Representation of continuous linear functionals on a subspace of a countable Cartesian product of Banach spaces

by

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Abstract. In a wide variety of applications (e.g., to harmonic analysis, ergodic theory, approximation theory, summability) spaces of the following kind occur. We are given a sequence $\mathcal{A} = (B_k)$ of Banach spaces and a solid Banach sequence space $E$ with a Schauder basis, and we consider elements from the Cartesian product of the $B_k$, namely sequences $x := (x_k)$ such that $x_k \in B_k$ for each $k$, such that the sequence of norms $(\|x_k\|_p)$ lies in the given space $E$. The "composed" set $EB$ of such sequences $x$ is a subset of the Cartesian product and may be given the obvious norm topology. It is then of considerable interest to be able to determine the form of the general continuous linear functional on the space $EB$, namely to identify the functional dual $(EB)^*$, and that is the object of this paper. Some relaxations are possible (e.g., we may take $p$-norms or semi-norms instead of norms in some places). Several examples from widely different sources are given in the final part of the paper.

We shall suppose in our theorems that the index $k$ of our spaces $B_k$ ranges over $\mathbb{Z}^+$, the set of non-negative integers; we could equally well take the set $Z$ of all integers, but we choose the former as yielding slightly simpler notation.

Given a sequence $\mathbf{x} = (x_k)_{k \in \mathbb{Z}^+}$ with $x_k \in B_k$ ($k = 0, 1, \ldots$), we shall denote the $n$-th section of $x$ by

$$x^{[n]} := (x_k)^{[n]} := (x_0, x_1, \ldots, x_n, 0, 0, \ldots),$$

where $\theta_k$ is the zero element of $B_k$ ($k \in \mathbb{Z}^+$). Similarly, for a sequence $\omega := (\omega_k)_{k \in \mathbb{Z}^+}$, where $\omega$ is the space of all complex-valued sequences, its $n$th section is

$$\omega^{[n]} := (\omega_k)^{[n]} := (\omega_0, \omega_1, \ldots, \omega_n, 0, 0, \ldots) = \sum_{k=0}^{n} \omega_k \theta_k,$$

where $\theta^* := (0, \ldots, 0, 1, 0, 0, \ldots)$, with 1 in the $k$th place. A sequence space is a linear subspace of $\omega$.

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Theorem 1. Let $\mathcal{B} := (B_n)_{n \geq 1}$ be a sequence of complete linear $p$-normed spaces, with the $p$-norms $\| \cdot \|_{B_n}$ (where $0 < p \leq 1$). Let $E$ be a sequence space, complete with respect to the $q$-norm $\| \cdot \|_q$ (where $0 < q \leq 1$), with continuous coordinate projections, and such that

1. $\| e_n \|_{B_n} = 0$ for all $n$.
2. $\lim_{n \to \infty} \| u - u^{(n)} \| = 0$ for all $u \in E$.

Define the composed space

3. $E\mathcal{B} := \{ (x_k)_{k \geq 1} : x_k \in B_k \ (\forall k \geq 1) \}$, where $(\| x_k \|_{B_k})_{k \geq 1}$.

and

4. $\forall x \in E\mathcal{B}, \| x \|_{E\mathcal{B}} := \sum_{k \geq 1} \| x_k \|_{B_k}$.

Then $E\mathcal{B}$ is a linear space, complete with respect to the $p\cdot q$-norm $\| \cdot \|_{E\mathcal{B}}$, and

5. $\forall x \in E\mathcal{B}, \lim_{n \to \infty} \| x_n \|_{E\mathcal{B}} = 0$.

We remark that property (1) says that $(e_k)_{k \geq 1}$ is a Schauder basis for $E$, or that $E$ has the AK-property. While property (2) (i) says that $E$ is solid. An obvious consequence of (2) is

6. $\forall x \in E, \| x_n \|_{B_n} = \| x \|_{E\mathcal{B}}$.

