STUDIA MATHEMATICA, T. LXXII. (1982)

Isomorphism of regular Morse dynamical systems

by

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Abstract. A necessary and sufficient condition for two Morse dynamical systems satisfying some properties to be metrically isomorphic is given. The main result is the following: if $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$, are continuous Morse sequences such that $|b^i| = |\beta^i| = \lambda_i$, $\lambda_i < r$, $i \geqslant 0$ and x, y are regular sequences, then the shift dynamical systems induced by x and y are metrically isomorphic iff $b^i = \beta^i$ for sufficiently large i.

Introduction. First we introduce notions, definitions and notations used in the paper. A sequence $B = (b_1 \ldots b_k)$ of zeros and ones is called a *block*. We put $|B| = k$ and call it the *length* of B. For i, j such that $1 \leqslant i \leqslant j \leqslant k$ we put $B[i, j] = (b_i \ldots b_j)$. We shall write $B[i]$ instead of $B[i, i]$. Let us denote by $\tilde{B}$ the block $(\tilde{b}_1 \ldots \tilde{b}_k)$, where $\tilde{b}_i = 1 - b_i$, $i = 1, \ldots, k$. If $C = (c_1 \ldots c_m)$ is another block, then BC means the block $(b_1 \ldots b_k c_1 \ldots c_m)$. Further, we define

$$B \times C = B^{c_1} B^{c_2} \ldots B^{c_m}, \tag{1}$$

where $B^0 = B$ and $B^1 = \tilde{B}$. If $|B| \leqslant |C|$, then $\mathrm{fr}(B, C)$ will denote the frequency B in C, i.e.

$$\mathrm{fr}(B, C) = \mathrm{card}\{1 \leqslant j \leqslant |C| - |B| + 1,\ C[j, j + |B| - 1] = B\}.$$

Now we recall the definition of a generalized continuous Morse sequence introduced by Keane in [4]. Let $b^0, b^1, \ldots$ be finite blocks of 0's and 1's with the length at least two starting with 0 and let

$$x = b^0 \times b^1 \times \ldots \tag{2}$$

Next, let

$$r_i = \min\left\{\frac{1}{\lambda_i} \cdot \mathrm{fr}(0, b^i), \frac{1}{\lambda_i} \mathrm{fr}(1, b^i)\right\}, \qquad \lambda_i = |b^i|, i \geqslant 0.$$

The sequence x is called a *generalized Morse sequence* (or shortly a *Morse sequence*) if $\sum_{i=0}^{\infty} r_i = \infty$ and if an infinity of the b^i are different from $00\ldots0$, and an infinity are different from $0101 \ldots 010$.

Keane [4] has shown that if x is a continuous Morse sequence, then each block B has the average relative frequency

$$\mu_x(B) = \lim_{n\to\infty} \frac{1}{n} \operatorname{fr}\big((B, x)[1, n]\big) \quad \text{in} \quad x,$$

$$\mu_x(0) = \mu_x(1) = 1/2 \quad \text{and} \quad \mu_x(B) = \mu_x(\tilde{B}).$$

Obviously, μ_x is an invariant measure with respect to the shift T on the space $X = \prod_{-\infty}^{\infty}\{0, 1\}$. We shall say that μ_x is a *Morse measure defined by* x and $(X, \mu_x, T) = \theta(x)$ is a *Morse dynamical system induced by* x. It turns out in [4] that $\theta(x)$ possesses partly continuous and partly discrete spectrum, and the group of eigenvalues of $\theta(x)$ conincides with the group Λ of all n_i-roots of unity, where $n_i = \lambda_0 \cdot \lambda_1 \ldots \lambda_i$, $i \geqslant 0$. The maximal spectral type of $\theta(x)$ has been calculated in [5].

Kakutani in [3] considered the problem of the spectral isomorphism in the class of Morse systems $\theta(x)$, where x is a sequence constructed from blocks $b^i = 00$ or $b^i = 01$, $i \geqslant 0$. He has shown that such systems $\theta(x)$ and $\theta(y)$, $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$, are spectrally isomorphic iff $\beta^i = b^i$ for sufficiently large i.

Now, let $x = b^0 \times b^1 \times \ldots$ be a Morse sequence and let $x_t = b^t \times b^{t+1} \times \ldots$, $t \geqslant 0$. We denote by μ_t the Morse measure defined by x_t, $t \geqslant 0$. The sequence x is said to be a *regular Morse sequence* if there exists a number $\varrho > 0$ such that

$$\varrho \leqslant p_t, \qquad q_t \leqslant 1/2 - \varrho,$$

where

$$p_t = \mu_t(00) = \mu_t(11), \qquad q_t = \mu_t(01) = \mu_t(10), \qquad t \geqslant 0.$$

In this paper we study the problem of the metric isomorphism in the class of regular Morse systems. In order to obtain our main results we prove at first that every Morse system is metrically isomorphic to a skew product automorphism, for which the base automorphism is an automorphism with discrete spectrum with Λ as the group of eigenvalues, and the fiber automorphisms are permutations of $\{0, 1\}$.

Now, let us suppose $\theta(x)$ and $\theta(y)$ are regular Morse systems, where $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$, $|b^i| = |\beta^i| = \lambda_i$, $i \geqslant 0$. Let μ_t and $\bar{\mu}_t$ denote the Morse measures defined by x_t and y_t, respectively and let $\bar{d}(\mu_t, \bar{\mu}_t)$ mean the Ornstein distance between μ_t and $\bar{\mu}_t$, $t \geqslant 0$. If $\theta(x)$ and $\theta(y)$ are metrically isomorphic, then $\lim_{t\to\infty} \bar{d}(\mu_t, \bar{\mu}_t) \to 0$. If $\theta(x)$ and $\theta(y)$ satisfy in addition the property $\sup \lambda_i < \infty$, then they are metrically isomorphic iff $b^i = \beta^i$ for sufficiently large i. The above theorem easily implies that in the class of Morse systems considered by Kakutani the metric and the spectral isomorphisms coincide. In the forthcoming paper [6] we show that this assertion is false in the class of all regular Morse systems.

§ 1. Invariant measures induced by Morse sequences. In this section we establish exactly formulas on $\mu_x(B)$, which will be needed in the sequel. First, we calculate $\mu_x(B)$, where $|B| = 2$.

Let us denote $c_t = b^0 \times b^1 \times \ldots \times b^t$, $n_t = \lambda_0 \cdot \lambda_1 \ldots \lambda_t$, $t \geqslant 0$. It is clear that $n_t = |c_t|$. If $y = \{y_i\}_{i\in Z} \in X$, then $y[i, k]$ will denote the block $(y_i \ldots y_k)$, $i \leqslant k$. Further, let

$$\begin{aligned}
(\mathrm{fr})_t(00, 11) &= \operatorname{fr}(00, c_t) + \operatorname{fr}(11, c_t),\\
(\mathrm{fr})_t(01, 10) &= \operatorname{fr}(01, c_t) + \operatorname{fr}(10, c_t),\\
h_t(00, 11) &= \operatorname{fr}(00, b^t) + \operatorname{fr}(11, b^t),\\
h_t(01, 10) &= \operatorname{fr}(01, b^t) + \operatorname{fr}(10, b^t).
\end{aligned}$$

Using the obvious equalities $\operatorname{fr}(00, c_t) = \operatorname{fr}(11, \tilde{c}_t)$, $\operatorname{fr}(11, c_t) = \operatorname{fr}(00, \tilde{c}_t)$, $\operatorname{fr}(01, c_t) = \operatorname{fr}(10, \tilde{c}_t)$, $\operatorname{fr}(10, c_t) = \operatorname{fr}(01, \tilde{c}_t)$ it is easy to obtain

$$\begin{aligned}
(\mathrm{fr})_{t+1}(00, 11) &= (\mathrm{fr})_t(00, 11)\cdot\lambda_{t+1} + \begin{cases} h_{t+1}(00, 11), & t \in \mathrm{I},\\ h_{t+1}(01, 10), & t \in \mathrm{II},\end{cases}\\
(\mathrm{fr})_{t+1}(01, 10) &= (\mathrm{fr})_t(01, 10)\cdot\lambda_{t+1} + \begin{cases} h_{t+1}(01, 10), & t \in \mathrm{I},\\ h_{t+1}(00, 11), & t \in \mathrm{II},\end{cases}
\end{aligned} \tag{3}$$

where $\mathrm{I} = \{t \geqslant 0,\ c_t[n_t] = 0\}$, $\mathrm{II} = \{t \geqslant 0,\ c_t[n_t] = 1\}$. If we put

$$\hat{\mu}_t(00, 11) = \frac{1}{n_t}\cdot(\mathrm{fr})_t(00, 11), \qquad \hat{\mu}_t(01,10) = \frac{1}{n_t}\cdot(\mathrm{fr})_t(01, 10),$$

then (3) implies

$$\begin{aligned}
\hat{\mu}_{t+1}(00, 11) &= \hat{\mu}_t(00, 11) + \begin{cases} h_{t+1}(00, 11)/n_{t+1}, & t \in \mathrm{I},\\ h_{t+1}(01, 10)/n_{t+1}, & t \in \mathrm{II},\end{cases}\\
\hat{\mu}_{t+1}(01, 10) &= \hat{\mu}_t(01, 10) + \begin{cases} h_{t+1}(01, 10)/n_{t+1}, & t \in \mathrm{I},\\ h_{t+1}(00, 11)/n_{t+1}, & t \in \mathrm{II}.\end{cases}
\end{aligned} \tag{4}$$

Since $\mu_x(00) = \mu_x(11) = \frac{1}{2}\cdot\lim_{t\to\infty}\hat{\mu}_t(00, 11)$, $\mu_x(01) = \mu_x(10) = \frac{1}{2}\cdot\lim_{t\to\infty}\hat{\mu}_t(01, 10)$, from (4) follows

$$\begin{aligned}
\mu_x(00) &= \mu_x(11)\\
&= \tfrac{1}{2}\cdot\Big[h_0(00, 11)/\lambda_0 + \sum_{\substack{i=1\\ i-1\in\mathrm{I}}}^{\infty} h_i(00, 11)/n_i + \sum_{\substack{i=1\\ i-1\in\mathrm{II}}}^{\infty} h_i(01, 10)/n_i\Big],\\
\mu_x(01) &= \mu_x(10)\\
&= \tfrac{1}{2}\cdot\Big[h_0(01, 10)/\lambda_0 + \sum_{\substack{i=1\\ i-1\in\mathrm{I}}}^{\infty} h_i(01, 10)/n_i + \sum_{\substack{i=1\\ i-1\in\mathrm{II}}}^{\infty} h_i(00, 11)/n_i\Big].
\end{aligned} \tag{5}$$

Observe that the series in (5) are convergent, because

$$h_i(00,11)/n_i \leqslant (\lambda_i-1)/(\lambda_i\cdot n_{i-1}) < 1/2^{i-1}$$

and similarly

$$h_i(01,10)/n_i < 1/2^{i-1},$$

for $i\geqslant 1$. Thus, we have established formulas on $\mu_x(B)$ for $|B| = 2$.

In order to establish formulas on $\mu_x(B)$, where B is an arbitrary block, we use graphs $\overline{Y}_t$, $t\geqslant 0$. First, for each $n\geqslant 1$ we define a graph Y_n as follows: the vertices of Y_n are all blocks A, $|A| = n$, and two blocks $A_1 = (i_1i_2\ldots i_n)$, $A_2 = (j_1j_2\ldots j_n)$ are joined by the oriented arrow (A_1, A_2) iff $(i_2\ldots i_n) = (j_1\ldots j_{n-1})$. We remark that the arrows of Y_n may be identified with blocks of the length $n+1$, namely the arrow (A_1, A_2) determines the block $(i_1i_2\ldots i_nj_n)$.

Now, we describe closed paths of Y_n which will be used below. Let A be a vertice of Y_n. Then, the successive arrows (blocks of the length $n+1$) appearing in AA form a closed path $\gamma(AA)$ in Y_n. The number of all pairwise different arrows of γ will be called the *length of* γ and denoted by $|\gamma|$. It is easy to verify that $|\gamma(AA)|$ is the smallest period of the one-sided sequence AAA ... (or shortly the smallest period od A). In the same way we can define the path $\gamma(\tilde{A}\tilde{A})$. It is clear that $|\gamma(AA)| = |\gamma(\tilde{A}\tilde{A})|$. Further, denote by $\gamma(A\tilde{A}A)$ a path formed by the successive arrows appearing in the block $A\tilde{A}A$. Now, we can prove

Lemma 1. *If A is a block of the length n, then*

(a) $k = |\gamma(AA)| = |\gamma(\tilde{A}\tilde{A})|$ *divides* n,

(b) *either* $\gamma(AA) = \gamma(\tilde{A}\tilde{A})$ *or* $\gamma(AA)\cap\gamma(\tilde{A}\tilde{A}) = \emptyset$,

(c) $\gamma(AA) = \gamma(\tilde{A}\tilde{A})$ *iff A has a form* $A = B\tilde{B}\ldots B\tilde{B}$ *with* $|B| = \frac{1}{2}\cdot k$,

(d) $|\gamma(A\tilde{A}A)|$ *is an even number, say* $2\cdot m$ *and* $m|n$,

(e) m *is the smallest number such that there exists a block B with* $|B| = m$ *and* $A = B\tilde{B}B\tilde{B}\ldots B\tilde{B}B$,

(f) $\gamma(A\tilde{A}A)\cap[\gamma(AA)\cup\gamma(\tilde{A}\tilde{A})] = \emptyset$, *i.e. there exists no arrow appearing in* $\gamma(A\tilde{A}A)$ *and in* $\gamma(AA)\cup\gamma(\tilde{A}\tilde{A})$,

(g) $\gamma(AA)$, $\gamma(\tilde{A}\tilde{A})$ *and* $\gamma(A\tilde{A}A)$ *are paths not having any loops.*

Proof. (a): Observe that n is a period of the sequence $AA\ldots$, and k is the smallest period of this sequence. Thus $k|n$.

(b): First remark that if $B = (b_1b_2\ldots b_n)$ is a vertice of $\gamma(AA)$, then $\gamma(AA)$ is exactly determined by B. In fact, B is preceded by two arrows $0B$ and $1B$, but $\gamma(AA)$ contains only one of them, namely b_nB. Similarly, $\gamma(AA)$ contains only Bb_1 from among two arrows $B0$ and $B1$ following after B. It is obvious that the path $\gamma(\tilde{A}\tilde{A})$ has the same property. Thus, if a block $C = (c_1c_2\ldots c_nc_{n+1})$ is a common arrow of $\gamma(AA)$ and $\gamma(\tilde{A}\tilde{A})$, then $B = (c_1c_2\ldots c_n)$ is a common vertice of those paths. It follows from the above remarks that either $\gamma(AA) = \gamma(\tilde{A}\tilde{A})$ or $\gamma(AA)\cap\gamma(\tilde{A}\tilde{A}) = \emptyset$.

