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A calculus of symbols and convolution semigroups on the Heisenberg group

by

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Abstract. The main result of this paper is the following:

THEOREM. *Let $a \in C^\infty(\mathbf{R}^N \times \mathbf{R}^N)$ be a function of polynomial growth satisfying*

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < C_{\alpha,\beta}(1+|x|)^{-\delta|\alpha|}(1+|\xi|)^{-\varrho|\beta|}|a(x, \xi)|$$

and

$$|a(x, \xi)| > C(1+|x|)^K(1+|\xi|)^k$$

for some $K, k, \delta, \varrho > 0$. Then the spectrum of the operator

$$Au(x) = \int e^{ix\xi} a(x, \xi)\,\hat{u}(\xi)\,d\xi$$

is either the whole plane $\mathbf{C}$ or it is discrete and

$$\sum_{0 \neq \lambda \in \mathrm{Sp}A} |\lambda|^{-p} < \infty$$

for $p > 1$ and $p > \max(1/K, 1/k)$.

As an application we obtain that the semigroup of measures generated by the operator $X^{(r)} + Y^{(s)}$ on the Heisenberg group consists of absolutely continuous measures with square integrable densities.

Introduction. Let G be the 3-dimensional Heisenberg group and $\{X, Y, Z\}$ a basis of its right-invariant Lie algebra such that $[X, Y] = Z$ and Z is central. Let $(\mu_t)_{t>0}$ and $(\nu_t)_{t>0}$ be the semigroups of measures generated by X^2 and Y^2, respectively. Consider a sublaplacian

$$\mathscr{L} = X^2 + Y^2$$

and $(m_t)_{t>0}$, the semigroup of measures generated by $\mathscr{L}$. By the Hörmander theorem ([8]), $\mathscr{L} - \partial/\partial t$ is hypoelliptic and this implies that $m_t = p_t dx$, where $p_t \in C^\infty(G)$ for $t > 0$ and thus $p_t \in L^1(G)$ (see [5], 3.1, Theorem). In this paper we consider the following generalization of $\mathscr{L}$: For real numbers $0 < r, s \leqslant 2$, let us write

$$X^{(r)} f = \int_0^\infty t^{-1-r/2}(\mu_t * f - f)\,dt,$$

$$Y^{(s)} f = \int_0^\infty t^{-1-s/2}(\nu_t * f - f)\,dt$$

for compactly supported $f \in C^\infty(G)$. Now set

$$\mathscr{L}_{r,s} = X^{(r)} + Y^{(s)}.$$

Note that $\mathscr{L}_{r,s}$ is no longer a differential operator unless $r = s = 2$ and $\mathscr{L}_{2,2} = \mathscr{L}$. However, $\mathscr{L}_{r,s}$ is an infinitesimal generator of a semigroup of measures $(m_t^{r,s})_{t>0}$. The operator $\mathscr{L}_{2,s}$, $0 < s \leqslant 2$, was investigated by A. Hulanicki in [11] and it was announced there that $m_t^{2,s} = p_t^{2,s}dx$, where $p_t^{2,s} \in L^2(G) \cap L^1(G)$. We extend this result to the case of arbitrary $0 < r, s \leqslant 2$. As a consequence one obtains that the spectra of $\mathscr{L}_{r,s}$ on different $L^p(G)$, $1 \leqslant p < \infty$, coincide. Another corollary due to A. Hulanicki is that the semigroup generated by $\mathscr{L}_{r,s}$ on $L^1(G)$ is holomorphic, but this is beyond the scope of the present paper. The main tool we use is a kind of calculus of symbols for certain classes of operators on $\mathscr{S}(\boldsymbol{R}^N)$. We consider a class of symbols which are different from "pseudodifferential symbols" of e.g. [1]. Namely, we treat variables x, ξ separately and symmetrically as our symbols satisfy

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant C_{\alpha,\beta}\langle x\rangle^{K-|\alpha|\delta}\langle \xi\rangle^{k-|\beta|\varrho}, \quad 0 \leqslant \delta,\ \varrho \leqslant 1.$$

In a sense the most similar (but more restrictive than ours) is a class of pseudodifferential operators with symbols satisfying (cf. [6], [17]):

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant C_{\alpha,\beta}\langle w\rangle^{m-(|\alpha|+|\beta|)\delta}, \quad 0 \leqslant \delta \leqslant 1,$$

where $w = (x, \xi)$. Nevertheless the dependence of our calculus on the existent ones is apparent.

The main result of this paper, which enables us to prove the above statements, is the following:

THEOREM. *Let $a \in C^\infty(\boldsymbol{R}^N \times (\boldsymbol{R}^N)^*)$ be a function of polynomial growth satisfying*

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant C_{\alpha,\beta}\langle x\rangle^{-\delta|\alpha|}\langle \xi\rangle^{-\varrho|\beta|}|a(x, \xi)|$$

and

$$|a(x, \xi)| \geqslant C\langle x\rangle^{K_0}\langle \xi\rangle^{k_0}$$

for some K_0, k_0, δ, $\varrho > 0$. Then the spectrum of the operator

$$Au(x) = \int e^{ix\xi} a(x, \xi)\hat{u}(\xi)\,d\xi, \quad u \in \mathscr{S}(X),$$

is either the whole plane $\boldsymbol{C}$ or it is discrete and

$$\sum_{0 \neq \lambda \in \operatorname{Sp} A} |\lambda|^{-p} < \infty$$

for $p \geqslant 1$ and $p > \max(1/K_0, 1/k_0)$.

This paper is substantially influenced by [6]. We adopt from [6] the point of view on operators on $\mathscr{S}(\boldsymbol{R}^N)$ as coming from the Heisenberg group. Also the idea of the proof of Theorem 3.1 is essentially presented in [6].

Notation. In this paper we apply the following notation. By $\boldsymbol{C}$, $\boldsymbol{R}$ and $\boldsymbol{Z}$ we denote complex, real and integer numbers, respectively. Let $N \in \boldsymbol{Z}$, $N \geqslant 1$. For a vector $x = (x_1, \ldots, x_N) \in \boldsymbol{R}^N$, let us denote:

$$|x| = \text{Euclidean norm of } X, \quad \langle x\rangle = (|x|^2+1)^{1/2},$$

$$x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_N^{\alpha_N} \quad \text{for} \quad \alpha = (\alpha_1, \ldots, \alpha_N) \in \boldsymbol{Z}^N.$$

We shall consider the following function and distribution spaces on $\boldsymbol{R}^N$ (or on a group):

$\mathscr{K}(\boldsymbol{R}^N)$ = continuous functions with compact support;
$\mathscr{C}_0(\boldsymbol{R}^N)$ = continuous functions vanishing at infinity;
$\mathscr{C}^\infty(\boldsymbol{R}^N)$ = infinitely differentiable functions;
$\mathscr{D}(\boldsymbol{R}^N)$ = infinitely differentiable functions with compact support;
$\mathscr{S}(\boldsymbol{R}^N)$ = Schwartz functions;
$L^p(\boldsymbol{R}^N)$ = measurable functions integrable with pth power with respect to Lebesgue measure $1 \leqslant p < \infty$;
$L^\infty(\boldsymbol{R}^N)$ = measurable, essentially bounded functions;
$M(\boldsymbol{R}^N)$ = $\boldsymbol{C}_0^*(\boldsymbol{R}^N)$ = bounded measures;
$\mathscr{S}^*(\boldsymbol{R}^N)$ = temperate distributions.

All these spaces are Banach (or Fréchet) spaces with usual norms (or locally convex topologies).

For $x \in \boldsymbol{R}^N$, $\xi \in (\boldsymbol{R}^N)^*$, denote by ξx the action of ξ on x. Next, denote by dx a Lebesgue measure on $\boldsymbol{R}^N$, which is self-dual with respect to group duality $(x, \xi) \to e^{ix\xi}$. The Fourier transforms for $f \in \mathscr{S}(\boldsymbol{R}^N)$ are denoted by

$$\mathscr{F}f(\xi) = \hat{f}(\xi) = \int e^{-i\xi x} f(x)\,dx, \quad \xi \in (\boldsymbol{R}^N)^*,$$

$$\overline{\mathscr{F}}f(\xi) = \check{f}(\xi) = \int e^{i\xi x} f(x)\,dx, \quad \xi \in (\boldsymbol{R}^N)^*,$$

$$\mathscr{F}_j f(\xi) = \int e^{-i\xi_j x_j} f(x)\,dx_j, \quad \xi = (\xi_1, \ldots, \xi_N) \in (\boldsymbol{R}^N)^*,$$

$$\overline{\mathscr{F}}_j f(\xi) = \int e^{i\xi_j x_j} f(x)\,dx_j, \quad \xi = (\xi_1, \ldots, \xi_N) \in (\boldsymbol{R}^N)^*.$$

The Fourier transforms extend by duality to the whole $\mathscr{S}^*(\boldsymbol{R}^N)$.

Let E, F be locally convex topological vector spaces. The space of all continuous linear operators from E to F is denoted by $\mathscr{L}(E, F)$. If $E = F$, we shall write $\mathscr{L}(E, E) = \mathscr{L}(E)$. Suppose now E to be a Banach space and L a densely defined, closeable operator on E. Then by $\mathscr{D}_E(L) = \mathscr{D}(L)$, $\operatorname{Sp} L = \operatorname{Sp}_E L$ and $\varrho(L) = \varrho_E(L)$ we denote a domain of L, its spectrum and its resolvent set, respectively. For $\lambda \in \varrho(L)$ by $R(L, \lambda)$ we denote the bounded inverse of $\lambda - L$.

We shall use multi-indices, i.e. N-tuples of non-negative integers $\alpha, \beta, \gamma, \ldots$ If $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\beta = (\beta_1, \ldots, \beta_N)$ are multi-indices, then

$$|\alpha| = \sum_{j=1}^{N} \alpha_j, \quad \alpha! = \prod_{j=1}^{N} \alpha_j!, \quad \binom{\alpha}{\beta} = \prod_{j=1}^{N} \binom{\alpha_j}{\beta_j},$$

$$\alpha \leqslant \beta \quad \text{if} \quad \alpha_j \leqslant \beta_j, \quad j = 1, \ldots, N,$$

$$\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_N - \beta_N), \quad \beta_j \leqslant \alpha_j, \quad j = 1, \ldots, N.$$

Differentiations will be denoted by:

$$\partial_x^\alpha \partial_\xi^\beta f = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N} \partial_{\xi_1}^{\beta_1} \ldots \partial_{\xi_N}^{\beta_N} f, \quad f_{(\alpha)}^{(\beta)} = (-i\partial_x)^\alpha \partial_\xi^\beta f$$

for multi-indices α, β and $f \in C^\infty(\boldsymbol{R}^N \times (\boldsymbol{R}^N)^*)$. If $P = \sum c_\alpha x^\alpha$ is a polynomial in N variables, then $P(D)$ denotes the differential operator: $P(D)f = \sum c_\alpha(-i\partial_x)f$. We shall refer to the Leibniz formula

$$(f \cdot g)_{(\beta)}^{(\alpha)} = \sum_{\substack{0 \leqslant \iota \leqslant \alpha \\ 0 \leqslant \varkappa \leqslant \beta}} \binom{\alpha}{\iota}\binom{\beta}{\varkappa} f_{(\varkappa)}^{(\iota)} g_{(\beta-\varkappa)}^{(\alpha-\iota)}$$

for $f, g \in C^\infty(\boldsymbol{R}^N \times (\boldsymbol{R}^N)^*)$.

1. The Heisenberg group and operators on $\mathscr{S}(R^N)$. In this section we give a review of some basic facts concerning the Heisenberg group. If there is no explicit reference the reader is referred to [6] for the details.

