GL(X)$ is a Lie subgroup in $GL(X)$.

First of all we will prove some theorems on isometric representations of type I factors on Banach spaces.

1. Isometric representations of type I factors on Banach spaces. Throughout this section let X denote a complex Banach space, K a complex Hilbert space and $r\colon \mathscr{L}(K)\to\mathscr{L}(X)$ an isometric algebra homomorphism such that $r(I_K) = I_X$.

1.1. Lemma. *Let $\{P_i;\ i\in I\}$ be a family of mutually orthogonal equivalent non-zero projections in $\mathscr{L}(K)$. Choose $j\in I$, denote $P_j = P$ and let $V_i\in\mathscr{L}(K)$ be such that $V_i^*V_i = P$ and $V_iV_i^* = P_i$ $(i\neq j)$. Let f be a vector of norm 1 in $r(P)X$ and let $f_i = r(V_i)f$ $(i\neq j)$, $f_j = f$. If $\{t_i;\ i\in I\}$ is a set of scalars such that $\sum |t_i|^2$ $(i\in I)$ converges, then $\sum t_if_i$ $(i\in I)$ converges.*

Denote $Z_f = \{\sum t_if_i;\ \sum|t_i|^2<\infty\}$. Then Z_f is a Hilbert subspace of X (meaning that Z_f is a Hilbert space in the norm it inherits from X) and $\{f_i;\ i\in I\}$ is a Hilbert basis of X. If $f, g\in r(P)X$ are linearly independent, then $Z_f\cap Z_g = \{0\}$.

Proof. Remember that $P_iV_iP = V_i$ and $V_i^*V_k = 0$ for $i\neq k$. Consequently $r(P_i)f_i = r(P_iV_i)f = f_i$. Let $t_i\in C$ be such that $1 = \sum|t_i|_2$ $(i\in I)$. If J is a finite subset of I and $T_J = \sum t_iV_i$ $(i\in J)$, $a_J = \sum|t_i|^2$ $(i\in J)$, then $T_J^*T_J = a_JP$. It follows that $\sum t_iV_i$ $(i\in I)$ converges to some $T\in\mathscr{L}(X)$ and that $T^*T = P$. So T is a partial isometry. Since r is isometric, $r(T) = \sum t_ir(V_i)$ $(i\in I)$. Now

$$\begin{aligned}1 = \|f\| = \|r(P)f\| &= \|r(T^*)r(T)f\| \leqslant \|r(T^*)\|\,\|r(T)f\|\\ &= \|T^*\|\,\|r(T)f\| = \|r(T)f\| \leqslant \|r(T)\|\,\|f\| = 1.\end{aligned}$$

But $r(T)f = \sum t_ir(V_i)f = \sum t_if_i$. It follows that $\sum t_if_i$ converges and that $\|\sum t_if_i\| = 1 = \sum|t_i|^2$. So $Z_f = \{\sum t_if_i;\ \sum|t_i|^2<\infty\}$ is isometrically isomorphic to the space l^2 over I.

Let $f, g\in r(P)X$ be linearly independent. If $x\in Z_f\cap Z_g$, then $x = \sum s_if_i = \sum t_ig_i$ and so $r(P_i)X = s_if_i = t_ig_i = s_ir(V_i)f = t_ir(V_i)g$. We apply $r(V_i^*)$ on both sides to find $s_if = t_ig$. Hence $s_i = t_i = 0$ for all i and $x = 0$.

1.2. Proposition. *Let $\{P_i;\ i\in I\}$ be a family of mutually orthogonal one-dimensional projections in $\mathscr{L}(K)$ with $I_K = \sum P_i$ $(i\in I)$. Suppose that*

$r(P_i)$ is a one-dimensional projection for each $i\in I$ and that $x = \sum r(P_i)x$ $(i\in I)$ for all $x\in X$ (unordered convergence). Then X is a Hilbert space, isometrically isomorphic to K, and r is unitarily equivalent to the identity representation.

Proof. Let $e_i\in K$ be such that $P_ie_i = e_i$ and $\|e_i\| = 1$, so that $\{e_i;\ i\in I\}$ is a Hilbert basis of K. Now let j, f, f_i, V_i, Z_f be as in Lemma 1.1. If $x\in X$, then $r(P_i)x = t_if_i$ $(t_i\in C)$. It follows that $x = \sum r(P_i)x = \sum t_if_i$ $(i\in I)$. Since Z_f is complete, $x\in Z_f$. So $X = Z_f$ is a Hilbert space and $\{f_i;\ i\in I\}$ is a Hilbert basis of X.