Definition. Let $(E, \| \cdot \|_E)$, $E \subset C$, be a $p$-normed sequence space (0 < $q$ < 1), and let 0 < $p$ < 1. Define the $p$-dual of $E$ to be

$$E^p := \{ \phi \in C : \sup_{\| x \|_E \leq 1} \| \phi \cdot x \|_E < \infty \}.$$ 

For $p = 1$ we get the well-known $\alpha$-dual of $E$; it follows by elementary arguments (as for the $\alpha$-dual) that:

If $E$ is a $q$-normed sequence space (0 < $q$ < 1), and 0 < $p$ < 1, then $E^p$ is a $p$-normed linear space and

7. $\forall x \in E, \forall \phi \in E^p, \sum_{k \geq 1} \| x_k \|_{B_k} \| \phi \|_{E^p}$

8. If $(e_k)_{k \geq 1} \subset E$ then $E^p$ is a Banach space.
and hence, \( \forall r \in \mathbb{Z}^+, \forall m \geq N \),

\[
(15) \quad \| (a^m - a^r) \|^2_{\mathbb{B}} \leq \| (a^m - a^r) \|^2_{\mathbb{B}} + \| (a^m - a^r) \|^2_{\mathbb{B}} \\
\leq \epsilon + \sum_{m \neq r} \| a^m - a^r \|_{\mathbb{B}}^2 |a^m| \times |a^r| \\
\rightarrow \epsilon \text{ as } n \rightarrow \infty \quad \text{by (14)}.
\]

Since \( \| a^m \|_{\mathbb{B}} \rightarrow 0 \) as \( \epsilon \rightarrow 0 \), we have, by (1), that \( \forall \epsilon > 0, \exists N = M(\epsilon, m) \) such that

\[
(16) \quad \| (a^m) - (a^n) \|^2_{\mathbb{B}} \leq \epsilon \quad (\forall r, s \geq N).
\]

Choose a fixed \( m \) in (15), say \( m = N \), and any \( r, s \geq M(\epsilon, N) \) in (16), and we then get, using (2) and the triangle inequality in \( \mathbb{B} \), in the form

\[
\| (a^m - a^n) \|^2_{\mathbb{B}} - \| (a^m - a^n) \|^2_{\mathbb{B}} \leq \| (a^m - a^n) \|^2_{\mathbb{B}} + \| (a^m - a^n) \|^2_{\mathbb{B}} \\
\leq 3\epsilon \quad \text{by (16) and (15)}.
\]

Thus \( \| (a^m) \|^2_{\mathbb{B}} \) is a Cauchy sequence in \( E \). But \( E \) is complete and has continuous coordinate projections, so the limit in \( E \) must be \( \| a^m \|_{\mathbb{B}} e^m \); that is, \( a^* \in \mathbb{B} \).

Finally, by (5) and (15),

\[
\| a^m - a^n \|^2_{\mathbb{B}} = \lim_{r \rightarrow \infty} \| (a^m - a^n) \|^2_{\mathbb{B}} \leq \epsilon \quad (\forall m \geq N),
\]

so \( (a^m) \) is convergent to \( a^* \) in \( (E, \mathbb{B}) \), and \( \mathbb{B} \) is therefore complete.

Proof of Theorem 2. (a) Let \( \varphi : = \langle \varphi_k \rangle_{k \in \mathbb{Z}} + (E^\mathbb{B})^* \).

Then, for each \( e \in \mathbb{B} \),

\[
\sum_{k \in \mathbb{Z}^+} |\varphi_k(e_k)| \leq \sum_{k \in \mathbb{Z}^+} \| e_k \|_{\mathbb{B}} \| \varphi_k \|_{(E^\mathbb{B})^*} \quad \text{by (7)}
\]

\[
\| \varphi \|_{(E^\mathbb{B})^*} \leq \| \varphi \|_{(E^\mathbb{B})^*} \quad \text{by definition (4)}.
\]

Hence the functional \( \varphi \) defined by (10) satisfies \( \varphi \in (E\mathbb{B})^* \) and

\[
(17) \quad \| \varphi \|_{(E^\mathbb{B})^*} \leq \| \varphi \|_{(E^\mathbb{B})^*}.
\]

(b) Suppose \( \varphi \in (E\mathbb{B})^* \). For each \( h \in \mathbb{Z}^+ \), write \( \mathbb{B}_h : = \langle \varphi_k \rangle_{k \in \mathbb{Z}} + 1 \).