Let X be N-dimensional real vector space and X^* its dual. Then $X \times X^* \times \boldsymbol{R}$ may be considered as an underlying manifold of $(2N+1)$-dimensional Heisenberg group G. Let us choose coordinates in G in such a way that the group law is given by

$$(1.1) \qquad (x, \xi, t)(y, \eta, s) = (x+y, \xi+\eta, t+s-\eta x),$$

where $x, y \in X$; $\xi, \eta \in X^*$; $t, s \in \boldsymbol{R}$. The choice of coordinates automatically gives us a standard basis in $\mathfrak{G}$, a right invariant Lie algebra of G. Set

$$(1.2) \qquad X_i f = \frac{\partial}{\partial x_j} f(0), \quad \Xi_i f = \frac{\partial}{\partial \xi_j} f(0), \quad Ef = \frac{\partial}{\partial t} f(0), \quad i = 1, \ldots, N,$$

for $f \in C^\infty(G)$. Then the differential operators $\{L_{x_i}, L_{\Xi_i}, L_E\}_{i=1}^N$, where L_T denotes the operator of left convolution with T, form a basis for $\mathfrak{G}$ and the rules of commutation are

$$[L_{X_i}, L_{\Xi_j}] = \delta_{ij} L_E, \quad [L_{X_i}, L_E] = [L_{\Xi_j}, L_E] = 0,$$

where $1 \leqslant i, j \leqslant N$ and δ_{ij} denotes the Kronecker delta. G is a 2-step

nilpotent, connected, simply connected Lie group. It is amenable and its group algebra $L^1(G)$ is symmetric ([13], Satz 5), which implies

$$(1.3) \qquad \nu(f) = \|\sigma_f\|_{\mathscr{L}(L^2(G))} \quad \text{for} \quad f \in L^1(G),$$

where $\nu(f)$ is a spectral radius of f in a Banach algebra $L^1(G)$ and σ is the left regular representation of G on $L^2(G)$ (see [16]). The Hilbert space $L^2(G)$ admits the decomposition

$$L^2(G) = \oplus \int_{\boldsymbol{R}\setminus\{0\}} L^2(W_\lambda) d\lambda,$$

where $W_\lambda = W = X \times X^*$ for $\lambda \in \boldsymbol{R}\setminus\{0\}$. We have also

$$\sigma = \oplus\int \sigma^\lambda d\lambda$$

where σ^λ acts on $L^2(W_\lambda)$ by

$$(1.4) \qquad \sigma^\lambda_{(y,\eta,t)} f(x, \xi) = e^{i\lambda t} e^{i\lambda y(\eta-\xi)} f(x-y, \xi-\eta)$$

for $f \in L^2(W_\lambda) = L^2(W)$. For each $\lambda \in \boldsymbol{R}\setminus\{0\}$ we have a natural "projection" of $\mathscr{S}(G)$ onto $\mathscr{S}(W)$:

$$(1.5) \qquad s^\lambda(f)(x, \xi) = \overline{\mathscr{F}}_{2N+1} f(x, \xi, \lambda), \quad f \in \mathscr{S}(G).$$

As s^λ maps $\mathscr{S}(G)$ onto $\mathscr{S}(W)$ it transports the convolution and the involution from $\mathscr{S}(G)$ to $\mathscr{S}(W)$. Denote by $f \overset{\lambda}{\#} g$ and $f^{\overset{\lambda}{\#}}$ this new "convolution" and new "involution" in $\mathscr{S}(W)$. It is not hard to find explicit formulas:

$$(1.6) \qquad f \overset{\lambda}{\#} g(x, \xi) = \iint g(x-y, \xi-\eta) f(y, \eta) e^{i\lambda y(\eta-\xi)} dy d\eta$$
$$= \iint f(x-y, \xi-\eta) g(y, \eta) e^{i\lambda\eta(y-x)} dy d\eta,$$

$$(1.7) \qquad f^{\overset{\lambda}{\#}}(x, \xi) = e^{i\xi x} \overline{f(-x, -\xi)},$$

for $f, g \in \mathscr{S}(W)$. Of course, formulas (1.5), (1.6), (1.7) apply to much more general classes of functions and distributions. For example, we may convolve L^1-functions with L^p-functions (the result being an L^p-function) for $1 \leqslant p \leqslant \infty$. In particular, $L^1(W)$ is an algebra with respect to $\overset{\lambda}{\#}$. What is more interesting also $L^2(W)$ is closed under the convolution $\overset{\lambda}{\#}$. Moreover, we may convolve temperate distributions with Schwartz functions and with compactly supported distributions, and so on. Also (1.7) extends by duality to all temperate distributions on W. From (1.5) and (1.7) we obtain

$$(1.8) \qquad \sigma_f^\lambda \cdot g = s^\lambda(f) \overset{\lambda}{\#} g \quad \text{for} \quad f \in L^1(G), \ g \in L^2(W).$$

Irreducible unitary representations of G are either one-dimensional or, in coordinates (1.1), have the form:

$$\pi^\lambda_{(x,\xi,t)}\varphi(u) = e^{i\lambda t}e^{i\lambda\xi u}\varphi(u-x), \tag{1.9}$$

where $\varphi \in L^2(X)$ is the Hilbert space of π^λ and $\lambda \in \boldsymbol{R}\setminus\{0\}$. Integrating (1.9), we obtain for $f \in L^1(G)$

$$\pi^\lambda_f\cdot\varphi(u) = \iint s^\lambda(f)(x,\xi)\, e^{i\lambda\xi u}\varphi(u-x)\,dx\,d\xi$$

and thus $\pi^\lambda_f = \pi^\lambda_g$ if $s^\lambda(f) = s^\lambda(g)$. Hence the following diagram is commutative:

$$\begin{array}{ccc} L^1(G) & \xrightarrow{\ \pi^\lambda\ } & \mathscr{L}(L^2(X)) \\ & {\scriptstyle s^\lambda}\searrow \qquad \nearrow{\scriptstyle \pi_\lambda} & \\ & L^1(W) & \end{array} \tag{1.10}$$

where $\pi_\lambda(f)\cdot\varphi(u) = \iint f(x,\xi)\,e^{i\lambda\xi u}\varphi(u-x)\,dx\,d\xi$. Note that π_λ is a representation of the algebra $\big(L^1(W), \overset{\lambda}{\#}\big)$. As (1.8) shows, s^λ is closely related to σ^λ, which itself is related to π^λ by

$$\sigma^\lambda = \bigoplus_{n=1}^{\infty} \pi^\lambda, \tag{1.11}$$

that is to say, σ^λ is a direct sum of a countable many copies of π^λ.

If $f \in L^1(W)$, then a direct computation shows that $\pi_\lambda(f)$ is an operator on $L^2(X)$ with an integral kernel. More exactly:

$$\pi_\lambda(f)\cdot\varphi(x) = \int k_\lambda(f)(x-y, x)\varphi(y)\,dy,$$

where $\varphi \in L^2(X)$ and

$$k_\lambda(f)(x, y) = \overline{\mathscr{F}}_\xi f(x, \lambda y). \tag{1.12}$$

By application of the Plancherel theorem we deduce from (1.12) the following important facts:

(1.13) $\mathscr{S}(W)$ is isomorphic by π_λ to the space of all integral operators on X, which have kernels in $\mathscr{S}(X\times X)$. (We shall call such operators *smoothing*.)

(1.14) $L^2(W)$ is isometric by an extension of π_λ to the space of Hilbert–Schmidt operators on $L^2(X)$.

(1.15) $\mathscr{S}^*(W)$ is isomorphic by an extension of π_λ to the space of all the operators on X, which have kernels in $\mathscr{S}^*(X\times X)$, that is (in view of the Schwartz kernel theorem [14]), with all the continuous linear operators from $\mathscr{S}(X)$ to $\mathscr{S}^*(X)$.

Let us return to $L^1(W)$. For $\lambda \in \boldsymbol{R}\setminus\{0\}$ it is a Banach $*$-algebra with respect to convolution $f \overset{\lambda}{\#} g$, invoultion $f^{\overset{\lambda}{\#}}$ and L^1-norm. Sometimes it is called a ***Weyl algebra***. By [12], Proposition 2, $L^1(W)$ is simple and symmetric. Its only irreducible unitary representation is π_λ (see (1.10)). Let us denote by τ_λ its left regular representation on $L^2(W)$:

$$\tau_\lambda(f)\cdot g = f \overset{\lambda}{\#} g, \quad f \in L^1(W),\ g \in L^2(G).$$

Then

$$\tau_\lambda = \bigoplus_{n=1}^{\infty} \pi_\lambda,$$

and hence by [16] and (1.14)

$$\nu(f) = \|\pi_\lambda(f)\|_{\mathscr{L}(L^2(X))} = \|\tau_\lambda(f)\|_{\mathscr{L}(L^2(W))} \tag{1.16}$$

for $f \in L^1(W)$. Here $\nu(f)$ denotes the spectral radius of f in the Banach algebra $L^1(W)$.

There is one more structure on W, which will be useful for our purposes. Namely, W has a natural symplectic form:

$$W\times W \ni \big((x,\xi),(y,\eta)\big) \to \langle (x,\xi),(y,\eta)\rangle = \xi y - \eta x. \tag{1.17}$$

Using this form, we define a "symplectic Fourier transform" on $\mathscr{S}(W)$ by

$$f^{\wedge}(w) = \int_W f(v)\,e^{i\langle v,w\rangle}\,dv. \tag{1.18}$$

It is evidently a topological linear isomorphism of $\mathscr{S}(W)$ and it extends by duality to a topological isomorphism of $\mathscr{S}^*(W)$. Let us remark that the transform "$^\wedge$" is "compound" of usual N-dimensional Fourier transforms as it can easily be seen for simple tensors $f\otimes g$. In this case

$$(f\otimes g)^{\wedge}(x,\xi) = \check{g}(x)\cdot\hat{f}(\xi).$$

Therefore the properties of "$^\wedge$" are similar to those of $2N$-dimensional Fourier transform. For example, if we denote by $*$ an abelian convolution on W, then

$$(f*g)^{\wedge} = f^{\wedge}\cdot g^{\wedge}, \quad f, g \in L^1(W). \tag{1.19}$$

Moreover, the map $f \to f^{\wedge}$ is an isometry of $L^2(W)$. But, on the other hand, "$^\wedge$" is of order two:

$$(f^{\wedge})^{\wedge} = f \quad \text{for} \quad f \in \mathscr{S}^*(W). \tag{1.20}$$

If δ_w denotes a Dirac delta at w, we have

$$\delta_w^{\wedge}(v) = e^{i\langle v,w\rangle}. \tag{1.21}$$

We also have

$$\left(\frac{\partial}{\partial x_j}\right)^{\wedge} = i\xi_j, \qquad \left(\frac{\partial}{\partial \xi_j}\right)^{\wedge} = \frac{1}{i} x_j, \tag{1.22}$$

where $\partial/\partial x_j$, $\partial/\partial \xi_j$ are regarded as distributions supported at origin.

2. A calculus of symbols. Let us first briefly summarize the results of the first section. For a given finite-dimensional real vector space X we form a phase space $W = X \oplus X^*$ of even dimension. On W we consider the space $\mathscr{S}^*(W)$ of all temperate distributions and its subspaces $\mathscr{S}(W)$, $L^1(W)$, $L^p(W)$, etc. An involution $D \to D^{\#}$ in $\mathscr{S}^*(W)$ and a convolution $\underset{\lambda}{\#}$ in $L^1(W)$ were defined. (We shall use $\underset{\lambda}{\#}$ and $\overset{\#}{\lambda}$ only for $\lambda = 1$, so we denote them simply by $\#$ and $^{\#}$.) Of course, in $L^1(W)$ there is also a usual abelian convolution which will be denoted by $*$. Then the representation π (we again set $\pi_\lambda = \pi$ for $\lambda = 1$) was introduced. It establishes an isomorphism between $\mathscr{S}^*(W)$ and $\mathscr{L}(\mathscr{S}(X), \mathscr{S}^*(X))$ and maps $L^2(W)$ isometrically onto $\mathrm{HS}(L^2(X))$, the space of Hilbert–Schmidt operators on $L^2(X)$. Moreover, there is a natural symplectic form $\langle\cdot,\cdot\rangle$ on W, which defines "a symplectic Fourier transform" on $\mathscr{S}^*(W)$.

By (1.14) for a given operator $T \in \mathscr{L}(\mathscr{S}(X), \mathscr{S}^*(X))$ there exists a unique distribution D on W such that $T = \pi(D)$. It is convenient and useful sometimes to investigate operators T by means of symplectic transforms of corresponding distributions on W. This leads to a pseudodifferential calculus as presented, e.g., in [6], [1]. A basic notion is the following:

2.1. Definition. Let $T \in \mathscr{L}(\mathscr{S}(X), \mathscr{S}^*(X))$. If $D \in \mathscr{S}^*(W)$ is such that $\pi(D^{\wedge}) = T$, then it is called a *symbol of* T.

2.2. Remark. Various kinds of "symbols" are being used in the theory of pseudodifferential operators (cf. [17]). The one we have defined above could be called a "left polarized symbol" ([17], [6]).

Motivated by Definition 2.1, we shall sometimes call temperate distributions on W just *symbols*. If a symbol a is a locally integrable function of polynomial growth, then the operator $T = \pi(a^{\wedge})$ (for which a is a symbol) is given by the formula

$$Tu(x) = \int e^{ix\xi} a(x,\xi)\hat{u}(\xi)\,d\xi; \qquad u \in \mathscr{S}(X). \tag{2.1}$$

It follows directly from (1.12), (1.18) (cf. [6]). For such a symbol a we shall sometimes denote T by $a(x, D)$. Our aim in this section is to develop (by modification of standard cases) a kind of calculus for a certain class of smooth symbols of polynomial growth. As usually such a calculus may be applied to symbols with singularities in a compact set. In our case we are able to treat also some symbols with unbounded sets of singularities.