(c): It is easy to see that $\gamma(AA) = \gamma(\tilde{A}\tilde{A})$ iff there exists a number l, with $1 < l\leqslant|\gamma(AA)|$, such that $\tilde{A} = AA[l, l+n-1]$. In this way $2\cdot l-2$ is a period of A. Therefore, $|\gamma(AA)| = k|2\cdot(l-1)$ and from the inequality $2\cdot(l-1) < 2\cdot k$ follows $2\cdot(l-1)k$. Hence k is an even number and $l = \frac{1}{2}\cdot k+1$. Thus, if $A = CC\ldots C$, where $|C| = k$, then $C = B\tilde{B}$ with $|B| = \frac{1}{2}\cdot k$.

(d): It is not difficult to verify that the length of $\gamma(A\tilde{A}A)$ is the smallest period of the one-sided sequence $A\tilde{A}A\tilde{A}\ldots$ Let us observe that if $\gamma(A\tilde{A}A)$ contains an arrow C, then γ contains $\tilde{C}$. Hence $|\gamma(A\tilde{A}A)|$ is an even number. Putting $2\cdot m = |\gamma(A\tilde{A}A)|$ we have $2\cdot m|2\cdot n$ because $2\cdot n$ is a period of $A\tilde{A}A\tilde{A}\ldots$ So we obtain $m|n$.

(e): Let us suppose A has the form

$$A = B\tilde{B}B\tilde{B}\ldots B\tilde{B}B, \quad |B| = m'. \tag{6}$$

Then $2\cdot m'$ is a period of $A\tilde{A}A\tilde{A}\ldots$ Hence $m'\geqslant m$ because $2\cdot m|2\cdot m'$. Thus it remains to show that A has the form (6) with $|B| = m$. If $m = n$, then putting $B = A$ we obtain (6). Suppose $m < n$. Since $m|n$, then $A = B\tilde{B}B\ldots\tilde{B}B$ or $A = B\tilde{B}\ldots B\tilde{B}$, with $|B| = m$. If $A = B\tilde{B}\ldots B\tilde{B}$, then we should have $A\tilde{A} = B\tilde{B}\ldots B\tilde{B}\tilde{B}B\ldots\tilde{B}B$, what is a contradiction to the fact that $2\cdot m$ is a period of $A\tilde{A}A\tilde{A}\ldots$ Thus A has the form (6).

(f): If $C = (c_1c_2\ldots c_nc_{n+1})$ is an arrow appearing in $\gamma(AAA)$, then either $(c_1c_{n+1}) = 01$ or 10. If C appears in $\gamma(AA)\cup\gamma(\tilde{A}\tilde{A})$, then $(c_1c_{n+1}) = 00$ or 11. Thus $\gamma(A\tilde{A}A)\cap[\gamma(AA)\cup\gamma(\tilde{A}\tilde{A})] = \emptyset$.

(g): If $B = (b_1b_2\ldots b_n)$ is a vertice of $\gamma(A\tilde{A}A)$, then γ contains only Bb_1 from among two arrows $B0$ and $B1$ following after B, and $\gamma(A\tilde{A}A)$ contains only b_nB from among two arrows $0B$, $1B$ preceding B. Therefore, $\gamma(A\tilde{A}A)$ has not any loops. Since the paths $\gamma(AA)$ and $\gamma(\tilde{A}\tilde{A})$ have the above property, $\gamma(AA)$ and $\gamma(\tilde{A}\tilde{A})$ have not any loops.

Definition 1. A number m will be called a *mirror period of the block* A iff $2\cdot m$ is a period of one-sided sequence $A\tilde{A}A\tilde{A}\ldots$ A numer m will be called the *smallest mirror period of* A iff $2\cdot m$ is the smallest period of $A\tilde{A}A\tilde{A}\ldots$

Now, we can give formulas on $\mu_x(B)$, if $|B| = n_t+1$, $t\geqslant 0$. For the remaining blocks B, $\mu_x(B)$ may be calculated by using the condition of the consistency: if $|B| = n$, $n_{t-1} < n\leqslant n_t+1$, then

$$\mu_x(B) = \sum_{C,|BC|=n_t+1}\mu_x(BC).$$

Let us denote $x_t = b^t\times b^{t+1}\times\ldots$, $t\geqslant 0$. Observe that $x = c_t\times x_{t+1}$ what means that x is constructed from the blocks c_t and $\tilde{c}_t$. Therefore, if B is a block having the length n_t+1 and B appears in x, then B appears in one of the four blocks c_tc_t, $c_t\tilde{c}_t$, $\tilde{c}_tc_t$, $\tilde{c}_t\tilde{c}_t$. Thus the measure μ_x is concentrated on those blocks B, $|B| = n_t+1$, which appear in the blocks c_tc_t, $c_t\tilde{c}_t$, $\tilde{c}_tc_t$, $\tilde{c}_t\tilde{c}_t$. We use the following notations:

$\gamma_t(00)$ — the path $\gamma(c_tc_t)$,

$\gamma_t(11)$ — the path $\gamma(\tilde{c}_t\tilde{c}_t)$,

$\gamma_t(01)$ — the set of those arrows $C \in \gamma(c_t\tilde{c}_tc_t)$ which have the form

$$C = c_t\tilde{c}_t[l+1, l+n_t], \qquad l = 1, 2, \ldots, \tfrac{1}{2}\cdot|\gamma(c_t\tilde{c}_tc_t)|,$$

$\gamma_t(10)$ — the second half of $\gamma(c_t\tilde{c}_tc_t)$, i.e. $\gamma_t(10) = \gamma(c_t\tilde{c}_tc_t)\setminus\gamma_t(01)$,

$k_t = |\gamma_t(00)| = |\gamma_t(11)|$, $m_t = |\gamma_t(01)| = |\gamma_t(10)|$,

$\overline{Y}_t = \gamma_t(00)\cup\gamma_t(11)\cup\gamma_t(01)\cup\gamma_t(10)$.

Thus $\overline{Y}_t$ is a subgraph of Y_{n_t}. By Lemma 1 (b) we have two possibilities: $\gamma_t(00)\cap\gamma_t(11) = \emptyset$ or $\gamma_t(00) = \gamma_t(11)$. We may imagine $\overline{Y}_t$ as follows:

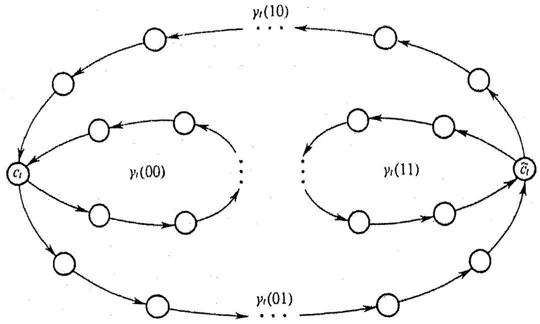

Figure 1

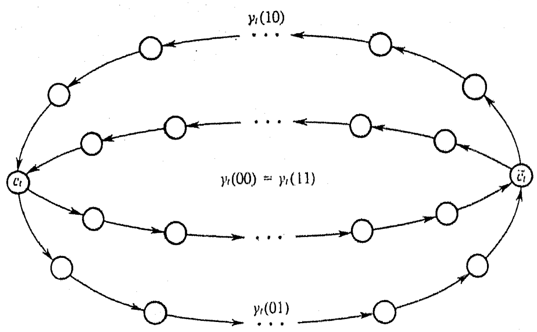

Figure 2

Remark 1. It is possible that $\gamma_t(00)\cup\gamma_t(11)$ may have common vertices with $\gamma_t(01)\cup\gamma_t(10)$ (see Figure 3 with $b^0 = 01$, $b^1 = 001$).

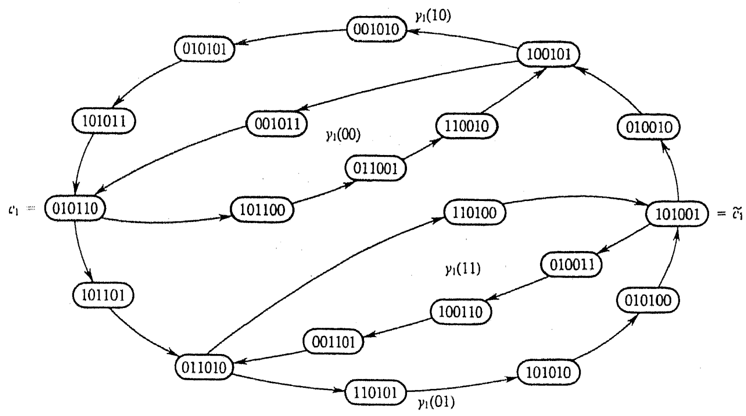

Figure 3

Let μ_t be a Morse measure on X defined by the sequence x_t, $t \geqslant 0$. Put

$$p_t = \mu_t(00) = \mu_t(11), \qquad q_t = \mu_t(01) = \mu_t(10).$$

It is clear that $p_t+q_t = \frac{1}{2}$.

THEOREM 1. *If C is an arrow of $\overline{Y}_t$, then*

$$\mu_x(C) = \begin{cases} q_{t+1}/m_t & \text{if} \quad C \in \gamma_t(01)\cup\gamma_t(10), \\ p_{t+1}/k_t & \text{if} \quad C \in \gamma_t(00)\cup\gamma_t(11) \text{ and } \gamma_t(00\cap\gamma_t(11) = \emptyset, \\ 2\cdot p_{t+1}/k_t & \text{if} \quad C \in \gamma_t(00)\cup\gamma_t(11) \text{ and } \gamma_t(00) = \gamma_t(11). \end{cases}$$

Proof. Suppose $\gamma_t(00)\cap\gamma_t(11) = \emptyset$ and let $C \in \gamma_t(00)$. By Lemma 1((b), (f)) it follows that C is a subblock of c_tc_t and C does not appear in $c_t\tilde{c}_t$, $\tilde{c}_tc_t$, $\tilde{c}_t\tilde{c}_t$. Further, it is clear that the frequency of the occurrence C in c_tc_t is n_t/k_t.

Take a positive integer n and consider the block $x[1, n\cdot n_t]$. Then we have

$$\mathrm{fr}(C, x[1, n\cdot n_t]) = \frac{n_t}{k_t}\cdot\mathrm{fr}(00, x_{t+1}[1, n]),$$

what gives

$$\frac{1}{n\cdot n_t}\cdot\mathrm{fr}(C, x[1, n\cdot n_t]) = \frac{1}{n}\cdot\mathrm{fr}(00, x_{t+1}[1, n])\cdot\frac{1}{k_t}.$$

Since

$$\lim_{n\to\infty} \frac{\mathrm{fr}(00, x_{t+1}[1, n])}{n} = p_{t+1}$$

and

$$\lim_{n\to\infty} \frac{\mathrm{fr}(C, x[1, n\cdot n_t])}{n\cdot n_t} = \mu_x(C),$$

we obtain $\mu_x(C) = p_{t+1}/k_t$. Using the same arguments we have

$$\mu_x(C) = p_{t+1}/k_t \quad \text{if} \quad C \in \gamma_t(11).$$

Now suppose $\gamma_t(00) = \gamma_t(11)$ and let $C \in \gamma_t(00)$. Using Lemma 1((b), (c), (f)) we conclude that C is a subblock of $c_t c_t$ and $\tilde{c}_t\tilde{c}_t$ but C does not appear in $c_t\tilde{c}_t$ and $\tilde{c}_t c_t$. Then we have

$$\mathrm{fr}(C, x[1, n\cdot n_t]) = \mathrm{fr}(00, x_{t+1}[1, n])\cdot\frac{n_t}{k_t} + \mathrm{fr}(11, x_{t+1}[1, n])\cdot\frac{n_t}{k_t}.$$

Further,

$$\frac{1}{n\cdot n_t}\cdot\mathrm{fr}(C, x[1, n\cdot n_t]) = \frac{1}{n}\cdot\mathrm{fr}(00, x_{t+1}[1, n])\cdot\frac{1}{k_t} + \frac{1}{n}\cdot\mathrm{fr}(11, x_{t+1}[1, n])\cdot\frac{1}{k_t}.$$

Taking the limit as n tends to infinity we obtain

$$\mu_x(C) = \frac{2\cdot p_{t+1}}{k_t}.$$

Let $C \in \gamma_t(01)$. In this case C may to appear only in $c_t\tilde{c}_t$ and $\tilde{c}_t c_t$. Assuming that $c_t = B\tilde{B}B\ldots B\tilde{B}B$, $|B| = m_t$ and using Lemma 1(e) we obtain $C = c_t\tilde{c}_t[l, l+n_t]$ with $1 \leqslant l \leqslant m_t$. Hence it is clear that

$$\mathrm{fr}(C, c_t\tilde{c}_t) = \tfrac{1}{2}\cdot(n_t/m_t+1), \qquad \mathrm{fr}(C, \tilde{c}_t c_t) = \tfrac{1}{2}\cdot(n_t/m_t-1),$$

and so

$$\mathrm{fr}(C, x[1, n\cdot n_t]) = \tfrac{1}{2}\cdot(n_t/m_t+1)\cdot\mathrm{fr}(01, x_{t+1}[1, n]) + \tfrac{1}{2}(n_t/m_t-1)\cdot\mathrm{fr}(10, x_{t+1}[1, n]).$$

Thus

$$\mu_x(C) = \big(1/(2\cdot m_t)\big)\cdot q_{t+1} + q_{t+1}/(2\cdot n_t) + q_{t+1}/(2\cdot m_t) - q_{t+1}/(2\cdot n_t) = q_{t+1}/m_t.$$

Finally, if $C \in \gamma_t(10)$, then

$$\mathrm{fr}(C, x[1, n\cdot n_t]) = \tfrac{1}{2}(n_t/m_t-1)\cdot\mathrm{fr}(01, x_{t+1}[1, n]) + \tfrac{1}{2}\cdot(n_t/m_t+1)\cdot\mathrm{fr}(10, x_{t+1}[1, n]).$$

In this case we obtain also

$$\mu_x(C) = q_{t+1}/m_t,$$

what completes the proof.

We shall assume in the sequel of this paper that x is a continuous Morse sequence.

§ 2. Period and mirror period of c_t. t-sectors. Now, we establish a connection between the numbers k_t, m_t, k_{t+1}, m_{t+1}. For given two blocks $B = (b_1\ldots b_n)$, $C = (c_1\ldots c_n)$, $n \geqslant 1$, we define

$$d(B, C) = (1/n)\,\mathrm{card}\{1 \leqslant j \leqslant n,\ b_j \neq c_j\}.$$

THEOREM 2. *Let m_t, m_{t+1}, l_{t+1} be the smallest mirror periods of c_t, c_{t+1}, b^{t+1}, respectively. Then we have*

$$m_{t+1} = \begin{cases} m_t & \text{if } b^{t+1} = 0101\ldots010, \\ l_{t+1}\cdot n_t & \text{otherwise}. \end{cases}$$

Proof. Let us observe that the number $l_{t+1}\cdot n_t$ is a mirror period of c_{t+1}. Then by Definition 1 we have

(7) $$m_{t+1} \leqslant l_{t+1}\cdot n_t.$$

In order to establish the converse inequality we shall show that $n_t | m_{t+1}$, whenever $b^{t+1} \neq 0101\ldots010$.