If \( y \in \mathbb{B}_h \) then \( y \in \mathbb{B} \) and

\[
\| y \|_{\mathbb{B}} = \| y \|_{\mathbb{B}} \leq \| y \|_{\mathbb{B}} = \| y \|_{\mathbb{B}} \| a^m \|_{\mathbb{B}}.
\]

Hence \( \mathbb{A}_h \) and \( \mathbb{B}_h \) are isomorphic as complete \( p \)-normed spaces, and \( \mathbb{A}_h \) is a closed subspace of \( \mathbb{B}_h \). Thus the restriction of \( \varphi \) to \( \mathbb{A}_h \) yields a continuous linear functional on \( \mathbb{A}_h \), so that

\[
(18) \quad \exists y \in \mathbb{A}_h : \forall y \in \mathbb{A}_h, \varphi(y) = \varphi_k(y_k).
\]

Also, given \( \epsilon > 0 \) satisfying \( 0 < \epsilon \leq 1 \),

\[
(19) \quad \exists g \in \mathbb{B}_h : \| g \|_{\mathbb{B}_h} = 1 \quad \text{and} \quad \varphi_k(g_k) \geq (1 - \epsilon) \| \varphi_k \|_{(E^\mathbb{B})^*} > 0,
\]

because, for any \( a \in \mathbb{B}_h \),

\[
\varphi_k(a^m) = \epsilon^k \varphi_k(a) \quad \text{and} \quad \| \varphi_k(a) \|_{(E^\mathbb{B})^*} = \| a \|_{\mathbb{B}}.
\]

Now by (5) of Theorem 1, for any \( x \in \mathbb{B} \), we have, since \( \varphi \in (E\mathbb{B})^* \),

\[
(20) \quad \varphi(x) = \lim_{n \rightarrow \infty} \sum_{k \in \mathbb{Z}} \varphi_k(x_k) = \sum_{k \in \mathbb{Z}} \varphi_k(x_k) \quad \text{by (18)}.
\]

Let \( (g_k) \) be chosen as in (19) and, given \( u \in \mathbb{B} (u \neq 0) \), define

\[
\lambda := (a^m)_{a \in \mathbb{A}}, \quad \lambda := |u|^2 \varphi_k \quad (\forall k \in \mathbb{Z}^+).
\]

Then

\[
\| \lambda \|_{\mathbb{B}} \leq \| (u^m) \|_{\mathbb{B}} + \lambda \quad \text{since each \( \mathbb{B}_k \) is \( p \)-normed}
\]

\[
\| (u^m) \|_{\mathbb{B}} + \| \lambda \|_{(E^\mathbb{B})^*} \quad \text{since \( |u|^2 \varphi_k \geq 0 \) by (19)}
\]

\[
= \| u \|_{\mathbb{B}} \quad \text{by (8)}
\]

Thus \( \lambda \in \mathbb{B} \) and we now have

\[
\| \varphi \|_{(E^\mathbb{B})^*} = \| \varphi \|_{(E^\mathbb{B})^*}
\]

\[
\geq \| \varphi(h) \|_{(E^\mathbb{B})^*} \quad \text{since, by Theorem 1, \( \mathbb{B} \) is \( p \)-normed}
\]

\[
= \sum_{k \in \mathbb{Z}} \varphi_k(a_k) \quad \text{by (20)}
\]

\[
= \sum_{k \in \mathbb{Z}} |u_k|^2 |\varphi_k|_{(E^\mathbb{B})^*} \quad \text{since \( \varphi_k \) is linear and \( \varphi_k(g_k) \geq 0 \)}
\]

\[
\geq (1 - \epsilon) \sum_{k \in \mathbb{Z}} |u_k|^2 |\varphi_k|_{(E^\mathbb{B})^*} \quad \text{by (19)}.
\]

Letting \( \epsilon \rightarrow 0 \), we get

\[
\| \varphi \|_{(E^\mathbb{B})^*} \geq \sum_{k \in \mathbb{Z}} |u_k|^2 |\varphi_k|_{(E^\mathbb{B})^*}.
\]
thus \((\|y\|_g)_{g \in B^+} \in K^{\infty}\) (that is, \(\varphi \in K^{\infty}\)) and, taking the supremum over all \(u \in B\) (\(u \neq 0\)) and using (6), we get

\[
\|F\| \geq \|\langle \psi, y \rangle\|_{L^\infty} = \|\psi\|_{K^{\infty}}
\]

which, when combined with (17), yields (12).