2.3. Definition. Let $K, k \in \mathbf{R}$ and $0 \leqslant \delta, \varrho \leqslant 1$. By $H^{(K,k)}_{(\delta,\varrho)}$ we denote the space of all smooth complex functions a on W such that

$$|a^{(\beta)}_{(\alpha)}| \leqslant C_{\alpha,\beta}\langle x\rangle^{K-\delta|\alpha|}\langle\xi\rangle^{k-\varrho|\beta|}, \tag{2.2}$$

all α, β. Moreover, set

$$H^{(K,k)} = \bigcup_{\substack{0\leqslant\delta\leqslant 1\\ 0\leqslant\varrho\leqslant 1}} H^{(K,k)}_{(\delta,\varrho)} = H^{(K,k)}_{(0,0)}, \qquad H = \bigcup_{K,k\in\mathbf{R}} H^{(K,k)}.$$

Note that each of the spaces $H^{(K,k)}_{(\delta,\varrho)}$ is a Fréchet space with a family of seminorms defined by:

$$|a|_{(\alpha,\beta)} = \inf\{C_{\alpha,\beta}\colon C_{\alpha,\beta} \text{ satisfies } (2.2)\}.$$

2.4. Remark. It follows directly from the definition that

(i) if $a \in H^{(K,k)}_{(\delta,\varrho)}$, $b \in H^{(K',k')}_{(\delta,\varrho)}$, then $a\cdot b \in H^{(K+K',k+k')}_{(\delta,\varrho)}$ and $a+b \in H^{(K_0,k_0)}_{(\delta,\varrho)}$, where $K_0 = \max(K, K')$, $k_0 = \max(k, k')$;

(ii) if $a \in H^{(K,k)}_{(\delta,\varrho)}$, then $a^{(\beta)}_{(\alpha)} \in H^{(K-\delta|\alpha|,k-\varrho|\beta|)}_{(\delta,\varrho)}$.

In what follows we shall make use of the following simple facts:

2.5. Proposition. *If $a \in H^{(K,k)}$, then $A = \pi(\hat{a})$ is a continuous endomorphism of $\mathscr{S}(X)$. More exactly: for any seminorm $|\cdot|$ in $\mathscr{S}(X)$ there exists a seminorm $|\cdot|'$ in $\mathscr{S}(X)$ such that*

$$|Au| \leqslant C|u|', \qquad u \in \mathscr{S}(X)$$

and the constant C depends only on seminorms of a in $H^{(K,k)}$ ([1], Proposition 3.11, [2], Lemma 1).

2.6. Corollary. *Let (a_n) be a bounded sequence in $H^{(K,k)}$ converging pointwise to $a \in H^{(K,k)}$. Then $(\pi(a_n^{\wedge}))$ converges to $\pi(a^{\wedge})$ strongly on $\mathscr{S}(X)$.* ([2], Corollary to Lemma 1).

2.7. Proposition. *Let (a_n) be a bounded sequence in $H^{(K,k)}$ and let $A_n = \pi(a_n^{\wedge})$ converge to A strongly on $\mathscr{S}(X)$. Then A has a symbol $a \in H^{(K,k)}$.*

Proof follows immediately from the Ascolli theorem.

2.8. Remark. *For any $a \in H^{(K,k)}$ there exists a sequence $a_n \in \mathscr{D}(W)$ bounded in $H^{(K,k)}$ such that $(a_n)^{(\beta)}_{(\alpha)}$ converges pointwise to $a^{(\beta)}_{(\alpha)}$ for every α, β.*

Proof. Let $f \in \mathscr{D}(W)$ be such that $f(w) = 1$ for $|w| \leqslant 1$. Set $a_n(w) = f\left(\frac{1}{n}w\right)a(w)$.

Let a, b be symbols of class H. Then by Proposition 2.5 a composition of the operators $a(x, D)$ and $b(x, D)$ is a continuous endomorphism of $\mathscr{S}(X)$, so it has a symbol which is *a priori* a temperate distribution. Denote it by $a\circ b$. For $a, b \in \mathscr{D}(W)$ we may write down $a\circ b$ explicitely:

$$a\circ b(x,\xi) = \iint h(x,\eta,y,\xi)a(x,\eta)b(y,\xi)\,dy\,d\eta, \tag{2.3}$$

where $h(x, \xi, y, \eta) = e^{i(x-y)(\eta-\xi)}$ (see [2]). For $a, b \in H$, let us define

$$(2.4)\qquad r_n(a, b) = a\circ b - \sum_{|\alpha|<n} \frac{1}{\alpha!} a^{(\alpha)} b_{(\alpha)}, \qquad n = 0, 1, 2, \ldots$$

Then for $a, b \in \mathcal{D}(W)$ and $n \geqslant 1$ we have an explicit formula:

$$(2.5)\quad r_n(a, b)(x, \xi) = \sum_{|\alpha|=n} \int_0^1 \int\int h a^{(\alpha)}\big(x, \eta + t(\xi-\eta)\big) b_{(\alpha)}(y, \xi)\, dy\, d\eta\, dt,$$

where h is as defined in (2.3). By an application of the Leibniz formula and integration by parts we get from (2.3)

$$(2.6)\qquad (a\circ b)^{(\beta)}_{(\alpha)} = \sum_{\substack{0\leqslant \varkappa \leqslant \alpha\\ 0\leqslant \iota\leqslant \beta}} \binom{\alpha}{\varkappa}\binom{\beta}{\iota} a^{(\iota)}_{(\varkappa)} b^{(\beta-\iota)}_{(\alpha-\varkappa)}$$

for $a, b \in \mathcal{D}(W)$. Hence by (2.4)

$$(2.7)\qquad r_n(a, b)^{(\beta)}_{(\alpha)} = \sum_{\substack{0\leqslant \iota \leqslant \alpha\\ 0\leqslant \varkappa\leqslant \beta}} \binom{\alpha}{\iota}\binom{\beta}{\varkappa} r_n\big(a^{(\varkappa)}_{(\iota)} b^{(\beta-\varkappa)}_{(\alpha-\iota)}\big)$$

for $a, b \in \mathcal{D}(W)$.

Now, we are ready to prove the formulas for composition and taking adjoint in the class of operators with symbols in H. We obtain them, adopting a lemma of R. Beals and C. Fefferman ([2], Lemma 2),

2.9. Lemma. *Let $b \in C^\infty(\boldsymbol{R}^{4N})$. Denote the variables in $\boldsymbol{R}^{4N}$ by (x, η, y, ξ) and assume that b has a compact support in (η, y) independent of (x, ξ). Let b satisfy*

$$|\partial_y^\alpha \partial_\eta^\beta b| \leqslant C_{\alpha,\beta} \langle y-x\rangle^K \langle \xi-\eta\rangle^k$$

for some $K, k \in \boldsymbol{R}$ and all α, β. Set

$$a(x, \xi) = \int\int b(x, \eta, y, \xi) e^{i(x-y)(\eta-\xi)} dy\, d\eta.$$

Then there exists a constant C depending only on $C_{\alpha,\beta}$ such that

$$|a(x, \xi)| \leqslant C.$$

Proof. Let $g(x, \eta, y, \xi) = \langle x-y\rangle^2 + \langle \xi-\eta\rangle^2$ and $L\varphi = (2 - \Delta_y - \Delta_x)$ for $\varphi \in C^\infty(\boldsymbol{R}^{4N})$. For h defined in (2.3) we have $L\cdot h = g\cdot h$. Write

$$b_j = (g^{-1}L)^j b, \qquad j = 0, 1, 2, \ldots$$

Then the following estimates hold:

$$|b_j| \leqslant C_j' \langle y-x\rangle^{K-j} \langle \xi-\eta\rangle^{k-j}, \qquad j = 0, 1, 2, \ldots,$$

where C_j' depend only on $C_{\alpha,\beta}$. It is immediate by induction and the Leibniz formula for differentiation of product. Thus for $j_0 \geqslant \max(K+N+1, k+N+1)$ we get

$$|a(x, \xi)| = \Big|\int\int h\cdot b(x, \eta, y, \xi)\, dy\, d\eta\Big| = \Big|\int\int (g^{-1}L)^{j_0} h\cdot b\Big| = \Big|\int\int h\cdot b_{j_0}\Big|$$
$$\leqslant C_{j_0}' \int\int \langle y-x\rangle^{-(N+1)} \langle \xi-\eta\rangle^{-(N+1)} dy\, d\eta = C < \infty.$$

The lemma is proved.

2.10. Proposition. *Let $a \in H^{(K,k)}_{(\delta,\varrho)}$, $b \in H^{(K',k')}_{(\delta,\varrho)}$. Then*

$$r_n(a, b) \in H^{(K+K'-\delta n, k+k'-\varrho n)}_{(\delta,\varrho)}, \qquad n = 0, 1, 2, \ldots$$

Proof. Suppose first that a and b have compact supports and fix some $n \geqslant 0$. Set

$$K_0 = K + K' - \delta n, \qquad k_0 = k + k' - \varrho n,$$

$$C_t^\alpha = \begin{cases} \langle x\rangle^{-K_0} \langle \xi\rangle^{-k_0} a^{(\alpha)}\big(x, \eta + t(\xi-\eta)\big) b_{(\alpha)}(y, \xi) & \text{if} \quad n \geqslant 1,\\ \langle x\rangle^{-K_0} \langle \xi\rangle^{-k_0} a^{(\alpha)}(x, \eta) b_{(\alpha)}(y, \xi) & \text{if} \quad n = 0,\end{cases}$$

and

$$a_t^\alpha(x, \xi) = \int\int h\cdot C_t^\alpha(x, \eta, y, \xi)\, dy\, d\eta.$$

Since C_t^α satisfies the assumptions of Lemma 2.9 (independently of $0 \leqslant t \leqslant 1$), we obtain

$$|a_t^\alpha(x, \xi)| \leqslant C,$$

where C depends only on seminorms of a and b in $H^{(K,k)}_{(\delta,\varrho)}$ and $H^{(K',k')}_{(\delta,\varrho)}$, respectively. Notice that

$$r_n(a, b) = \langle x\rangle^{K_0} \langle \xi\rangle^{k_0} \sum_{|\alpha|=n} \frac{1}{\alpha!} \int_0^1 a_t^\alpha(x, \xi)\, dt.$$

Hence

$$(2.8)\qquad |r_n(a, b)| \leqslant C\langle x\rangle^{K+K'-\delta n} \langle \xi\rangle^{k+k'-\varrho n}.$$

Now, consider arbitrary $a \in H^{(K,k)}_{(\delta,\varrho)}$ and $b \in H^{(K',k')}_{(\delta,\varrho)}$. By Remark 2.8 there exist bounded sequences (a_j) in $H^{(K,k)}_{(\delta,\varrho)}$ and (b_j) in $H^{(K',k')}_{(\delta,\varrho)}$ such that $a_j, b_j \in \mathcal{D}(W)$ and

$$(2.9)\qquad (a_j)^{(\beta)}_{(\alpha)} \to a^{(\beta)}_{(\alpha)}, \qquad (b_j)^{(\beta)}_{(\alpha)} \to b^{(\beta)}_{(\alpha)}$$

pointwise for all α, β. Thus, by Corollary 2.6 and Proposition 2.5, $\pi(a_j^{\hat{}})\pi(b_j^{\hat{}})$ converges strongly on $\mathcal{S}(X)$ to $\pi(a^{\hat{}})\pi(b^{\hat{}})$. In view of (2.9) and (2.4) this implies that

$$\pi\big((r_n(a_j, b_j))^{\hat{}}\big) \to \pi\big((r_n(a, b))^{\hat{}}\big) \quad \text{strongly on } \mathcal{S}(X).$$

Hence by Proposition 2.7 and (2.8) $r_n \in H^{(K+K'-\delta n, k+k'-\varrho n)}$ and satisfies the estimate (2.8). We need yet the estimates for derivatives of $r_n(a, b)$. To obtain them it is sufficient to apply (2.7) to (2.8) (which we just have proved to be valid for all $a \in H^{(K,k)}_{(\delta,\varrho)}$ and $b \in H^{(K',k')}_{(\delta,\varrho)}$).

2.11. COROLLARY. *Let* $a \in H^{(K,k)}_{(\delta,\varrho)}$, $b \in H^{(K',k')}_{(\delta,\varrho)}$. *Then* $a \circ b \in H^{(K+K',k+k')}_{(\delta,\varrho)}$ *and the following composition formula holds*

$$(2.10) \qquad a\circ b = \sum_{|\alpha|<n} \frac{1}{\alpha!} a^{(\alpha)} b_{(\alpha)} + r_n(a, b), \qquad n = 1, 2, \ldots,$$

where $r_n(a, b) \in H^{(K+K'-\delta n, k+k'-\varrho n)}_{(\delta,\varrho)}$.

