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Representation of continuous linear functionals on a subspace of a countable Cartesian product of Banach spaces*

by

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Abstract. In a wide variety of applications (e.g. to harmonic analysis, almost periodic functions, ergodic theory, approximation theory, summability) spaces of the following kind occur. We are given a sequence $\mathscr{B} := (B_k)$ of Banach spaces and a solid Banach sequence space E with a Schauder basis, and we consider elements from the Cartesian product of the B_k, namely sequences $x := (x_k)$ such that $x_k \in B_k$ for each k, such that the sequence of norms $(\|x_k\|_{B_k})$ lies in the given space E. The "composed" set $E\mathscr{B}$ of such sequences x is a subset of the Cartesian product and may be given the obvious norm topology. It is then of considerable interest to be able to determine the form of the general continuous linear functional on the space $E\mathscr{B}$, namely to identify the functional dual $(E\mathscr{B})^*$, and that is the object of this paper. Some relaxations are possible (e.g. we may take p-norms or semi-norms instead of norms in some places). Several examples from widely different sources are given in the final part of the paper.

We shall suppose in our theorems that the index k of our spaces B_k ranges over Z^+, the set of non-negative integers; we could equally well take the set Z of all integers, but we choose the former as yielding slightly simpler notation.

Given a sequence $x := (x_k)_{k\in Z^+}$ with $x_k \in B_k$ $(k = 0, 1, \ldots)$, we shall denote the *n-th section of* x by

$$x^{[n]} := (x_k)^{[n]} := (x_0, \ldots, x_n, \theta_{n+1}, \theta_{n+2}, \ldots),$$

where θ_k is the zero element of B_k $(k \in Z^+)$. Similarly, for a sequence $u := (u_k)_{k\in Z^+} \in \omega$, where ω is the space of all complex-valued sequences, its nth section is

$$u^{[n]} := (u_k)^{[n]} := (u_0, \ldots, u_n, 0, 0, \ldots) = \sum_{k=0}^{n} u_k e^k,$$

where $e^k := (0, \ldots, 0, 1, 0, 0, \ldots)$, with 1 in the kth place. A *sequence space* is a linear subspace of ω.

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We begin by stating our first theorem, which concerns only the elementary algebraic and topological properties of our composed space.

THEOREM 1. *Let* $\mathscr{B} := (B_k)_{k\in Z^+}$ *be a sequence of complete linear* p*-normed spaces, with the* p*-norms* $\|\cdot\|_{B_k}$ *(where* $0 < p \leqslant 1$*). Let* E *be a sequence space, complete with respect to the* q*-norm* $\|\cdot\|_E$ *(where* $0 < q \leqslant 1$*), with continuous coordinate projections, and such that*

(1) $$\{e^k\}_{k\in Z^+} \subset E \quad \text{and} \quad \forall u \in E,\ \lim_{n\to\infty} \|u - u^{[n]}\|_E = 0;$$

(2) *whenever* $u \in E$ *and* $v \in \omega$, *with* $|v_k| \leqslant |u_k|$ $(\forall k \in Z^+)$, *we have* (i) $v \in E$, (ii) $\|v\|_E \leqslant \|u\|_E$.

Define the composed space

(3) $$E\mathscr{B} := \{x = (x_k)_{k\in Z^+} |\ x_k \in B_k\ (\forall k \in Z^+),\quad (\|x_k\|_{B_k})_{k\in Z^+} \in E\}$$

and

(4) $$\forall x \in E\mathscr{B} \quad \text{denote} \quad \|x\|_{E\mathscr{B}} := \|(\|x_k\|_{B_k})_{k\in Z^+}\|_E.$$

Then $E\mathscr{B}$ *is a linear space, complete with respect to the* pq*-norm* $\|\cdot\|_{E\mathscr{B}}$, *and*

(5) $$\forall x \in E\mathscr{B},\quad \lim_{n\to\infty} \|x - x^{[n]}\|_{E\mathscr{B}} = 0.$$

It is easy to restate and prove Theorem 1 in the case where E has a family of q-semi-norms $(\|\cdot\|_j)$ in place of a single q-norm $\|\cdot\|_E$. With the obvious modifications in hypothesis and notation, $E\mathscr{B}$ is then a complete linear space with respect to a family of pq-semi-norms $(\|\cdot\|_{j,E\mathscr{B}})$.