It remains to show that the \(\varphi\) in the representation (10) is unique. Suppose that \(\varphi\) and \(\psi\) are two such choices satisfying (10); then

\[
\sum_{k \geq n} \langle \varphi_k(s_k) - y_k(s_k) \rangle = 0 \quad (\forall s \in B).
\]

Now for any \(s \in B\), we have \(\langle \varphi(s), \varphi(s) \rangle, \langle \psi(s), \varphi(s) \rangle \in B\), whence \(\varphi(s) - \varphi(s) = 0 \quad (\forall s \in B)\), and so \(\varphi = \psi\).

Example 1. The Wiener space \(T_1\).

\[
T_1 := \{f \in C(B), \|f\|_{L^\infty} := \sum_{k \geq 0} \max_{s \in B} |f(s)| < +\infty\}.
\]

This space was introduced by Wiener [13], p. 27, in the formulation of one of his tauberian theorems; the functional dual \(T_1^*\) was determined by Goldberg [5], thus enabling him to give a quick proof of the tauberian theorem.

Now \(T_1\) is a closed linear subspace of \(L_1([k, k+1])\) (this in turn is a subspace of the \(L_1\) of Example 8 below, \(\varphi(s) = s\)), so by the Hahn-Banach Theorem, if \(F \in T_1^*\) then \(F\) can be extended to a continuous linear functional on \(L_1([k, k+1])\) with the same norm. By the Riesz Representation Theorem, \(F \in C([k, k+1])^*\) implies

\[
\exists V \in BV([k, k+1]): F(f) = \int_k^{k+1} f(s) dV(s) \quad (\forall f \in C([k, k+1])),
\]

where the \(V\) are normalized. Thus by Theorem 2, if \(F \in T_1^*\) then

\[
F(f) = \int_k^{k+1} f(s) dV(s) = \int_k^{k+1} dV(s) \quad (\forall f \in T_1),
\]

where we must, in order to get an unambiguous definition of \(V(t)\) at the integers, define

\[
V(t) := \begin{cases} V_k(t) + \sum_{r \neq t} V_{r-1}(r), & k \leq t < k+1, \ k \geq 0; \\ V_k(t) - \sum_{r \neq t} V_{r-1}(r), & k \leq t < k+1, \ k < 0. \end{cases}
\]

Then \(V\) is a normalized function of bounded variation in each interval \([k, k+1]\), and

\[
\|F\| = \sup_{k \geq 0} \int_k^{k+1} |dV(s)| = \sup_{k \geq 0} \int_k^{k+1} |dV| < +\infty.
\]

This representation was obtained in a different way by Goldberg [5], Theorem G, but with only the inequality \(\|F\| \geq \sup_{k \geq 0} \int_k^{k+1} |dV|\).

Actually, more is true. For suppose that the given \(f \in T_1^*\) has the above representation. Take \(0 < \delta < 1\) and any \(k \in Z\), and define

\[
f_{k, \delta}(t) := 1 - \delta|t-k| \quad (t \in [k-\delta, k+\delta]), \quad f_{k, \delta}(t) = 0 \quad \text{otherwise};
\]

then \(f_{k, \delta} \in T_1\) and \(F(f_{k, \delta}) = \int f_{k, \delta} dV = V(k+1) - V(k-1)\) as \(\delta \to 0\).

Thus the jumps of \(V\) at the points \(k\) are uniquely determined by \(F\). Write

\[
U_r(t) := \begin{cases} V(r+1) - V(r) & \text{for } (r \leq t < r+1), \\ V(r) & \text{for } (t = r), \end{cases}
\]

Now suppose, if possible, that for some \(r \in Z\), \(\int_r^{r+1} |dU_r| > 0\). Choose \(0 < \epsilon < \int_r^{r+1} |dU_r|\). Then since \(U_r\) is continuous at \(r\) and \(r+1\), there exists \(\delta > 0\) such that

\[
g(r) := g(r+1) := 0 \quad \text{and} \quad \int_r^{r+1} g(t) dt > \int_r^{r+1} |dU_r| - \epsilon > 0.
\]

Define \(f_1(t) := g(t) \cdot f(t)\) on \([r, r+1]\), \(f_1(t) = 0\) otherwise; then \(f_1 \in T_1\), but \(F(f_1) = \int f_1 dU_r \neq 0\), contradiction. Thus \(U_r(t) = 0\).