Now, let $A \in \mathcal{L}(\mathcal{S}(X))$. We call a *formal adjoint* to A the operator A^+ defined by

$$(2.11) \qquad \langle A^+ u, v\rangle = \langle u, Av\rangle \quad \text{for} \quad u, v \in \mathcal{S}(X),$$

where $\langle u, v\rangle = \int u(x)\overline{v(x)}\,dx$. Evidently, $A^+ \in \mathcal{L}(\mathcal{S}, \mathcal{S}^*)$, so it has a symbol. If a is a symbol we denote by a^+ the symbol of $\pi(a^{\hat{}})^+$. If moreover $a \in \mathcal{D}(W)$, then

$$(2.12) \qquad a^+(x, \xi) = \iint h\cdot \bar{a}(y, \eta)\,dy\,d\eta,$$

where h is as in (2.3) (see [2]). For $a \in H$ let us write

$$(2.13) \qquad S_n(a) = a^+ - \sum_{|\alpha|<n} \frac{1}{\alpha!} \bar{a}^{(\alpha)}_{(\alpha)}.$$

Again, for $a \in \mathcal{D}(W)$ we have the explicit formula:

$$(2.14) \qquad S_n(a) = \sum_{|\alpha|=n} \frac{1}{\alpha!} \int_0^1 \iint h \bar{a}^{(\alpha)}_{(\alpha)}(y, \eta + t(\xi-\eta))\,dy\,d\eta\,dt$$

(see [2]). From (2.13) we get immediately that for $a \in \mathcal{D}(W)$

$$(2.15) \qquad S_n(a^{(\beta)}_{(\alpha)}) = S_n(a)^{(\beta)}_{(\alpha)}.$$

2.12. PROPOSITION. *Let* $a \in H^{(K,k)}_{(\delta,\varrho)}$. *Then* $a^+ \in H^{(K,k)}_{(\delta,\varrho)}$ *and*

$$a^+ = \sum_{|\alpha|<n} \frac{1}{\alpha!} \bar{a}^{(\alpha)}_{(\alpha)} + S_n(a), \qquad n = 1, 2, \ldots,$$

where $S_n(a) \in H^{(K-\delta n, k-\varrho n)}_{(\delta,\varrho)}$.

Proof uses Lemma 2.9 and formulas (2.12)–(2.15). Since it is quite parallel to the proof of Proposition 2.10, we omit it.

2.13. DEFINITION. A function $f \in C^\infty(\boldsymbol{R}^m)$ is said to satisfy the *Hörmander condition with a constant* $0 < \delta \leqslant 1$ if

$$(2.16) \qquad |\partial_x^\alpha f(x)| \leqslant C_\alpha \langle x\rangle^{-\delta|\alpha|} |f(x)|, \quad \text{all } \alpha,$$

for all x outside a compact subset of $\boldsymbol{R}^m$.

2.14. DEFINITION. A symbol $a \in C^\infty(W)$ is said to satisfy the *weak Hörmander condition* if there exist $K, k, K_0, k_0, \delta, \varrho > 0$ such that

$$(2.17) \qquad |a^{(\beta)}_{(\alpha)}(x, \xi)| \leqslant C_{\alpha,\beta} \langle x\rangle^{-\delta|\alpha|} \langle \xi\rangle^{-\varrho|\beta|} |a(x, \xi)|, \quad \text{all } \alpha, \beta,$$

$$(2.18) \qquad C_1 \langle x\rangle^{K_0} \langle \xi\rangle^{k_0} \leqslant |a(x, \xi)| \leqslant C_2 \langle x\rangle^{K} \langle \xi\rangle^{k}$$

for (x, ξ) outside a compact subset of W.

Estimates of type (2.16)–(2.18) are frequently being used in the theory of differential and pseudodifferential operators (cf. [1], [6], [7], [17]).

2.15. Remarks. (i) If f is a polynomial in m variables, then the differential operator $f(D)$ is hypoelliptic if and only if f satisfies (2.16) for some $0 < \delta \leqslant 1$ ([7]).

(ii) If $a \in C^\infty(W)$ $(f \in C^\infty(\boldsymbol{R}^m))$ satisfies (2.17) and (2.18) ((2.16)) outside a compact set, then there exists $a_1 \in C^\infty(W)$ $(f_1 \in C^\infty(\boldsymbol{R}^m))$ such that $a_1(f_1)$ satisfies (2.17) and (2.18) ((2.16)) everywhere and $a - a_1 \in \mathcal{D}(W)$ $(f - f_1 \in \mathcal{D}(\boldsymbol{R}^m))$.

(iii) If $a \in C^\infty(W)$ satisfies the weak Hörmander condition, then it belongs to $H^{(K,k)}_{(\delta,\varrho)}$.

(iv) If $a \in C^\infty(W)$ satisfies (2.17), (2.18) for all $(x, \xi) \in W$, then $1/a \in H^{(-K_0,-k_0)}_{(\delta,\varrho)}$. This can directly be checked by differentiation.

(v) If f is a polynomial in $2N$ variables of degree n satisfying the Hörmander condition with a constant δ (Def. 2.13), then it satisfies the weak Hörmander condition with constants $K = n$, $k = n$, $K_0 = \frac{1}{2}\delta n$, $k_0 = \frac{1}{2}\delta n$, δ, $\varrho = \delta$ (Def. 2.14).

2.16. LEMMA. *Let a satisfy the weak Hörmander condition with constants* $K, k, K_0, k_0, \delta, \varrho$. *Then there exists* $b_0 \in H^{(-K_0,-k_0)}_{(\delta,\varrho)}$ *such that for* $n \geqslant 1$

$$r_n(a, b_0) \in H^{(-n\delta, -\varrho n)}_{(\delta,\varrho)}.$$

Proof. By Remarks 2.15 (ii)–(iv) we may assume without loss of generality that $a \in H^{(K,k)}_{(\delta,\varrho)}$ and $1/a \in H^{(-K_0,-k_0)}_{(\delta,\varrho)}$ for the constants (K, k) and (K_0, k_0) occurring in (2.18). Hence, by Proposition 2.10, $r_j(a, 1/a) \in H^{(K_j,k_j)}_{(\delta,\varrho)}$, where K_j, k_j tend to $-\infty$. Choose m such that $k_m \leqslant -\delta n$, $k_m \leqslant -\varrho n$. We have

$$r_n\left(a, \frac{1}{a}\right) = \sum_{n\leqslant|\alpha|<m} \frac{1}{\alpha!} a^{(\alpha)} \left(\frac{1}{a}\right)_{(\alpha)} + r_m\left(a, \frac{1}{a}\right).$$

By direct differentiation and application of (2.17) we check that the first term on the right-hand side is in $H^{(-\delta n, -\varrho n)}_{(\delta,\varrho)}$. Since we know that $r_m(a, 1/a) \in H^{(-\delta n, -\varrho n)}_{(\delta,\varrho)}$, our proof is complete.

2.17. PROPOSITION. *Let $K_j, k_j \searrow -\infty$ and let $a_j \in H^{(K_j, k_j)}_{(\delta,\varrho)}$ for some $0 \leqslant \delta, \varrho \leqslant 1$. Then there exists a symbol $a \in H^{(K_0, k_0)}_{(\delta,\varrho)}$ such that*

$$a - \sum_{j=0}^{n} a_j \in H^{(K_{n+1}, k_{n+1})}_{(\delta,\varrho)}, \quad n \geqslant 0. \tag{2.19}$$

To denote that (2.19) holds we shall write $a \sim \sum_{j=0}^{\infty} a_j$.

Proof (cf. [1], Theorem 4.13). We may assume that $K_{j+1} < K_j$ and $k_{j+1} < k_j$. (If not, we sum up suitable groups of terms.) Let $h \in C^\infty(\boldsymbol{R})$ be such that $h(t) = 0$ for $t \leqslant 1$ and $h(t) = 1$ for $t \geqslant 2$. Set

$$b_j(x, \xi) = h(\varepsilon_j \langle x\rangle^{K_{j-1}-K_j} \langle \xi\rangle^{k_{j-1}-k_j})\, a_j(x, \xi), \quad \text{where} \quad 0 < \varepsilon_j \in \boldsymbol{R}.$$

First notice that $(b_j)^{(\beta)}_{(\alpha)} = (a_j)^{(\beta)}_{(\alpha)}$ except where $\langle x\rangle^{K_{j-1}-K_j} \langle \xi\rangle^{k_{j-1}-k_j} \leqslant 2/\varepsilon_j$, that is, outside a compact set. Hence

$$b^{(\beta)}_{j(\alpha)} - a^{(\beta)}_{j(\alpha)} \in \mathcal{D}(W) \tag{2.20}$$

and in particular $b_j \in H^{(K_j, k_j)}_{(\delta,\varrho)}$.

Now we show that the series $\sum_{j=n}^{\infty} b_j$ is convergent in $H^{(K_n, k_n)}_{(\delta,\varrho)}$. In fact, $b^{(\beta)}_{j(\alpha)} = 0$ outside the set where $\langle x\rangle^{K_j} \langle \xi\rangle^{k_j} \leqslant \varepsilon_j \langle x\rangle^{K_{j-1}} \langle \xi\rangle^{k_{j-1}}$, so

$$|b^{(\beta)}_{j(\alpha)}| \leqslant C^{\alpha,\beta}_j \langle x\rangle^{K_j - \delta|\alpha|} \langle \xi\rangle^{k_j - \varrho|\beta|} \leqslant \varepsilon_j C^{\alpha,\beta}_j \langle x\rangle^{K_{j-1}-\delta|\alpha|} \langle \xi\rangle^{k_{j-1}-\varrho|\beta|}$$
$$\leqslant \varepsilon_j C^{\alpha,\beta}_j \langle x\rangle^{K_n - \delta|\alpha|} \langle \xi\rangle^{k_n - \varrho|\beta|} \quad \text{for} \quad j \geqslant n.$$

Thus we have to choose ε_j in such a way that for each α, β $\sum_{j=0}^{\infty} \varepsilon_j C^{\alpha,\beta}_j < \infty$, where $C^{\alpha,\beta}_j$ are seminorms of b_j in $H^{(K_j, k_j)}_{(\delta,\varrho)}$. This can be done by selecting ε_j inductively for $j \geqslant n$ and $|\alpha| + |\beta| \leqslant n$. Write $a = \sum_{j=0}^{\infty} b_j \in H^{(K_0, k_0)}_{(\delta,\varrho)}$. By the above and (2.20) a satisfies (2.19), so the proof is completed.

We use Lemma 2.16 and Proposition 2.17 to construct for a symbol satisfying the weak Hörmander condition a substitute of the parametrix.

2.18. THEOREM. *Let a be a symbol satisfying the weak Hörmander condition with constants $\delta, \varrho, K, k, K_0, k_0$. Then there exists a symbol $b \in H^{(-K_0, -k_0)}_{(\delta,\varrho)}$ such that*

$$b \circ a - 1 \in \mathcal{S}(W) \quad \textit{and} \quad a \circ b - 1 \in \mathcal{S}(W). \tag{2.21}$$

Proof. By Lemma 2.16 there exist $b_0 \in H^{(-K_0, -k_0)}_{(\delta,\varrho)}$ and $r \in H^{(-\delta, -\varrho)}_{(\delta,\varrho)}$ such that

$$b_0 \circ a = 1 - r.$$

Set

$$b_n = r^{2^n - 1} \circ \sum_{0 \leqslant j < n} b_j, \quad \text{where} \quad r^m = \underbrace{r \circ \ldots \circ r}_{m \text{ times}}.$$

By induction we check that

$$\Big(\sum_{j=0}^{n} b_j\Big) \circ a = 1 - r^{2^n}.$$

Note that, by Proposition 2.10, $r^{2^n} \in H^{(-2^n\delta, -2^n\varrho)}_{(\delta,\varrho)}$ and $b_n \in H^{(-K_0 - 2^n\delta, -k_0 - 2^n\varrho)}_{(\delta,\varrho)}$ for $n \geqslant 1$. Then $b \sim \sum_{j=0}^{\infty} b_j$ (see Proposition 2.17) satisfies

$$b \circ a - 1 \in \mathcal{S}(W).$$

By the same argument there exists $b' \in H^{(-K_0, -k_0)}_{(\delta,\varrho)}$ such that $a \circ b' - 1 \in \mathcal{S}(W)$. But then $b - b' \in \mathcal{S}(W)$, and hence b satisfies (2.21).

We end this section with

2.19. EXAMPLES. (i) Let P, Q be polynomials in N variables of degrees n, m. Assume that $P(u) \neq 0 \neq Q(u)$ for $u \in \boldsymbol{R}^N$. Set

$$a(x, \xi) = |P(x)|^p + |Q(\xi)|^q, \quad b(x, \xi) = |P(x)|^p \cdot |Q(\xi)|^q$$

for some $p, q > 0$. Then $a, b \in H^{(pn, qm)}_{(1,1)}$.

(ii) If, moreover, P, Q satisfy the Hörmander condition (Def. 2.13) with constants δ, ϱ, then a, b satisfy the weak Hörmander condition (Def. 2.14) with constants $\delta, \varrho, K = pn, k = qm, K_0 = \delta n, k_0 = \varrho m$ for b and $K_0 = \frac{1}{2}\delta n$, $k_0 = \frac{1}{2}\varrho m$ for a.

(iii) Our class H contains the symbols which are called *pseudo-Toeplitz symbols* in [6], 4.2.16, or *symbols of Γ^m_ϱ type* in [17], def. 23.1. These symbols are required to satisfy the estimates

$$|a^{(\beta)}_{(\alpha)}| \leqslant C_{\alpha,\beta} \langle w\rangle^{m - \varrho(|\alpha| + |\beta|)}, \quad \text{where} \quad w = (x, \xi).$$

Of course, such symbols belong to $H^{(m,m)}_{(\varrho,\varrho)}$.