Let $U\colon K\to X$ be the isometric isomorphism defined by $Ue_i = f_i$. Now $r(P_i)Ue_k = r(P_i)f_k = \delta_{ik}f_i = \delta_{ik}Ue_i$, hence $U^{-1}r(P_i)Ue_k = \delta_{ik}e_i$. It follows that $U^{-1}r(P_i)U = P_i$, or equivalently

$$r(P_i) = UP_iU^{-1}.$$

Denote $e_j = e$. We may assume that $V_ie = e_i$ $(i\neq j)$. (V_i was any partial isometry giving the equivalence between P and P_i.) Now $V_i\big((PK)^\perp\big) = 0$, hence $V_iP_k = 0$ for $k\neq j$. It follows that $r(V_i)r(P_k) = r(V_i)UP_kU^{-1} = 0$ and so $\big(U^{-1}r(V_i)U\big)P_k = 0$ for $i, k\neq j$. Also $r(V_i)Ue = r(V_i)f = f_i = Ue_i$, hence $U^{-1}r(V_i)Ue = e_i$. This implies that $U^{-1}r(V_i)U = V_i$, or equivalently,

$$r(V_i) = UV_iU^{-1}.$$

In the same fashion we can prove that

$$r(V_i^*) = UV_i^*U^{-1}.$$

Let $A\in\mathscr{L}(K)$. Since $P_iV_iP = V_i$ for all V_i, it follows that $V_k^*AV_i = PV_k^*AV_iP$. Now, P being one-dimensional, we see that for some $t_{ki}\in C$

$$V_k^*AV_i = t_{ki}P$$

and $r(V_k^*)r(A)r(V_i) = t_{ki}r(P)$. Thus $UV_k^*U^{-1}r(A)UV_iU^{-1} = t_{ki}UPU^{-1}$, hence

$$V_k^*\big(U^{-1}r(A)U\big)V_i = t_{ki}P.$$

But $V_k^*AV_ie = V_k^*Ae_i = t_{ki}e$. Thus $P_kAe_i = V_kV_k^*AV_ie = t_{ki}V_ke = t_{ki}e_k$. Therefore the numbers $\{t_{ki};\ i, k\in I\}$ determine A completely. Consequently $U^{-1}r(A)U = A$ or

$$r(A) = UAU^{-1}$$

and U gives us the desired equivalence.

1.3. Theorem. *Let $\{P_i;\ i\in I\}$ be a family of mutually orthogonal equivalent one-dimensional projections in $\mathscr{L}(K)$ with $I_K = \sum P_i$ $(i\in I)$. Let $I_X = \sum r(P_i)$ $(i\in I)$ in the strong operator topology. If $P\in\{P_i;\ i\in I\}$*

and $\{f_c;\ c \in C\}$ is any algebraic basis of the space $r(P)X$, then there exists a family $\{Z_c;\ c \in C\}$ of Hilbert subspaces of X, all isometrically isomorphic to K and invariant for r, so that their algebraic direct sum is dense in X. For each c, $r|Z_c$ is unitarily equivalent to the identity representation of the algebra $\mathscr{L}(K)$.

Proof. Once more we use Lemma 1. If $i,\ k \in I$, then clearly $r(P_i)X$ and $r(P_k)X$ are isometrically isomorphic. First we prove that the spaces Z_f are r-invariant. In fact it suffices to prove that, for any $A \in \mathscr{L}(K)$, $r(A)f_i \in Z_f$ for each $i \in I$. Since for $i \neq j$, $f_i = r(V_i)f$, it is even enough to prove that $x = r(A)f \in Z_f$.

Since $V_iP = V_i$, $r(V_i^*)x = r(PV_i^*)r(A)r(P)f = r(PV_i^*AP)f$; P being one-dimensional, $PV_i^*AP = t_iP$ for some $t_i \in \boldsymbol{C}$. Hence $r(V_i^*)x = t_ir(P)f$. Now $r(P_i)x = r(V_iV_i^*)x = r(V_i)r(V_i^*)x = t_ir(V_iP)f = t_ir(V_i)f = t_if_i \in Z_f$. Since $x = \sum r(P_i)x$ and Z_f is complete, $x \in Z_f$. We observe that the action of r on Z_f is such as in 1.2. Hence, on Z_f, r is equivalent to the identity representation.