We remark that property (1) says that $(e^k)_{k\in Z^+}$ is a *Schauder basis* for E, or that E has the *AK-property*; while property (2) (i) says that E is *solid*. An obvious consequence of (2) is

(6) $$\forall u \in E,\quad \|(|u_k|)_{k\in Z^+}\|_E = \|u\|_E.$$

DEFINITION. Let $(E, \|\cdot\|_E)$, $E \subset \omega$, be a q-normed sequence space $(0 < q \leqslant 1)$, and let $0 < p \leqslant 1$. Define the *ap-dual of* E to be

$$E^{ap} := \Big\{v \in \omega \Big|\ \|v\|_{E^{ap}} := \sup_{\|u\|_E=1} \sum_{k\in Z^+} |u_k|^{1/p}|v_k| < +\infty\Big\}.$$

For $p = 1$ we get the well-known α*-dual* of E; it follows by elementary arguments (as for the α-dual) that:

If E *is a* q*-normed sequence space* $(0 < q \leqslant 1)$, *and* $0 < p \leqslant 1$, *then* E^{ap} *is a normed linear space and*

(7) $$\forall u \in E,\ \forall v \in E^{ap},\quad \sum_{k\in Z^+} |u_k|^{1/p}|v_k| \leqslant \|u\|_E^{1/pq}\|v\|_{E^{ap}};$$

(8) *if* $\{e^k\}_{k\in Z^+} \subset E$ *then* E^{ap} *is a Banach space.*

THEOREM 2. *Let* $0 < p \leqslant 1$, $0 < q \leqslant 1$. *For each* $k \in Z^+$, *let* B_k *be a complete* p*-normed space with functional-dual space* B_k^*; *write* $\mathscr{B} := (B_k)_{k\in Z^+}$ *and* $\mathscr{B}^* := (B_k^*)_{k\in Z^+}$. *Let* E *be a complete* q*-normed sequence space with continuous coordinate projections, satisfying* (1) *and* (2). *Then:*

(a) *For each* $\varphi := (\varphi_k)_{k\in Z^+} \in E^{ap}\mathscr{B}^*$ *we have*

(9) $$\sum_{k\in Z^+} |\varphi_k(x_k)| < +\infty \quad (\forall x := (x_k)_{k\in Z^+} \in E\mathscr{B})$$

and the functional F *defined on* $E\mathscr{B}$ *by*

(10) $$F(x) := \sum_{k\in Z^+} \varphi_k(x_k) \quad (\forall x \in E\mathscr{B})$$

satisfies

(11) $$F \in (E\mathscr{B})^*.$$

(b) *To each* $F \in (E\mathscr{B})^*$ *corresponds a unique sequence* $\varphi := (\varphi_k)_{k\in Z^+} \in E^{ap}\mathscr{B}^*$ *such that* (10) *holds; moreover,*

(12) $$\|F\| = \|\varphi\|_{E^{ap}\mathscr{B}^*} := \big\|(\|\varphi_k\|_{B_k^*})_{k\in Z^+}\big\|_{E^{ap}}.$$

Proof of Theorem 1. It is easy to prove by straightforward arguments that $E\mathscr{B}$ is a linear space and that $\|\cdot\|_{E\mathscr{B}}$ (defined in (4)) is a pq-norm: we simply use the facts that each B_k is p-normed and that E is linear, q-normed, and satisfies (2). Property (1) is needed only to establish (5), as follows: if $x \in E\mathscr{B}$ then, by (4) and (1),

$$\|x - x^{[n]}\|_{E\mathscr{B}} = \big\|(\|x_k\|_{B_k})_{k\in Z^+} - (\|x_k\|_{B_k})^{[n]}\big\|_E \to 0 \quad \text{as} \quad n \to \infty.$$

We shall prove that $E\mathscr{B}$ is complete. Suppose that $\{x^m\}_{m\in Z^+} := \{(x_k^m)_{k\in Z^+}\}_{m\in Z^+}$ is a Cauchy sequence in $E\mathscr{B}$, so that

(13) $$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \text{ such that } \|x^m - x^n\|_{E\mathscr{B}} < \varepsilon \quad (\forall m, n \geqslant N).$$

Then for $m, n \geqslant N$ and any $j \in Z^+$ we have, by (2), (4), and since E is q-normed,

$$\|x_j^m - x_j^n\|_{B_j}^q \|e^j\|_E = \big\|\|x_j^m - x_j^n\|_{B_j} e^j\big\|_E \leqslant \big\|(\|x_k^m - x_k^n\|_{B_k})_{k\in Z^+}\big\|_E < \varepsilon.$$

Hence, for each j, $(x_j^m)_{m\in Z^+}$ is a Cauchy sequence in B_j; since B_j is complete, $\exists x^* := (x_k^*)_{k\in Z^+}$ such that

(14) $$\forall k \in Z^+,\quad x_k^* \in B_k \text{ and } \lim_{m\to\infty} \|x_k^m - x_k^*\|_{B_k} = 0.$$