3. Operators on $L^2(X)$. The purpose of this section is to investigate operators on $L^2(X)$ using the calculus developped in Section 2.

Let a be a symbol of class H (Def. 2.3). Consider $A = \pi(a^\wedge) \in \mathcal{L}(\mathcal{S}(X))$. A has a natural and unique extension to a continuous operator on $\mathcal{S}^*(X)$, which we denote also by A. It is given by

$$\langle A \cdot U, v\rangle = \langle U, A^+ v\rangle; \quad v \in \mathcal{S}(X), \ U \in \mathcal{S}^*(X), \tag{3.1}$$

where A^+ is a formal adjoint to A (see (2.11)). We may also regard A as an operator on $L^p(X)$ for $1 \leqslant p < \infty$ with a dense domain $D_p(A) = \mathcal{S}(X) \subset L^p(X)$. Since A^* acting on $L^{p'}(X)$ $(1/p + 1/p' = 1)$ contains A^+, its domain is dense. This implies that A is closeable on $L^p(X)$. We are going now to study certain properties of operators A on a Hilbert space $L^2(X)$. First let us recall the definition of J_p classes of operators on a Hilbert space $\mathscr{H}$. For $0 \leqslant T \in \mathscr{L}(\mathscr{H})$ let us denote by $\mathrm{Tr}(T)$ the trace of T, which is either a non-negative number or infinity. For arbitrary $T \in \mathscr{L}(\mathscr{H})$, set

$$|T|_{J_p} = \begin{cases} (\mathrm{Tr}\,|T|^p)^{1/p} & \text{for} \quad 1 \leqslant p < \infty, \\ \|T\|_{\mathscr{L}(\mathscr{H})} & \text{for} \quad p = \infty. \end{cases} \tag{3.2}$$

For every $1 \leqslant p \leqslant \infty$ the space

$$J_p(\mathscr{H}) = \{T \in \mathscr{L}(\mathscr{H}) \colon |T|_{J_p} < \infty\}$$

is a Banach space with the norm $|\cdot|_{J_p}$. In particular, $J_1(\mathscr{H})$ equals the space of trace class operators, $J_2(\mathscr{H}) = \mathrm{HS}(\mathscr{H})$ and $J_\infty(\mathscr{H}) = \mathscr{L}(\mathscr{H})$. If $T \in J_p(\mathscr{H})$ for some $p < \infty$, then it is compact and $\big(\sum_{\lambda \in \mathrm{Sp}\,T} |\lambda|^p\big)^{1/p} \leqslant |T|_{J_p}$. Our first theorem shows the relation between L^p-properties of symbols and J_p-properties of corresponding operators on $L^2(X)$.

3.1. THEOREM. *Let $a \in C^\infty(W)$ and let $1 \leqslant p \leqslant \infty$. If $a_{(\alpha)}^{(\beta)} \in L^p(W)$ for $|\alpha|+|\beta| \leqslant 2(N+1)$, then $\pi(a^\frown)$ extends to an operator from $J_p(L^2(X))$ and the following estimate holds:*

$$|\pi(a^\frown)|_{J_p} \leqslant C_p \max_{0 \leqslant |\alpha|+|\beta| \leqslant 2(N+1)} \|a_{(\alpha)}^{(\beta)}\|_{L^p(W)}. \tag{3.3}$$

3.2. Remarks. In the case $p = \infty$ this is a version of the Calderón–Vaillencourt $(0, 0)$ boundedness theorem. The case $p = 1$ was proved in [17] for symbols of Γ_ϱ^m type, $m < 2N$. Note that for $p = 2$ we have much more better result, see (1.14).

Since our proof uses an interpolation method, let us consider first the extreme cases, i.e. $p = 1$, $p = \infty$.

3.3. LEMMA. *If $\varphi \in \mathscr{S}(W)$, then $\pi(\varphi)$ is of trace class and the map*

$$\varphi(W) \ni \varphi \to \pi(\varphi) \in J_1\big(L^2(X)\big)$$

is continuous.

Proof. Let

$$a(x, \xi) = (x^2+\xi^2+1)^N;$$

a is a symbol of $H_{(1,1)}^{(2N,2N)}$ class and satisfies the Hörmander condition (Def. 2.13) with $\delta = 1$ and so, by Theorem 2.18,

$$a \circ b = 1 - d,$$

where $b \in L^2(W)$ and $d \in \mathscr{S}(W) \subset L^2(W)$. Hence

$$\varphi = (\varphi \# a^\frown) \# b^\frown + \varphi \# d^\frown$$

so that by Corollary 2.11, the Plancherel theorem (see Sec. 1) and (1.14), $\pi(\varphi)$ is of trace class. The linear map

$$\mathscr{S}(W) \ni \varphi \to \pi(\varphi) \in \mathscr{L}\big(L^2(X)\big)$$

is continuous. Thus by the above and the closed graph theorem it is also continuous as a map into $J_1\big(L^2(X)\big)$.

3.4. Remark. Let $\varphi \in \mathscr{S}(W)$ and $1 \leqslant p \leqslant \infty$. Set $\varphi_w(v) = \varphi(v-w)$ for $v, w \in W$. Then

$$|\pi(\varphi_w)|_{J_p} = |\pi(\varphi)|_{J_p}$$

for any $w \in W$. In fact, let $w = (x, \xi)$. Then

$$\pi(\varphi_w) = \pi_{(x,0)}\pi(\varphi)\pi_{(0,\xi)}.$$

Since π_v is unitary for any $v \in W$, the inequality

$$|\pi(\varphi_w)|_{J_p} \leqslant |\pi(\varphi)|_{J_p}$$

is obvious. Hence, by symmetry we obtain the required equality.

3.5. LEMMA. *Let $f \in C^\infty(W)$ be supported in a compact K. Then the following estimate holds:*

$$|\pi(f)|_{J_1} \leqslant C_k \|f^\frown\|_{L^1(W)},$$

where C_k depends only on K.

Proof. Fix U, a neighbourhood of K with compact closure and $\varphi \in \mathscr{S}(W)$ such that $\varphi(w) \neq 0$ for $w \in U$. Then

$$\pi(\varphi f) = \int f(w)\varphi(w)\pi_w dw$$
$$= \int f^\frown(v) \int e^{i\langle v,w\rangle}\varphi(w)\pi_w dw dv = \int f^\frown(v)\pi(\varphi_v)\,dv.$$

The $J_1\big(L^2(X)\big)$-valued function $v \to \pi(\varphi_v)$ is continuous and bounded (Lemma 3.3, Remark 3.4) and f is integrable so

$$|\pi(\varphi f)|_{J_1} \leqslant \int |f^\frown(v)|\, |\pi(\varphi_v)|_{J_1} dv = |\pi(\varphi)|_{J_1} \|f^\frown\|_{L^1(W)} \quad \text{(Remark 3.4).}$$

Since f is supported in K and φ is invertible on U, we finally get

$$|\pi(f)|_{J_1} \leqslant C_k \|f^\frown\|_{L^1(W)}.$$

3.6. LEMMA. *Let $f \in C^\infty(W)$ be supported in a compact K. Then the following estimate holds:*

$$\|\pi(f)\|_{\mathscr{L}(L^2(X))} \leqslant C_k \|f^\frown\|_{L^\infty(W)},$$

where C_k depends only on K.

This is exactly Theorem 3.1.1 from [6] and the proof goes as follows:

For $f \in L^1(W)$ and $T \in \mathrm{HS}(L^2(X))$ set

$$(*) \qquad \pi^\circ(f)T = \int_W f(w)\pi_w T \pi_w^{-1} dw,$$

where $\pi_w = \pi(\delta_w)$ (cf. 1.15). This formula defines a *-representation of the algebra $L^1(W)$ on a Hilbert space $\mathrm{HS}(L^2(X))$. By (1.14) we can realize this representation on $L^2(W)$ with π being the intertwining operator. It follows from (*) and (1.6) that $\pi^\circ(f)$ acts on $L^2(W)$ by

$$\pi^\circ(f)\cdot g = f^\wedge \cdot g, \qquad g \in L^2(W),$$

so we get an estimation

$$(**) \qquad \|\pi^\circ(f)\|_{\mathscr{L}(\mathrm{HS})} \leqslant \|f^\wedge\|_{L^\infty(W)}.$$

Now, let $u(x) = e^{-\frac{1}{2}|x|^2}$ (u is an element of $L^2(X)$ of norm one). Moreover, let $\varphi_u(w) = \langle u, \pi_w \cdot u\rangle$. Define the maps:

$$\alpha_u\colon L^2(X) \to \mathrm{HS}(L^2(X)),$$
$$\beta_u\colon \mathrm{HS}(L^2(X)) \to L^2(X)$$

by

$$\alpha_u(v)\cdot x = \langle x, u\rangle v, \qquad v \in L^2(X),\ x \in L^2(X),$$
$$\beta_u(T) = Tu, \qquad T \in \mathrm{HS}(L^2(X)).$$

Note that both the maps α_u, β_u are contractions.

We claim that for $v \in L^2(X)$ and $f \in L^1(W)$

$$(*{*}*) \qquad \pi(\varphi_u \cdot f)v = \beta_u(\pi^\circ(f)\alpha_u(v)).$$

In fact,

$$\pi_w \alpha_u(v)\pi_w^{-1}\cdot u = \pi_w\langle \pi_w^{-1}\cdot u, u\rangle v = \varphi_u(w)\pi_w \cdot v$$

so by (*) (see also Sec. 1)

$$\beta_u(\pi^\circ(f)\alpha_u(v)) = \pi^\circ(f)\alpha_u(v)\cdot u = \int f(w)\pi_w\alpha_u(v)\pi_w^{-1}\cdot u\,dw$$
$$= \int f(w)\varphi_u(w)\pi_w\cdot v = \pi(\varphi_u f)\cdot v.$$

By an application of (**) and (***) we get

$$\|\pi(f)\cdot v\|_{L^2(X)} \leqslant \|\pi^\circ(\varphi_u^{-1}f)\alpha_u(v)\|_{\mathrm{HS}(L^2(X))}$$
$$\leqslant \|(\varphi_u^{-1\wedge} f)\|_{L^\infty(W)}\|v\|_{L^2(X)} \qquad (\text{note that } \varphi_u(w) \neq 0,\ w \in W).$$

Therefore if $\mathrm{supp} f \subset K$, where K is a compact subset of W and $h_k \in \mathscr{D}(W)$ equals 1 in a neighbourhood of K then $f = h_k f$ and so

$$\|\pi(f)v\|_{L^2(X)} \leqslant \|(\varphi_u^{-1\wedge} h_k)\|_{L^1(W)}\|f^\wedge\|_{L^\infty(W)}\|v\|_{L^2(X)}.$$

Since $C_k = \|(\varphi_u^{-1\wedge} h_k)\|_{L^1(W)}$ does not depend on f, our lemma is proved.

Now, by interpolation we extend the results of Lemma 3.5 and Lemma 3.6 to other p.

3.7. LEMMA. *Let $K \subset W$ be compact and $1 \leqslant p \leqslant \infty$. Then there exists a constant C_k such that*

$$|\pi(f)|_{J_p} \leqslant C_k\|f^\wedge\|_{L^p(W)}$$

for $f \in C^\infty(W)$ supported in K.

Proof. Choose a neighbourhood U of K with compact closure and set

$$V = \{f \in \mathscr{S}(W)\colon \mathrm{supp} f^\wedge \subset U\}.$$

In the vector space V we introduce the following norms: $\|\cdot\|^{(0)} = \|\cdot\|_{L^1(W)}$, $\|\cdot\|^{(1)} = \|\cdot\|_{L^\infty(W)}$. Similarly in $S(X)$, the space of smoothing operators which is contained in $J_1(L^2(X))$ (see (1.13) and Lemma 3.3), we shall consider two norms: $\|\cdot\|^{(0)} = |\cdot|_{J_1}$, $\|\cdot\|^{(1)} = |\cdot|_{J_\infty}$. The norms $\|\cdot\|^{(0)}$, $\|\cdot\|^{(1)}$ in V (and also in $S(X)$) are consistent (see e.g. [15]) and the interpolation norms are equal:

$$\|\cdot\|^{(t)} = \begin{cases} \|\cdot\|_{L^{p_t}(W)} & \text{in } V, \\ |\cdot|_{J_{p_t}} & \text{in } S(X), \end{cases}$$

where $p_t = 1/(1-t)$ and $0 < t < 1$ ([15]).

Now, consider the map

$$V \ni f \to \pi(f^\wedge) \in S(X).$$

In virtue of Lemma 3.5, Lemma 3.6, and application of the interpolation theorem ([15], Theorem IX.20) yields:

$$|\pi(f^\wedge)|_{J_p} \leqslant C_k\|f^\wedge\|_{L^p(W)}$$

for $f \in V$ and the constant C_k depends only on K.

