related to, but simpler than the problem above, which we feel are interesting even without their relation to Ulam's conjecture.

4.1. *Approximate isometries and measure.* We move the problem of §3 to a more amenable environment, hoping that a solution to the new problem would help in the old situation.

CONJECTURE. There is a constant $c \geqslant 1$ such that if S is a subset of some $\boldsymbol{R}^n$, $g: S \to \boldsymbol{R}^n$ is a δ-isometry, and $T = \{x \in \boldsymbol{R}^n \mid d(x, g(S)) \leqslant c\delta\}$, then $\lambda_n(T) \geqslant \lambda_n(S)$.

4.2. *Neighborhoods in bricks.* Let B be a brick (rectangular parallelepiped) in $\boldsymbol{R}^n$, S and T isometric subsets of B. It would seem that not too much more of S than of T can be near the boundary of B. Precisely:

CONJECTURE. There is a $c_n > 0$ depending only on n such that for any $\delta > 0$

$$\lambda_n(\{x \in B \mid d(x, S) \leqslant \delta\}) \leqslant \lambda_n(\{x \in B \mid d(x, T) \leqslant c_n \delta\}).$$

In fact we guess that the best value for c_n is $1+\sqrt{n}$, corresponding to S a small corner of the brick B.

To see the connection between this conjecture and the problem described at the beginning of the section, let S' be a subset of the brick B and $g: S' \to B$ an approximate isometry. Let $f: S' \to \boldsymbol{R}^n$ be an isometry within, say, δ of g. Set $T = f(S') \cap B$ and $S = f^{-1}(T)$. Then the above conjecture implies that the $(1+c_n)\delta$ neighborhood of $g(S')$ in B has measure at least equal to the measure of S'.

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A characterization of BMO and BMO_ϱ

by

CARL MUELLER (Berkeley, Cal.)

Abstract. The purpose of this paper is to use probabilistic characterizations of BMO and BMO_ϱ to solve two problems. Our first result is a characterization of BMO_ϱ in terms of Carleson measures when ϱ is regular. This result was conjectured by Sarason [9], and is similar to Fefferman and Stein's characterization of BMO [2]. Secondly, we give a probabilistic proof of a criterion for BMO due to Hayman and Pommerenke [4].

1. Introduction. Probability has recently become an important tool in complex analysis. Using Brownian motion, Burkholder, Gundy, and Silverstein have characterized the H^p spaces in terms of maximal functions, thereby solving a longstanding problem of Hardy and Littlewood.

Currently, the BMO functions are receiving attention. Fefferman and Stein [2] have shown that BMO is the dual of H^1. An important step in their proof was the characterization of BMO in terms of Carleson measures. The BMO_ϱ spaces were introduced by Spanne [10] as generalizations of BMO. These have also aroused interest (see Sarason [9]).

The purpose of this paper is to use probabilistic characterizations of BMO and BMO_ϱ to solve two problems. Our first result is a characterization of BMO_ϱ in terms of Carleson measures when ϱ is regular. This result was conjectured by Sarason [9], and is similar to, Fefferman and Stein's characterization of BMO [2]. Secondly, we give a probabilistic proof of a criterion for BMO due to Hayman and Pommerenke [4].

Let D be the unit disc in the complex plane, and let I be a subarc of ∂D. We will consider holomorphic functions f in H^1, which are the Poisson integrals of their boundary values. Let

$$I(f) = \frac{1}{|I|} \int_I f(x)\,dx.$$

The space BMO consists of all functions f for which the BMO norm

$$\|f\|_* = \sup_{I \subseteq \partial D} I(|f - I(f)|)$$

is finite. If $\varrho(x)$ is an increasing function satisfying $\varrho(0) = 0$, then the space BMO_ϱ consists of all functions f for which

$$I(|f-I(f)|) = O(\varrho(|I|)).$$

Spanne [10] has shown that if $\varrho(x) = x^a$, $0 < a \leqslant 1$, then BMO_ϱ coincides with the space of functions which are Lipschitz continuous of order a.