This means that if \(F \in T_1^*\), then the function \(V(t)\) is determined uniquely, up to a constant, in each interval \((r, r+1)\) \((r \in Z)\). Of course, the converse is also true, in the sense that each normalized function of bounded variation with sup \(\int_0^{k+1} |dV| < +\infty\) defines an \(F \in T_1^*\) such that

\[
F(f) := \int f dV \quad (\forall f \in T_1), \quad \text{with} \quad |F| = \sup_{k \geq 0} \int_k^{k+1} |dV|.
\]

Example 2. The space \(T_p^p, 1 \leq p < \infty\), \(\alpha \in Z^+\).

\[
T_p^p := \{f | f \in C([0, \infty)), \|f\|_{L^p} := \left(\sum_{k=0}^{\infty} \max_{s \in [k, k+1]} |f(s)|^p\right)^{1/p} < +\infty\}.\]
Here $\| \cdot \|$ is the Euclidean norm. If, for each $k \in \mathbb{Z}^+$, $A_k$ denotes the annulus $\{ x \in \mathbb{R}^n | k - \frac{1}{2} \leq \| x \|_2 \leq k + 1 \}$, then $T_0^k$ is a closed linear subspace of $l_2^k(\mathcal{A}_k)_{k \geq 1}$. Also $T_0^k$ is the Wiener space $T_0$ of Example 1, while $T_0^1$ was also considered by Goldberg [5]. The definition of $T_0^{q,n}$ ($1 \leq p < \infty$, $n \in \mathbb{Z}^+$) was given by Nguyen Phuong Can [10], who obtained the general form of the continuous linear functionals on $T_0^{q,n}$.

**Example 3. Stepanoff spaces.**

In the theory of almost-periodic functions, the spaces

$$
\mathcal{S}_2^p := \left\{ f \in \mathcal{S} \left| \sup_{x \in \mathbb{R}} \frac{\| f \|_{\mathcal{S}_2^p}}{1 + e^{-x}} \right\| \right\} ^{1/p} < +\infty \quad (1 \leq p < \infty)
$$

are of considerable importance; e.g., see Stepanoff [12], Basilevitch [3, Chapter II], Amerio and Pruneri [1], Chapter 4, § 7. For any two fixed non-zero values of $d$, the spaces are the same (say $\mathcal{S}_p$), and have equivalent norms which are also equivalent to the norm $\| \cdot \|_{\mathcal{S}_p}$, where

$$
\mathcal{S}_p := \left\{ f \in \mathcal{L}_p \left| \sup_{x \in \mathbb{R}} \frac{\| f \|_{\mathcal{S}_p}}{1 + e^{-x}} \right\| \right\} ^{1/p} < +\infty \quad (1 \leq p < \infty).
$$

Consequently, by Theorem 2, for $1 < p < \infty$, $\mathcal{S}_p \cong (l_1, (l_0^p, \mathcal{L}_p), \mathcal{L}_p)^\mathcal{S}_p$.

Amerio (see [1], p. 55, where further references are given) considers almost-periodic functions with values in the space $l_p(\mathbb{R})$, $1 < p < \infty$, where $\mathbb{R}$ is a Banach space. For these spaces the functional dual is isomorphic to $l_p(\mathbb{R})$, a fact which is established by Köthe [15, p. 359].

**Example 4. The space $\mathcal{S}_p^d$, $0 < p < \infty$.**

The space $\mathcal{S}_p^d$ is the space of complex-valued sequences, strongly $(\mathcal{C}_1, 1)$-sumnable to zero with index $p$, defined by

$$
\mathcal{S}_p^d := \left\{ x \in \mathcal{C}_1, \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \right\} \quad (0 < p < \infty).
$$

Wilansky and Zeller [14] considered this space for $p = 1$, Borwein [4] for $p \geq 1$, and Maddox [9] for $p > 0$. For each $k \in \mathbb{Z}^+$, write $\mathcal{B}_k := \mathcal{L}_k(\mathbb{R})$, namely

$$
\mathcal{B}_k := \left\{ x \in \mathbb{R}^k, \frac{1}{2^k} \sum_{n \in \mathbb{Z}^+} |x_n|^p = 0 \right\}.
$$

Also $\mathcal{B}_k$ is the $\| \cdot \|$-norm of $\mathcal{S}_p^d$. The norm of $\mathcal{S}_p^d$ is $\| x \|_{\mathcal{S}_p^d} := \left\{ \frac{1}{2^k} \sum_{n \in \mathbb{Z}^+} |x_n|^p \right\}^{1/p}$, $1 \leq p < \infty$ (norm); $\| x \|_{\mathcal{S}_p^d} := \left\{ \frac{1}{2^k} \sum_{n \in \mathbb{Z}^+} |x_n|^p \right\}^{1/p}$, $0 < p < 1$ (p-norm).