We now prove that $x^* \in E\mathscr{B}$. For all $r \in Z^+$ and all $m \geqslant N$, $n \geqslant N$, we have, by (4), (2), (13), and since $\{e^k\}_{k\in Z^+} \subset E$,

$$\|(x^m - x^n)^{[r]}\|_{E\mathscr{B}} = \big\|(\|x_k^m - x_k^n\|_{B_k})^{[r]}\big\|_E \leqslant \big\|(\|x_k^m - x_k^n\|_{B_k})_{k\in Z^+}\big\|_E < \varepsilon;$$

and hence, $\forall r \in Z^+$, $\forall m \geqslant N$,

$$(15)\quad \|(x^m - x^*)^{[r]}\|_{E\mathscr{B}} \leqslant \|(x^m - x^n)^{[r]}\|_{E\mathscr{B}} + \|(x^n - x^*)^{[r]}\|_{E\mathscr{B}}$$
$$< \varepsilon + \big\|(\|x_k^n - x_k^*\|_{B_k})^{[r]}\big\|_E$$
$$\leqslant \varepsilon + \sum_{k=0}^{r} \|x_k^n - x_k^*\|_{B_k}^q \|e^k\|_E$$
$$\to \varepsilon \text{ as } n \to \infty \text{ by } (14).$$

Since $(\|x_k^m\|_{B_k})_{k\in Z^+} \in E$, we have, by (1), that $\forall \varepsilon > 0$, $\exists M = M(\varepsilon, m)$ such that

$$(16)\quad \big\|(\|x_k^m\|_{B_k})^{[r]} - (\|x_k^m\|_{B_k})^{[s]}\big\|_E < \varepsilon \quad (\forall r, s \geqslant M).$$

Choose a fixed m in (15), say $m = N$, and any $r, s \geqslant M(\varepsilon, N)$ in (16), and we then get, using (2) and the triangle inequality in B_k in the form $|\,\|a\| - \|b\|\,| \leqslant \|a - b\|$,

$$\big\|(\|x_k^*\|_{B_k})^{[r]} - (\|x_k^*\|_{B_k})^{[s]}\big\|_E$$
$$\leqslant \big\|(\|x_k^N\|_{B_k})^{[r]} - (\|x_k^N\|_{B_k})^{[s]}\big\|_E + \big\|(\|x_k^N - x_k^*\|_{B_k})^{[r]}\big\|_E + \big\|(\|x_k^N - x_k^*\|_{B_k})^{[s]}\big\|_E$$
$$< 3\varepsilon \quad \text{by (16) and (15).}$$

Thus $\{(\|x_k^*\|_{B_k})^{[r]}\}_{r\in Z^+}$ is a Cauchy sequence in E. But E is complete and has continuous coordinate projections, so the limit in E must be $(\|x_k^*\|_{B_k})_{k\in Z^+}$; that is, $x^* \in E\mathscr{B}$.

Finally, by (5) and (15),

$$\|x^m - x^*\|_{E\mathscr{B}} = \lim_{r\to\infty} \|(x^m - x^*)^{[r]}\|_{E\mathscr{B}} \leqslant \varepsilon \quad (\forall m \geqslant N),$$

so $(x^m)_{m\in Z^+}$ is convergent to x^* in $\|\cdot\|_{E\mathscr{B}}$, and $E\mathscr{B}$ is therefore complete.

Proof of Theorem 2. (a) Let $\varphi := (\varphi_k)_{k\in Z^+} \in E^{\alpha p}\mathscr{B}^*$. Then, for each $x \in E\mathscr{B}$,

$$\sum_{k\in Z^+} |\varphi_k(x_k)| \leqslant \sum_{k\in Z^+} \|\varphi_k\|_{B_k^*} \|x_k\|_{B_k}^{1/p} \quad \text{since each } B_k \text{ is } p\text{-normed}$$
$$\leqslant \big\|(\|\varphi_k\|_{B_k^*})_{k\in Z^+}\big\|_{E^{\alpha p}} \big\|(\|x_k\|_{B_k})_{k\in Z^+}\big\|_E^{1/pq} \quad \text{by (7)}$$
$$= \|\varphi\|_{E^{\alpha p}\mathscr{B}^*} \|x\|_{E\mathscr{B}}^{1/pq} \quad \text{by definition (4).}$$

Hence the functional F defined by (10) satisfies $F \in (E\mathscr{B})^*$ and

$$(17)\quad \|F\| \leqslant \|\varphi\|_{E^{\alpha p}\mathscr{B}^*}.$$

(b) Suppose $F \in (E\mathscr{B})^*$. For each $k \in Z^+$ write

$$\mathscr{B}_k := \{y := (y_r)_{r\in Z^+} \mid y_r = \theta_r (r \neq k),\ y_k \in B_k\}.$$

If $y \in \mathscr{B}_k$ then $y \in E\mathscr{B}$ and

$$\|y\|_{E\mathscr{B}} = \big\|\,\|y_k\|_{B_k} e^k\big\|_E = \|y_k\|_{B_k}^q \|e^k\|_E.$$

Hence $\mathscr{B}_k$ and B_k are isomorphic as complete p-normed spaces, and $\mathscr{B}_k$ is a closed subspace of $E\mathscr{B}$. Thus the restriction of F to $\mathscr{B}_k$ yields a continuous linear functional on $\mathscr{B}_k$, so that

$$(18)\quad \exists \varphi_k \in B_k^*:\ \forall y \in \mathscr{B}_k,\ F(y) = \varphi_k(y_k).$$