Let f be holomorphic in D. Let $B(t)$ be Brownian motion in D, and let σ be the first exit time of $B(t)$ from D. A theorem of Paul Lévy states that $f(B(t))$ is Brownian motion with a new time scale $T_t = \int_0^t |f'(B(t))|^2 dt$. Let $\tilde{B}(t)$ be the motion obtained from this time change, and let $\tilde{\sigma}$ be the image of σ. That is, $\tilde{\sigma} = T_\sigma$. By an abuse of notation, $E^x\tilde{\sigma}$ will mean that $B(0) = x$ (so $\tilde{B}(0) = f(x)$).

$\varrho(x)$ is said to be *regular* if

$$\theta \int_0^\infty (\varrho(t)/t^2)\, dt = O(\varrho(\theta)).$$

For example, $\varrho(x) = x^a$, $0 < a < 1$ is regular.

The following two theorems are not too difficult, and their proofs will be omitted.

THEOREM 1. $f \in \mathrm{BMO}$ *iff*

$$\sup_{x\in D} E^x\tilde{\sigma} < \infty$$

and this supremum is comparable to $\|f\|_*^2$.

THEOREM 2. *Let* $\varrho(x)$ *be regular.* $f \in \mathrm{BMO}_\varrho$ *iff*

$$E^x\tilde{\sigma} = O(\varrho(1-|x|))^2.$$

These theorems were motivated by a theorem of Burkholder [1] stating that $f \in H^p$ iff

$$E^0\tilde{\sigma}^{p/2} < \infty.$$

For $I \subseteq \partial D$ let $R(I)$ be a square in polar coordinates with one edge coinciding with I. Thus,

$$R(I) = \{(r, \theta)\colon e^{i\theta} \in I,\ 1-|I|/2\pi \leqslant r \leqslant 1\}.$$

A measure μ on D is said to be a *Carleson measure* if

$$\mu(R(I)) = O(|I|).$$

Fefferman [2] showed that $f \in \mathrm{BMO}$ iff $d\mu = |f'(z)|^2(1-|z|)\,dx\,dy$ is a Carleson measure. In Section 2, we prove the following

THEOREM 3. *Let* $\varrho(x)$ *be regular. Then* $f \in \mathrm{BMO}$ *iff* $d\mu = |f'(z)|^2(1-|z|)\,dx\,dy$ *satisfies*

$$\frac{1}{|I|}\mu(R(I)) = O(\varrho(|I|)^2).$$

This theorem was conjectured by Sarason [9].

Probability often clarifies potential theory. In Section 3, we prove the following potential theoretic criterion for BMO, due to Hayman and Pommerenke [4].

THEOREM 4. *The domain* $G \subseteq C$ *has the property that every function* $f(z)$ *analytic in* D *with values in* G *belongs to* BMO, *iff there exist constants* R, $\delta > 0$ *such that for all* $\omega_0 \in G$,

$$\operatorname{cap}(G^c \cap \{|\omega-\omega_0| \leqslant R\}) > \delta.$$

2. Proof of Theorem 3. Let $G(x, y)$ be the Green's function for the disc. From potential theory [5], we know that $G(x, y)\,dy$ is the expected amount of time that Brownian motion starting at x spends in dy. Since $\tilde{B}(t)$ is obtained from $f(B(t))$ by the time change $T_t = \int_0^t |f'(s)|^2 ds$, we can write

$$E^x\tilde{\sigma} = \int_D |f'(y)|^2 G(x, y)\, dy.$$

Therefore, by Theorem 2, it suffices to show:

LEMMA. *Let* ϱ *be regular. Then*

$$\int_D |f'(y)|^2 G(x, y)\, dy = O(\varrho(1-|x|)^2) \tag{1}$$

iff

$$\frac{1}{|I|}\int_{R(I)} |f'(z)|^2(1-|z|)\, dz = O(\varrho(|I|)^2). \tag{2}$$

Proof of (1) ⇒ (2). This part of the lemma is true even without the assumption of regularity. The result is known (Sarason [9]), and we will omit the proof.

Proof of (2) ⇒ (1). There will be two steps. In Step 1, we will modify the Green's function $G(x, y)$. In Step 2, we will approximate the modified Green's function by the function $(1-|y|)$ on a union of Carleson rectangles.