Let $m_{t+1} = s\cdot n_t + r$ with $0 \leqslant r < n_t$. Using Lemma 1(d) we obtain $m_{t+1} | n_{t+1}$ what implies $s \leqslant \lambda_{t+1}$. Put $b^{t+1} b^{t+1}[s+1, s+2] = ij$ and next suppose that $r > 0$. Then $s < \lambda_{t+1}$. From Lemma 1(e) follows

(8) $$c_{t+1}\tilde{c}_{t+1}[m_{t+1}+1, m_{t+1}+n_{t+1}] = \tilde{c}_{t+1}.$$

The last equation implies

(9)
$$b^{t+1}\tilde{b}^{t+1}[s+u, s+u+1] = ij \quad \text{whenever} \quad b^{t+1}[u] = 0, \quad u = 1, \ldots, \lambda_{t+1},$$
$$b^{t+1}\tilde{b}^{t+1}[s+u, s+u+1] = \tilde{i}\tilde{j} \quad \text{whenever} \quad b^{t+1}[u] = 1, \quad u = 1, \ldots, \lambda_{t+1}.$$

Let us remark that (9) implies $j = \tilde{i}$. Indeed, if we suppose $j = i$, then from (9) follows

$$b^{t+1}\tilde{b}^{t+1}[s+1, \lambda_{t+1}+s+1] = \underbrace{ii\ldots i}_{\lambda_{t+1}+1}$$

and $b^{t+1} = 00\ldots0$ what is contradiction. It is easy to see that for $j = \tilde{i}$ we have $b^{t+1} = 0101\ldots010$.

Thus we obtain $m_{t+1} = s\cdot n_t$. The last equality and (8) imply that s is a mirror period of b^{t+1} and so $s \geqslant l_{t+1}$. Therefore $m_{t+1} \geqslant l_{t+1}\cdot n_t$. In view of (7) we have $m_{t+1} = l_{t+1}\cdot n_t$ whenever $b^{t+1} \neq 0101\ldots010$. One can easy to check that in the case $b^{t+1} = 01010\ldots010$ the equality $m_{t+1} = m_t$ holds. This remark completes the proof.

THEOREM 3. *If k_t, k_{t+1}, s_{t+1} are the smallest periods of the blocks c_t, c_{t+1}, b^{t+1}, respectively, then*

$$k_{t+1} = \begin{cases} k_t & \text{if} \quad b^{t+1} = 00\ldots0, \\ 2\cdot m_t & \text{if} \quad b^{t+1} = 0101\ldots01, \\ s_{t+1}\cdot n_t & \text{otherwise}. \end{cases}$$

Proof. The proof of this theorem is similar to the proof of Theorem 2. First we have

$$k_{t+1} \leqslant s_{t+1}\cdot n_t. \tag{10}$$

Further we may establish in the same way as in the proof of the Theorem 2 that $n_t | k_{t+1}$ whenever $b^{t+1} \neq 00\ldots0$ and $b^{t+1} \neq 0101\ldots01$. If $k_{t+1} = s\cdot n_t$, then s is a period of b^{t+1} what gives $s \geqslant s_{t+1}$ and by (10) we obtain $k_{t+1} = s_{t+1}\cdot n_t$.

If $b^{t+1} = 00\ldots0$, then obviously $k_{t+1} = k_t$. Now, suppose that $b^{t+1} = 0101\ldots01$. In this case $2\cdot n_t$ is a period of c_{t+1} so that $k_{t+1} | 2\cdot n_t$. If $k_{t+1} < 2\cdot n_t$, then $k_{t+1} \leqslant n_t$. By the condition $c_{t+1}c_{t+1}[k_{t+1}+1, k_{t+1}+n_{t+1}] = c_{t+1}$ it follows $c_t\tilde{c}_t[k_{t+1}+1, n_t+k_{t+1}] = c_t$, i.e. $\frac{1}{2}\cdot k_{t+1}$ is a mirror period of c_t (see Def. 1 and Lemma 1(d)). On the other hand, if m is a mirror period of c_t, then $2\cdot m$ is a period of $c_{t+1} = c_t \times (0101\ldots01)$. Therefore $m_t = \frac{1}{2}\cdot k_{t+1}$, i.e. $k_{t+1} = 2\cdot m_t$. Thus Theorem 3 is proved.

Now, we can describe mappings from $\overline{Y}_{t+s}$ to $\overline{Y}_t$ for any $t = 0, 1, \ldots$ and $s = 1, 2, \ldots$ First we take $s = 1$. Let $C = (c_1 \ldots c_{n_{t+1}+1})$ be an arrow of $\overline{Y}_{t+1}$. We put

$$f_t(C) = (c_1 \ldots c_{n_t+1}). \tag{11}$$

Remark that if $C \in \overline{Y}_{t+1}$, then there exists k, with $1 \leqslant k \leqslant n_{t+1}$, such that $C = c_{t+1}^i c_{t+1}^j [k, k+n_{t+1}]$, where $i, j = 0, 1$. If $k = (q-1)\cdot n_t + r$, $1 \leqslant q \leqslant \lambda_{t+1}$, $1 \leqslant r \leqslant n_t$, then $f_t(C) = c_t^{i_q} c_t^{i_{q+1}}[r, r+n_t]$, where $(i_q i_{q+1})$ is a pair appearing in the block $\big((b^{t+1})^i\big)j$ on the q th place. Thus $f_t(C) \subset \overline{Y}_t$ and so (11) defines a mapping from $\overline{Y}_{t+1}$ to $\overline{Y}_t$. Further, the image of f_t consists of those arrows of $\overline{Y}_t$ which appear in $c_t^i c_t^j$, where (ij) runs over all pairs of $b^{t+1}0$, $b^{t+1}1$, $\tilde{b}^{t+1}0$, $\tilde{b}^{t+1}1$. Since the blocks 00, 01, 10, 11 appear in $b^{t+1}0$, $b^{t+1}1$, $\tilde{b}^{t+1}0$, $\tilde{b}^{t+1}1$, we have $f_t(\overline{Y}_{t+1}) = \overline{Y}_t$. Therefore (11) defines a mapping onto $\overline{Y}_t$.

As a consequence of above considerations we obtain a sequence of graphs $\overline{Y}_t$ and mappings f_t, $t \geqslant 0$,

$$\overline{Y}_0 \overset{f_0}{\leftarrow} \overline{Y}_1 \overset{f_1}{\leftarrow} \overline{Y}_2 \overset{f_2}{\leftarrow} \ldots$$

Put

$$f_{t,s} = f_t \cdot f_{t+1} \ldots f_{t+s-1}, \quad t \geqslant 0, \; s = 1, 2, \ldots$$

Thus $f_{t,s}$ is a mapping from $\overline{Y}_{t+s}$ onto $\overline{Y}_t$.

Now, we mark out subsets of $\overline{Y}_{t+s}$ which we shall call t-sectors. Each t-sector will be a subset of one of the four paths $\gamma_{t+s}(00)$, $\gamma_{t+s}(01)$, $\gamma_{t+s}(10)$, $\gamma_{t+s}(11)$. Take a path $\gamma_{t+s}(ij) \subset \overline{Y}_{t+s}$ and suppose that $|\gamma_{t+s}| = n_t\cdot l_s(ij)$, with $l_s = l_s(ij) > 1$. Let $C_1, C_2, \ldots, C_{l_s\cdot n_t}$ be the successive arrows of $\gamma_{t+s}(ij)$.

DEFINITION 2. Each group of arrows of $\gamma_{t+s}(ij)$ having the form $C_{q\cdot n_t+1}$, $C_{q\cdot n_t+2}, \ldots, C_{(q+1)\cdot n_t}$, $0 \leqslant q < l_s(ij)$, will be called a *t-sector*.

THEOREM 4. *For any $t \geqslant 0$ there exists $s \geqslant 1$ such that*

(a) *each of the paths $\gamma_{t+s}(00)$, $\gamma_{t+s}(01)$, $\gamma_{t+s}(10)$, $\gamma_{t+s}(11)$ may be divided on t-sectors,*

(b) *if $S = (C_1, C_2, \ldots, C_{n_t})$ is a t-sector of $\overline{Y}_{t+s}$, then $\tilde{S} = (\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_{n_t})$ is a t-sector of this graph,*

(c) *there exists a pair (u, v), $u, v = 0, 1$, such that*

$$f_{t,s}(C_j) = c_t^u c_t^v [j, j+n_t], \quad j = 1, 2, \ldots, n_t.$$

Proof. (a): Take $t \geqslant 0$ and put $\delta_{t,s} = b^{t+1} \times \ldots \times b^{t+s}$, $s \geqslant 1$. The definition of operation (1) immediately implies the following facts:

(α) $\delta_{t,s} = \underbrace{010\ldots10}_{\lambda_{t+1}\cdots\lambda_{t+s}}$ iff $b^{t+1} = \underbrace{010\ldots10}_{\lambda_{t+1}}, \ldots, b^{t+s} = \underbrace{010\ldots10}_{\lambda_{t+s}}$,

(β) $\delta_{t,s} = \underbrace{000\ldots00}_{\lambda_{t+1}\cdots\lambda_{t+s}}$ iff $b^{t+1} = \underbrace{00\ldots0}_{\lambda_{t+1}}, \ldots, b^{t+s} = \underbrace{00\ldots0}_{\lambda_{t+s}}$,

(γ) $\delta_{t,s} = \underbrace{0101\ldots01}_{\lambda_{t+1}\cdots\lambda_{t+s}}$ iff

either $b^{t+1} = \underbrace{01\ldots01}_{\lambda_{t+1}}, b^{\,+2} = \underbrace{00\ldots0}_{\lambda_{t+2}}, \ldots, b^{t+s} = \underbrace{00\ldots0}_{\lambda_{t+s}}$,

or $b^{t+1} = \underbrace{01\ldots010}_{\lambda_{t+1}}, \ldots, b^{t+s-1} = \underbrace{01\ldots010}_{\lambda_{t+s-1}}, b^{t+s} = \underbrace{01\ldots01}_{\lambda_{t+s}}$,

$b^{t+s+1} = \underbrace{00\ldots0}_{\lambda_{t+s+1}}, \ldots, b^{t+s} = \underbrace{00\ldots0}_{\lambda_{t+s}}$.

Using (α), (β), (γ) and the fact that x is a continuous Morse sequence, we conclude that there exists $s \geqslant 1$ such that $\delta_{t,s} \neq 010\ldots10$, $\delta_{t,s} \neq 00\ldots0$, $\delta_{t,s} \neq 0101\ldots01$. Further, applying Lemma 1(e) we see that the smallest mirror period of $\delta_{t,s}$ is equal to 1 iff $\delta_{t,s} = 010\ldots10$ and the smallest period of $\delta_{t,s}$ is equal to 1 iff $\delta_{t,s} = 00\ldots0$. Denoting by m'_s the smallest mirror period of $\delta_{t,s}$ and by k'_s the smallest period of $\delta_{t,s}$ we have $m'_s > 1$ and $k'_s > 1$. Next, we can apply Theorems 2 and 3 to the blocks c_t and $c_{t+s} = c_t \times \delta_{t,s}$. As a consequence we obtain $m_{t+s} = n_t \cdot m'_s$ and $k_{t+s} = n_t \cdot k'_s$. The last equalities imply (a).

(c): From Lemma 1(e) it follows that $\beta_{t+s} = A_s \tilde{A}_s A_s \ldots \tilde{A}_s A_s$, where $|A_s| = m'_s$. Then we have

$$c_{r+s}\tilde{c}_{t+s} = (c_t \times A_s)(c_t \times \tilde{A}_s)(c_t \times A_s) \ldots$$
$$\ldots (c_t \times A_s)(c_t \times \tilde{A}_s)(c_t \times A_s)(c_t \times \tilde{A}_s) \ldots (c_t \times \tilde{A}_s)$$

and $|c_t \times A_s| = m_{t+s}$. Thus t-sectors of $\gamma_{t+s}(01)$ may be identified with the successive pairs (u, v) appearing in $A_s 1$. In a similar way all t-sectors of $\gamma_{t+s}(10)$ may be identified with the pairs appearing in $\tilde{A}_s 0$. Next, representing β_{t+s} as $B_s B_s \ldots B_s$, where $|B_s| = k'_s$, we see that all t-sectors of $\gamma_{t+s}(00)$ $(\gamma_{t+s}(11))$ may be identified with the blocks 00, 01, 10, 11 appearing in $B_s 0 (\tilde{B}_s 1)$. The above assertions imply (c).

(b): First assume $\gamma_{t+s}(00) \cap \gamma_{t+s}(11) = \emptyset$. Using some arguments from (c) we may easy check that an arrow C appears in $\gamma_{t+s}(00)$ $(\gamma_{t+s}(01))$ iff $\tilde{C}$ appears in $\gamma_{t+s}(11)$ $(\gamma_{t+s}(10))$.

Now let $\gamma_{t+s}(00) = \gamma_{t+s}(11)$ and let S be a t-sector of $\gamma_{t+s}(00)$. Lemma 1(c) implies $c_{t+s} = B\tilde{B} \ldots B\tilde{B}$ with $|B| = \frac{1}{2}\cdot k_{t+s}$. If $C = c_{t+s}c_{t+s}[k, k+n_{t+s}]$ is an arrow of $\gamma_{t+s}(00)$ with $1 \leqslant k \leqslant \frac{1}{2}\cdot k_{t+s}$, then $\tilde{C} = c_{t+s}c_{t+s}[k+\frac{1}{2}\cdot k_{t+s}, k+\frac{1}{2}\cdot k_{t+s}+n_{t+s}]$ is an arrow of $\gamma_{t+s}(00)$. It remains to show that $n_t \mid \frac{1}{2}\cdot k_{t+s}$. In order to do this we choose a number s in the same manner as in (a) and such that $b^{t+s} \neq 00\ldots0$, $n_t \mid m_{t+s-1}$. It follows from Lemma 1(c) that

$$\tfrac{1}{2}\cdot k_{t+s} = \min\{1 \leqslant l \leqslant \tfrac{1}{2}\cdot n_{t+s},\ c_{t+s}c_{t+s}[l+1, l+n_{t+s}] = \tilde{c}_{t+s}\}.$$

Reasoning as in the proof of Theorems 2 and 3, we can establish

$$(12)\qquad \tfrac{1}{2}\cdot k_{t+s} = \begin{cases} m_{t+s-1} & \text{if}\quad b^{t+s} = 0101\ldots01,\\ n_{t+s-1}\cdot u_{t+s} & \text{if}\quad b^{t+s} \neq 0101\ldots01,\end{cases}$$

where

$$u_{t+s} = \min\{1 \leqslant l \leqslant \tfrac{1}{2}\cdot\lambda_{t+s},\ \tilde{b}^{t+s} = b^{t+s}b^{t+s}[l+1, l+\lambda_{t+s}]\}.$$

Therefore (12) implies that $n_t \mid \frac{1}{2}\cdot k_{t+s}$, what completes the proof.