Suppose $\{f_c;\ c \in C\}$ is a Hamel basis of the space $r(P)X$. Denote $Z_c = Z_{f_c}$. From 1.2 we learn that for $c \neq d$, $Z_c \cap Z_d = 0$. Since for each $i \in I$, $r(V_i)$ maps $r(P)X$ isometrically onto $r(P_i)X$, $\{r(V_i)f_c;\ c \in C\}$ is a Hamel basis for $r(P_i)X$. Thus if $x \in X$, $r(P_i)x = \sum_c s_{ic}r(V_i)f_c$. But $r(V_i)f_c \in Z_c$, hence $r(P_i)x \in \oplus \sum Z_c$.

2. W^*-algebras on certain Banach spaces. Let A be a W^*-algebra. Then there exists a Banach space A_* (called the *predual of* A and uniquely determined up to isometric isomorphism) such that $A_*^* = A$. We will make no difference between A_* and its canonical image in $A_*^{**} = A^*$. By σ we will denote the weak *-topology on A. Let T denote the set of all normal positive functionals on A. For $f \in T$ define $a_f(x) = f(x^*x)^{1/2}$. Then the seminorms $\{a_f;\ f \in T\}$ define a locally convex topology on A, called the *s-topology*. The properties of the σ-topology (also called *weak topology*) and the s-topology are presented in [8].

We will need some facts on hermitian operators. The reference is [1]. If X is a complex Banach space and $T \in \mathscr{L}(X)$, then the spatial numerical range of T is denoted by $V(T)$ and defined as

$$V(T) = \{f(Tx);\ f \in X^*, x \in X, \|f\| = \|x\| = f(x) = 1\}.$$

An operator $H \in \mathscr{L}(X)$ is hermitian if and only if $V(H) \subset \boldsymbol{R}$. Also $V(H) = \{0\}$ implies $H = 0$. If P is a hermitian projection (i.e., a non-zero hermitian idempotent), then $\|P\| = 1$. We call an operator $A \in \mathscr{H}(X)$ *positive* if $V(A) \subset \boldsymbol{R}^+$ or, equivalently, if $\sigma(A) \subset \boldsymbol{R}^+$. Clearly $\mathscr{H}(X)$ and the set of all positive operators are closed in the weak operator topology.

2.1. Theorem. *Let X be a complex Banach space and $\{P_i;\ i \in I\}$ an increasing net of hermitian projections on X such that $x = \sum P_ix$ $(i \in I)$ for all $x \in X$. For each $i \in I$ let $\mathscr{H}(P_iX) + i\mathscr{H}(P_iX)$ be a finite dimensional algebra. Denote $\mathscr{A} = \mathscr{H}(X) + i\mathscr{H}(X)$. Then the following are true:*

(a) *$\mathscr{A}$ is a type* I *W^*-algebra. On the unit ball of $\mathscr{A}$ the strong operator topology* (s.o. *topology) and the s-topology coincide.*

(b) (Decomposition of $\mathscr{A}$.) *There exists a family $\{G_z;\ z \in Z\}$ of mutually orthogonal central projections in $\mathscr{A}$ such that $G_z\mathscr{A}G_z$ is a type* I *factor for each z and such that $\mathscr{A}$ is a l^∞-direct sum of the ideals $G_z\mathscr{A}G_z$.*

(c) *If $G \in \{G_z;\ z \in Z\}$ and $S, S' \in G\mathscr{A}G$ are minimal projections, then the spaces SX, $S'X$ are isometrically isomorphic. If n is the dimension of the vector space SX, then there exists a family $\{Z_c;\ c \in C\}$ of Hilbert subspaces of GX such that $n = \operatorname{card} C$, $Z_c \cap Z_d = \{0\}$ for $c \neq d$, the algebraic direct sum $\oplus \sum Z$ is dense in GX, and such that each Z_c is invariant for $G_z\mathscr{A}G_z$ and $A \mapsto A|Z_c$ is a *-isomorphism of $G_z\mathscr{A}G_z$ onto $\mathscr{L}(Z_c)$.*

(d) *If $V\colon X \to X$ is a surjective linear isometry, then V permutes the subspaces $G_zX = X_z$. In case $VX_z = X_z$ we can write $V|X_z = UW$, where U is a unitary element of $G_z\mathscr{A}G_z|X_z$ and W commutes with every element of $G_z\mathscr{A}G_z|X_z$. (Thus W is completely determined by $W|QX$ where Q is any minimal projection in $G_z\mathscr{A}G_z$.)*