Following [6], we introduce a partition of unity on W. To do this choose a lattice W_0 in W, which contains the points (x, ξ) with integral coordinates:

$$(3.4) \qquad W_0 = \{(x, \xi) \in W\colon x_j,\ \xi_j \in \mathbf{Z}\}.$$

Let $K = \{(x, \xi) \in W\colon |x_j|, |\xi_j| \leqslant 1;\ j = 1, \ldots, N\}$ and let $\psi \in \mathscr{D}(W)$ be positive and such that $\int \psi(w)dw = 1$. If 1_K denotes a characteristic function of K and $\varphi = \psi * 1_K$, then the family

$$(3.5) \qquad \{\varphi_w\}_{w \in W_0}, \quad \varphi_w(v) = \varphi(v-w) \quad \text{for} \quad v \in W$$

forms the partition of unity on W which consists of functions of class $\mathscr{D}(W)$.

3.8. LEMMA (cf. [6], Lemma 3.1.2). *Let $\varphi \in \mathcal{D}(W)$ and $f \in \mathcal{S}(W)$. Then for every $k \geqslant 0$ and $1 \leqslant p \leqslant \infty$ the following estimate holds*

$$\|(\varphi_w f)^\wedge\|_{L^p(W)} \leqslant C \sum_{|\alpha|+|\beta|\leqslant 2k} \|(f^\wedge)^{(\beta)}_{(\alpha)}\|_{L^p(W)} \langle w\rangle^{-2k}, \quad w \in W,$$

where C depends only on φ and k (and not on w).

Proof. Since $\varphi_w^\wedge(v) = e^{i\langle w,v\rangle}\varphi^\wedge(v)$, we have

$$(\varphi_w f)^\wedge(v) = \varphi_w^\wedge * f^\wedge(v) = \int f^\wedge(v-u)\varphi^\wedge(u)e^{i\langle w,v\rangle}du.$$

Using the fact that $(1-\Delta_u)^k e^{i\langle w,u\rangle} = \langle w\rangle^{2k} e^{i\langle w,u\rangle}$ and then, integrating by parts and applying the Leibniz formula, we obtain

$$(\varphi_w f)^\wedge(v) = \langle w\rangle^{-2k}\int (1-\Delta_u)^k f^\wedge(v-u)\varphi^\wedge(u)e^{i\langle w,u\rangle}du$$
$$= \langle w\rangle^{-2k} \sum_{|\alpha|+|\beta|\leqslant 2k} \int (f^\wedge)^{(\beta)}_{(\alpha)}(v-u)P^{(\beta)}_{(\alpha)}(D)\varphi^\wedge(u)e^{i\langle w,u\rangle}du,$$

where $P(u) = (1+|u|^2)^k$. Hence

$$\|(\varphi_w f)^\wedge\|_{L^p(W)} \leqslant \langle w\rangle^{-2k} \sum_{|\alpha|+|\beta|\leqslant 2k} \frac{1}{\alpha!\beta!} \|(f^\wedge)^{(\beta)}_{(\alpha)}\|_{L^p(W)} \cdot \|P^{(\beta)}_{(\alpha)}(D)\varphi\|_{L^1(W)}$$
$$\leqslant C_{\varphi,k}\langle w\rangle^{-2k} \sum_{|\alpha|+|\beta|\leqslant 2k} \|(f^\wedge)^{(\beta)}_{(\alpha)}\|_{L^p(W)}.$$

The lemma is proved.

Now we are ready to give the proof of Theorem 3.1. In fact, it is enough to show that (3.3) holds for $a \in \mathcal{S}(W)$. Let $f = a^\wedge$, $1 \leqslant p \leqslant \infty$ and K, W_0, φ be as in (3.4), (3.5). Since $\operatorname{supp}\varphi f_w \subset \sup\varphi$, there exists by Lemma 3.7 and Remark 3.4 a constant C such that

$$|\pi(\varphi_w \cdot f)|_{J_p} \leqslant C\|\varphi_w^\wedge * f^\wedge\|_{L^p(W)}$$

for all $f \in \mathcal{S}(W)$ and $w \in W$. By an application of Lemma 3.8 with $k = N+1$ we obtain

$$(3.6) \qquad |\pi(\varphi_w f)|_{J_p} \leqslant C_0 \sum_{|\alpha|+|\beta|\leqslant 2(N+1)} \|(f)^{\wedge}{}^{(\beta)}_{(\alpha)}\|_{L^p(W)}\langle w\rangle^{-2(N+1)}$$

so the series

$$\sum_{w\in W_0} \pi(\varphi_w f)$$

is absolutely convergent in J_p. But on the other hand, $f = \sum_{w\in W_0}\varphi_w f$ with L^1-norm convergence, so

$$\pi(f) = \sum_{w\in W_0} \pi(\varphi_w f).$$

Thus by (3.6) our assertion follows.

Using Theorem 3.1, we now prove the following:

3.9. PROPOSITION. *Let a be a symbol of type $H^{(K,k)}$ with $K, k > 0$ satisfying the weak Hörmander condition with K_0, $k_0 > 0$. Consider $A = \pi(a^\wedge)$ as an operator on $L^2(X)$ with a domain $\mathcal{D}(A) = \mathcal{S}(X)$. Denote by $\bar{A}$ the closure of A. Then*

$$\bar{A} = (A^+)^*,$$

where A^+ is a formal adjoint to A.

Proof. We need only to show that $\mathcal{D}((A^+)^*) \subset \mathcal{D}(\bar{A})$. Let b be a "parametrix" for a (Theorem 2.18) and $B = \pi(b^\wedge)$. Thus $S = AB-I$, $T = BA-I$ are smoothing operators and by Theorem 3.1 B is a bounded operator on $L^2(X)$.

Let $u \in \mathcal{D}((A^+)^*) \subset L^2(X)$ and $u^0 = Au \in L^2(X)$ (here A is considered as an operator on $\mathcal{S}^*(X)$; see (3.1)). Take a sequence $u^0_n \in \mathcal{S}(X)$ such that $u^0_n \to u^0$ in $L^2(X)$ and set $u_n = Bu^0_n \in \mathcal{S}(X)$. Then

$$(3.7) \qquad Au_n = ABu^0_n = u^0_n + Su^0_n \xrightarrow{L^2(X)} u^0 + Su^0.$$

As S is smoothing $Su^0 \in \mathcal{S}(X)$. On the other hand, since B is bounded on $L^2(X)$,

$$(3.8) \qquad u_n = Bu^0_n \xrightarrow{L^2(X)} Bu^0 = BAu = u + Tu,$$

where $Tu \in \mathcal{S}(X)$. (3.7) and (3.8) imply $u \in \mathcal{D}(\bar{A})$.

Now, we are going to prove our main result. First let us state a lemma.

3.10. LEMMA. *Let L be a densely defined closed operator in a Hilbert space $\mathcal{H}$ and let $1 \leqslant p \leqslant \infty$. Assume that there exist $Q, S \in J_p$ such that*

$$(3.9) \qquad LQ = I + S.$$

Then for $\lambda \in \varrho(L)$ $R(L,\lambda) \in J_p$.

Proof. Set $R = R(L,\lambda)$. Then applying R to both sides of (3.9), we get

$$-Q + \lambda Q = R + RS.$$

Hence

$$R = (\lambda-1)Q - RS.$$

Since J_p is an ideal in $\mathcal{L}(\mathcal{H})$, our lemma is proved.

3.11. THEOREM. *Let $a \in C^\infty(W)$ be a symbol of polynomial growth satisfying the weak Hörmander condition with constants $K_0, k_0, \delta, \varrho > 0$. Then the spectrum of $A = \pi(a^\wedge)$ on $L^2(X)$ is either the whole plane $\boldsymbol{C}$ or it is discrete and*

$$\sum_{0 \neq \lambda \in \mathrm{Sp} A} |\lambda|^{-p} < \infty \quad \text{for} \quad p > \max\left(\frac{N}{K_0}, \frac{N}{k_0}\right), \quad p \geqslant 1. \tag{3.10}$$

Proof. Let b be a "parametrix" for a (Theorem 2.18). Then $b \in H^{(-K_0, -k_0)}_{(\delta,\varrho)}$ and thus $b^{(\beta)}_{(\alpha)} \in L^p$ for p satisfying our assumptions and all α, β. By Theorem 3.1 $\pi(b^\wedge) \in J_p$. On the other hand, by (2.21) $\pi(a^\wedge)\pi(b^\wedge) - I$ is a smoothing operator, so by Theorem 3.1 it is also in J_p. Set $Q = \pi(b^\wedge)$ and $S = \pi(a^\wedge)\pi(b^\wedge) - I$. Then

$$AQ = I + S$$

and if $\lambda \in \varrho(A)$, then by Lemma 3.10 $R(A, \lambda) \in J_p$. Our assertion follows immediately.

3.12. COROLLARY. *Let P and Q be polynomials in N variables of degrees n, m, respectively. Suppose they satisfy the Hörmander condition* (Def. 2.13) *with constants $\delta, \varrho > 0$ and $P(u) > 0$, $Q(u) > 0$ for $u \in \boldsymbol{R}^N$. Then the differential operators*

$$A = P(x)Q(D), \qquad B = P(x) + Q(D)$$

are essentially self-adjoint. Moreover, they have discrete spectra and

$$\sum_{\lambda \in \mathrm{Sp} A} |\lambda|^{-p} < \infty \quad \textit{for} \quad p > \max\left(\frac{N}{n\delta}, \frac{N}{m\varrho}\right), \ p \geqslant 1,$$

$$\sum_{\lambda \in \mathrm{Sp} B} |\lambda|^{-p} < \infty \quad \textit{for} \quad p > \max\left(\frac{2N}{n\delta}, \frac{2N}{m\varrho}\right), \ p \geqslant 1.$$

Proof. Since the symbol of A is $a(x, \xi) = P(x)Q(\xi)$ and the symbol of B is $b(x, \xi) = P(x) + Q(\xi)$, the corollary follows from Theorem 3.11, Proposition 3.9, Examples 2.19 (ii), and the fact that 0 is an eigenvalue neither for A nor for B.

3.13. COROLLARY. *Under the same assumptions on a, Theorem 3.11 is true also for the operator*

$$A = \pi(a^\wedge) + T,$$

where T is a bounded operator on $L^2(X)$.

Proof. Let Q, S be the operators which appeared in the proof of Theorem 3.11. We then have

$$AQ = I + S + TQ.$$

As J_p is an ideal in $\mathscr{L}(L^2(X))$, $S + TQ \in J_p$. Therefore we may apply Lemma 3.10. The rest of the proof is the same as that of Theorem 3.11.

For the further examples we shall use the following lemma.

3.14. LEMMA. *Let $h \in C^\infty(\boldsymbol{R}^N)$ fulfil the Hörmander condition* (Def. 2.13). *Then for every $r > 0$*

$$|h(x)|^r = a(x) + b(x)$$

where $a \in C^\infty(\boldsymbol{R}^N)$ satisfies the Hörmander condition (with the same δ) and $b \in \mathscr{K}(\boldsymbol{R}^N)$.

Proof. Take a natural n such that $r/2n < 1$ and $0 \leqslant f \in \mathscr{D}(\boldsymbol{R}^N)$ such that $|h(x)|^{2n} + f(x) \geqslant 1$ (Remarks 2.15 (ii)). Set

$$a(x) = \left(|h(x)|^{2n} + f(x)\right)^{r/2n}$$

and

$$b(x) = |h(x)|^r - a(x).$$

Then $a \in C^\infty(\boldsymbol{R}^N)$ and satisfies the Hörmander condition with the same δ as h, what can be checked directly. Moreover,

$$|b(x)| = a(x) - |h(x)|^r \leqslant f(x)^{r/2n}.$$

The lemma is proved.

3.15. COROLLARY. *Let A be an operator on $L^2(X)$ given by*

$$Au(x) = \int e^{ix\xi} |Q(\xi)|^s \hat{u}(\xi)\, d\xi + |P(x)|^r u(x)$$

for $u \in \mathscr{S}(X)$, where P and Q are polynomials in N variables of degrees n, m and satisfying the Hörmander condition with constants δ, ϱ; and $r, s > 0$. Then A is essentially self-adjoint and has a discrete spectrum. The eigenvalues of A satisfy

$$\sum_{\lambda \in \mathrm{Sp} A} |\lambda|^{-p} < \infty, \qquad p > \frac{2N}{nr\delta}, \frac{2N}{ms\varrho} \ \textit{and} \ p \geqslant 1.$$

Proof. By (2.1) the symbol of A equals

$$a(x, \xi) = |P(x)|^r + |Q(\xi)|^s.$$

Applying Lemma 3.14 to both summands of a separately, we find $b \in H^{(K,k)}_{(\delta,\varrho)}$, $K = nr/2$, $k \in ms/2$, and a bounded operator T on $L^2(X)$ such that

$$A = \pi(b^\wedge) + T. \tag{3.11}$$

Since A is symmetric by Proposition 3.9 it is self-adjoint. In particular, $\mathrm{Sp} A \neq \boldsymbol{C}$ as it is real and so by (3.11) and Corollary 3.13 $\mathrm{Sp} A$ is discrete and satisfies (3.10) for suitable p. To end the proof notice that A is a sum of two strictly positive operators, so it is injective. Thus $0 \notin \mathrm{Sp} A$.