§3. Standard representation of Morse dynamical systems. In this section we construct special dynamical systems, which will be isomorphic to Morse dynamical systems.

Let $Z_{\lambda_i} = \{0, 1, \ldots, \lambda_i - 1\}$, $i \geqslant 0$, $Z = \sum_0^\infty Z_{\lambda_i}$ and let $\overline{\mathscr{F}}$ be the σ-field of all borelian subsets of Z. Let p_i be the measure on Z_{λ_i} defined by $p_i(z_i) = 1/\lambda_i$, $z_i \in Z_{\lambda_i}$, $i \geqslant 0$. The triple $(Z, \overline{\mathscr{F}}, \overline{p})$ is a measure space, where

$\overline{p}$ is the product measure of $p_0, p_1, \ldots$ We denote by Λ the group of all n_i-roots of the unity. Then Z may be identified with the dual group of Λ, where the addition in Z is defined in a similar way as in the set of p-adic numbers. Let $\bar{Z}$ be the subset of Z consisting of $\bar{z} = (z_0, z_1, \ldots) \in Z$ with the property: an infinity of the z_i are different from 0 and an infinity different from $\lambda_i - 1$. It is clear that $\overline{p}(\bar{Z}) = 1$. We define a transformation $\bar{S}\colon \bar{Z} \to \bar{Z}$, defined by the formula

$$\bar{S}(z_0, z_1, \ldots) = (0, \ldots, 0, z_i + 1, z_{i+1}, \ldots),$$

where i is the smallest number for which $z_i < \lambda_i - 1$. It is easy to check that $(\bar{Z}, \overline{\mathscr{F}}, \overline{p}, \bar{S})$ is a dynamical system with discrete spectrum and the point spectrum of this system is Λ. Observe that $Z \setminus \bar{Z}$ is the trajectory of zero in Z.

Consider two copies of $\bar{Z}$, namely $M_0 = (\bar{Z}, 0)$ and $M_1 = (\bar{Z}, 1)$. Put $M = M_0 \cup M_1$. Denote by $\mathscr{F}$ the natural σ-field of subsets of M defined by $\overline{\mathscr{F}}$ on M_0 and M_1. Let p^* be a measure on $\mathscr{F}$ given by $p^*(F) = \frac{1}{2}\cdot\overline{p}(F)$, $F \subset M_0$ or $F \subset M_1$. Now, we may define a transformation S^* on M. Take $\bar{z} = (z_0, z_1, \ldots) \in \bar{Z}$. Let j be the smallest number such that $z_j < \lambda_j - 1$. Define

$$S^*(\bar{z}, i) = \big(\bar{S}(\bar{z}), \varepsilon(i, \bar{z})\big),$$

where $i = 0, 1$ and

$$\varepsilon(i, \bar{z}) = \varepsilon(i)$$
$$= \begin{cases} i & \text{if}\quad j = 0,\\ i & \text{if}\quad j > 0 \text{ and } p(b^1[\lambda_1]\ldots b^{j-1}[\lambda_{j-1}]b^j[z_j+1]b^j[z_j+2]) = 0,\\ 1-i & \text{if}\quad j < 0 \text{ and } p(b^1[\lambda_1]\ldots b^{j-1}[\lambda_{j-1}]b^j[z_j+1]b^j[z_j+2]) = 1,\end{cases}$$

where $p(\varepsilon_1 \ldots \varepsilon_n) = 0$ if the number of 1's in $\varepsilon_1 \ldots \varepsilon_n$ is even and $p(\varepsilon_1 \ldots \varepsilon_n) = 1$ otherwise.

It is easy to show that S^* is measurable and preserves the measure p^*. The dynamical system $\theta^*(x) = (M, \mathscr{F}, p^*, S^*)$ will be called a *standard dynamical system* for the Morse dynamical system $\theta(x)$. Now, we shall show that the systems $\theta(x)$ and $\theta^*(x)$ are isomorphic. First we define a mapping f from M to X and next we shall show that f is an isomorphism between $\theta^*(x)$ and $\theta(x)$. Take $\bar{z} = (z_0, z_1, \ldots) \in \bar{Z}$ and set

$$i_t = i_t(\bar{z}) = z_0 + z_1 \cdot n_0 + \ldots + z_t \cdot n_{t-1},\quad t > 0,\quad i_0 = z_0.$$

We have $i_t \leqslant n_t - 1$. Put $j_t = n_t - i_t - 1$. It is obvious that $\bar{z} \in \bar{Z}$ implies

$$(13)\qquad i_{t+1} \geqslant i_t,\quad j_{t+1} \geqslant j_t,\quad i_t \to \infty,\quad j_t \to \infty \text{ as } t \to \infty.$$

Now, we define a two-sided sequence $y = f(\bar{z}, 0)$ as follows:

$$(14)\qquad y[-i_0, j_0] = b^0,$$

and in general

$$(15)\qquad y[-i_t, j_t] = \begin{cases} y[-i_{t-1}, j_{t-1}]\times b^t & \text{if} \quad b^t[z_t+1] = 0, \\ y[-i_{t-1}, j_{t-1}]\times \tilde{b}^t & \text{if} \quad b^t[z_t+1] = 1, \end{cases}$$

$t = 1, 2, \ldots$ It is not difficult to verity that (14) and (15) well define a two-sided sequence y. Next, we define $y = f(\bar{z}, 1)$ as

$$(16)\qquad y[-i_0, j_0] = \tilde{b}^0,$$

and $y[-i_t, j_t]$ as in (15), $t = 1, 2, \ldots$

THEOREM 5. *The mapping f defined by* (14), (15) *and* (16) *is an isomorphism between the dynamical systems $\theta^*(x)$ and $\theta(x)$.*

Proof. We shall show the following facts:

(a) f is one-to-one mapping,

(b) f is measurable and $\mu_x = p^*\circ f^{-1}$,

(c) $f\circ S^* = T\circ f$.

We prove the points (a), (b), (c) successively.

(a): From the definition of $y = f(\bar{z}, i)$ it follows that

$$y[-i_t+l\cdot n_t, j_t+l\cdot n_t] = c_t \text{ or } \tilde{c}_t \quad \text{for} \quad l = 0, \pm 1, \pm 2, \ldots,\quad t = 0, 1, \ldots$$

Therefore, for any $t\geqslant 0$ there exists a pair (r_t, s_t), $r_t, s_t = 0, 1$, with

$$(17)\qquad y[0, n_t] = c_t^{r_t}\tilde{c}_t^{s_t}[i_t+1, n_t+i_t+1].$$

This means that $y[0, n_t]$ is an arrow of Y_t. Take $\overline{w}\in\bar{Z}$, $\overline{w} = (w_0, w_1, \ldots)$ with $\overline{w}\neq\bar{z}$. We shall show that $f(\bar{z}, i)\neq f(\overline{w}, j)$ for any $i, j = 0, 1$.

Let $i'_t = i_t(\overline{w})$, $j'_t = n_t - i'_t - 1$, $t\geqslant 0$. The condition $\overline{w}\neq\bar{z}$ implies that there exists a number t_0 with $i_{t_0}\neq i'_{t_0}$. We choose a number s satisfying (a), (b) and (c) of Theorem 4, with $t = t_0$. If $f(\bar{z}, i)[0, n_{t_0+s}]$ and $f(\overline{w}, j)[0, n_{t_0+s}]$ are contained in different t_0-sectors, then obviously $f(\bar{z}, i)[0, n_{t_0+s}]$, $w\neq f(\overline{w}, j)[0, n_{t_0+s}]$, i.e. $f(\bar{z}, i)\neq f(w, j)$. If $f(\bar{z}, i)[0, n_{t_0+s}]$ and $f(\overline{w}, j)[0, n_{t_0+s}]$ are contained in the same t_0-sector S, then $f(\bar{z}, i)[0, n_{t_0+s}]$ is $(i_{t_0}+1)$th arrow of S (see (c) of Theorem 4 and (17)) and $f(\overline{w}, j)[0, n_{t_0+s}]$ is the $(i'_{t_0}+1)$th arrow of S. So $i_{t_0}\neq i'_{t_0}$ implies $f(\bar{z}, i)[0, n_{t_0+s}]\neq f(\overline{w}, j)[0, n_{t_0+s}]$ that is $f(\bar{z}, i)\neq f(\overline{w}, j)$. In order to finish the proof of (a) we remark that $f(\bar{z}, 0) = \overline{f(\bar{z}, 1)}$ for any $\bar{z}\in\bar{Z}$.

(b): Denote by X_1 a subset of X consisting of those two-sided sequences, which are constructed from the blocks B, $|B| = n_t+1$, appearing in $\bar{Y}_t$, $t\geqslant 0$. X_1 is an intersection of countable many cylinders and so X_1 is a closed subset of X. It is clear that $\mu_x(X_1) = 1$. Now, we define a subset X_0 of X_1 such that $\mu_x(X_0) = 0$.

Applying Theorem 4 we may choose a sequence of numbers $t_0 < t_1 < t_2\ldots$, such that the paths of $\bar{Y}_{t_j}$ may be divided on j-sectors, $j = 0, 1, 2, \ldots$ Each j-sector $S = (C_1, C_2, \ldots, C_{n_j})$ may be divided on λ_j of $(j-1)$-sectors in the following way:

$$S_1^{j-1} = (C_1, C_2, \ldots, C_{n_{j-1}}), \ldots, S_{\lambda_t}^{j-1} = (C_{(\lambda_j-1)n_{j-1}+1}, \ldots, C_{n_j}).$$

We write $S = [S_1^{j-1}, S_2^{j-1}, \ldots, S_{\lambda_j}^{j-1}]$. Further, we put

$$\overline{\overline{X}}(j) = \{y\in X_1,\ y[0, n_{t_j}]\in S_{\lambda_j}^{j-1},\ S \text{ is any } j\text{-sector of } \bar{Y}_{t_j}\},$$

$$\underline{\underline{X}}(j) = \{y\in X_1,\ y[0, n_{t_j}]\in S_1^{j-1},\ S \text{ is any } j\text{-sector of } \bar{Y}_{t_j}\},$$

$$\bar{X}(j) = \bigcap_{k=j}^{\infty}\overline{\overline{X}}(k),\qquad \underline{X}(j) = \bigcap_{k=j}^{\infty}\underline{\underline{X}}(k),\qquad X_0 = \bigcup_{j=0}^{\infty}\bar{X}(j)\cup\bigcup_{j=0}^{\infty}\underline{X}(j).$$

We shall show $\mu_x(X_0) = 0$. In fact, let $\overline{\overline{X}}(j, l) = \bigcap_{k=j}^{j+l}\overline{\overline{X}}(k)$, $j = 0, 1, \ldots$, $l = 1, 2, \ldots$ Each $(j+l)$-sector of $\bar{Y}_{t_{j+l}}$ may be divided on $\lambda_{j+1}\ldots\lambda_{j+l}$ j-sectors. It is easy to verify that

$$\overline{\overline{X}}(j, l) = \{y\in X_1, y[0, n_{t_{j+l}}] \text{ belongs to the last } j\text{-sector from among } \lambda_{j+1}\ldots\lambda_{j+l}\ j\text{-sectors of any } (j+l)\text{-sector of } Y_{t_{j+l}}\}.$$

If $k_{t_{j+l}} = n_{j+l}\cdot s_l$, $m_{t_{j+l}} = n_{j+l}\cdot s'_l$ and if $\gamma_{t_{j+l}}(00)\cap\gamma_{t_{j+l}}(11) = \emptyset$ $(\gamma_{t_{j+l}}(00) = \gamma_{t_{j+l}}(11))$, then $2\cdot s_l$ (or s_l) is the number of all $(j+l)$-sectors of $\gamma_{t_{j+l}}(00)\cup\gamma_{t_{j+l}}(11)$. Similarly $2\cdot s'_l$ is the number of all $(j+l)$-sectors of $\gamma_{t_{j+l}}(01)\cup\gamma_{t_{j+l}}(10)$. Using Theorem 1 and the equality $p_{t_{j+l}}+q_{t_{j+l}} = \frac{1}{2}$ we have

$$\mu_x(\overline{\overline{X}}(j, l)) = 2\cdot(p_{t_{j+l}}/k_{t_{j+l}})\cdot s_l\cdot n_j + 2(q_{t_{j+l}}/m_{t_{j+l}})\cdot s'_l\cdot n_j$$
$$= (2^{p_{t_{j+l}}}/(n_{j+l}\cdot s_l))\cdot s_l\cdot n_j + (2^{q_{t_{j+l}}}/(n_{j+l}\cdot s'_l))\cdot s'_l\cdot n_j = 1/(\lambda_{j+1}\ldots\lambda_{j+l}).$$

The last equality implies $\mu_x(\bar{X}(j)) = 0$. In the same way we obtain $\mu_x(\underline{X}(j)) = 0$, $j = 1, 2, \ldots$, what implies $\mu_x(X_0) = 0$.

In order to prove (b) we show the following facts:

(b1) $f(M_0\cup M_1)$ is a measurable set and $\mu_x(f(M_0\cup M_1)) = 1$,

(b2) f is measurable mapping,

(b3) f transports the measure p^* on μ_x.

(b1): We show that $X_1\setminus X_0\subset f(M_0\cup M_1)\subset X_1$. Take $y\in X_1\setminus X_0$. For any $j = 0, 1, \ldots$, $y[0, n_{t_j}]$ is an arrow of $\bar{Y}_{t_j}$. Then there exists exactly one j-sector S^j such that $y[0, n_{t_j}]\in S^j$. Writting $S^j = [S_1^{j-1}, S_2^{j-1}, \ldots, S_{\lambda_j}^{j-1}]$, where $S_1^{j-1}, \ldots, S_{\lambda_j}^{j-1}$ are $(j-1)$-sectors, we choose a number z'_j, $1\leqslant z'_j\leqslant\lambda_j$ such that $y[0, n_{t_j}]\in S_{z'_j}^{j-1}$. Put $z_j = z'_j - 1$. In this way we obtain a sequence of numbers $z_0, z_1, z_2, \ldots$ The condition $y\notin X_0$ implies that an infinity of z_i is different from 0 and an infinity of z_i is different from $\lambda_i - 1$. Hence $\bar{z} = (z_0, z_1, \ldots)\in\bar{Z}$. As previously, we put $i_j = z_0 +$

$+z_1\cdot n_0+\ldots+z_j\cdot n_{j-1}$. By the definition of the numbers $z_0, z_1, \ldots$, it follows that $y[0, n_{t_j}]$ is (i_j+1)th arrow of S^j.