Step 1. Note that for r close to 1,

$$G(x, re^{i\theta}) \approx \mathscr{P}(x, e^{i\theta})(1-r)$$

where $\mathscr{P}$ is the Poisson kernel. Here "$\approx$" means "comparable to". With each point $x \in D\setminus\{0\}$ we will associate an interval $I_x \subset \partial D$. Let I_x be the interval centered at $x/|x|$ with length $|I_x| = 2\pi(1-|x|)$. Let $\delta = (1/2)|I_x|$, and assume without loss that I_x is centered at 1. Let N be the largest integer such that $2^N - \delta < \pi$. Define the intervals I_n, $0 \leqslant n \leqslant N+1$ by

$$I_0 = I_x, \quad I_n = [-2^n \delta, 2^n \delta], \quad n \leqslant N, \quad I_N = \partial D.$$

In their proof of duality for BMO, Fefferman and Stein approximated $\mathscr{P}(x, e^{i\theta})$ by the indicator functions of the intervals I_n. We intend to approximate $G(x, y)$ by $(1-r)$ times the indicator functions of $R(I_n)$. This procedure will give the desired result, near the edge of D. In this step, we will "hollow out the center" of $G(x, y)$ using the following idea. Note that

(A) $$|f'(x)|^2 = |\int_{\partial D} (f'(y))^2 \mathscr{P}(x, y)\,dy| \leqslant \int_{\partial D} |f'(y)|^2 \mathscr{P}(x, y)\,dy.$$

So, in the integral $\int_D |f'(y)|^2 G(x, y)\,dy$ we can "project" $|f'(y)|^2 G(x, y)$ onto a region near the boundary.

Let m be a measure with support Q contained in a disc $D(c, r)$ with center c and radius r. Let $\mathscr{P}_{c,r}(x, y)$, $y \in \partial D(c, r)$ be the Poisson kernel with respect to $\partial D(c, r)$. The "projection" of m onto $\partial D(c, r)$ will mean the measure μ on $\partial D(c, r)$ defined by

$$\mu(dy) = dy \int_Q \mathscr{P}_{c,r}(x, y)\, m(dx).$$

"Hollowing out" Q means replacing m by its "projection". Often, we shall speak of "projecting" a function $f(x)$ and "hollowing out" its domain. In these cases, we are referring to the measure $f(x)\,dx$. Also, the "mass" of a function is its Lebesgue integral.

Let $p(dc, dr)$ be a probability measure with support on the set

$$\{(c, r):\ Q \subset D(c, r)\}.$$

Let $\mu_{c,r}$ be the "projection" of m onto $\partial D(c, r)$. The "projection" of m with respect to p will mean the measure μ satisfying

$$\mu = \iint \mu_{c,r}\, p(dc, dr).$$

Most of the time, both m and μ will have continuous densities $f(x)$ and $h(x)$, respectively. Then we shall speak of $h(x)$ as the "projection" of $f(x)$. As in the previous part of the proof, assume that x lies on the positive real axis, and

$$x = R, \quad y = re^{i\theta}.$$

Let $m_x(y)$ be as before. Now

$$\frac{\partial}{\partial r} G(x, y) = \frac{1}{|m_x(y)|} \frac{\partial}{\partial r} |m_x(y)| \leqslant \frac{1}{|m_x(y)|} |m_x'(y)| = \left| \frac{1 - x\bar{x}}{(y-x)(1-y\bar{x})} \right|$$
$$= \frac{1 - R^2}{\sqrt{(r^2 + R^2 - 2rR\cos\theta)(1 + r^2R^2 - 2rR\cos\theta)}}.$$

When $r = 1$, equality holds in the above formula. Fix $K > 1$. We wish to find a region in D for which

$$\left| \frac{\partial}{\partial r} G(x, re^{i\theta}) \right| \leqslant -K \frac{\partial}{\partial r} G(x, re^{i\theta}) \Big|_{r=1}.$$

That is, the radial derivative of G should be bounded by K times its value on the boundary. We will compare each of the expressions

(a) $$1 + r^2R^2 - 2rR\cos\theta,$$

(b) $$r^2 + R^2 - 2rR\cos\theta$$

with its value on the boundary. First, expression (a):

$$\begin{aligned} 1 + r^2R^2 - 2rR\cos\theta &= 1 + R^2 - 2R\cos\theta + R(1-r)\big(2\cos\theta - R(1+r)\big) \\ &\leqslant [(1-R)^2 + 2R(1-\cos\theta)] + \\ &\quad + [c_1(1-r)(1-R) + c_2(1-r)^2 + c_3(1-r)(1-\cos\theta)]. \end{aligned}$$