Also, given ε satisfying $0 < \varepsilon \leqslant 1$,

$$(19)\quad \exists g_k \in B_k:\ \|g_k\|_{B_k} = 1 \text{ and } \varphi_k(g_k) \geqslant (1-\varepsilon)\|\varphi_k\|_{B_k^*} \geqslant 0,$$

because, for any $\alpha_k \in R$,

$$\varphi_k(e^{i\alpha_k} g_k) = e^{i\alpha_k}\varphi_k(g_k) \quad \text{and} \quad \|e^{i\alpha_k} g_k\|_{B_k} = \|g_k\|_{B_k}.$$

Now by (5) of Theorem 1, for any $x \in E\mathscr{B}$, we have, since $F \in (E\mathscr{B})^*$,

$$(20)\quad F(x) = \lim_{n\to\infty} F(x^{[n]}) = \lim_{n\to\infty} \sum_{k=0}^{n} \varphi_k(x_k) = \sum_{k\in Z^+} \varphi_k(x_k), \quad \text{by (18).}$$

Let (g_k) be chosen as in (19) and, given $u \in E$ $(u \neq \theta)$, define

$$h := (h_k)_{k\in Z^+}, \quad \text{where} \quad h := |u_k|^{1/p} g_k \ (\forall k \in Z^+).$$

Then

$$\|h\|_{E\mathscr{B}} = \|(|u_k|^{1/p} g_k)_{k\in Z^+}\|_{E\mathscr{B}}$$
$$= \big\|(|u_k|\ \|g_k\|_{B_k})_{k\in Z^+}\big\|_E \quad \text{since each } B_k \text{ is } p\text{-normed}$$
$$= \big\|(|u_k|)_{k\in Z^+}\big\|_E \quad \text{since } \|g_k\|_{B_k} = 1 \text{ by (19)}$$
$$= \|u\|_E \quad \text{by (6).}$$

Thus $h \in E\mathscr{B}$ and we now have

$$\|F\|\ \|u\|_E^{1/pq} = \|F\|\ \|h\|_{E\mathscr{B}}^{1/pq}$$
$$\geqslant |F(h)| \quad \text{since, by Theorem 1, } E\mathscr{B} \text{ is } pq\text{-normed}$$
$$= \Big|\sum_{k\in Z^+} \varphi_k(h_k)\Big| \quad \text{by (20)}$$
$$= \sum_{k\in Z^+} |u_k|^{1/p}\varphi_k(g_k) \quad \text{since } \varphi_k \text{ is linear and } \varphi_k(g_k) \geqslant 0$$
$$\geqslant (1-\varepsilon) \sum_{k\in Z^+} |u_k|^{1/p} \|\varphi_k\|_{B_k^*} \quad \text{by (19).}$$

Letting $\varepsilon \to 0$, we get

$$\|F\| \geqslant \sum_{k\in Z^+} (|u_k| / \|u\|_E^{1/q})^{1/p} \|\varphi_k\|_{B_k^*};$$

thus $(\|\varphi_k\|_{B_k^*})_{k\in Z^+}\in E^{ap}$ (that is, $\varphi\in E^{ap}\mathscr{B}^*$) and, taking the supremum over all $u\in E$ ($u\neq\theta$) and using (6), we get

$$\|F\|\geqslant \big\|(\|\varphi_k\|_{B_k^*})_{k\in Z^+}\big\|_{E^{ap}} := \|\varphi\|_{E^{ap}\mathscr{B}^*}$$

which, when combined with (17), yields (12).

It remains to show that the φ in the representation (10) is unique. Suppose that φ and ψ are two such choices satisfying (10); then

$$\sum_{k\in Z^+}\{\varphi_k(x_k)-\psi_k(x_k)\}=0\quad (\forall x\in E\mathscr{B}).$$

Now for any $z\in B_j$ we have $(\theta_0,\ldots,\theta_{j-1},z,\theta_{j+1},\ldots)\in E\mathscr{B}$, whence $\varphi_j(z)-\psi_j(z)=0$ ($\forall z\in B_j$), and so $\varphi=\psi$.

Example 1. *The Wiener space* T_1.

$$T_1:=\Big\{f\mid f\in C(R),\ \|f\|_{T_1}:=\sum_{k\in Z}\max_{[k,k+1]}|f|<+\infty\Big\}.$$

This space was introduced by Wiener [13], p. 27, in the formulation of one of his tauberian theorems; the functional dual T_1^* was determined by Goldberg [5], thus enabling him to give a quick proof of the tauberian theorem.