Proof. Our first step will be to prove that $\mathscr{A}$ is an algebra. Let $A \in \mathscr{H}(X)$. We choose arbitrary $t \in \boldsymbol{R}$, $x \in X$ and claim that $\|\exp(itA^2)x\| = \|x\|$. First we must see that $A_i = P_iAP_i|P_iX \in \mathscr{H}(P_iX)$. In fact

$$\begin{aligned}\|I_{P_iX} + itA_i\| &= \|P_i + itP_iAP_i\| = \|P_i(I+itA)P_i\| \\ &\leqslant \|I+itA\| = 1 + o(t) \qquad (t \in \boldsymbol{R},\ t \to 0).\end{aligned}$$

Since $\mathscr{H}(P_iX) + i\mathscr{H}(P_iX)$ is an algebra, it is a C^*-algebra (Vidav–Palmer) and thus $A_i^2 \in \mathscr{H}(P_iX)$. The fact that $\|P_i\| = 1$ for all i implies that $\exp\big(it(P_iAP_i)^2\big)$ converges to $\exp(itA^2)$ in the strong operator (s.o.) topology. But

$$\begin{aligned}\big\|\exp\big(it(P_iAP_i)^2\big)x\big\| &= \big\|\exp\big(it(P_iAP_i)^2\big)P_ix + (I-P_i)x\big\| \\ &= \|\exp(itA_i^2)P_ix + (I-P_i)x\| \leqslant \|P_ix\| + \|(I-P_i)x\| \\ &\leqslant \|x\| + \|x - P_ix\|.\end{aligned}$$

Hence $\|\exp(itA^2)x\| \leqslant \|x\|$ for all $t \in \boldsymbol{R}$ which implies that $A^2 \in \mathscr{H}(X)$.

As we remarked in the introduction this suffices to prove that $\mathscr{A}$ is an algebra and thus a C^*-algebra by the Vidav–Palmer theorem. We also observe that $B \mapsto B|P_iX$ defines a*-isomorphism of the C^*-algebra $P_i\mathscr{A}P_i$ into $\mathscr{H}(P_iX) + i\mathscr{H}(P_iX)$. Since the latter algebra is finite dimensional, $P_i\mathscr{A}P_i$ is a finite dimensional W^*-algebra.

Sublemma 1. *Every increasing bounded net $\{A(b);\ b \in B\}$ of positive elements in $\mathscr{A}$ converges in the* s.o. *topology to its* l.u.b.

Proof. We may assume that $A(b) \leqslant I_X$ for all $b \in B$. Since for $b' \geqslant b$

$$P_i(A(b')-A(b))P_i = (P_i(A(b')-A(b))^{1/2})(P_i(A(b')-A(b))^{1/2})^*,$$

$\{P_iA(b)P_i;\ b\in B\}$ is an increasing net bounded by P_i. But this net is contained in the finite dimensional algebra $P_i\mathscr{A}P_i$. Hence its l.u.b. exists and is equal to the weak limit and also to the norm limit.

Since $\|A(b')-A(b)\|\leqslant\|A(b')\|\leqslant 1$, it follows that

$$\|(A(b')-A(b))^{1/2}\|\leqslant 1.$$

Now observe that

$$\|P_iA(b')P_i-P_iA(b)P_i\| = \|(A(b')-A(b))^{1/2}P_i\|^2.$$

The inequality

$$\|(A(b')-A(b))P_i\|\leqslant\|(A(b')-A(b))^{1/2}\|\,\|(A(b')-A(b))^{1/2}P_i\|$$
$$\leqslant\|(A(b')-A(b))^{1/2}P_i\|,$$

therefore shows that $\{A(b)P_i;\ b\in B\}$ is a Cauchy net in $\mathscr{A}$ and as such convergent. Since $\{P_ix;\ i\in I,\ x\in X\}$ is dense in X and $\{A(b)P_ix;\ b\in B\}$ converges, it follows from the Banach–Steinhaus theorem that $\lim A(b)x$ exists for every $x\in X$. Denote this limit by Ax. Then A is a bounded linear operator. Since the cone $\mathscr{A}^+$ of positive elements of $\mathscr{A}$ is closed in the weak operator topology, $A\in\mathscr{A}^+$. It remains to show that $A=$ l.u.b. $\{A(b);\ b\in B\}$.