The order of magnitude of the second term is bounded by the order of magnitude of the first term if

$$(1-r)^2 \leqslant c\max\big(1-\cos\theta, (1-R)^2\big).$$

That is,

(*) $$1 - r < C\max(\theta, 1-R).$$

The argument for expression (b) is similar:

$$\begin{aligned} r^2 + R^2 - 2rR\cos\theta &= 1 + R^2 - 2R\cos\theta + (1-r)\big(2R\cos\theta - (1+r)\big) \\ &\leqslant [(1-R)^2 + 2R(1-\cos\theta)] + \\ &\quad + [c_1(1-r)(1-R) + c_2(1-r)^2 + c_3(1-r)(1-\cos\theta)]. \end{aligned}$$

This again leads to (*). Let Q be the region determined by (*). Next, we will find the regions $Q_n = m_{1-\delta}^{-1}\{x:\ |x| \leqslant 1 - 2^{-n}\}$. The boundary of Q_n is a circle, one of whose diameters has endpoints p_{n1}, p_{n2} on the real axis. We will estimate the location of these points. Let y be real, and

$$1 - y = \varepsilon, \quad 1 - x = \Delta.$$

Then

$$\begin{aligned} m_x(y) = (y-x)/(1-yx) &= (\Delta - \varepsilon)/(\Delta + \varepsilon - \Delta\varepsilon) \\ &= \begin{cases} 1 - (\varepsilon/\Delta)(2-\Delta) + O(\varepsilon/\Delta)^2 & \text{if } \varepsilon < \Delta, \\ -1 + (\Delta/\varepsilon)(2-\varepsilon) + O(\Delta/\varepsilon)^2 & \text{if } \varepsilon > \Delta. \end{cases} \end{aligned}$$

So, the endpoints in question are

$$p_{n1} = 1 - 2^{-(n+1)}\delta + O\big(2^{-n}\delta(\delta + 2^{-n})\big),$$
$$p_{n2} = 1 - 2^{n+1}\delta + O\big(2^n\delta(\delta + 2^{-n})\big).$$

Let S_n be the circle with center $1-2^n\delta$ and radius $2^{n-1}\delta$. Then, for n large enough, $S_n \subset Q_n$. Note that for C large enough, $\bigcup_{n=1}^{N} S_n$ covers $R^c = \{re^{i\theta}: 1-r \geqslant C\max(1-R, \theta)\}$. Let M be a constant to be specified later. Now, for $y \notin Q_M$

$$G(x, y) < K_M \sum_{n=M}^{\infty} 2^{-n} l(y \in Q_n).$$

Since the S_n cover Q, by changing K_M, we find, for $y \notin S_M$,

$$G(x, y) < K_M \sum_{n=M}^{\infty} 2^{-n} l(y \in S_n \cap R).$$

We wish to use inequality (A) to "project" $2^{-n} l(y \in S_n)$ onto $D \setminus S_n$. Let s_n be the center of S_n. Now, $2^{-n} l(y \in S_n)$ has mass $C2^n\delta^2$. We claim that the projection of $2^{-n} l(y \in S_n)$ is the same as if the mass were concentrated at s_n.

By symmetry, both $2^{-n} l(y \in S_n)$ and the mass concentrated at s_n have the same projection onto ∂S_n. The projection onto a larger circle is the same if we first project onto ∂S_n. This fact holds because a projection is the hitting density of Brownian motion. If τ_1 is the first hitting time of ∂S_n and τ_2 is the first hitting time of the larger circle,

$$P\{B(\tau_2) \in dz\} = \int_{x \in \partial S_n} P\{B(\tau_1) \in dx\} P^x\{B(\tau_2) \in dz\}.$$

Let $\varepsilon > 0$ be a small constant which we will specify later. For $0 < u < \varepsilon$, let $D_u = \{|x| = 1-u\}$, and "project" $(2/\varepsilon^2)u\,du$ of the mass onto the circles $m_{s_n}^{-1}(D_t)$: $t \in (u, u+du)$. Since $m_{s_n}^{-1}$ is a conformal transformation, the density of mass will be, for $u < \varepsilon$, comparable to $(u/\varepsilon^2)|m'_{s_n}(y)|^2\, 2^n\delta^2$. Now, in the region

$$Q_{s_n} = \{re^{i\theta}: 1-r < C\max(1-|s_n|, \theta)\}$$

we have seen that $|m'_{s_n}(re^{i\theta})|$ and $|m'_{s_n}(e^{i\theta})|$ are comparable. Since the latter equals $\mathscr{P}(s_n, e^{i\theta})$, the density of mass is comparable to $(1-r) \times \times \mathscr{P}(s_n, e^{i\theta})^3 2^n \delta^2$ for $re^{i\theta} \notin Q_{s_n}$.