Now T_1 is a closed linear subspace of $l_1\{C[k,k+1]\}_{k\in Z}$ (this in turn is a subspace of the $L_{[1]}$ of Example 8 below, $x_k=k$), so by the Hahn–Banach Theorem, if $F\in T_1^*$ then F can be extended to a continuous linear functional on $l_1\{C[k,k+1]\}_{k\in Z}$ with the same norm. By the Riesz Representation Theorem, $F\in C[k,k+1]^*$ implies

$$\exists V_k\in BV[k,k+1]:\ F(f)=\int_k^{k+1} f\,dV_k\quad (\forall f\in C[k,k+1]),$$

where the V_k are normalized. Thus by Theorem 2, if $F\in T_1^*$ then

$$F(f)=\sum_{k\in Z}\int_k^{k+1} f\,dV_k=\int_R f\,dV\quad (\forall f\in T_1),$$

where we must, in order to get an unambiguous definition of $V(t)$ at the integers, define

$$V(t):=\begin{cases} V_k(t)+\sum_{r=1}^{k}V_{r-1}(r), & k\leqslant t\leqslant k+1,\ k\geqslant 0;\\ V_k(t)-\sum_{r=k+1}^{0}V_{r-1}(r), & k\leqslant t\leqslant k+1,\ k<0.\end{cases}$$

Then V is a normalized function of bounded variation in each interval $[k,k+1]$, and

$$\|F\|=\sup_{k\in Z}\int_k^{k+1}|dV_k|=\sup_{k\in Z}\int_k^{k+1}|dV|<+\infty.$$

This representation was obtained in a different way by Goldberg [5], Theorem G, but with only the inequality $\|F\|\geqslant\sup_{k\in Z}\int_k^{k+1}|dV|$.

Actually, more is true. For suppose that the given $f\in T_1^*$ has the above representation. Take $0<\delta<1$ and any $k\in Z$, and define

$$f_{k,\delta}(t):=1-\delta^{-1}|t-k|\quad\text{for}\quad t\in(k-\delta,k+\delta),\qquad f_{k,\delta}(t)=0\quad\text{otherwise};$$

then $f_{k,\delta}\in T_1$ and $F(f_{k,\delta})=\int_R f_{k,\delta}dV\to V(k+0)-V(k-0)$ as $\delta\to 0$. Thus the jumps of V at the points k are uniquely determined by F. Write

$$U_r(t)=0\quad(\text{for } t=r),\qquad U_r(t)=V(t)-V(r+0)\quad(\text{for } r<t<r+1),$$
$$U_r(t)=V(r+1-0)-V(r+0)\quad(\text{for } t=r+1).$$

Now suppose, if possible, that for some $r\in Z$, $\int_r^{r+1}|dU_r|>0$. Choose $0<\varepsilon<\int_r^{r+1}|dU_r|$. Then since U_r is continuous at r and $r+1$, $\exists g\in C[r,r+1]$ such that

$$g(r):=g(r+1):=0\quad\text{and}\quad \Big|\int_r^{r+1} g\,dU_r\Big|\geqslant\int_r^{r+1}|dU_r|-\varepsilon>0.$$

Define $f_1(t):=g(t)$ on $[r,r+1]$, $f_1(t):=0$ otherwise; then $f_1\in T_1$ but $F(f_1)=\int_r^{r+1} g\,dU_r\neq 0$, contradiction. Thus $U_r(t)\equiv 0$.

This means that if $F\in T_1^*$, then the function $V(t)$ is determined uniquely, up to a constant, in each interval $(r,r+1)$ $(r\in Z)$. Of course, the converse is also true, in the sense that each normalized function of bounded variation with $\sup_{k\in Z}\int_k^{k+1}|dV|<+\infty$ defines an $F\in T_1^*$ such that

(21) $$F(f):=\int_R f\,dV\quad(\forall f\in T_1),\qquad\text{with}\qquad \|F\|=\sup_{k\in Z}\int_k^{k+1}|dV|.$$

Example 2. *The space* $T_n^{(p)}$, $1\leqslant p<\infty$, $n\in Z^+$.

$$T_n^{(p)}:=\Big\{f\mid f\in C(R^n),\ \|f\|_{T_n^{(p)}}:=\Big(\sum_{k=0}^{\infty}\big(\max_{k\leqslant\|x\|_{R^n}\leqslant k+1}|f(x)|\big)^p\Big)^{1/p}<+\infty\Big\}.$$

Here $\|\cdot\|_{R^n}$ is the Euclidean norm. If, for each $k \in Z^+$, A_k denotes the annulus $\{x \in R^n \mid k \leqslant \|x\|_{R^n} \leqslant k+1\}$, then $T_n^{(p)}$ is a closed linear subspace of $l_p\{C(A_k)\}_{k \in Z^+}$. Also $T_1^{(1)}$ is the Wiener space T_1 of Example 1, while $T_2^{(1)}$ was also considered by Goldberg [5]. The definition of $T_n^{(p)}$ $(1 \leqslant p < \infty,\ n \in Z^+)$ was given by Nguyen Phuong Các [10], who obtained the general form of the continuous linear functionals on $T_n^{(p)}$.