Then Borwein and Maddox showed that an equivalent definition of $\mathcal{S}_p^d$ is

$$
\mathcal{S}_p^d := \mathcal{L}^1(\mathbb{R})^{\mathcal{S}_p^{d+1}} + \mathcal{L}^1(\mathbb{R})^{\mathcal{S}_p^{d+1}} + \cdots + \mathcal{L}^1(\mathbb{R})^{\mathcal{S}_p^{d+1}} + \mathcal{L}^1(\mathbb{R})^{\mathcal{S}_p^{d+1}}, \quad \mathcal{L}^1(\mathbb{R})^{\mathcal{S}_p^{d+1}} = \sup_{x \in \mathcal{S}_p^{d+1}} \| x \|_{\mathcal{S}_p^{d+1}}.
$$

The continuous linear functionals $\varphi_x$ on $\mathcal{B}_k$ have the form

$$
\varphi_x(x) = \sum_{n \in \mathbb{Z}^+} x_n a_n, \quad \| \varphi_x \|_{\mathcal{B}_k} := \left( \sum_{n \in \mathbb{Z}^+} |x_n|^p \right)^{1/p} \quad (1 < p < \infty),
$$

$$
\| \varphi_x \|_{\mathcal{B}_k} := \max_{1 \leq n \leq k} |x_n|, \quad 0 < p \leq 1.
$$

(The continuous dual of $\mathcal{L}_p$ is isomorphic to $\mathcal{L}_p$ for $1 < p < \infty$, and to $l_p$ for $0 < p \leq 1$.) Also $(\mathcal{C}_1, p) = l_1$, $\mathcal{S}_p \cong (l_1, \mathcal{L}_p, \mathcal{L}_p)^\mathcal{S}_p$. By Theorem 2, for each $\mathcal{F} \in (\mathcal{S}_p^d)^*$ there exists a unique sequence $a \in \mathcal{S}_p^d$ such that

$$
\mathcal{F}(x) = \sum_{n \in \mathbb{Z}^+} a_n x_n, \quad \| \mathcal{F} \| = \sum_{n \in \mathbb{Z}^+} |a_n|^p.
$$

This representation was obtained by Maddox [9], p. 290, for $0 < p < \infty$. Borwein's representation ([4], Theorem 1), holds for $1 < p < \infty$. A different representation for $\mathcal{F} \in (\mathcal{S}_p^d)^*$ was given by Wilansky and Zeller [13, Satz 5]. The space $\mathcal{S}_p^d$ also occurs in ergodic theory in connection with weak mixing (e.g., see Halms [6, p. 38]).

**Example 5. The space $\mathcal{S}_p^{d+1}$.**

Here $0 < p < \infty$, $d := (d_1, d_2, \ldots, d_n)$, $\mathcal{B}_d := \mathcal{B}_1 + \cdots + \mathcal{B}_d$, $\mathcal{S}_p^d := \mathcal{B}_d + \cdots + \mathcal{B}_d$, $\mathcal{S}_p^{d+1} := \mathcal{B}_d + \cdots + \mathcal{B}_d + \cdots + \mathcal{B}_d$.

$$
\mathcal{S}_p^{d+1} := \left\{ x \in \mathcal{S}_p^{d+1}, \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \right\}
$$

is the space of sequences strongly $(\mathcal{S}_p^{d+1})$-sumnable to zero, with index $p$. This generalization of $\mathcal{S}_p^d$ (take $d_1 = 1$) was considered by Jakimowski and Livno [17, § 4], who obtained an equivalent definition and norm (or $p$-norm) expressing $\mathcal{S}_p^{d+1}$ in the form required by Theorems 1 and 2 above. The representation for $\mathcal{F} \in (\mathcal{S}_p^{d+1})^*$ obtained from Theorem 2 then tallies with that of [7], Theorem 4.4.