Clearly $P_iA(b)P_i\underset{b}{\to}P_iAP_i$ in the s.o. topology, which implies that

$$P_iAP_i = \text{l.u.b.}\{P_iA(b)P_i;\ b\in B\}.$$

Thus $P_i(A-A(b))P_i = P_iAP_i-P_iA(b)P_i\in\mathscr{A}^+$ for all $i\in I$. Since the strong limit of the net $\{P_i(A-A(b))P_i;\ i\in I\}$ is equal to $A-A(b)$, A is an upper bound for $\{A(b);\ b\in B\}$. If $A(b)\leqslant D\in\mathscr{A}^+$ for all $b\in B$, then for each $i\in I$, $P_iA(b)P_i\leqslant P_iDP_i$. It follows that $P_iDP_i\geqslant P_iAP_i$ and, as before, $D\geqslant A$. ∎

Now we are ready to prove that $\mathscr{A}$ is a W^*-algebra. Denote

$$\Pi(X) = \{(x,f)\in X\times X^*;\ \|x\| = \|f\| = f(x) = 1\}.$$

For $a=(x,f)\in\Pi(X)$ let $g_a\in\mathscr{A}^*$ be defined by $g_a(A)=f(Ax)$. Since $\{g_a(A);\ a\in\Pi(X)\} = V(A)$, the spatial numerical range of a, each g_a is a positive linear functional on $\mathscr{A}$. If $V(A)=\{0\}$, then $A=0$. Hence $\{g_a;\ a\in\Pi(X)\}$ is a total family on $\mathscr{A}$.

Suppose now that $\{A(b);\ b\in B\}$ is an increasing bonded net of positive elements in $\mathscr{A}$ and A its l.u.b. Let $a=(x,f)\in\Pi(X)$ and $g=g_a$. By Sublemma 1, $Ax=\lim A(b)x$. Hence

$$g(A) = f(Ax) = \lim g(A(b)) = \text{l.u.b.}\{g(A(b))\}.$$

We conclude that on $\mathscr{A}$ every increasing bounded net of positive elements has the l.u.b. and that there exists a total family of positive linear functionals on $\mathscr{A}$ which "preserve" such suprema. By a theorem of Kadison [6] $\mathscr{A}$ is a W^*-algebra.

As we remarked, $P_i\mathscr{A}P_i$ is a finite dimensional algebra for each $i\in I$. We choose a family $S_{i,1},\ldots,S_{i,n(i)}$ of minimal mutually orthogonal projections in $P_i\mathscr{A}P_i$ such that $S_{i,1}+\ldots+S_{i,n(i)}=P_i$. Let $\{S_t;\ t\in T\}$ be the set of all $S_{i,k}$ $(i\in I,\ k=1,\ldots,n(i))$. Clearly each S_t is a minimal projection in $\mathscr{A}$ and thus $S_t\mathscr{A}S_t$ is a one-dimensional algebra. Also, $x=\sum S_tx$ $(t\in T)$ for all $x\in X$.

By G_t we denote the central support of S_t. If $G\in\{G_t;\ t\in T\}$, we claim that $G\mathscr{A}G$ is a type I factor. Indeed, let Q be a non-zero central projection in $G\mathscr{A}G$. Then Q is central in $\mathscr{A}$, too. Now there exists $t\in T$ such that $QS_t\neq 0$. Since S_t is minimal, $QS_t=S_t$. But $Q\leqslant G$, hence $Q=G$.

If $G_tG_s\neq 0$, we can find $w\in T$ such that $G_tG_sS_w\neq 0$. As before, $G_sS_w=S_w$ and $G_tS_w=S_w$, hence $G_s=G_t$. Let $\{G_z;\ z\in Z\}$ be a subset of $\{G_t;\ t\in T\}$ such that

$$G_zG_y = 0\quad (y\neq z;\ y,z\in Z),$$
$$x = \sum G_zx\quad (z\in Z)\ (x\in X).$$

SUBLEMMA 2. *For each $z\in Z$ let $A_z\in G_z\mathscr{A}G_z$. If* l.u.b. $\{\|A_z\|;\ z\in Z\}<\infty$, *there exists one and only one $A\in\mathscr{A}$ such that $G_z\mathscr{A}G_z=A_z$ for all $z\in Z$. In this case $\|A\|=$* l.u.b. $\{\|A_z\|;\ z\in Z\}$ *and A is hermitian if and only if all of A_z are hermitian.*

Proof. Let $K=\{z_1,\ldots,z_n\}$ be a finite subset of Z. We denote $G_{z_i}=G_i$ and $A_{z_i}=A_i$, $A_K=\sum A_i$, $G_K=\sum G_i$ $(i=1,\ldots,n)$. Since $A_i^*A_j=0$ for $i\neq j$,

$$\|A_K\|^2 = \|A_K^*A_K\| = \Big\|\sum_{i=1}^{n}A_i^*A_i\Big\|.$$

Let $M=$ l.u.b. $\{\|A_z\|;\ z\in Z\}$ and $Q_K=A_K^*A_K$. For any $m\in N$

$$Q_K^m = \sum_{i=1}^{n}(A_i^*A_i)^m$$

and therefore $\|Q_K^m\|\leqslant nM^{2m}$. Hence $\|Q_K^{m}\|\leqslant M^2n^{1/m}$ and it follows that $\|A_K\|\leqslant M$.