Note that "hollowing out" S_n puts some mass onto S_{n+m}, $m > 1$. We will show that this fact is of no consequency.

Let $y = re^{i\theta}$, $1-r = \Delta$. We know that $1 - s_n = 2^n\delta$. Now

$$|m'_{s_n}(y)| = \frac{1-R_n^2}{(1-rR_n)^2 + 2rR_n(1-\cos\theta)} \leqslant (2^n\delta/\Delta^2)\big(1 + O(2^n\delta/\Delta)\big).$$

Now, if $y \in S_{n+m}$, then $\Delta \approx 2^{m+n}\delta$ and

$$|m'_{s_n}(y)| \leqslant C/2^{2m+n}\delta.$$

Therefore, the density of mass in S_{n+m}, for m large, is less than

$$Cu/\varepsilon^2 2^{3m+n}.$$

(We must use a uniform bound in S_{n+m}, so that the mass can be concentrated at s_{n+m}.) Since S_{n+m} has area $C2^{2(n+m)}\delta^2$, the total mass transferred to S_{n+m} is less than

$$Cu2^n\delta^2/\varepsilon^2 2^m.$$

Now for large m, u is comparable to 2^{-m}. If $\varepsilon \approx C2^{-M}$, then no mass will be transferred to S_{n+m} for $m < M$. So, the above quantity is less than

$$(C2^{-m})(2^{-3(m-M)})(2^{n+m}\delta^2) = K2^{-3(m-M)}(\text{mass of } S_{n+m}).$$

Choose M so large that $(2^{-3M}/K)^{1/M} \leqslant 1/2$ and $K < e^{-M/2}$. Suppose that $(r-1)M < m < rM$. We wish to estimate the total mass added to S_{n+m} from S_n. Such mass could reach S_{n+m} directly, or after first landing in j intermediate circles. The mass arising from the latter case is less than

$$\big(m^{(r-j)M}/[(r-j)M]!\big)K^j 2^{-3(m-jM)}(\text{mass of } S_{n+m}).$$

Summing over j, the above is bounded by

$$C2^M K^{m/M}\exp\big(m(2^{-3M}/K)^{1/M}\big)(\text{mass of } S_{n+m}) \leqslant C(Ke^{M/2})^{m/M}(\text{mass of } S_{n+m}).$$

When summed over m, the above converges. This establishes the claim that "hollowing out" S_n does not put too much mass on the larger circles.

Step 2. We will now show that the "hollowed out" Green's function is bounded by

$$\text{(B)} \qquad C\sum_{n=1}^{N} \frac{(1-|y|)}{|I_n|}\, l\big(y \in R(I_n)\big)$$

where $I_n = \{\theta: -2^n\delta < \theta/2\pi < 2^n\delta\}$, as in Section 3.

Note that for $m \geqslant n$, there exists a constant K such that

$$Q_{s_n}^c \cap \{re^{i\theta}: e^{i\theta} \in KI_m\} \subset R(I_m).$$

For each interval I_n, all the circles S_m with $m > n$ will contribute to the modified Green's function on I_n. It suffices to consider the endpoints $-2^n\delta/2\pi$, $2^n\delta/2\pi$. Using the approximation of (B) by $\mathscr{P}(x, e^{i\theta})(1-r)$, we need to show that if $\theta_n = 2^n\delta/2\pi$, then

$$\sum_{k=0}^{n} \mathscr{P}(s_k, e^{i\theta_n})^3 2^k\delta^2 \leqslant \mathscr{P}(x, e^{i\theta_n}).$$

Now

$$\mathscr{P}(x, e^{i\theta_n}) \approx \delta/(\delta^2 + \theta_n^2),$$

$$\mathscr{P}(s_k, e^{i\theta_n}) \approx 2^k\delta/(2^{2k}\delta^2 + \theta_n^2) \leqslant 2^k \mathscr{P}(x, e^{i\theta_n}).$$