**Example 6. The space $\mathcal{O}_A$.**

Given $1 < p < \infty$, and a non-negative matrix $A = (a_{ij})$ with no zero rows and columns, write, as in Balsm, Juckat, and Peyerimhoff [3],

$$
\mathcal{O}_A := \left\{ x \in \mathcal{S}_p^d, \lim_{n \to \infty} \sum_{k=1}^{n} a_{ik} |x_k|^p = 0 \right\}.
$$
Let $E_k^i := \{ i \in Z^+ \mid a_{ik} \neq 0 \}$ and take $B_k$ to be the weighted $l_p$ spaces $B_k := \{ x = (x_i)_{i \in Z^+} \mid \| x \|_{B_k} := \left( \sum_{i \in E_k^i} |x_i|^p \right)^{1/p} < +\infty \} \,(k \in Z^+)$. Then

$$o(A_k) = o(B_k)_{ax^+}, \quad \| x \|_{o(A_k)} = \sup_{k \in Z^+} \| x \|_{B_k}.$$ 

Let $p^{-1} + (p')^{-1} = 1$. For each $k \in Z^+$, $\phi_k \in B_k^*$ if and only if $\exists (\eta_k)_{k \in Z^+}$ (unique) such that

$$\phi_k(x) = \sum_{i \in E_k} h_i a_{ik}^p x_i \quad (\forall x \in B_k), \quad \| \phi_k \|_{B_k^*} = \| (\eta_k)_{k \in Z^+} \|_{l_p} < \infty.$$ 

In order to complete the matrix $H$ and preserve its uniqueness, define $h_i = 0$ if $i \not\in E_k^i$ (that is, when $a_{ik} = 0$). Then, by Theorem 2, $F \in o(A_k^*) = (o(B_k))^*_{ax^+}$ if and only if

$$F(x) = \sum_{k \in Z^+} \phi_k(x) \quad (\forall x \in o(A_k)), \quad \| F \|_{o(A_k^*)} = \sum_{k \in Z^+} \| \phi_k \|_{B_k^*} < \infty.$$ 

On the other hand, since $o(A_k)$ has $AX$ (by (8)), we also have, $\forall x \in o(A_k) F(x) = \sum_{i \in E_k} x_i$ where $e_i = F(e_i)$; and $\sum_{i \in E_k} |x_i| < \infty$, because $o(A_k)$ is solid. Hence

$$e_i = F(e_i) = \sum_{k \in Z^+} \phi_k(e_i) = \sum_{k \in Z^+} h_i a_{ik}^p$$

and we get the representation of $o(A_k^*)$ given in (2), Theorem 7, (a) and (b) — note that the regularity hypothesis on $A$ stated there is not required for this form of the functional.

Example 7. The space $W_p^*$, $1 < p < \infty$.

$$W_p^* := \left\{ f \mid f \text{ is measurable on } [1, \infty], \lim_{t \to +\infty} \frac{1}{t} \int_1^t |f|^p = 0 \right\}.$$ 

Take

$$(B_k, \| \cdot \|_{B_k}) := L_p(2^k, 2^{k+1}), \quad \| f \|_{B_k} := \left( \int_{2^k}^{2^{k+1}} |f|^p \right)^{1/p} \quad (\forall k \in Z^+).$$ 

Then an equivalent definition of $W_p^*$ (Borwein [4]) is

$$W_p^* := c''(B_k)_{ax^+}, \quad \| f \|_{W_p^*} := \sup_{k \in Z^+} \| f \|_{B_k}.$$ 

The continuous linear functionals $\phi_k$ on $B_k$ have the form

$$\phi_k(f) = \int_1^{2^k} f(x) \, dx, \quad \| \phi_k \|_{B_k^*} = 2^{k/p} \| f \|_{L_p(2^k, 2^{k+1})}, \quad p^{-1} + (p')^{-1} = 1,$$

and by Theorem 2, $F \in (W_p^*)^*$ implies that $\exists g$ (unique) such that

$$F(f) = \int_1^\infty g(x) \, dx \quad (\forall f \in W_p^*), \quad \| F \| = \sum_{k \in Z^+} 2^{k/p} \| g \|_{L_p(2^k, 2^{k+1})} < +\infty,$$

which is equivalent to the representation obtained by Borwein [4], Theorem 2.