Example 3. *Stepanoff spaces.*

In the theory of almost-periodic functions, the spaces

$$S_d^p := \Big\{ f \Big| \ \|f\|_{S_d^p} := \sup_{x \in R} \Big(d^{-1} \int_x^{x+d} |f(t)|^p dt \Big)^{1/p} < +\infty \Big\} \qquad (1 \leqslant p < \infty)$$

are of considerable importance; e.g. see Stepanoff [12], Besicovitch [3], Chapter II, Amerio and Prouse [1], Chapter 4, § 7. For any two fixed non-zero values of d, the spaces are the same (say S^p), and have equivalent norms which are also equivalent to the norm $\|\cdot\|_{S^p}$, where

$$S^p := l_\infty\{L_p[k, k+1)\}_{k \in Z}, \quad \|f\|_{S^p} := \sup_{k \in Z} \|f\|_{L_p[k,k+1)} \qquad (1 \leqslant p < \infty).$$

Consequently, by Theorem 2, for $1 < p < \infty$, $p^{-1} + (p')^{-1} = 1$, $S^p \cong (l_1\{L_{p'}[k, k+1)\}_{k \in Z})^*$.

Amerio (see [1], p. 55, where further references are given) considers almost-periodic functions with values in the space $l_p\{X_n\}_{n \geqslant 1}$ $(1 \leqslant p < \infty)$, where each X_n is a Banach space. For these spaces the functional dual is isomorphic to $l_{p'}\{X_n^*\}_{n \geqslant 1}$, a fact which is established by Köthe [15], p. 359.

Example 4. *The space* w_p^0, $0 < p < \infty$.

The space w_p^0 is the space of complex-valued sequences, strongly $(C, 1)$-summable to zero with index p, defined by

$$w_p^0 := \Big\{ x \in \omega \Big| \ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \Big\} \qquad (0 < p < \infty).$$

Wilansky and Zeller [14] considered this space for $p = 1$, Borwein [4] for $p \geqslant 1$, and Maddox [9] for $p > 0$. For each $k \in Z^+$, write $(B_k, \|\cdot\|_{B_k}) = l_p[2^k, 2^{k+1})$, namely

$$B_k := \{x \mid x = (x_i)_{2^k \leqslant i < 2^{k+1}}\},$$

$$\|x\|_{B_k} := \begin{cases} \Big(\dfrac{1}{2^k} \displaystyle\sum_{2^k \leqslant i < 2^{k+1}} |x_i|^p \Big)^{1/p}, & 1 \leqslant p < \infty \quad \text{(norm)}; \\ \dfrac{1}{2^k} \displaystyle\sum_{2^k \leqslant i < 2^{k+1}} |x_i|^p, & 0 < p < 1 \quad (p\text{-norm}). \end{cases}$$

Then Borwein and Maddox showed that an equivalent definition of w_p^0 is

$$w_p^0 := c^0\{B_k\}_{k \in Z^+}, \qquad \text{with} \quad \|x\|_{w_p^0} := \big\| (\|x\|_{B_k})_{k \in Z^+} \big\|_{c^0} := \sup_{k \in Z^+} \|x\|_{B_k}.$$

The continuous linear functionals φ_k on B_k have the form

$$\varphi_k(x) = \sum_{2^k \leqslant i < 2^{k+1}} a_i x_i, \qquad \|\varphi_k\|_{B_k^*} = \begin{cases} 2^{k/p} \Big(\displaystyle\sum_{2^k \leqslant i < 2^{k+1}} |a_i|^{p'} \Big)^{1/p'}, & 1 < p < \infty, \\ & \dfrac{1}{p} + \dfrac{1}{p'} = 1; \\ 2^{k/p} \displaystyle\max_{2^k \leqslant i < 2^{k+1}} |a_i|, & 0 < p \leqslant 1. \end{cases}$$

(The continuous dual of l_p is isomorphic to $l_{p'}$ for $1 < p < \infty$, and to l_∞ for $0 < p \leqslant 1$). Also $(c^0)^{\alpha q} = l_1$, $q = \min(p, 1)$, and so, by Theorem 2, for each $F \in (w_p^0)^*$ there exists a unique sequence $a \in \omega$ such that

$$F(x) = \sum_{i=1}^{\infty} a_i x_i \quad (\forall x \in w_p^0), \qquad \|F\| = \sum_{k=0}^{\infty} \|\varphi_k\|_{B_k^*}.$$

This representation was obtained by Maddox ([9], p. 290), for $0 < p < \infty$. Borwein's representation ([4], Theorem 1), holds for $1 \leqslant p < \infty$. A different representation for $F \in (w_1^0)^*$ was given by Wilansky and Zeller ([13], Satz 5). The space w_1^0 also occurs in ergodic theory in connection with weak mixing (e.g. see Halmos [6], p. 38).