Assume that S^j is contained in a path $\gamma_{t_j}(kl)$. Then the arrow $y[0, n_{t_j}]$ is preceded by at least i_j of arrows of $\gamma_{t_j}(kl)$. If

$$y[0, n_{t_j}] = c_{t_j}^k c_{t_j}^l [q\cdot n_j+i_j+1,\ q\cdot n_j+i_j+1+n_{t_j}],\qquad 0\leqslant q<|\gamma_{t_j}(kl)|/n_j,$$

then i_j arrows immediately preceding the arrow $y[0, n_{t_j}]$ have the form

$$c_{t_j}^k c_{t_j}^l [q\cdot n_j+r,\ q\cdot n_j+r+n_{t_j}],\qquad \text{where}\quad r = 1, 2, \ldots, i_j.$$

Thus $y[-i_j, n_j-i_j-1] = c_{t_j}^k c_{t_j}^l [q\cdot n_j+1, (q+1)\cdot n_j]$. Taking into consideration the equality $c_{t_j} = c_j\times(b^{j+1}\times\ldots\times b^{t_j})$ we obtain $y[-i_j, n_j-i_j-1] = c_j$ or $\tilde{c}_j$ for $j = 0, 1, \ldots$

We put

$$(18)\qquad i = \begin{cases} 0 & \text{if}\quad y[-i_0, n_0-i_0-1] = b^0,\\ 1 & \text{if}\quad y[-i_0, n_0-i_0-1] = \tilde{b}^0.\end{cases}$$

Now, we may show $f(\bar{z}, i) = y$. The block $y[-i_{j+1}, n_{j+1}-i_{j+1}-1]$ may be divided on λ_{j+1} equal parts and the equality $i_{j+1} = i_j+z_{j+1}\cdot n_j$ implies the $(z_{j+1}+1)$th part of $y[-i_{j+1}, n_{j+1}-i_{j+1}-1]$ is $y[-i_j, n_j-i_j-1]$. In order to show $f(\bar{z}, i) = y$ it remains to verify that y satisfies (14), (15) and (16).

Consider the following possibilities:

(α) $\quad y[-i_{j+1}, n_{j+1}-i_{j+1}-1] = c_{j+1},\qquad y[-i_j, n_j-i_j-1] = c_j,$

(β) $\quad y[-i_{j+1}, n_{j+1}-i_{j+1}-1] = c_{j+1},\qquad y[-i_j, n_j-i_j-1] = \tilde{c}_j,$

(γ) $\quad y[-i_{j+1}, n_{j+1}-i_{j+1}-1] = \tilde{c}_{j+1},\qquad y[-i_j, n_j-i_j-1] = c_j,$

(δ) $\quad y[-i_{j+1}, n_{j+1}-i_{j+1}-1] = \tilde{c}_{j+1},\qquad y[-i_j, n_j-i_j-1] = \tilde{c}_j.$

If (α) holds, then $b^{j+1}[z_{j+1}+1] = 0$ and

$$\begin{aligned} y[-i_{j+1}, n_{j+1}-i_{j+1}-1] &= c_{j+1} = c_j\times b^{j+1}\\ &= y[-i_j, n_j-i_j-1]\times b^{j+1}.\end{aligned}$$

In case (β) we have $b^{j+1}[z_{j+1}+1] = 1$. Further, we have

$$\begin{aligned} y[-i_{j+1}, n_{j+1}-i_{j+1}-1] &= c_{j+1} = c_j\times b^{j+1} = \tilde{c}_j\times\tilde{b}^{j+1}\\ &= y[-i_j, n_j-i_j-1]\times\tilde{b}^{j+1}.\end{aligned}$$

In case (γ) $b^{j+1}[z_{j+1}+1] = 1$ and therefore

$$\begin{aligned} y[-i_{j+1}, n_{j+1}-i_{j+1}-1] &= \tilde{c}_{j+1} = c_j\times\tilde{b}^{j+1}\\ &= y[-i_j, n_j-i_j-1]\times\tilde{b}^{j+1}.\end{aligned}$$

In case (δ) we have $b^{j+1}[z_{j+1}+1] = 0$ and

$$\begin{aligned} y[-i_{j+1}, n_{j+1}-i_{j+1}-1] &= \tilde{c}_{j+1} = \tilde{c}_j\times b^{j+1}\\ &= y[-i_j, n_j-1]\times b^{j+1}.\end{aligned}$$

The above consideration, (14), (15), (16) and (18) imply $f(\bar{z}, i) = y$.

In this way we have proved $X_1\setminus X_0\subset f(M_0\cup M_1)$. It is proved in part (a) that $f(M_0\cup M_1)\subset X_1$ what implies $f(M_0\cup M_1)$ is a measurable set and $\mu_x(f(M_0\cup M_1)) = 1$.

(b2): In order to prove f is a measurable mapping we take a cylinder in $M_0\cup M_1$, say $M(z_0, z_1, \ldots, z_j)$ corresponding to the $z_0, z_1, \ldots, z_j$, $0\leqslant z_i\leqslant\lambda_i-1$, $i = 1, \ldots, j$. Let $S_1, S_2, \ldots, S_{k'}$ be all j-sectors of $\gamma_{t_j}(00)$ and let $S_1', S_2', \ldots, S_{m'}'$ be all j-sectors of $\gamma_{t_j}(01)$. Applying Theorem 4(b) we see that $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_{k'}$ are all j-sectors of $\gamma_{t_j}(11)$ and $\tilde{S}_1', \tilde{S}_2', \ldots, \tilde{S}_{m'}'$ are all j-sectors of $\gamma_{t_j}(10)$ (if $\gamma_{t_j}(00) = \gamma_{t_j}(11)$, then k' is an even number and $S_{k'/2}+1 = \tilde{S}_i$, $i = 1, 2, \ldots, k'/2$). Further we denote by $C(S)$ the cylinder set corresponding to (i_j+1)-arrow of j-sector S. Thus the definition of the mapping f and $f(M_0\cup M_1)\subset X_1$ imply

$$(19)\qquad \begin{aligned} f(M(z_0, \ldots, z_j)) &\subset \bigcup_{i=1}^{k'} C(S_i)\cup\bigcup_{i=1}^{k'} C(\tilde{S}_i)\cup\bigcup_{i=1}^{m'} C(S_i')\cup\bigcup_{i=1}^{m'} C(\tilde{S}_i')\\ &= X(z_0, \ldots, z_j)\end{aligned}$$

if $\gamma_{t_j}(00)\cap\gamma_{t_j}(11) = \emptyset$ and

$$(20)\qquad f(M(z_0, \ldots, z_j))\subset\bigcup_{i=1}^{k'} C(S_i)\cup\bigcup_{i=1}^{m'} C(S_i')\cup\bigcup_{i=1}^{m'} C(\tilde{S}_i') = X(z_0, \ldots, z_j),$$

if $\gamma_{t_j}(00) = \gamma_{t_j}(11)$.

Now the properties:

(1) f is one-to-one,

(2) $\bigcup_{z_0,\ldots,z_j} M(z_0, \ldots, z_j) = M$; $\bigcup_{z_0,\ldots,z_j} X(z_0, \ldots, z_j) = X_1$,

(3) the sets $M(z_0, \ldots, z_j)$ are pairwise disjoint,

(4) the sets $X(z_0, \ldots, z_j)$ are pairwise disjoint,

(5) $\mu_x(f(M_0\cup M_1)) = 1$,

imply $f(M(z_0, \ldots, z_j)) = X(z_0, \ldots, z_j)$. It is not difficult to see that $f(M(z_0, \ldots, z_j)\cap M_0)$ consists of those $C(S)$ in (19) and (20) that the corresponding (i_j+1)-arrow define b^0 between the places $-i_0$ and n_0-i_0-1. This implies that $f(M(z_0, \ldots, z_j)\cap M_0)$ and $f(M(z_0, \ldots, z_j)\cap M_1)$ are measurable subsets of X_1.

(b3): since $S_1', S_2', \ldots, S_{m'}', \tilde{S}_1', \tilde{S}_2', \ldots, \tilde{S}_{m'}'$ are all j-sectors of $\gamma_{t_j}(01)\cup\gamma_{t_j}(10)$, then exactly m' of them define b^0 on the places between $-i_0$ and n_0-i_i-1 and m' define $\tilde{b}^0$. The same property have the sectors S_1, $S_2, \ldots, S_{k'}, \tilde{S}_2, \ldots, \tilde{S}_{k'}$ if $\gamma_{t_j}(00)\cap\gamma_{t_j}(11) = \emptyset$ and $S_1, S_2, \ldots, S_{k'}$ if $\gamma_{t_j}(00)$

$= \gamma_{t_j}(11)$. This implies

(21) $$\mu_x\big(f(M(z_0,\ldots,z_j)\cap M_0)\big) = \mu_x\big(f(M(z_0,\ldots,z_j)\cap M_1)\big) = \tfrac{1}{2}\mu_x\big(f(M(z_0,\ldots,z_j))\big).$$

It remains to calculate $\mu_x\big(f(M(z_0,\ldots,z_j))\big)$. If $\gamma_{t_j}(00)\cap\gamma_{t_j}(11) = \emptyset$, then by Theorem 1 we have

$$\mu_x(C(S_i)) = \mu_x(C(\tilde{S}_i)) = p_{t_j}/k_{t_j}, \quad i = 1, 2, \ldots, k',$$
$$\mu_x(C(S'_i)) = \mu_x(C(\tilde{S}'_i)) = q_{t_j}/m_{t_j}, \quad i = 1, 2, \ldots, m'.$$

Further the equality $f(M(z_0,\ldots,z_j)) = X(z_0,\ldots,z_j)$ implies

$$\mu_x\big(f(M(z_0,\ldots,z_j))\big) = 2\cdot k'\cdot(p_{t_j}/k_{t_j}) + 2\cdot m'\cdot(q_{t_j}/m_{t_j}) = 1/n_j = p^*\big(M(z_0,\ldots,z_j)\big),$$

because $k_{t_j} = k'\cdot n_j$ and $m_{t_j} = m'\cdot n_j$.

If $\gamma_{t_j}(00) = \gamma_{t_j}(11)$, then by the same arguments we obtain

$$\mu_x\big(f(M(z_0,\ldots,z_j))\big) = 1/n_j = p^*\big(M(z_0,\ldots,z_j)\big).$$

In view of (21) we obtain

$$\mu_x\big(f(M(z_0,\ldots,m_j)\cap M_0)\big) = \mu_x\big(f(M(z_0,\ldots,z_j)\cap M_1)\big) = 1/(2\cdot n_j) = p^*\big(M(z_0,\ldots,z_j)\cap M_0\big) = p^*\big(M(z_0,\ldots,z_j)\cap M_1\big).$$

The last equalities mean that $\mu_x = p^*\circ f^{-1}$.

(c): Take $y = f(\bar{z}, i)$, $\bar{z}\in Z$, $i = 0, 1$. We shall show that there exists a number t such that

(22) $$Ty[-i_s-1, j_s-1] = f(\bar{z}+1, \varepsilon(i))[-i_s-1, j_s-1] \quad \text{for} \quad s \geqslant t.$$

First suppose $z_0 < \lambda_0 - 1$. In this case $\bar{z}+1 = (z_0+1, z_1, z_2, \ldots)$ and $\varepsilon(i) = i$. Putting $i'_s = i_s(\bar{z}+1)$, $s = 0, 1, \ldots$, we obtain $i'_0 = i_0+1$, $i'_1 = i_1+1, \ldots$ and so on. It is easy to remark the following fact:

(23) $f(\bar{z}, i) = y$

$$\text{iff} \quad y[-i_s, j_s] = \begin{cases} c_s & \text{if} \quad p(ib^1[z_1+1]\ldots b^s[z_s+1]) = 0, \\ \tilde{c}_s & \text{if} \quad p(ib^1[z_1+1]\ldots b^s[z_s+1]) = 1, \end{cases}$$

$s = 0, 1, \ldots$ Applying (23) for $f(\bar{z}+1, i)$ we obtain (22) with $t = 0$.

Next suppose $z_0 = \lambda_0 - 1$. We choose the first number t, $t \geqslant 1$, such that $z_t < \lambda_t - 1$ and $z_s = \lambda_s - 1$, $0 \leqslant s < t$. Then we have

$$\bar{z}+1 = (0, \ldots, 0, z_t+1, z_{t+1}, \ldots) \quad \text{and} \quad i'_s = i_s+1,\ j'_s = j_s-1 \quad \text{for} \quad s \geqslant t.$$

Using (23) we find

$$Ty[-i'_t, j'_t] = \begin{cases} c_t & \text{if} \quad p(ib^1[\lambda_1]\ldots b^{t-1}[\lambda_{t-1}]b^t[z_t+1]) = 0, \\ \tilde{c}_t & \text{otherwise}. \end{cases}$$

Similarly using (23) for $\bar{y} = f(\bar{z}+1, \varepsilon(i))$ we obtain

$$\bar{y}[-i'_t, j'_t] = \begin{cases} c_t & \text{if} \quad p(\varepsilon(i)\underbrace{0\ldots 0}_{t-1}b^t[z_t+2]) = 0, \\ \tilde{c}_t & \text{if} \quad p(\varepsilon(i)\underbrace{0\ldots 0}_{t-1}b^t[z_t+2]) = 1. \end{cases}$$

From the definition of the function p it follows that

$$p(ib^1[\lambda_1]\ldots b^{t-1}[\lambda_{t-1}]b^t[z_t+1]) = p(j\underbrace{0\ldots 0}_{t-1}b^t[z_t+2])$$

iff $j = i$ whenever $p(b^1[\lambda_1]\ldots b^{t-1}[\lambda_{t-1}]b^t[z_t+1]b^t[z_t+2]) = 0$ and $j = 1-i$ otherwise, iff $j = \varepsilon(i)$. This remark implies $Ty[-i'_t, j'_t] = \bar{y}[-i'_t, j'_t]$. We may repeat the above considerations for any $s \geqslant t$, so we obtain $Ty[-i'_s, j'_s] = \bar{y}[-i'_s, j'_s]$. Therefore (22) is true. But (22) means $T\circ f = f\circ S^*$. Thus (c) holds and the theorem is proved.

In the sequel we denote $X(x) = f(M_0\cup M_1)$.

Definition 3. A two-sided sequence y is called a *generic sequence for* μ_x iff for any block B we have

$$\mu_x(B) = \lim_{n\to\infty}(1/n)\,\mathrm{fr}(B, y[1, n]).$$

Remark 2. The automorphism S^* is a skew product of the automorphism $\bar{S}$ with a family $S(\bar{z})$ of permutations of the two points sets $\{0, 1\}$, each with mass $\frac{1}{2}$. If $\varepsilon(i, \bar{z}) = i$, then $S(\bar{z})$ is the identity of $\{0, 1\}$ and if $\varepsilon(i, \bar{z}) = 1-i$, then $S(\bar{z})$ is the permutation of $\{0, 1\}$ changing zero to one and vice-versa.

Theorem 6. *Each* $y\in X(x)$ *is a generic sequence for the measure* μ_x.