For each $z\in Z$ let $A_z=B_z+iC_z$ where B_z, C_z are hermitian elements of $G_z\mathscr{A}G_z$. Moreover, let

$$B_z = B_z^+-B_z^-\quad (B_z^+,B_z^-\geqslant 0,\ B_z^++B_z^- = |B_z|).$$

Clearly $\{B_z^+;\ z\in Z\}$ satisfies the requirements of the Sublemma. For every finite subset K of Z let $B_K^+=\sum B_z^+$ $(z\in K)$ as before. If we order the finite subsets of Z by inclusion, $\{B_K^+\}$ becomes an increasing bounded net. By Sublemma 1 there exists $B^+ \in \mathscr{A}$ such that $B^+ = \text{l.u.b.}\, B_K^+$. Also, $B^+x = \lim B_K^+ x$ ($x \in X$), in particular

$$B^+G_zx = \lim_K B_K^+ G_z x \quad (x \in X).$$

If $z \in K$, then $B^+G_zx = B_z^+x$. Therefore $B^+G_z = B_z^+$.

Let $A = \lim A_K$ in the s.o. topology (or equivalently, in the σ-topology). Then $AG_z = A_z$ ($z \in Z$). If $A' \in \mathscr{A}$ and $A'G_z = AG_z$ for all $z \in Z$, then clearly $B = A$. Also the facts that $\|A_K\| \leqslant M$ for all K and that the unit ball of $\mathscr{A}$ is closed in the σ-topology imply that $\|A\| \leqslant M$. It follows that $\|A\| = M$. As for the last statement, observe that $\mathscr{H}(X)$ is closed in the s.o. topology. This completes the proof of (b).

We proceed with the proof of the theorem. Choose an arbitrary $G \in \{G_z;\ z \in Z\}$. Let $S, S' \in G\mathscr{A}G$ be any minimal projections. Since $G\mathscr{A}G$ is a type I factor, S and S' are equivalent, i.e., there exists $V \in G\mathscr{A}G$ such that $V^*V = S$ and $VV^* = S'$. It is easy to see that V maps SX isometrically onto $S'X$.

Let $J = \{t \in T;\ S_t \leqslant G\}$. Then $x = \sum S_jx$ ($j \in J$) for all $x \in GX$. We can apply Theorem 1.3 to end the proof of (c).

Now let us return to (a) and prove that on the unit ball of $\mathscr{A}$ the s.o. topology and the s-topology coincide. Let $\{A_a;\ a \in B\}$ be a bounded net in $\mathscr{A}$, converging to 0 in the s-topology. Then $\{P_iA_a^*A_aP_i;\ a \in B\}$ also converges to 0 in the s-topology. Since $P_i\mathscr{A}P_i$ is a finite dimensional algebra, this amounts to the fact that $P_iA_a^*A_aP_i \xrightarrow[a]{} 0$ in the norm topology, or equivalently, that $A_aP_i \xrightarrow[a]{} 0$ in the norm topology. It follows that $A_ax \to 0$ for every $x \in X$.

Conversely, suppose that $\{A_a;\ a \in B\}$ is a bounded net in $\mathscr{A}$, converging to 0 in the s.o. topology. Let $f \in \mathscr{A}^*$ be a normal positive functional. We are to show that $f(A_a^*A_a) \to 0$. First we observe that $f \in \mathscr{A}_*$, the predual of $\mathscr{A}$. But $\mathscr{A}_*$ is the l^1-direct sum of the spaces $(G_z\mathscr{A}G_z)_*$. Thus we have a decomposition

$$f = (f_z)_{z\in Z}$$

in the sense that

$$f(A) = \sum f_z(G_zAG_z) \quad (z \in Z)$$

for all $A \in \mathscr{A}$, where $f_z \in (G_z\mathscr{A}G_z)_*$ and $\|f\| = \sum \|f_z\|$ ($z \in Z$). Apparently it suffices to prove that $f_z(G_zA_a^*A_aG_z) \to 0$ for every $z \in Z$. Let $Z_c \subset G_zX$ be as in statement (c) of the theorem. Since f_z is a normal positive functional on $G_z\mathscr{A}G_z$, there exist vectors $x_i \in Z_c$ ($i \in N$) with $\sum \|x_i\|^2 < \infty$ and such that

$$f_z(G_zA_a^*A_aG_z) = \sum_{i=1}^{\infty} \|A_ax_i\|^2.$$

Since $A_ax \to 0$ for every $x \in X$, this concludes the proof of our statement and thus the proof of (a).