Example 5. *The space* $w_p^0(d)$.

Here $0 < p < \infty$, $d := (d_k)_{k \geqslant 1}$, $d_k > 0$ $(\forall k)$, $D_k := d_1 + \ldots + d_k \to +\infty$, and

$$w_p^0(d) := \Big\{ x \in \omega \Big| \ \lim_{n \to \infty} (D_n)^{-1} \sum_{k=1}^{n} d_k |x_k|^p = 0 \Big\}$$

is the space of sequences strongly $(\overline{N}, d)$-summable to zero, with index p. This generalization of w_p^0 (take $d_k = 1$) was considered by Jakimovski and Livne ([7], § 4), who obtained an equivalent definition and norm (or p-norm) expressing $w_p^0(d)$ in the form required by Theorems 1 and 2 above. The representation for $F \in (w_p^0(d))^*$ obtained from Theorem 2 then tallies with that of [7], Theorem 4.4.

Example 6. *The space* $o[A]_p$.

Given $1 \leqslant p < \infty$, and a non-negative matrix $A = (a_{ki})$ with no zero rows and columns, write, as in Balser, Jurkat, and Peyerimhoff [2],

$$o[A]_p := \Big\{ x \in \omega \Big| \ \lim_{k \to \infty} \sum_{i=0}^{\infty} a_{ki} |x_i|^p = 0 \Big\}.$$

Let $Z_k^+ := \{i \in Z^+ | \; a_{ki} \neq 0\}$ and take B_k to be the weighted l_p spaces

$$B_k := \Big\{x = (x_i)_{i\in Z_k^+} \Big| \; \|x\|_{B_k} := \Big(\sum_{i\in Z_k^+} a_{ki}|x_i|^p\Big)^{1/p} < +\infty\Big\} \quad (k \in Z^+);$$

then

$$o[A]_p = c^0\{B_k\}_{k\in Z^+}, \quad \|x\|_{o[A]_p} = \sup_{k\in Z^+} \|x\|_{B_k}.$$

Let $p^{-1}+(p')^{-1} = 1$. For each $k \in Z^+$, $\varphi_k \in B_k^*$ if and only if $\exists (h_{ki})_{i\in Z_k^+}$ (unique) such that

$$\varphi_k(x) = \sum_{i\in Z_k^+} h_{ki} a_{ki}^{1/p} x_i \quad (\forall x \in B_k), \quad \|\varphi_k\|_{B_k^*} = \|(h_{ki})_{i\in Z_k^+}\|_{l_{p'}} < \infty.$$

In order to complete the matrix H and preserve its uniqueness, define $h_{ki} = 0$ if $i \notin Z_k^+$ (that is, when $a_{ki} = 0$). Then, by Theorem 2, $F \in o[A]_p^* = (c^0\{B_k\})^*$ if and only if

$$F(x) = \sum_{k\in Z^+} \varphi_k(x) \quad (\forall x \in o[A]_p), \quad \|F\| = \sum_{k\in Z^+} \|\varphi_k\|_{B_k^*} = \sum_{k\in Z^+} \|(h_{ki})_{i\in Z^+}\|_{l_{p'}} < \infty.$$

On the other hand, since $o[A]_p$ has AK (by (5)), we also have, $\forall x \in o[A]_p$, $F(x) = \sum_{i\in Z^+} \varepsilon_i x_i$, where $\varepsilon_i = F(e^i)$; and $\sum_{i\in Z^+} |\varepsilon_i x_i| < \infty$, because $o[A]_p$ is solid. Hence

$$\varepsilon_i = F(e^i) = \sum_{k\in Z^+} \varphi_k(e^i) = \sum_{i\in Z^+} h_{ki} a_{ki}^{1/p}$$

and we get the representation of $o[A]_p^*$ given in [2], Theorem 7, (a) and (b) — note that the regularity hypothesis on A stated there is not required for this form of the functional.

Example 7. *The space* W_p^0, $1 \leqslant p < \infty$.

$$W_p^0 := \Big\{f \Big| \; f \text{ is measurable on } [1, \infty), \lim_{T\to+\infty} \frac{1}{T}\int_1^T |f|^p = 0\Big\}.$$

Take

$$(B_k, \|\cdot\|_{B_k}) := L_p[2^k, 2^{k+1}), \quad \|f\|_{B_k} := \Big(2^{-k}\int_{2^k}^{2^{k+1}} |f|^p\Big)^{1/p} \quad (\forall k \in Z^+).$$

Then an equivalent definition of W_p^0 (Borwein [4]) is

$$W_p^0 := c^0\{B_k\}_{k\in Z^+}, \quad \|f\|_{W_p^0} := \sup_{k\in Z^+} \|f\|_{B_k}.$$

The continuous linear functionals φ_k on B_k have the form

$$\varphi_k(f) = \int_{2^k}^{2^{k+1}} fg, \quad \|\varphi_k\|_{B_k^*} = 2^{k/p}\|g\|_{L_{p'}[2^k, 2^{k+1})}, \quad p^{-1}+(p')^{-1} = 1,$$

and by Theorem 2, $F \in (W_p^0)^*$ implies that $\exists g$ (unique) such that

$$F(f) = \int_1^\infty fg \quad (\forall f \in W_p^0), \quad \|F\| = \sum_{k=0}^\infty 2^{k/p}\|g\|_{L_{p'}[2^k, 2^{k+1})} < +\infty,$$

which is equivalent to the representation obtained by Borwein [4], Theorem 2.