Proof. In order to prove the theorem we shall show that $\theta^*(x)$ is uniquely ergodic system. In real, suppose that ν is an ergodic measure on M and $\nu \neq p^*$. Then there exist S^*-invariant sets $A, B\subset M$ such that $p^*(A) = 1$, $\nu(B) = 1$ and $A\cap B = \emptyset$. Put $A_0 = M_0\cap A$, $A_1 = M_1\cap A$. Each of the sets A_0, A_1 has a form $A_0 = (C, 0)$, $A_1 = (D, 1)$, where $C, D\subset \bar{Z}$. By the definition of the measure p^* it follows that $\bar{p}(\bar{C}) = 1$, where $\bar{C} = C\cap D$. We may assume that $\bar{C}$ is an $\bar{S}$-invariant set. Further, the measure ν may be transferred by the natural projection to $\bar{Z}$.

Thus it is clear that $\nu(\bar{Z}\setminus\bar{C}) = 1$ and ν is an $\bar{S}$-invariant ergodic measure on $\bar{Z}$. Since $\bar{S}$ is a translation on $\bar{Z}$, the system $(\bar{Z}, \bar{\mathscr{F}}, \bar{p}, \bar{S})$ (and so $\theta^*(x)$) is uniquely ergodic. Therefore $(X(x), \mathscr{F}, \mu_x, T)$ is uniquely ergodic. Proposition (5.15) from [2] implies that every $y\in X(x)$ is generic.

§ 4. Isomorphism theorems. In this section we give sufficient and necessary conditions for two regular Morse dynamical systems to be isomorphic.

Let $x = b^0\times b^1\times\ldots$, $y=\beta^0\times\beta^1\times\ldots$ be continuous Morse sequences. We assume $|\beta^i| = |b^i| = \lambda_i$, $i = 0, 1, \ldots$ Let $x_t = b^t\times b^{t+1}\times\ldots$, $y_t = \beta^t\times\beta^{t+1}\times\ldots$, and let μ_t, $\bar{\mu}_t$ be measures defined by x_t and y_t, respectively (we will write μ_y and μ_x instead of $\bar{\mu}_0$ and μ_0, respectively). Further, let P_x and P_y be the partitions of $X(x)$ and $X(y)$, obtained as the restrictions of the standard partition P to $X(x)$ and $X(y)$, respectively. Denote by $\overline{Y}_t(y)$, $t\geqslant 0$, the graphs defined by y in the same way as $\overline{Y}_t$ was defined by x. First we describe two-element partitions Q such that $Q\subset\bigvee_0^{n_t} T^{-i}P_y$ for fixed $t\geqslant 0$.

Suppose that $Q = (Q_0, Q_1)$ is a measurable partition of $X(y)$ such that $Q\subset\bigvee_0^{n_t} T^{-i}P_y$. Then Q define a partition of all atoms of $\bigvee_0^{n_t} T^{-i}P_y$ on two parts: one part is designed by 0 and the other by 1. In this way we obtain a code of atoms of $\bigvee_0^{n_t} T^{-i}P_y$, i.e. a code of arrows of $\overline{Y}_t(y)$. Thus we obtain four blocks $A'(Q)$, $B'(Q)$, $C'(Q)$, $D'(Q)$, where $A'(Q)$ is a block obtained by the coding of $\gamma_t(00)$ (we write $\gamma_t(00)\sim A'(Q)$) and $B'(Q)\sim\gamma_t(01)$, $C'(Q)\sim\gamma_t(10)$, $D'(Q)\sim\gamma_t(11)$. In the case $\gamma_t(00) = \gamma_t(11)$ k_t is an even number and then

$$D'(Q) = E_2E_1 \quad\text{whenever}\quad A'(Q) = E_1E_2 \tag{24}$$

with $|E_1| = |E_2| = \frac{1}{2}\cdot k_t$. Put

$$\begin{aligned}
A(Q) &= \underbrace{A'(Q)A'(Q)\ldots A'(Q)}_{n_t/k_t}\\
B(Q) &= \underbrace{B'(Q)B'(Q)\ldots B'(Q)}_{n_t/m_t}\\
C(Q) &= \underbrace{C'(Q)C'(Q)\ldots C'(Q)}_{n_t/m_t}\\
D(Q) &= \underbrace{D'(Q)D'(Q)\ldots D'(Q)}_{n_t/k_t}.
\end{aligned} \tag{25}$$

It is obvious that if blocks A, B, C, D satisfy (24) and (25), then they determine a partition of $\bigvee_0^{n_t} T^{-i}P_y$.

Now, we define a distance between two measurable partitions of $X(y)$. Suppose $Q = (Q_0, Q_1)$ and $R = (R_0, R_1)$ be such partitions. Define

$$|Q-R| = \mu_y(Q_0\cap R_1)+\mu_y(Q_1\cap R_0).$$

Fix $t\geqslant 0$ and take two partitions $Q, \overline{Q}$ such that $Q, \overline{Q}\subset\bigvee_0^{n_t} T^{-1}P_y$. Applying Theorem 1 to μ_y and $\overline{Y}_t(y)$ it is easy to verify

$$\begin{aligned}|Q-\overline{Q}| = \bar{p}_{t+1}\cdot d\big(A(Q), A(\overline{Q})\big)+\bar{q}_{t+1}\cdot d\big(B(Q), B(\overline{Q})\big)+\\ +\bar{q}_{t+1}\cdot d\big(C(Q), C(\overline{Q})\big)+\bar{p}_{t+1}\cdot d\big(D(Q), D(\overline{Q})\big),\end{aligned} \tag{26}$$

where

$$\bar{p}_t = \bar{\mu}_t(00),\qquad \bar{q}_t = \bar{\mu}_t(01)$$

and

$$d(E, \overline{E}) = (1/n_t)\operatorname{card}\{1\leqslant i\leqslant n_t,\ E[i]\neq\overline{E}[i]\},$$

E, $\overline{E}$ are blocks of the length n_t.

We shall use in the sequel the definition of the Ornstein distance $\bar{d}$ of invariant measures (see for example [1]).

Take two invariant ergodic measures φ, ψ on X and denote by X_φ, X_ψ the set of all generic sequences for φ and ψ, respectively. Define

$$\bar{d}(\varphi, \psi) = \inf_{y\in X_\varphi, \bar{y}\in X_\psi}\ \overline{\lim}\ \frac{1}{n}\sum_{i=0}^{n-1}\varrho(y[i], \bar{y}[i]), \tag{27}$$

where

$$\varrho(u, v) = \begin{cases}0, & (uv) = 00 \text{ or } 11,\\ 1, & (uv) = 01 \text{ or } 10.\end{cases}$$

DEFINITION 4. We say that $x = b^0\times b^1\times\ldots$ is a *regular Morse sequence* if there exists $\varrho>0$ such that $\varrho\leqslant p_t$, $q_t\leqslant\frac{1}{2}-\varrho$, $t = 0, 1, \ldots$

Further we assume that x and y are regular Morse sequences with the same ϱ. Now, we may prove the following

THEOREM 7. *If the Morse dynamical systems $\theta(x)$ and $\theta(y)$ are isomorphic, then $\bar{d}(\mu_t, \bar{\mu}_t)\to 0$ as $t\to\infty$.*

Proof. Suppose that $h\colon X(x)\to X(y)$ is an isomorphism between $\theta(x)$ and $\theta(y)$. Then $Q = h(P_x) = (Q_0, Q_1)$ is a measurable partition of $X(y)$. The partition P_y is a strong generator because $\theta(y)$ is a dynamical system with zero entropy (see [4] and [8]). Thus there exists a sequence $Q^t = (Q_0^t, Q_1^t)$ of partitions of $X(y)$ such that

$$|Q-Q^t|\xrightarrow[t\to\infty]{}0 \quad\text{and}\quad Q^t\subset\bigvee_0^{n_t} T^{-i}P_y. \tag{28}$$

Let $A(Q^t) = A_t$, $B(Q^t) = B_t$, $C(Q^t) = C_t$, $D(Q^t) = D_t$ be the codes of Q^t, $t\geqslant 0$. We shall show (Lemma 2 below) that

$$d(A_t, B_t)\xrightarrow[t\to\infty]{}0,\qquad d(C_t, D_t)\xrightarrow[t\to\infty]{}0. \tag{29}$$

At present we assume that the sequence Q^t satisfies (29).

Consider the stationary processes (T, Q^t, μ_y) and (T, Q, μ_y) on $X(y)$. As a simple consequence of the definition of the Ornstein distance $\bar{d}$ [7] we obtain the following inequality

$$\bar{d}\big((T, Q^t, \mu_y), (T, Q, \mu_y)\big) \leqslant |Q^t - Q|. \tag{30}$$

The processes (T, Q^t, μ_y), $t \geqslant 0$, determine invariant measures ν_t on X as follows: if $C = \{w \in X,\ w[0] = c_0, \ldots, w[n] = c_n\}$, $n \geqslant 1$, then $\nu_t(C) = \mu_y\big(\bigcap_{j=0}^{n} T^{-j} Q^t_{c_j}\big)$, $c_j = 0, 1$, $j = 0, \ldots, n$. The process (T, Q, μ_y) determines the measure μ_x since (T, Q, μ_y) is equivalent to the process (T, P_x, μ_x) on $X(x)$. Thus (28) and (30) give

$$\bar{d}(\nu_t, \mu_x) \underset{t\to\infty}{\to} 0. \tag{31}$$

Observe that the dynamical systems (X, T, ν_t), $t \geqslant 0$, are homomorphic images of $(X(y), T, \mu_y)$. In fact, the partitions Q^t define homomorphisms g_t, $t \geqslant 0$, from $(X(y), T, \mu_y)$ to (X, T, ν_t) in the following way:

$$g_t(u)[-i_t + k\cdot n_t, j_t + k\cdot n_t] = \begin{cases} A_t & \text{if} \quad u[-i_t + k\cdot n_t, -i_t + (k+2)\cdot n_t - 1] = d_t d_t, \\ B_t & \text{if} \quad u[-i_t + k\cdot n_t, -i_t + (k+2)\cdot n_t - 1] = d_t \tilde{d}_t, \\ C_t & \text{if} \quad u[-i_t + k\cdot n_t, -i_t + (k+2)\cdot n_t - 1] = \tilde{d}_t d_t, \\ D_t & \text{if} \quad u[-i_t + k\cdot n_t, -i_t + (k+2)\cdot n_t - 1] = \tilde{d}_t \tilde{d}_t, \end{cases}$$

where $k = 0, 1, 2, \ldots$, $u = f(\bar{z}, i) \in X(y)$ and $d_t = \beta^0 \times \beta^1 \times \ldots \times \beta^t$, $t \geqslant 0$. It is not difficult to check that $\mu_y \cdot g_t^{-1} = \nu_t$. Putting $W_t = g_t(X(y))$ we have $\nu_t(W_t) = 1$. Further we observe that $\bar{\mu}_t, \mu_t$, $t \geqslant 0$, are ergodic measures because they are continuous Morse measures [4] and ν_t are ergodic measures because ν_t are homomorphic images of μ_y. Now, take $\varepsilon > 0$, $\varepsilon < \varrho^2$, $\varepsilon < 1/9$, and choose t_0 such that for $t \geqslant t_0$

$$\bar{d}(\nu_t, \mu_x) < \varepsilon, \qquad d(A_t, B_t) < \sqrt{\varepsilon}, \qquad d(C_t, D_t) < \sqrt{\varepsilon}. \tag{32}$$

Fix $t \geqslant t_0$. Then (32) and (27) imply that there exist sequences $v \in X(x)$, $g_t(u) \in W_t$ $(u \in X(y))$ which are generic for μ_x and ν_t and such that

$$\overline{\lim_n}\,(1/n)\,\mathrm{card}\{i,\ 0 \leqslant i < n,\ v[i] \neq g_t(u)[i]\} < \varepsilon. \tag{33}$$

Since $v \in X(x)$ and $u \in X(y)$, $v = f(\bar{z}, i)$, $u = f^*(\bar{w}, j)$, where $\bar{z}, \bar{w} \in \bar{Z}$, $i, j = 0, 1$ and f^* is a function defined by y in the same way as f by x. Next, we define two sequences $\bar{v}$ and $\bar{u}$ as follows:

$$\bar{v}[k] = \begin{cases} 0 & \text{if} \quad v[-i_t + k\cdot n_t, j_t + k\cdot n_t] = c_t, \\ 1 & \text{if} \quad v[-i_t + k\cdot n_t, j_t + k\cdot n_t] = \tilde{c}_t, \end{cases}$$

$$\bar{u}[k] = \begin{cases} 0 & \text{if} \quad u[-i'_t + k\cdot n_t, j'_t + k\cdot n_t] = d_t, \\ 1 & \text{if} \quad u[-i'_t + k\cdot n_t, j'_t + k\cdot n_t] = \tilde{d}_t, \end{cases}$$

where $k = 0, \pm 1, \pm 2, \ldots$, $i_t = i_t(\bar{z})$, $i'_t = i_t(\bar{w})$. It follows from the construction of the sets $X(x), X(y), X(x_{t+1}), X(y_{t+1})$ that $\bar{v} \in X(x_{t+1})$, $\bar{u} \in X(y_{t+1})$. Therefore Theorem 6 implies that $\bar{u}, \bar{v}$ are generic sequences for $\bar{\mu}_{t+1}$ and μ_{t+1}, respectively. We shall show that if $v^* = \bar{v}$ or $T\bar{v}$ or $\tilde{\bar{v}}$ or $T\tilde{\bar{v}}$, then

$$\overline{\lim_n}\,(1/n)\,\mathrm{card}\{i,\ 0 \leqslant i \leqslant n,\ \bar{u}[i] \neq v^*[i]\} < \sqrt{\varepsilon}. \tag{34}$$

We may assume $i'_t = 0$ because in the contrary case we may shift the sequences v and u. Put

$$Z_0 = \{j \geqslant 0,\ d\big(g_t(u)[j\cdot n_t, (j+1)\cdot n_t - 1], v[j\cdot n_t, (j+1)\cdot n_t - 1]\big) < \sqrt{\varepsilon}\}. \tag{35}$$

Further we have

$$\begin{aligned} &\frac{1}{l\cdot n_t}\cdot \mathrm{card}\{j,\ 0 \leqslant j \leqslant l\cdot n_t - 1,\ g_t(u)[j] \neq v[j]\} \\ &\quad = \frac{1}{l}\cdot \sum_{s=0}^{l-1} \frac{1}{n_t}\cdot \mathrm{card}\{j,\ s\cdot n_t \leqslant j \leqslant (s+1)\cdot n_t - 1,\ g_t(u)[j] \neq v[j]\} \\ &\quad = \frac{1}{l}\cdot \sum_{s=0}^{l-1} d\big(g_t(u)[s\cdot n_t, s\cdot n_t + n_t - 1], v[s\cdot n_t, (s+1)\cdot n_t - 1]\big). \end{aligned}$$

Using the above equality and (33) we obtain

$$\lim_{l\to\infty} (1/l)\,\mathrm{card}\{j,\ 0 \leqslant j < l,\ j \in Z_0\} > 1 - \sqrt{\varepsilon}. \tag{36}$$