We pass to (d). Let $V\colon X \to X$ be a surjective linear isometry. If A is a hermitian operator on X, then VAV^{-1} is hermitian, too. Thus $A \mapsto VAV^{-1}$ is a *-automorphism on $\mathscr{A}$. Let $G \in \{G_z;\ z \in Z\}$. It is easy to see that $VGV^{-1} = G'$ where $G' \in \{G_z;\ z \in Z\}$. Thus $V(GX) = G'(VX) = G'X$.

Suppose now that $G' = G$, i.e. $VG = GV$. Then $A \mapsto VAV^{-1}$ is a *-automorphism of $G\mathscr{A}G$, which is a type I factor. Hence this is an inner automorphism and there exists a unitary operator U in $G\mathscr{A}G|GX$ such that $VAV^{-1} = UAU^{-1}$ for all $A \in G\mathscr{A}G|GX$. Let $W = U^*(V|GX)$. Then $WA = AW$ for all $A \in G\mathscr{A}G|GX$, as required.

If $Q \in G\mathscr{A}G$ is a minimal projection, there exists a family $\{Q_j;\ j \in J\}$ of mutually orthogonal equivalent projections in $G\mathscr{A}G$, with $Q_k = Q$ for some $k \in J$, and such that $G = \sum Q_j$ ($j \in J$). By Sublemma 1, $Gx = \sum Q_jx$ for all $x \in GX$. Let $V_j \in G\mathscr{A}G$ be such that $V_j^*V_j = Q$ and $V_jV_j^* = Q_j$ ($j \neq k$). If $x \in Q_jX$, then $Wx = WQ_jx = WV_jV_j^*x = V_jWV_j^*x$. Since $V_j^*x \in QX$, we see that W is completely determined by $W|QX$. ∎

Let $\{e_i;\ i \in N\}$ be a hyperorthogonal basis of a complex Banach space X (i.e., $\{e_i\}$ is a Schauder basis of X and $\|\sum_{i=1}^{\infty} a_ie_i\| = \|\sum_{i=1}^{\infty} b_ie_i\|$ if $|a_i| = |b_i|$ for all $i \in N$). Define $E_i \in \mathscr{L}(X)$ in the following way: if $x = \sum_{i=1}^{\infty} a_ie_i$, then $E_ix = a_ie_i$. Clearly E_i is a hermitian projection on X. The space $Y = (E_1+\ldots+E_n)X$ is a finite dimensional subspace with a hyperorthogonal basis. It follows from [9] or ([10], Theorem 9) that $\mathscr{H}(Y)+i\mathscr{H}(Y)$ is an algebra. Therefore Theorem 2.1 applies. In fact (since E_i are one-dimensional) we can improve the results as follows:

2.2. THEOREM. *Let X be a complex Banach space with a hyperorthogonal basis. Denote $\mathscr{A} = \mathscr{H}(X)+i\mathscr{H}(X)$. Then*

(a) *$\mathscr{A}$ is a type I W^*-algebra and on the unit ball of $\mathscr{A}$ the s-topology and the strong operator topology coincide.*

(b) *There exists a family $\{G_z;\ z \in Z\}$ of mutually orthogonal central projections in $\mathscr{A}$ such that $G_zX = X_z$ is a Hilbert space for each $z \in Z$ and such that $A \mapsto \{G_zAG_z|X_z;\ z \in Z\}$ is a *-isomorphism of $\mathscr{A}$ onto the direct sum of the algebras $\mathscr{L}(X_z)$.*

(c) *If $V\colon X \to X$ is a surjective linear isometry, then V permutes the subspaces X_z ($z \in Z$). If V preserves X_z, then $V|X_z = U$ where U is a unitary element of $\mathscr{L}(X_z)$.*