4. Applications. Semigroups of measures. This section is devoted to some properties of semigroups of measures on the Heisenberg group. The result of Sections 2, 3 will be applied. First, let us recall some elementary facts.

Let G be a Lie group. A family of measures on G $(\mu_t)_{t\geqslant 0}$ is said to be a *semigroup of type* $\mathfrak{M}(G)$ (cf. [3], p. 229; [4], Def. III. 1) if the map $[0,\infty)\ni t\to\mu_t\in M(G)$ is *-weakly continuous and such that $\|\mu_t\|\leqslant 1$, $\mu_{t+s}=\mu_t*\mu_s$ for all $t,s\geqslant 0$ and $\mu_0=\delta$, the Dirac delta on G.

If $(\mu_t)_{t\geqslant 0}\in\mathfrak{M}(G)$, then for every $f\in\mathcal{D}(G)$ a limit

$$\lim_{t\to 0}\frac{1}{t}\langle\mu_t-\delta,f\rangle \tag{4.1}$$

exists and defines a distribution T which is dissipative, i.e. it satisfies

$$\operatorname{Re}\langle T,f\rangle\leqslant 0 \quad \text{for} \quad f\in\mathcal{D}(G);\quad f(e)=\|f\|_{L^\infty(G)}.$$

We shall call T an *infinitesimal generator* of $(\mu_t)_{t\geqslant 0}$.

Conversely, suppose that a dissipative distribution T on G is given. Then there exists exactly one semigroup $(\mu_t)_{t\geqslant 0}$ of type $\mathfrak{M}(G)$ such that (4.1) holds. Thus every dissipative distribution is an infinitesimal generator for a certain semigroup of type $\mathfrak{M}(G)$. ([3], Prop. 3, Prop. 4.)

Now, let (μ_t) be a semigroup of type $\mathfrak{M}(G)$, T its infinitesimal generator and $1\leqslant p<\infty$. $(\mu_t)_{t\geqslant 0}$ acts by left convolutions on $L^p(G)$ and in fact it gives a rise to a strongly continuous semigroup of contractions (P_t) (cf. [3]). Moreover an infinitesimal generator of (P_t) in a sense of Yosida–Hille ([18]) is nothing else but a closure on $L^p(G)$ of the operator L_T defined on $\mathcal{D}(G)$ by the convolutions with T. Conversely, if T is a dissipative distribution, then L_T is a closable operator on $L^p(G)$ and its closure is an infinitesimal generator in the sense of Yosida–Hille. The semigroup generated by L_T consists of operators acting by convolutions with $(\mu_t)_{t\geqslant 0}$, where $(\mu_t)_{t\geqslant 0}$ is a semigroup of type $\mathfrak{M}(G)$ generated by T (see [3], [19]). For dissipative distributions T on G we have the following decomposition. For every neighbourhood V of the unit element of G

$$T=S+\mu \tag{4.2}$$

for some $\mu\in M(G)$ and S a distribution supported by V ([3], Lemma 1; [4], Prop. II. 2). In particular, T is a sum of a measure and a distribution with a compact support. Thus for every strongly continuous representation π of G on a Banach space, $\pi(T)$ is well defined on a space of C^∞ vectors of π. Moreover, if $(\mu_t)_{t\geqslant 0}$ is a semigroup in $\mathfrak{M}(G)$, generated by T, then $(\pi(\mu_t))_{t\geqslant 0}$ is a strongly continuous semigroup of contractions on the space of representation and its infinitesimal generator is the closure of $\pi(T)$ ([3], § 12, Théorème).

Now, let us consider an operator A densely defined on a Banach space E and assume that A is an infinitesimal generator of a strongly continuous semigroup of contractions $(P_t)_{t\geqslant 0}$. Then for $0<s\leqslant 1$ the formula

$$A^{(s)}x=\frac{1}{\Gamma(-s)}\int_0^\infty\frac{1}{t^{s+1}}(I-P_t)x\,dt,\qquad x\in\mathcal{D}(A), \tag{4.3}$$

defines the operator $A^{(s)}$ on E such that its closure is an infinitesimal generator of a semigroup of contractions on E ([18], IX 11. (5)).

Let T be a dissipative distribution on $\boldsymbol{R}^N$. Then by (4.2) its Fourier transform is a continuous function. If $(\mu_t)_{t\geqslant 0}$ is a semigroup of measures generated by T, then

$$\hat{\mu}(\xi)=e^{t\hat{T}(\xi)}\quad\text{for}\quad \xi\in(\boldsymbol{R}^N)^*.$$

Moreover, if we denote by $T^{(s)}$ the distribution corresponding to $(L_T)^{(s)}$ (see (4.3)), then

$$\hat{T}^{(s)}(\xi)=-\big(-\hat{T}(\xi)\big)^s \tag{4.4}$$

(see [4]). By the above and an argument of [3], § 7, Example, Lemma 7 we obtain the following:

4.1. Proposition. *Let T be a dissipative distribution on G, the $(2N+1)$-dimensional Heisenberg group. Then*

(i) *T is an infinitesimal generator of a semigroup of measures $(\mu_t)_{t\geqslant 0}$ of type $\mathfrak{M}(G)$;*

(ii) *$(\sigma^1(\mu_t))_{t\geqslant 0}$ considered on $L^p(W)$, $1\leqslant p<\infty$, or on $C_0(W)$ is a strongly continuous semigroup of contractions which are in fact the operators of left convolution with measures $\nu_t=s^1(\mu_t)\in M(W)$. The infinitesimal generator of this semigroup is the closure of the operator $\sigma^1(T)$ with a domain $\mathcal{S}(W)$;*

(iii) *$(\pi^1(\mu_t))_{t\geqslant 0}$ is a strongly continuous semigroup of contractions on every $L^p(X)$, $1\leqslant p<\infty$, and $C_0(X)$. The infinitesimal generator of this semigroup is the closure of $\pi^1(T)$ defined on $\mathcal{S}(X)$.*

(iv) *For $0<s\leqslant 1$ we have*

$$\pi^1(T^{(s)})=\big(\pi^1(T)\big)^{(s)},\qquad \sigma^1(T^{(s)})=\big(\sigma^1(T)\big)^{(s)}.$$

From now on we assume G to be the $(2N+1)$-dimensional Heisenberg group. We fix an identification of G with Euclidean space $X\times X^*\times\boldsymbol{R}$ as described in Section 1 and a standard basis in $\mathfrak{G}$. Let X_j, Ξ_j be as in (1.2). If D and E are temperate distributions on $\boldsymbol{R}^N$, they can be considered as distributions on G through the above identification; the first acting on X, the other on X^*. More exactly: let $D'=D\otimes\delta_\xi\otimes\delta_t$, $E'=\delta_x\otimes E\otimes\delta_t$, where δ_ξ, δ_x, δ_t are Dirac deltas, respectively, on X^*, X, $\boldsymbol{R}$. Thus $D'+E'$ is a distribution on G and if moreover D, E are dissipative on $\boldsymbol{R}^N$ it is dissipative on G. A question arises what could be said about the semigroup of measures generated by $D'+E'$. We are going to consider this question for some rather special distributions D, E.

Namely, let P and Q be polynomials in N-variables satisfying:

(4.4)
(i) P and Q are of degree 2;
(ii) $P(u) \leqslant 0$, $Q(u) \leqslant 0$ for $u \in \boldsymbol{R}^N$;
(iii) P and Q satisfy the Hörmander condition with constants δ, ϱ.

Conditions (i) and (ii) imply that the distributions $\check{P}$, $\hat{Q} \in \mathcal{S}^*(\boldsymbol{R}^N)$ are dissipative (by the Taylor formula). We set for $0 < r, s \leqslant 1$

$$D = (\check{P})^{(r)}, \qquad E = (\hat{Q})^{(s)} \quad (\text{see } (4.4)). \tag{4.6}$$

Thus

$$\hat{D}(u) = -|P(u)|^r, \qquad \check{E}(u) = -|Q(u)|^s. \tag{4.7}$$

Note that if $P(u) \neq 0 \neq Q(u)$ for $u \in \boldsymbol{R}^N$, then (iii) and (ii) together imply that a symbol

$$a(x, \xi) = \hat{D}(x) + \check{E}(\xi) \tag{4.8}$$

satisfies the weak Hörmander condition with constants $K = 2r$, $k = 2s$, $K_0 = \delta r$, $k_0 = \varrho s$, δ, ϱ.

It is quite easy to produce a lot of examples of distributions satisfying (4.6). The simplest one is the following:

$$P(u) = Q(u) = -\sum_{j=1}^{N} u_j^2, \qquad 0 < r, s \leqslant 1.$$

Then

$$T = D' + E' = \Big(\sum_{j=1}^{N} X_j^2\Big)^{(r)} + \Big(\sum_{j=1}^{N} \Xi_j^2\Big)^{(s)}. \tag{4.9}$$

Our first and main application deals with the distribution (4.9).

4.2. Theorem. *Let T be given by* (4.9) *and let $0 < r, s \leqslant 1$ satisfy $(N-1)/(N+1) < r/s < (N+1)/N$. Then the semigroup $(\mu_t)_{t>0}$ generated by T consists of absolutely continuous measures. Their densities $(f_t)_{t>0}$ are square integrable on G and satisfy*:

$$\|f_t\|_{L^2(G)} \leqslant C \cdot t^{-(N+1)/2d}$$

for some constant C and $d = 2rs/(r+s)$.

Proof. Using the Plancherel theorem for G, it is enough to show that $\pi^\lambda(\mu_t)$ $(\lambda \in \boldsymbol{R}\setminus\{0\},\ t > 0)$ are of the Hilbert–Schmidt type and the

integral

$$\int_{-\infty}^{\infty} \|\pi^\lambda(\mu_t)\|_{\mathrm{HS}}^2 |\lambda|^N d\lambda \tag{4.10}$$

is convergent and its value is less then a constant times $t^{-(N+1)/d}$.

First note that $\pi^\lambda(T)$ is an operator on $L^2(X)$ whose symbol is

$$a_\lambda(x, \xi) = -\lambda^{2s}\Big(\sum x_i^2\Big)^s - \Big(\sum \xi_i^2\Big)^r.$$

Thus, by Corollary 3.15, $\pi^\lambda(T)$ has a discrete spectrum $\{z_n^\lambda\}$ $(z_n^\lambda \neq 0)$ and

$$\sum_{n=1}^{\infty} |z_n^\lambda|^{-p} < \infty \quad \text{for} \quad p > \max\Big(\frac{N}{r}, \frac{N}{s}\Big). \tag{4.11}$$

This implies that $\pi^\lambda(\mu_t)$ is of HS type. It was proved earlier by J. Cygan and A. Hulanicki that (4.11) is sufficient for (4.10) to hold. Here is their proof.

At first let us notice that the eigenvalues of $\pi^\lambda(T)$ satisfy

$$z_n^\lambda = |\lambda|^d z_n^1, \quad \text{where} \quad \frac{1}{d} = \frac{1}{2r} + \frac{1}{2s}.$$

In fact, it is easy to check that if $\varphi \in L^2(X)$ is an eigenvector for $\pi^1(T)$ corresponding to an eigenvalue z, then $x \to \varphi(|\lambda|^s x)$ is an eigenvector for $\pi^\lambda(T)$ corresponding to an eigenvalue $|\lambda|^d \cdot z$. Now, since

$$\|\pi^\lambda(\mu_t)\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty} e^{2tz_n^\lambda} = \sum_{n=1}^{\infty} e^{2t|\lambda|^d z_n}$$

(we put $z_n^1 = z_n$), it follows by (4.11) and (1.14) that $\pi^\lambda(\mu_t)$ are Hilbert–Schmidt operators and (4.10) equals

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} e^{2t|\lambda|^d z_n} |\lambda|^N d\lambda = 2\int_0^{\infty} \sum_{n=1}^{\infty} e^{2t\lambda^d z_n} \lambda^N d\lambda = 2\sum_{n=1}^{\infty} \int_0^{\infty} e^{2t\lambda^d z_n} \lambda^N d\lambda.$$

By the change of variables: $\nu = -2t\lambda^d z_n$ we get

$$2\sum_{n=1}^{\infty} \int_0^{\infty} e^{2t\lambda^d z_n} |\lambda|^N d\lambda = \frac{2}{d} \sum_{n=1}^{\infty} (2t|z_n|)^{-(N+1)/d} \int_0^{\infty} \nu^{(N+1)/d-1} e^{-\nu} d\nu$$
$$= C \cdot t^{-(N+1)/d} \sum_{n=1}^{\infty} |z_n|^{-(N+1)/d},$$

where $C = \frac{2}{d}\cdot\left(\frac{1}{2}\right)^{(N+1)/d}\Gamma\left(\frac{N+1}{d}\right)$. By assumptions on r and s

$$\frac{N+1}{d} = \frac{N+1}{2r}+\frac{N+1}{2s} > \max\left(\frac{N}{r},\frac{N}{s}\right) \quad \text{and} \quad \frac{N+1}{d}\geqslant 1.$$

Hence by (4.11) our proof is complete.