Next, we remark that for arbitrary $j \geqslant 0$, $v[j\cdot n_t, (j+1)\cdot n_t - 1]$ is one of the four blocks

$$\begin{aligned} \mathrm{I}_t &= c_t c_t[i_t + 1, i_t + n_t], & \mathrm{II}_t &= c_t \tilde{c}_t[i_t + 1, i_t + n_t], \\ \mathrm{III}_t &= \tilde{c}_t c_t[i_t + 1, i_t + n_t], & \mathrm{IV}_t &= \tilde{c}_t \tilde{c}_t[i_t + 1, i_t + n_t]. \end{aligned} \tag{37}$$

Further the definition of the distance d (see § 2) gives

$$\begin{gathered} d(\mathrm{I}_t, \mathrm{II}_t) = d(\mathrm{III}_t, \mathrm{IV}_t) = i_t/n_t, \qquad d(\mathrm{I}_t, \mathrm{III}_t) = d(\mathrm{II}_t, \mathrm{IV}_t) = 1 - i_t/n_t, \\ d(\mathrm{I}_t, \mathrm{IV}_t) = d(\mathrm{II}_t, \mathrm{III}_t) = 1. \end{gathered} \tag{38}$$

Now we put

$$\begin{aligned} Z_0^{(0)} &= \{j \in Z_0,\ g_t(u)[j\cdot n_t, (j+1)\cdot n_t - 1] = A_t \text{ or } B_t\} \\ &= \{j \in Z_0,\ \bar{u}[j] = 0\}, \\ Z_0^{(1)} &= \{j \in Z_0,\ g_t(u)[j\cdot n_t, (j+1)\cdot n_t - 1] = C_t \text{ or } D_t\} \\ &= \{j \in Z_0,\ \bar{u}[j] = 1\}. \end{aligned}$$

Then the conditions

$$\lim_{l\to\infty}(1/l)\operatorname{card}\{0\leqslant j<l,\ \bar{u}[j]=0\}=\bar{\mu}_{t+1}(0)=\tfrac{1}{2},$$

$$\lim_{l\to\infty}(1/l)\operatorname{card}\{0\leqslant j<l,\ \bar{u}[j]=1\}=\bar{\mu}_{t+1}(1)=\tfrac{1}{2},$$

and (36) imply

$$\overline{\lim_{l\to\infty}}(1/l)\operatorname{card}\{0\leqslant j<l,\ j\in Z_0^{(0)}\}>\tfrac{1}{2}-\sqrt{\varepsilon},$$

(39)

$$\overline{\lim_{l\to\infty}}(1/l)\operatorname{card}\{0\leqslant j<l,\ j\in Z_0^{(1)}\}>\tfrac{1}{2}-\sqrt{\varepsilon}.$$

Observe that the equality $v[j_1\cdot n_t,(j_1+1)\cdot n_t-1]=v[j_2\cdot n_t,(j_2+1)\cdot n_t-1]$ is not true for arbitrary $j_1,j_2\in Z_0^{(0)}$ $(Z_0^{(1)})$, $j_1\neq j_2$, because in the contrary case (39) and (37) imply either $p_{t+1}>\frac{1}{2}-\sqrt{\varepsilon}>\frac{1}{2}-\varrho$ or $q_{t+1}>\frac{1}{2}-\varrho$, i.e. x does not satisfy the condition of Definition 4. Further the equalities

$$v[j_1\cdot n_t,(j_1+1)\cdot n_t-1]=\mathrm{I}_t\ (\mathrm{II}_t)\quad\text{and}\quad v[j_2\cdot n_t,(j_2+1)\cdot n_t-1]=\mathrm{IV}_t\ (\mathrm{III}_t)$$

are not true for some $j_1,j_2\in Z_0^{(0)}$ $(Z_0^{(1)})$, because by (37) and (32) in the contrary case we should obtain

$$\begin{aligned}1=d(\mathrm{I}_t,\mathrm{IV}_t)&\leqslant d\big(\mathrm{I}_t,g_t(u)[j_1\cdot n_t,(j_1+1)\cdot n_t-1]\big)+\\&+d\big(g_t(u)[j_1\cdot n_t,(j_1+1)\cdot n_t-1],g_t(u)[j_2\cdot n_t,(j_2+1)\cdot n_t-1]\big)+\\&+d\big(g_t(u)[j_2\cdot n_t,(j_2+1)\cdot n_t-1],\mathrm{IV}_t\big)<3\cdot\sqrt{\varepsilon}\end{aligned}$$

what is contradiction with $\varepsilon<1/9$.

In this way the following cases are possible:

(a)
$$v[j\cdot n_t,(j+1)\cdot n_t-1]=\mathrm{I}_t\ \text{or}\ \mathrm{II}_t\qquad j\in Z_0^{(0)},$$
$$v[j\cdot n_t,(j+1)\cdot n_t-1]=\mathrm{III}_t\ \text{or}\ \mathrm{IV}_t\qquad j\in Z_0^{(1)},$$

(b)
$$v[j\cdot n_t,(j+1)\cdot n_t-1]=\mathrm{I}_t\ \text{or}\ \mathrm{III}_t,\qquad j\in Z_0^{(0)},$$
$$v[j\cdot n_t,(j+1)\cdot n_t-1]=\mathrm{II}_t\ \text{or}\ \mathrm{IV}_t,\qquad j\in Z_0^{(1)},$$

(c)
$$v[j\cdot n_t,(j+1)\cdot n_t-1]=\mathrm{II}_t\ \text{or}\ \mathrm{IV}_t,\qquad j\in Z_0^{(0)},$$
$$v[j\cdot n_t,(j+1)\cdot n_t-1]=\mathrm{I}_t\ \text{or}\ \mathrm{III}_t,\qquad j\in Z_0^{(1)},$$

(d)
$$v[j\cdot n_t,(j+1)\cdot n_t-1]=\mathrm{III}_t\ \text{or}\ \mathrm{IV}_t,\qquad j\in Z_0^{(0)},$$
$$v[j\cdot n_t,(j+1)\cdot n_t-1]=\mathrm{I}_t\ \text{or}\ \mathrm{II}_t,\qquad j\in Z_0^{(1)}.$$

If (a) holds, then $v[-i_t+j\cdot n_t,-i_t+(j+1)\cdot n_t-1]=c_t$ whenever $j\in Z_0^{(0)}$ and $v[-i_t+j\cdot n_t,-i_t+(j+1)\cdot n_t-1]=\tilde{c}_t$ whenever $j\in Z_0^{(1)}$. This

means that $\bar{v}[j]=0$ if $j\in Z_0^{(0)}$ and $\bar{v}[j]=1$ if $j\in Z_0^{(1)}$. Thus we obtain $\bar{v}[j]=\bar{u}[j]$ whenever $j\in Z_0$ what implies (34) with $v^*=\bar{v}$.

If (b) holds, then by the same arguments we obtain $\bar{v}[j+1]=\bar{u}[j]$ for each $j\in Z_0$. Thus we have (34) with $v^*=T\bar{v}$.

In case (c) we have $\bar{v}[j+1]\neq\bar{u}[j]$ whenever $j\in Z_0$ what implies (34) with $v^*=T\tilde{\bar{v}}$.

Finally, (d) implies $\bar{v}[j]=\bar{u}[j]$ for each $j\in Z_0$. Therefore we obtain (34) with $v^*=\tilde{\bar{v}}$.

Now, since each of the sequences $\bar{v}$, $T\bar{v}$, $\tilde{\bar{v}}$, $T\tilde{\bar{v}}$ is a generic sequence for μ_{t+1} and $\bar{u}$ is a generic sequence for $\bar{\mu}_{t+1}$, (34) and (27) imply

$$d(\bar{\mu}_{t+1},\mu_{t+1})<\sqrt{\varepsilon}\quad\text{for}\quad t\geqslant t_0.$$

The last inequality finishes the proof of the theorem.

Lemma 2. *Let Q^t, $t\geqslant 0$, be a sequence of partitions of $X(y)$ such that $|Q^t-Q|\underset{t\to\infty}{\to}0$, $Q^t\subset\bigvee_0^{n_t}T^{-i}P_y$ and let $A(Q^t)$, $B(Q^t)$, $C(Q^t)$, $D(Q^t)$ be the codes of Q^t, $t\geqslant 0$. Then*

$$\lim_{t\to\infty}d\big(A(Q^t),B(Q^t)\big)=0,\qquad\lim_{t\to\infty}d\big(C(Q^t),D(Q^t)\big)=0.$$

Proof. Take a positive number $\varepsilon>0$, $\varepsilon<\varrho$ and choose a number t_0 such that

(40) $$|Q^t-Q^s|<\varepsilon^2\quad\text{if}\quad s>t\geqslant t_0.$$

Next, we choose a number $s_0>t_0$ in such way that $n_{t_0}/n_{s_0}<\varepsilon$. Let $t=t_0+s$ with $s\geqslant s_0$. Since $Q^{t_0}\subset\bigvee_0^{n_t}T^{-i}P_y$, Q^{t_0} determines the blocks $A_t^{(1)}$, $B_t^{(1)}$, $C_t^{(1)}$, $D_t^{(1)}$ as the code of successive arrows appearing in d_td_t, $d_t\tilde{d}_t$, $\tilde{d}_td_t$, $\tilde{d}_t\tilde{d}_t$. It is not difficult to see that

$$A_t^{(1)}=(\beta^{t_0+1}\times\ldots\times\beta^t)0*\{A_{t_0},B_{t_0},C_{t_0},D_{t_0}\},$$
$$B_t^{(1)}=(\beta^{t_0+1}\times\ldots\times\beta^t)1*\{A_{t_0},B_{t_0},C_{t_0},D_{t_0}\},$$
$$C_t^{(1)}=(\widetilde{\beta^{t_0+1}\times\ldots\times\beta^t})0*\{A_{t_0},B_{t_0},C_{t_0},D_{t_0}\},$$
$$D_t^{(1)}=(\widetilde{\beta^{t_0+1}\times\ldots\times\beta^t})1*\{A_{t_0},B_{t_0},C_{t_0},D_{t_0}\},$$

where $E*\{A_{t_0},B_{t_0},C_{t_0},D_{t_0}\}$ (E is a block) denotes a block obtained with E by the substitution $00\sim A_{t_0}$, $01\sim B_{t_0}$, $10\sim C_{t_0}$, $11\sim D_{t_0}$. Using (26), we have

$$|Q^{t_0}-Q^t|=d(A_t,A_t^{(1)})\cdot\bar{p}_t+d(B_t,B_t^{(1)})\cdot\bar{q}_t+d(C_t,C_t^{(1)})\cdot\bar{q}_t+d(D_t,D_t^{(1)})\cdot\bar{p}_t.$$

Further, the inequalities $\overline{p}_t, \overline{q}_t \geqslant \varrho > \varepsilon$ (y satisfies the condition of Definition 4) and (40) imply

$$d(A_t^{(1)}, A_t) \leqslant 1/\overline{p}_t \cdot |Q^{t_0} - Q^t| < \varepsilon, \qquad d(B_t^{(1)}, B_t) \leqslant 1/\overline{q}_t \cdot |Q^{t_0} - Q^t| < \varepsilon$$

(41)

$$d(C_t^{(1)}, C_t) < \varepsilon, \qquad d(D_t, D_t^{(1)}) < \varepsilon.$$

It is obvious that

(42) $$d(A_t^{(1)}, B_t^{(1)}) \leqslant n_{t_0}/n_t \leqslant n_{t_0}/n_{s_0} < \varepsilon \quad \text{and} \quad d(C_t^{(1)}, D_t^{(1)}) < \varepsilon.$$

Finally (41) and (42) imply

$$d(A_t, B_t) \leqslant d(A_t, A_t^{(1)}) + d(A_t^{(1)}, B_t^{(1)}) + d(B_t^{(1)}, B_t) < 3\cdot\varepsilon,$$
$$d(C_t, D_t) \leqslant d(C_t, C_t^{(1)}) + d(C_t^{(1)}, D_t^{(1)}) + d(D_t^{(1)}, D_t) < 3\cdot\varepsilon,$$

whenever $t \geqslant t_0 + s_0$. Thus the above inequalities imply the thesis of the lemma.

THEOREM 8. *Let $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$, be two regular continuous Morse sequences with $|b^i| = |\beta^i| = \lambda_i$ satisfying $\lambda_i \leqslant r$, $i = 0, 1, \ldots$ The dynamical systems $\theta(x)$ and $\theta(y)$ are isomorphic iff there exists i_0 such that $b^i = \beta^i$ for $i \geqslant i_0$.*

Proof. *Sufficiency.* Putting $(b^0)' = b^0 \times \ldots \times b^{i_0}$, $(\beta^0)' = \beta^0 \times \ldots \times \beta^{i_0}$ we have

(43) $$x = (b^0) \times b^{i_0+1} \times \ldots, \qquad y = (\beta^0)' \times b^{i_0+1} \times \ldots$$

Starting with (43) we may construct the standard representations $\theta^*(x)$ and $\theta^*(y)$ of $\theta(x)$ and $\theta(y)$, respectively. Further we observe that the standard representation $\theta^*(x)$ is completely determined by $|(b^0)'|$ and by the remaining blocks $b^{i_0+1}, \ldots$ Therefore we obtain $\theta^*(x) = \theta^*(y)$ what implies that the systems $\theta(x)$ and $\theta(y)$ are isomorphic.

Necessity. As in the proof of Theorem 7 we construct a sequence of partitions $Q^t, t \geqslant 0$, such that $Q^t \subset \bigvee_0^{n_t} T^{-i} P_y$, $|Q^t - Q| \underset{t\to\infty}{\to} 0$, where $Q = h(P_x)$ and $h\colon X(x) \to X(y)$ is an isomorphism between $\theta(x)$ and $\theta(y)$. Let $A_t = A(Q^t)$, $B_t = B(Q^t)$, $C_t = C(Q^t)$, $D_t = D(Q^t)$ be the codes of Q^t, $t \geqslant 0$. We have shown that

(44) $$\lim_{t\to\infty} d(A_t, B_t) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(C_t, D_t) = 0.$$

Let A_t^* denote A_t or B_t, C_t^* denote C_t or D_t. We shall show

(45) $$\lim_{t\to\infty} d(A_t^*, C_t^*) = 1.$$

Let ε be an arbitrary positive number. We choose a positive integer t_0 such that (32) is satisfied and next for fixed $t \geqslant t_0$ choose sequences $u \in X(y)$, $v \in X(x)$ satisfying (33). Further, define the sets $Z_0, Z_0^{(0)}, Z_0^{(1)}$. In each case (a), (b), (c), (d) considered in the proof of Theorem 7 there exist $j_1 \in Z_0^{(0)}$, $j_2 \in Z_0^{(1)}$ such that

$$v[j_1 \cdot n_t, (j_1+1)\cdot n_t - 1] = \mathrm{I}_t \text{ (or } \mathrm{II}_t)$$

and

$$v[j_2 \cdot n_t, (j_2+1)\cdot n_t - 1] = \mathrm{IV}_t \text{ (or } \mathrm{III}_t)$$

or vice versa. Then

$$1 = d(\mathrm{I}_t, \mathrm{IV}_t) \leqslant d\big(I_t, g_t(u)[j_1\cdot n_t, (j_1+1)\cdot n_t - 1]\big) + d(A_t^*, C_t^*) + $$
$$+ d\big(g_t(u)[j_2\cdot n_t, (j_2+1)\cdot n_t - 1], \mathrm{IV}_t\big)$$
$$\leqslant d(A_t^*, C_t^*) + 2\cdot\sqrt{\varepsilon}.$$

Hence we obtain $d(A_t^*, C_t^*) \geqslant 1 - 2\cdot\sqrt{\varepsilon}$ whenever $t \geqslant t_0$. In this way (45) is established.