Example 8. *The space* $C_{p,x}^0$.

Let $x := (x_k)_{k\in Z}$ be a fixed real sequence, $-\infty \leftarrow \ldots < x_k < x_{k+1} < \ldots \ldots \to +\infty$, and write $X_k := (x_k, x_{k+1}]$, with length $|X_k| := x_{k+1} - x_k$. Let B_k be the L_p space of measurable functions on X_k, with

$$\|f\|_{B_k} := |X_k|^{-1/p}\|f\|_{L_p(X_k)} \quad (1 \leqslant p < +\infty, k \in Z).$$

Define

$$C_{p,x}^0 := c^0\{B_k\}_{k\in Z} = \{f| \lim_{|k|\to\infty} \|f\|_{B_k} = 0\}, \quad \|f\|_{C_{p,x}^0} := \sup_{k\in Z} \|f\|_{B_k}.$$

This space was considered by Jakimovski and Russell [8]. A representation for $F \in (C_{p,x}^0)^*$ is obtainable from Theorem 2 above, which coincides with that given in [8], Theorem 1. The representation is used in [8] to solve an interpolation problem, namely the existence of a function from a certain class which takes prescribed values y_k at the points x_k $(\forall k \in Z)$.

When $x_{k+1} - x_k = 1$ $(\forall k)$, Schoenberg [11] defines the space

$$L_{[1]} := \Big\{f| \; \|f\|_{L_{[1]}} := \sum_{k\in Z} \operatorname*{ess\,sup}_{(x_k, x_{k+1}]} |f| < +\infty\Big\}.$$

Thus $L_{[1]} = l_1\{L_\infty(x_k, x_{k+1}]\}_{k\in Z}$, and so his space is (isomorphic to) $(C_{1,x}^0)^*$.

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A remark on finite-dimensional P_λ-spaces

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Abstract. It is shown that finite-dimensional P_λ-spaces contain $l^\infty(m)$-subspaces, where m is proportional to the dimension of the P_λ-space.

Introduction. Let us recall that the Banach–Mazur distance $\Delta(E, F)$ of two normed linear spaces E and F of the same finite dimension is given by $\Delta(E,F) = \inf\{\|T\|\,\|T^{-1}\|;\ T\colon E\to F$ is a linear isomorphism$\}$.

We say that X is a *P_λ-space* provided for any Banach space Y in which X embeds isometrically, there exists a projection P from Y onto X with $\|P\|\leqslant\lambda$. As is well known, the space Y in the above definition may as well be replaced by the space l^∞. The characterization of P_λ-spaces is a rather old and still unsolved problem. One may hope for an affirmative solution to the following question, dealing with the finite-dimensional version of the problem:

Does there exist for all $\lambda<\infty$ some constant $c_\lambda<\infty$ such that $\Delta\big(E, l^\infty(d)\big)<c_\lambda$ holds, for any P_λ-space E of dimension d?

In [4], the existence is shown of a function $d(\lambda, m, \varepsilon)$ so that given a P_λ-space E, $\dim(E)\geqslant d(\lambda, m, \varepsilon)$, one can find a subspace F of E with $\dim(F) = m$ and $\Delta\big(F, l^\infty(m)\big)<1+\varepsilon$.

Our purpose is to show the following fact which, taking into account a related observation of [4], will improve the above result.

THEOREM 1. *Given $\lambda<\infty$, one can find a constant $c = c_\lambda<\infty$ such that given a finite-dimensional P_λ-space E, there exists a subspace F of E satisfying* $\dim(F) = m > c^{-1}\dim(E)$ *and* $\Delta\big(F, l^\infty(m)\big)\leqslant c$.

Proof of the result. We recall that if $T\colon X\to Y$ is an operator between Banach spaces X and Y, then T is (p, q)-*absolutely summing* if there exists a constant $M<\infty$ such that

$$(*)\qquad \Big(\sum_{i=1}^{n}\|T(x_i)\|^p\Big)^{1/p}\leqslant M\sup_{\|x^*\|\leqslant 1}\Big(\sum_{i=1}^{n}|\langle x_i, x^*\rangle|^q\Big)^{1/q}$$

holds, whenever $(x_i)_{1\leqslant i\leqslant n}$ is a finite sequence of vectors in X.

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