(d) *The group $\mathscr{G}$ of all isometries in $\mathrm{GL}(X)$ is a real Banach–Lie subgroup of $\mathrm{GL}(X)$. The principal component $\mathscr{G}_0$ of $\mathscr{G}$ is exactly the unitary group of $\mathscr{A}$.*

Proof. (d): Denote by $\mathscr{U}$ the unitary group of $\mathscr{A}$. Let $V \in \mathscr{G}$ and suppose that $VX_z = X_z$ for all $z \in Z$. Then $V|X_z = V_z$ is a unitary element in $\mathscr{L}(GX)$. There exists a selfadjoint operator $A_z \in \mathscr{L}(X_z)$ such that $\|A_z\| \leqslant 2\pi$ and $\exp(iA_z) = V_z$. Now there exists $A \in \mathscr{H}(X)$ such that $A|X_z = A_z$ for all z. Thus $\exp(iA) = V$. But V lies on the one-parameter group $\{\exp(itA);\ t \in \boldsymbol{R}\} \subset \mathscr{G}_0$. This implies that $\mathscr{U}$ is connected and thus $\mathscr{U} \subset \mathscr{G}_0$.

Suppose now that $V \in \mathscr{U}$ and $W \in \mathscr{G} - \mathscr{U}$. Then there exist $y, z \in Z$ such that $y \neq z$ and $WX_y = X_z$. If $x \in X_y$, then

$$\|G_y(V-W)x\| = \|G_yVx - G_yG_zWx\| = \|G_yVx\| = \|Vx\| = \|x\|.$$

This implies that $\|V-W\| \geqslant 1$ and proves that $\mathscr{U} = \mathscr{G}_0$. If

$$\mathscr{O} = \{A \in \mathrm{GL}(X);\ \|A-I\| < 1\},$$

then clearly $\mathscr{O} \cap \mathscr{G} \subset \mathscr{U}$. Let

$$\mathscr{K} = \{A \in \mathscr{L}(X);\ G_zAG_z|X_z \in \mathscr{H}(X_z) \text{ for all } z \in Z\}.$$

Clearly $\mathscr{K}$ is a real linear subspace of $\mathscr{L}(X)$. Since $\mathscr{H}(X_z)$ are closed subspaces, $\mathscr{K}$ is a (norm-) closed subspace in $\mathscr{L}(X)$. The real subspace $\mathscr{L} = i\mathscr{H}(X)$ is also closed in $\mathscr{L}(X)$ and apparently $\mathscr{L}(X) = \mathscr{K} \oplus \mathscr{L}$.

Since $(\mathscr{O}, \log, \mathscr{L}(X))$ is a chart for $\mathrm{GL}(X)$ at the identity and $\exp(\mathscr{L}) = \mathscr{U} \supset \mathscr{O} \cap \mathscr{G}$ as we have seen above,

$$\log|\mathscr{O}\cap\mathscr{G}\colon\ \mathscr{O}\cap\mathscr{G} \to \log(\mathscr{O}) \cap \mathscr{L}$$

is a homeomorphism. Hence $\mathscr{O}\cap\mathscr{G}$ is a submanifold in $\mathrm{GL}(X)$. By ([3], p. 101) $\mathscr{G}$ is a Lie subgroup in $\mathrm{GL}(X)$.

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References

[1] F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, Vol. I, Cambridge Univ. Press, Cambridge 1971.

[2] —, — *Numerical ranges*, Vol. II, Cambridge Univ. Press, Cambridge 1973.

[3] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. 2 & 3, Hermann, Paris 1972.

[4] R. J. Fleming and J. E. Jamison, *Hermitian and adjoint abelian operators on certain Banach spaces*, Pacific J. Math. 52 (1974), 67–85.

[5] —, — *Isometries on certain Banach spaces*, J. London Math. Soc. (2) 9 (1974), 121–127.

[6] R. V. Kadison, *Operator algebras with a faithful weakly closed representation*, Ann. of Math. 54 (1956), 175–181.

[7] N. J. Kalton and G. V. Wood, *Orthonormal systems in Banach spaces and their applications*, Math. Proc. Camb. Philos. Soc. 79 (1976), 493–510.

[8] S. Sakai, *C*-algebras and W*-algebras*, Springer, Erg. d. Math. 60, Berlin–Heidelberg–New York 1971.

[9] H. Schneider and R. E. L. Turner, *Matrices Hermitian for an absolute norm*, Linear and Multilinear Algebra 1 (1973), 9–31.

[10] I. Vidav, *The group of isometries and the structure of a finite dimensional normed space*, Linear Algebra and Appl. 14 (1976), 227–236.

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