Further, let $A_{t+1}^{(2)} = A_t \times \beta^{t+1}$, $t \geqslant 0$. Next, we shall establish that

(46) $$d(A_t, A_t^{(2)}) \underset{t\to\infty}{\to} 0.$$

For given $\varepsilon > 0$ we choose t_1 such that for $t \geqslant t_1$

(47) $$|Q^{t+1} - Q^t| < \varepsilon, \qquad d(A_t, B_t) < \varepsilon,$$
$$d(C_t, D_t) < \varepsilon, \qquad d(A_t^*, C_t^*) > 1 - \varepsilon.$$

Using (26) we have

(48) $$|Q^{t+1} - Q^t| = \overline{p}_{t+1}\cdot d(A_{t+1}, A_{t+1}^{(1)}) + \overline{q}_{t+1}\cdot d(B_{t+1}, B_{t+1}^{(1)}) + $$
$$+ \overline{q}_{t+1}\cdot d(C_{t+1}, C_{t+1}^{(1)}) + \overline{p}_{t+1}\cdot d(D_{t+1}, D_{t+1}^{(1)}),$$

where

$$A_{t+1}^{(1)} = \beta^{t+1} 0 * \{A_t, B_t, C_t, D_t\}, \qquad B_{t+1}^{(1)} = \beta^{t+1} 1 * \{A_t, B_t, C_t, D_t\},$$
$$C_{t+1}^{(1)} = \tilde{\beta}^{t+1} 0 * \{A_t, B_t, C_t, D_t\}, \qquad D_{t+1}^{(1)} = \tilde{\beta}^{t+1} 1 * \{A_t, B_t, C_t, D_t\}.$$

Now (47) implies

$$d(A_{t+1}^{(1)}, D_{t+1}^{(1)}) < \varepsilon, \qquad d(A_{t+1}^{(1)}, \tilde{C}_{t+1}^{(1)}) < \varepsilon, \qquad d(A_{t+1}^{(1)}, \tilde{D}_{t+1}^{(1)}) < \varepsilon,$$

(49)

$$d(A_{t+1}, \tilde{C}_{t+1}) < \varepsilon, \qquad d(A_{t+1}, \tilde{D}_{t+1}) < \varepsilon \quad \text{for} \quad t \geqslant t_1.$$

Thus (49) implies

$$d(B_{t+1}, B_{t+1}^{(1)}) \leqslant d(B_{t+1}, A_{t+1}) + d(A_{t+1}, A_{t+1}^{(1)}) + d(A_{t+1}^{(1)}, B_{t+1}^{(1)})$$
$$< d(A_{t+1}, A_{t+1}^{(1)}) + 2\cdot\varepsilon.$$

Similarly $d(A_{t+1}, A_{t+1}^{(1)}) < d(D_{t+1}, B_{t+1}^{(1)}) + 2\cdot\varepsilon$. Hence

$$|d(A_{t+1}, A_{t+1}^{(1)}) - d(B_{t+1}, B_{t+1}^{(1)})| < 2\cdot\varepsilon. \tag{50}$$

In the same way we can obtain

$$\begin{aligned} &|d(A_{t+1}, A_{t+1}^{(1)}) - d(C_{t+1}, C_{t+1}^{(1)})| < 2\cdot\varepsilon,\\ &|d(A_{t+1}, A_{t+1}^{(1)}) - d(D_{t+1}, D_{t+1}^{(1)})| < 2\cdot\varepsilon. \end{aligned} \tag{51}$$

Next, (48), (50) and (51) imply

$$\big|d(A_{t+1}, A_{t+1}^{(1)}) - |Q^{t+1} - Q^{t}|\big| < 2\cdot\varepsilon,$$

and by (47)

$$d(A_{t+1}, A_{t+1}^{(1)}) < 3\cdot\varepsilon \quad \text{for} \quad t \geqslant t_1. \tag{52}$$

Moreover, (47) implies

$$d(A_{t+1}^{(1)}, A_{t+1}^{(2)}) < \varepsilon.$$

By the above and by (52) we obtain

$$d(A_{t+1}, A_{t+1}^{(2)}) < 4\cdot\varepsilon,$$

whenever $t \geqslant t_1$. So (46) is established. Now, we may prove the thesis of the theorem.

Take $\varepsilon > 0$, $\varepsilon < 1/49\cdot r^2$; $\varepsilon < \varrho$, and choose t_0 such that for $t \geqslant t_0$ (47) holds and $d(A_t, A_t^{(2)}) < \varepsilon$. Fix $t \geqslant t_0$ and consider the cases (a), (b), (c) and (d) of the proof of Theorem 7. Suppose the case (a) holds. Then there exist $j_1, j_2 \in Z_0^{(0)}$ such that

$$v[j_1\cdot n_t, (j_1+1)\cdot n_t - 1] = \mathrm{I}_t, \qquad v[j_2\cdot n_t, (j_2+1)\cdot n_t - 1] = \mathrm{II}_t.$$

Moreover,

$$\begin{aligned} i_t/n_t = d(\mathrm{I}_t, \mathrm{II}_t) &\leqslant d(\mathrm{I}_t, \underbrace{g_t(u)[j_1\cdot n_t, (j_1+1)\cdot n_t - 1]}_{A_t \text{ or } B_t}) +\\ &+ d(\underbrace{g_t(u)[j_1\cdot n_t, (j_1+1)\cdot n_t - 1]}_{A_t \text{ or } B_t}, \underbrace{g_t(u)[j_2\cdot n_t, (j_2+1)\cdot n_t - 1]}_{A_t \text{ or } B_t}) +\\ &+ d(\underbrace{g_t(u)[j_2\cdot n_t, (j_2+1)\cdot n_t - 1]}_{A_t \text{ or } B_t}, \mathrm{II}_t) < 3\cdot\sqrt{\varepsilon}. \end{aligned}$$

Similarly, in the remaining cases (b), (c), (d) we can establish

$$i_t/n_t < 3\cdot\sqrt{\varepsilon}, \tag{53}$$

or

$$(n_t - i_t)/n_t < 3\cdot\sqrt{\varepsilon}. \tag{54}$$

Further (53) implies $i_t/n_{t-1} = (\lambda_t\cdot i_t)/(\lambda_t\cdot n_{t-1}) = (\lambda_t\cdot i_t)/n_t < 3\cdot r\cdot\sqrt{\varepsilon}$, and from (54) follows

$$(n_t - i_t)/n_{t-1} < 3\cdot r\cdot\sqrt{\varepsilon}.$$

Now, we choose $j \in Z_0$ such that $g_t(u)[j\cdot n_t, (j+1)\cdot n_t - 1] = A_t$. Then $E_t = v[j\cdot n_t, (j+1)\cdot n_t - 1]$ is one of the four blocks I_t, II_t, III_t, IV_t. By the definition of Z_0 we have $d(E_t, A_t) < \sqrt{\varepsilon}$ what implies

$$d(E_t, A_t^{(2)}) < 2\cdot\sqrt{\varepsilon} \tag{55}$$

because $d(A_t, A_t^{(2)}) < \varepsilon$. First we regard the possibility (53). In this case the sequences E_t and A_t have the following forms:

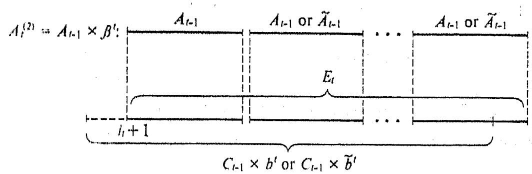

Figure 4

Remark that $E_t[1, n_{t-1}] = c_{t-1}^u c_{t-1}^v [i_t+1, i_t+n_{t-1}]$, where $u, v = 0, 1$. We shall show that

$$\begin{aligned} E_t[j\cdot n_{t-1}+1, (j+1)n_{t-1}] &= c_{t-1}^u c_{t-1}^{\square}[i_t+1, i_t+n_{t-1}] \quad \text{if} \quad \beta^t[j+1] = 0,\\ E_t[j\cdot n_{t-1}+1, (j+1)n_{t-1}] &= \tilde{c}_{t-1}^u c_{t-1}^{\square}[i_t+1, i_t+n_{t-1}] \quad \text{if} \quad \beta^t[j+1] = 1, \end{aligned} \tag{56}$$

where $0 \leqslant j \leqslant \lambda_t - 1$ and $\square$ denotes 0 or 1. Using (55) we have

$$d(A_{t-1}, E_t[1, n_{t-1}]) \leqslant \lambda_t\cdot d(A_t^{(2)}, E_t) < 2\cdot r\cdot\sqrt{\varepsilon}. \tag{57}$$

Now suppose $\beta^t[j+1] = 0$ and $E_t[j\cdot n_{t-1}+1, (j+1)n_{t-1}] = \tilde{c}_{t-1}^u c_{t-1}^{\square}[i_t+1, i_t+n_{t-1}]$. Then we should have

$$\begin{aligned} 1 - 3\cdot r\cdot\sqrt{\varepsilon} &< 1 - i_t/n_{t-1}\\ &\leqslant d(\underbrace{c_{t-1}^u c_{t-1}^v[i_t+1, i_t+n_{t-1}]}_{E_t[1, n_{t-1}]}, \underbrace{\tilde{c}_{t-1}^u c_{t-1}^{\square}[i_t+1, i_t+n_{t-1}]}_{E_t[j\cdot n_{t-1}+1, (j+1)\cdot n_{t-1}]})\\ &\leqslant d(E_t[1, n_{t-1}], A_{t-1}) + d(A_{t-1}, E_t[j\cdot n_{t-1}+1, (j+1)\cdot n_{t-1}])\\ &\leqslant d(A_{t-1}, E_t[j\cdot n_{t-1}+1, (j+1)\cdot n_{t-1}]) + 2\cdot r\cdot\sqrt{\varepsilon}, \end{aligned}$$

and therefore

$$d(A_t^{(2)}[j\cdot n_{t-1}+1, (j+1)\cdot n_{t-1}], E_t[j\cdot n_{t-1}+1, (j+1)\cdot n_{t-1}]) \geqslant 1 - 5\cdot r\cdot\sqrt{\varepsilon}.$$

The last inequality gives

$$d(A_t^{(2)}, E_t) \geqslant \frac{1}{\lambda_t} \cdot d(A_t^{(2)}[j\cdot n_{t-1}+1,\ (j+1)\cdot n_{t-1}], E_t[j\cdot n_{t-1}+1,\ (j+1)\cdot n_{t-1}])$$

$$\geqslant \frac{1}{r}\cdot(1-5\cdot r\cdot\sqrt{\varepsilon}).$$

what is a contradiction to the inequalities $d(E_t, A_t^{(2)}) < 2\cdot\sqrt{\varepsilon}$ and $\varepsilon < 1/49\cdot r^2$. Thus we obtain the first part of (56). In the same way we can show the second part of (56) Further we observe that (56) means the following

$$\beta^t[j] = 0 \quad \text{iff} \quad b^t[j] = u \ \ (\text{or } \tilde{b}^t[j] = u), \quad 1 \leqslant j \leqslant \lambda_t.$$

The last assertion it possible iff $b^t = \beta^t$, in view of the equality $\beta^t[1] = b^t[1] = 0$. By the similarly reasoning we may obtain $\beta^t = b^t$ if (54) holds and $t \geqslant t_0$. This completes the proof of the theorem.

Remark 3. Let $s_t, t = 0, 1, \ldots,$ be the greatest number $j \geqslant 0$ such that: either

$$b^t = 00\ldots0,\ b^{t+1} = 00\ldots0, \ldots, b^{t+j} = 00\ldots0,$$

or

$$b^t = 010\ldots10, \ldots, b^{t+s} = 010\ldots10,\ b^{t+s+1} = 01\ldots01, b^{t+s+2} = 00\ldots0,$$
$$b^{t+j} = 00\ldots0, \quad 0 \leqslant s \leqslant j,$$

or

$$b^t = 01\ldots01,\ b^{t+1} = 00\ldots0, \ldots, b^{t+j} = 00\ldots0.$$

If $b^t \neq 00\ldots0$ and $b^t \neq 010\ldots10$ and $b^t \neq 01\ldots01$, then we put $s_t = -1$. The assumption x is a continuous Morse sequence implies that s_t is well-defined for every $t \geqslant 0$. Using remarks (α), (β), (γ) considered in the proof of Theorem 4 we have

$$\delta_{t,s_t+1} \neq 00\ldots0, \qquad \delta_{t,s_t+1} \neq 0101\ldots010, \qquad \delta_{t,s_t+1} \neq 01\ldots01.$$

Assuming $\lambda_t \leqslant r$, $t \geqslant 0$, and applying (5) for x_t we obtain

$$p_t, q_t \geqslant 1/(2\cdot\lambda_t\ldots\lambda_{t+s_t+1}) \geqslant 1/(2\cdot r^{s_t+2}),$$

and

$$p_t \leqslant 1/2^{s_t+1} \quad \text{or} \quad q_t \leqslant 1/2^{s_t+1}.$$

The above inequalities imply that x is a regular Morse sequence iff $\{s_t\}$ is a bounded sequence.

KAKUTANI'S EXAMPLES. Putting $b^t = 00$ or $b^t = 01$, $t \geqslant 0$, we obtain a class of Morse sequences described in [3]. The assumption x is a continuous Morse sequence is equivalent to the fact that an infinity of such b^t is (01). Let $y = \beta^0 \times \beta^1 \times \ldots$, $\beta^t = 00$ or $\beta^t = 01$, $t \geqslant 0$, and let

$$\gamma_x = \tfrac{1}{2}\cdot \sum_{t=0}^{\infty} (b^t[2]/2^t), \qquad \gamma_y = \tfrac{1}{2}\cdot \sum_{t=0}^{\infty} (\beta^t[2]/2^t). \tag{58}$$

Kakutani has announced that Morse dynamical systems $\theta(x)$ and $\theta(y)$ are spectrally isomorphic iff $\gamma_x - \gamma_y$ is dyadic rational number. Since series (58) have an infinity of positive members, $\gamma_x - \gamma_y$ is a dyadic rational number iff $b^t = \beta^t$ for sufficiently large t. Now, using the sufficiency of Theorem 8, we obtain that the spectrally isomorphism is equivalent to metrically isomorphism in the class of Kakutani's examples.

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Received December 14, 1978
Revised version January 28, 1980 (1491)