Example 8. The space $C_{ox}^1$.

Let $x = (x_k)_{k \in Z^+}$ be a fixed real sequence, $-\infty < \cdots < x_k < x_{k+1} < \cdots \to + \infty$, and write $X_k := (x_k, x_{k+1})$, with length $|X_k| := x_{k+1} - x_k$. Let $B_k$ be the $B_k$ space of measurable functions on $X_k$ with

$$\| f \|_{B_k} := \| f \|_{L_p(X_k)} \quad (1 < p < +\infty, k \in Z).$$

Define

$$C_{ox}^1 := c''(B_k)_{ax^+} = (f \mid \lim_{k \to +\infty} \| f \|_{B_k} = 0), \quad \| f \|_{C_{ox}^1} := \sup_{k \in Z^+} \| f \|_{B_k}.$$ 

This space was considered by Jakimovski and Russell [8]. A representation for $F \in (C_{ox}^1)^*$ is obtainable from Theorem 2 above, which coincides with that given in [8], Theorem 1. The representation is used in [8] to solve an interpolation problem, namely the existence of a function from a certain class which takes prescribed values $y_k$ at the points $x_k$ ($\forall k \in Z$).

When $x_{k+1} - x_k = 1$ ($\forall k$), Schoenberg [11] defines the space

$$L_{ox}^1 := \left\{ f \mid \| f \|_{ox} := \sum_{k \in Z^+} \sup_{x_k < \cdot < x_{k+1}} |f(x)| < +\infty \right\}.$$ 

Thus $L_{ox}^1 = l_1(X, (x_k, x_{k+1}))_{ax^+}$, and so his space is (isomorphic to) $(C_{ox}^1)^*$.}

References


A remark on finite-dimensional $P_n$-spaces

by

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Abstract. It is shown that finite-dimensional $P_n$-spaces contain $P_n(m)$-subspaces, where $m$ is proportional to the dimension of the $P_n$-space.

Introduction. Let us recall that the Banach–Mazur distance $d(E, F)$ of two normed linear spaces $E$ and $F$ of the same finite dimension is given by $d(E, F) = \inf \{\|T\|\|T^{-1}\|; \; T: E \to F \text{ is a linear isomorphism}\}$.

We say that $X$ is a $P_n$-space provided for any Banach space $Y$ in which $X$ embeds isometrically, there exists a projection $P$ from $Y$ onto $X$ with $\|P\| \leq \lambda$. As is well known, the space $Y$ in the above definition may as well be replaced by the space $F^n$. The characterization of $P_n$-spaces is a rather old and still unsolved problem. One may hope for an affirmative solution to the following question, dealing with the finite-dimensional version of the problem:

Does there exist for all $1 < \infty$ some constant $c_0 < \infty$ such that $d(E, P^n(d)) < c_0$, holds for any $P_n$-space $E$ of dimension $d$?

In [4], the existence is shown of a function $d(\lambda, m, \epsilon)$ so that given a $P_n$-space $E$, $\dim(E) \geq d(\lambda, m, \epsilon)$, one can find a subspace $F$ of $E$ with $\dim(F) = m$ and $d(E, P^n(m)) < 1 + \epsilon$.

Our purpose is to show the following fact which, taking into account a related observation of [4], will improve the above result.

Theorem 1. Given $\lambda < \infty$, one can find a constant $c_0 < \infty$ such that given a finite-dimensional $P_n$-space $E$, there exists a subspace $F$ of $E$ satisfying $\dim(F) = m > c_0 \dim(E)$ and $d(F, P^n(m)) < c_0$.

Proof of the result. We recall that if $T: X \to Y$ is an operator between Banach spaces $X$ and $Y$, then $T$ is $(p, q)$-absolutely summing if there exists a constant $M < \infty$ such that

\[ \sum_{i=1}^{n} \|T(x_i)\|_{p} < M \sup_{\|\xi\| = 1} \left( \sum_{i=1}^{n} \langle x_i, \alpha_i \rangle \right)^{1/q}, \]

holds, whenever $(\alpha_i)_{i=1}^{n}$ is a finite sequence of vectors in $X$.