Also

$$\mathscr{P}(s_k, e^{i\theta_n}) \leqslant C2^k\delta/2^{2n}\delta^2 = 2^k/2^{2n}\delta.$$

So,

$$\sum_{k=0}^{n} |m'_{s_k}(e^{i\theta_n})|^2 2^k\delta^2 \leqslant C\mathscr{P}(x, e^{i\theta_n}) \sum_{k=0}^{n} 2^k(2^k/2^{2n}\delta)^2 2^k\delta^2 \leqslant C\mathscr{P}(x, e^{i\theta_n}).$$

Thus, we have shown that

$$\int_D |f'(y)|^2 G(x, y)\,dy \leqslant C\frac{1}{|I_x|}\int_{R(I_x)} |f'(z)|^2(1-|z|)\,dz.$$

This proves the lemma, and thus establishes the theorem.

3. Proof of Theorem 4. Given a kernel $K(x, y)$, the equilibrium measure of a region Q is the measure $m(dx)$, supported on Q, which has the greatest mass subject to the condition

$$\int_Q K(x, y)m(dy) \leqslant 1, \quad \text{all } x \in Q.$$

Let $D(\omega_0, R)$ be the disc of radius R with center ω_0, and let $E = G^c \cap D(\omega_0, R)$. Let $m(dx)$ be the equilibrium measure of E with respect to the logarithmic kernel $\log|1/(x-y)|$, and let $\hat{m}(dy)$ be the equilibrium measure of E with respect to

$$G(x, y) = (1/2\pi)\log|(2R - \bar{x}y/2R)/(x-y)|.$$

$G(x, y)$ is the Green's function of the region $D(\omega_0, R)$. Note that for $x, y \in D(\omega_0, R)$,

$$(*) \qquad (1/2\pi)\log\big((3/2)R\big) \leqslant G(x, y) - (1/2\pi)\log|1/(x-y)| \leqslant (1/2\pi)\log 2R.$$

Recall that

$$\operatorname{cap}(E) = \exp\big(-m(E)\big).$$

Suppose that $f(D) \subset G$ and that for all $\omega_0 \in G$,

$$\operatorname{cap}(E) > \delta.$$

Set

$$\mu(dx) = m(dx)/m(E), \qquad \hat{\mu}(dx) = \hat{m}(dx)/\hat{m}(E).$$

It is a standard fact that μ and $\hat{\mu}$ are the measures for which $\sup\limits_{x\in E}\sup\limits_{\nu(E)=1}\int K(x, y)\nu(dy)$ is smallest when $K(x, y)$ is the appropriate kernel. Using this fact and inequality $(*)$,

$$\begin{aligned} -\log\delta > 1/m(E) &= \sup_{x\in E}\int_E \log|1/(x-y)|\,\mu(dy) \\ &\geqslant \sup_{x\in E}\int_E \big(2\pi G(x, y) - \log 2R\big)\mu(dy) \\ &\geqslant \sup_{x\in E}\int_E 2\pi G(x, y)\hat{\mu}(dy) - \log 2R = \big(2\pi/\hat{m}(E)\big) - \log 2R. \end{aligned}$$

So,

$$\hat{m}(E) \geqslant 2\pi/\log(2R/\delta).$$

From probabilistic potential theory (see Hunt [5]), we know that

$$P^{\omega_0}\{B(t) \text{ hits } E \text{ before hitting } \partial D(\omega_0, 2R)\} = \int_E G(\omega_0, y)\hat{m}(dy).$$

Therefore, it follows that

$$\begin{aligned} P^{\omega_0}\{B(t) \text{ hits } G^c \text{ before hitting } \partial D(\omega_0, 2R)\} &\geqslant \int_E G(\omega_0, y)\hat{m}(dy) \\ &\geqslant (1/2\pi)\log 2\hat{m}(E) \geqslant \log 2/\log(2R/\delta). \end{aligned}$$

Let C_1 be the minimum of the last quantity and 1. The expected exit time from $D(\omega_0, R)$, for $B(0) = \omega_0$, is C_2R^2 for some constant C_2, since $B(xt)$ and $\sqrt{x}B(t)$ have the same distribution.