4.3. Remark. The assertion of Theorem 4.2 is valid in the following two cases:

(i) $N = 1$ and $0 < r, s \leqslant 1$,

(ii) N arbitrary and $r = s$.

4.4. Corollary. *The operator L_T with $\mathcal{D}_{L_T} = \mathcal{D}(G)$ for its domain considered on different $L^p(G)$ $(1 \leqslant p < \infty)$ has the same spectrum.*

Proof. By elementary properties of semigroups of measures and their generators there exists a measure μ such that

$$L_\mu \cdot f = R_{C_0(G)}(L_T, 1)\cdot f, \quad f \in C_0(G),$$

and moreover

$$L_\mu \cdot f = \int_0^\infty e^{-t} L_{\mu_t} f\, dt \quad \text{(cf. [18])}.$$

Hence

$$\langle \mu, f\rangle = \int_0^\infty e^{-t}\langle \mu_t, f\rangle\, dt, \quad f \in C_0(G).$$

Since $\|\mu_t\|_{M(G)} \leqslant 1$ and μ_t are absolutely continuous, the last formula implies that so is μ (cf. [10], Cor. 3.5). Consequently by [9], Proposition 2.5, and the symmetry of $L^1(G)$ (see (1.3)) we get

$$\mathrm{Sp}_{L^1(G)} L_\mu = \mathrm{Sp}_{L^p(G)} L_\mu, \quad 1 \leqslant p < \infty,$$

and since $1 \in \varrho_{L^p(G)}(L_T)$, this implies

$$\mathrm{Sp}_{L^1(G)} L_T = \mathrm{Sp}_{L^p(G)} L_T, \quad 1 \leqslant p < \infty.$$

The following two theorems tell something about the operators of type (4.5) acting in representations of G.

4.5. Theorem. *Let P, Q satisfy* (4.5) *and let*

$$a(x,\xi) = D^\wedge(x,\xi) = -|P(x)|^r - |Q(\xi)|^s, \quad 0 < r, s \leqslant 1.$$

Consider the operator L_D defined on $\mathcal{S}(W)$ as an unbounded operator on $L^p(W)$ for $1 \leqslant p < \infty$ and on $C_0(W)$. Denote by $L_D^{(p)}$ $(L_D^{(0)})$ its closure in $L^p(W)$ $(C_0(W))$. Then:

(i) *$L_D^{(p)}$ generates on $L^p(W)$ (or $C_0(W)$ for $p = 0$) a strongly continuous semigroup of contractions $(T_t^{(p)})_{t\geqslant 0}$ such that, for each $t > 0$, $T_t^{(p)} = L_{\mu_t}$, where $\mu_t \in M(W)$*

(ii) *For every $t > 0$, $\mu_t = f_t\,dw$, where $f_t \in L^2(W)$. Moreover, $\|f_t\|_{L^2(W)} \leqslant C_p t^{-p/2}$ for $p > \max(N/r\delta, N/s\varrho)$ and some $C_p > 0$.*

(iii) *$R(L_D^{(1)}, 1) = L_f$, where $f \in L^1(W)$.*

(iv) *The spectrum of L_D on $L^2(W)$ is discrete and for every $1 \leqslant p < \infty$ $\mathrm{Sp}_{L^p(W)} L_D = \mathrm{Sp}_{L^2(W)} L_D = \mathrm{Sp}_{C_0(W)} L_D$*

(v) *For every $\lambda \in \mathrm{Sp}_{L^1(W)} L_D$ there exists φ_λ such that for every $1 \leqslant p < \infty$, $p = 0$ $\varphi_\lambda \in \mathcal{D}(L_D^{(p)})$ and $L_D^{(p)}\varphi_\lambda = \lambda\varphi_\lambda$.*

Proof. It follows from (4.5) and the remarks preceding (4.5) that there exists a dissipative distribution F on G such that $\sigma^1(F) = L_D$. Thus Proposition 4.1 (ii) implies (i). Now, Corollary 3.15 implies that the operator $A = \pi(D)$ has a discrete spectrum on $L^2(X)$ and

$$\sum_{\lambda\in \mathrm{Sp}A} |\lambda|^{-p} \leqslant C < \infty \quad \text{for} \quad p > \max\left(\frac{N}{r\delta},\frac{N}{s\varrho}\right).$$

Hence

$$\|\pi(\mu_t)\|_{\mathrm{HS}}^2 = \sum_{\lambda\in\mathrm{Sp}A} e^{-2t|\lambda|} = \sum_{\lambda\in\mathrm{Sp}A}(e^{-2t|\lambda|/p})^p$$

$$\leqslant \sum_{\lambda\in\mathrm{Sp}A}\left(\frac{p}{2}\,\frac{1}{t|\lambda|}\right)^p \leqslant \left(\frac{p}{2}\right)^p\cdot Ct^{-p} \quad \text{if} \quad p > \max\left(\frac{N}{r\delta},\frac{N}{s\varrho}\right).$$

This together with (1.14) implies (ii). By the same argument as in the proof of Corollary 4.4 we show that (ii) implies (iii). Now, an application of (1.11) with $\lambda = 1$ yields that L_D acting on $L^2(W)$ is a direct sum of a countable number of copies of $A = \pi(D)$ acting on $L^2(X)$. Therefore,

$$\mathrm{Sp}_{L^2(W)} L_D = \mathrm{Sp}_{L^2(X)} A,$$

and hence $\mathrm{Sp}_{L^2(W)} L_D$ is a discrete subset of C. The equality of spectra in (iv) follows by (iii) and the symmetry of $L^1(W)$ (see Sec. 1), so (iv) is proved. At last, to prove (v) take $\lambda \in \mathrm{Sp}_{L^1(W)} L_D$. By (iv) there exists an eigenvector $\varphi \in L^1(W)$ corresponding to a given $\lambda \in \mathrm{Sp}_{L^1(W)} L_D$. Let $g, h \in \mathcal{D}(W)$. Then $\varphi_\lambda = \varphi \# g \# h$ satisfies (v).

4.6. Lemma. *Let D be a dissipative distribution on W and let $p \in [1, \infty) \cup \{0\}$. Denote by $\pi(D)^{(p)}$ the closure of $\pi(D)$ on $L^p(X)$ (or on $C_0(X)$ if $p = 0$). Then*

(i) *$\mathrm{Sp}_{L^p(X)}\pi(D) \subset \mathrm{Sp}_{L^1(W)} L_D$, $\mathrm{Sp}_{C_0(X)}\pi(D) \subset \mathrm{Sp}_{L^1(W)} L_D$.*

(ii) *If λ is an eigenvalue for $L_D^{(1)}$ then it is an eigenvalue for $\pi(D)^{(p)}$.*

Proof. (i) Let $\lambda \in \varrho(L_D^{(1)})$. Then there exists a measure μ_λ such that

$$(\lambda - D) \# \mu_\lambda = \mu_\lambda \# (\lambda - D) = \delta.$$

Suppose that $(\lambda - \pi(D)^{(p)})f = 0$ for some $f \in \mathscr{D}(\pi(D)^{(p)})$. Then $f \in \mathscr{D}(\pi(\mu_\lambda \# D)^{(p)})$ and

$$f = \pi(\mu_\lambda \# (\lambda - D))^{(p)} f = \pi(\mu_\lambda)\pi(\lambda - D)^{(p)} f = 0$$

so that $\lambda - \pi(D)^{(p)}$ is injective. Now, for any $f \in L^p(X)$ (or $f \in C_0(X)$ for $p = 0$) $\pi(\mu_\lambda) f \in \mathscr{D}(\pi(D)^{(p)})$ and

$$(\lambda - \pi(D)^{(p)}) \cdot \pi(\mu_\lambda) \cdot f = \pi((\lambda - D) \# \mu_\lambda)^{(p)} f = f.$$

This shows that $\lambda - \pi(D)^{(p)}$ is onto and ends the proof of (i).

(ii) Let φ_λ be an eigenvector of $L_D^{(1)}$ corresponding to λ. Then for any $u \in L^p(X)$ (or $u \in C_0(X)$)

$$\pi(D)^{(p)} \pi(\varphi_\lambda)^{(p)} \cdot u = \pi(D \# \varphi_\lambda)^{(p)} u = \lambda \pi(\varphi_\lambda)^{(p)} \cdot u.$$

Hence $\pi(\varphi_\lambda)^{(p)} \cdot u$ is an eigenvector for $\pi(D)^{(p)}$ corresponding to λ.

4.7. THEOREM. *Let P, Q be polynomials satisfying* (4.5) *and let A be an operator defined on $\mathscr{S}(X)$ by the symbol*

$$a(x, \xi) = -|P(x)|^r - |Q(\xi)|^s, \quad 0 < r, s \leqslant 1.$$

Denote by $A^{(p)}$ (resp. by $A^{(0)}$) the closure of A on $L^p(X)$, $1 \leqslant p < \infty$ (resp. on $C_0(X)$). Then:

(i) *For all $p \in [1, \infty) \cup \{0\}$ $A^{(p)}$ is an infinitesimal generator of a strongly continuous semigroup of contractions $(B_t^{(p)})_{t \geqslant 0}$.*

(ii) *$B_t^{(2)}$ are of trace class and*

$$\mathrm{Tr}\, |B_t^{(2)}| \leqslant C_p t^{-p}, \quad t > 0,$$

for $p > \max(N/r\delta, N/s\varrho)$ and some $C_p > 0$.

(iii) $\mathrm{Sp}\, A^{(p)} = \mathrm{Sp}\, A^{(q)}$ *for all $p, q \in [1, \infty) \cup \{0\}$ and this common spectrum is discrete.*

(iv) *For every λ from the common spectrum of $A^{(p)}$* (cf. (iii)) *there exists a common eigenvector $u_\lambda \in \bigcap_{p \in [1,\infty) \cup \{0\}} \mathscr{D}(A^{(p)})$.*

Proof. Note that $A = \pi(D)$, where $D \in \mathscr{S}^*(W)$ is as in Theorem 4.5. By a similar argument as in the proof of Theorem 4.5 (i), (ii) we get (i) and (ii). Then, (iii) follows immediately by Theorem 4.5 (iv) and Lemma 4.6. We have to prove (iv), yet. Let $\lambda \in \mathrm{Sp}\, A^{(0)}$. Then by Lemma 4.6 $\lambda \in \mathrm{Sp}\, L_D^{(1)}$ and by Theorem 4.5 (v) there exists $\varphi_\lambda \in L^1(W)$ which is an eigenvector for $L_D^{(1)}$ corresponding to λ. Now, take an arbitrary $u \in \mathscr{S}(X)$ and set $u_\lambda = \pi(\varphi_\lambda) \cdot u$. Then u_λ is a needed common eigenvector for all $A^{(p)}$ corresponding to λ.

References

[1] R. Beals, *A general calculus of pseudodifferential operators*, Duke Math. J. 42 (1975), 1–42.

[2] R. Beals, C. Fefferman, *Spatially inhomogeneous pseudodifferential operators I*, Comm. Pure Appl. Math. 27 (1974), 1–24.

[3] M. Duflo, *Representations de semi-groups de mesures sur un groupe localement compact*, Ann. Inst. Fourier, Grenoble 28. 3 (1978), 225–249.

[4] J. Faraut, *Semi-groupes de mesures complexes et calcul symbolique sur les générateurs infinitésimeaux de semi-groupes d'operateurs*, Ann. Inst. Fourier 20 (1970), 235–301.

[5] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Arkiv för matematik, 13. 2 (1975), 161–207.

[6] R. Howe, *Quantum mechanics and partial differential equations*, preprint.

[7] L. Hörmander, *Linear partial differential operators*, Springer-Verlag, New York 1963.

[8] — *Hypoelliptic second order differential equations*, Acta. Math. 119 (1967), 264–293.

[9] A. Hulanicki, *On the spectrum of convolution operators on groups with polynomial growth*, Inv. Math. 17 (1972), 135–142.

[10] — *Subalgebra of $L^1(G)$ associated with Laplacian on a Lie group*, Colloq. Math. 31.2 (1974), 259–287.

[11] — *A tauberian property of the convolution semi-group generated by $X^2 - |Y|^\gamma$ on the Heisenberg group*, in: *Harmonic analysis in Euclidean spaces*, Proc. Symp. Pure Math. 35.2 (1975), 403–406.

[12] H. Leptin, *On group algebras of nilpotent Lie groups*, Studia Math. 47 (1973), 37–49.

[13] — *Lokal kompakte Gruppen mit Symmetrischen Algebren*, Inst. Naz. Alta Math., Symp. Math. vol. 22 (1977).

[14] M. Reed, B. Simon, *Functional analysis*, Academic Press, New York–London 1972.

[15] —, — *Fourier analysis, self adjointness*, Academic Press, New York–San Francisco–London 1975.

[16] C. E. Rickart, *General theory of Banach algebras*, New York 1960.

[17] Šubin, *Pseudodifferential operators and spectral theory*, Moscow 1978 (in Russian).

[18] K. Yosida, *Functional analysis*, Berlin 1965.

[19] A. Hulanicki, *A class of convolution semi-groups of measures on a Lie group*, preprint.

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