Suppose that $B(0) = x_0$. Let x_1 be the first exit point of $B(t)$ from $D(x_0, 2R)$. Define x_n by induction. That is, x_{n+1} is the first exit point of $B(t)$ from $D(x_n, 2R)$. Then, if $\tilde{\sigma}$ is as before,

$$\begin{aligned} E^{x_0}\tilde{\sigma} &\leqslant \big(\text{expected exit time from } D(x_0, 2R)\big) \times \\ &\quad \times (\text{expected number of points } x_n \text{ which are hit before } \sigma) \\ &\leqslant C_2R^2\Big(\sum_{n=0}^{\infty}(1-C_1)^n\Big) = C_2R^2/C_1. \end{aligned}$$

So, $f \in \mathrm{BMO}$. Conversely, suppose that we can find sequences $R_n \nearrow \infty$, $\delta_n \searrow 0$ and $\omega_n \in G$ such that

$$\operatorname{cap}\big(G^c \cap D(\omega_n, R_n)\big) = \delta_n.$$

We wish to find $x_n \in \partial D(\omega_n, R_{n/2})$ such that for δ_n/R_n sufficiently small,

$$\text{(A)} \qquad P^{x_n}\{\text{hit } G^c \text{ before } \partial D(\omega_n, 2R_n)\} \leqslant C/\log(3R_n/2\delta_n).$$

It suffices to show that if $E = G^c \cap D(\omega_n, R_n)$ and if μ is uniform on $\partial D\,(\omega_n, (1/2)R_n)$,

$$P^{\mu}\{\text{hit } E \text{ before } \partial D(\omega_n, 2R_n)\} \leqslant C\log(3R_n/2\delta_n).$$

Let $G(x, y)$ be the Green's function for $D(\omega_n, 2R_n)$. Let m and $\hat{m}$ be as before. Also as before, we deduce from $(*)$ that

$$\hat{m}(E) \leqslant 2\pi/\log(3R_n/2\delta_n).$$

Let ν be uniform on $\partial D\big(\omega_n, (3/2)R_n\big)$. Then

$$\begin{aligned} P^{\nu}\{\text{hit } E \text{ before } \partial D(\omega_n, 2R_n)\} &\leqslant (1/2\pi)\log\big|1/4/\big(1-(1/2)\cdot(1/4)\big)\big|\hat{m}(E) \\ &\leqslant C/\log(3R_n/2\delta_n) = \varepsilon, \quad \text{say.} \end{aligned}$$

Thus, if τ is the first hitting time of $D(\omega_n, \frac{1}{2}R_n)$, and if

$$G = \{B(t) \text{ hits } E \text{ before } \partial D(\omega_n, 2R_n)\},$$

$$H = \{B(t) \text{ hits } E \text{ or } \partial D(\omega_n, 2R_n) \text{ before } \tau\},$$

then

$$\varepsilon \geqslant E^\nu[P^{B(\tau)}(G)l(H^c)] \geqslant E^\nu[P^{B(\tau)}(G)] - E^\nu[l(H)] = P^\mu(G) - \varepsilon.$$

Thus, inequality (A) is established and the required point x_n exists. Since $D(x_n, R_n/2) \subset D(\omega_n, R_n)$,

$$P^{x_n}\{\text{hit } G^c \text{ before } \partial D(x_n, R_n/2)\} \leqslant C/\log(3R_n/2\delta_n).$$

Let τ be the first exit time of Brownian motion from $D(x_n, R_n/2)$ starting from x_n. Suppose that n is so large that

$$C/\log(3R_n/2\delta_n) < 1/2.$$

Let $F(x)$ be the cumulative distribution function for τ. Let

$$k(n) = \int_{\{F(x)<1/2\}} (1-F(x))\,dx.$$

By the scaling properties of Brownian motion, $k(n) = c\sqrt{R_n}$, $c > 0$. Now $k(n)$ is the expectation of the smallest values of τ, so

$$E^{x_n}\tilde{\sigma} \geqslant E^{x_n}\tau l\{\text{hit } \partial D(x_n, R_n/2) \text{ before } G^c\} \geqslant k(R_n) = c\sqrt{R_n}.$$

Thus, $E^{x_n}\tilde{\sigma}$ is not bounded. If $F(z)$ maps D onto the covering space of G, then $F(z)$ is not in BMO.

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