STUDIA MATHEMATICA, T. LXXII. (1982)

Pointwise ergodic theorems and function classes M_p^α

by

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Abstract. A general treatment of pointwise ergodic theorems is given of positive linear operators which are at the same time contractions of L_p with $1 < p < \infty$ and of L_∞. For $1 \leqslant p < \infty$ and $0 \leqslant \alpha < \infty$, let M_p^α denote the function class consisting of all functions f such that $(|f|/t)^p(\log^+|f|/t)^\alpha$ is integrable over the set where $|f| > t$ for every $t > 0$. In our consideration the basic setting is just the class M_p^α. Some pointwise ergodic theorems (which are those as time tends to infinity and those as time tends to zero) are proved for products of positive linear contractions on L_p with $1 < p < \infty$ as well as on L_∞ and then extended to functions of M_p^α. Moreover, the continuous extensions of the Akcoglu–Chacon's convexity theorem and of the Akcoglu's ergodic theorem are also proved.

1. Introduction. The present paper is in essence concerned with two types of pointwise ergodic theorems for products of positive linear operators: those as time approaches infinity and those as time approaches zero. The usual setting in the study of ergodic theorems is the Banach space $L_p(X) = L_p(X, \mathscr{B}, \mu)$, $1 \leqslant p \leqslant \infty$, where $(X, \mathscr{B}, \mu)$ is a σ-finite measure space. A linear operator T: $L_p(X) \to L_p(X)$ is called *bounded* if $\|T\|_p \leqslant K$ for some constant $K > 0$, where $\|\cdot\|_p$ denotes the norm of $L_p(X)$. If $K = 1$ then T is said to be a *contraction*. In many studies of ergodic theorems, the boundedness of L_1-norm or L_1- and L_∞-norm of the operators in question, in addition to the signs of operators, was the indispensable conditions for various analyses of the limiting behaviors of operator averages, and has produced many good results. In 1956 Dunford and Schwartz [7] proved the so-called "non-commuting ergodic theorems" for products of linear operators on $L_1(X)$ which are at the same time contractions of $L_1(X)$ and of $L_\infty(X)$. The method of proof used by them consisted of majorizing the operator in question by a positive one, called the *linear modulus*. Then later Chacon and Krengel [6] also showed the existence of the linear modulus for an L_1-contracting linear operator without any additional hypotheses. The linear modulus does make it possible to extend the results for positive operators to those for non-positive operators in spaces of complex-valued functions. Therefore it is important to study ergodic theorems with the L_p-norm condition for $p > 1$ instead of the L_1-norm condition, and this problem is the starting point of the present work. In this communication, including an affirmative answer to this problem, a new approach to general treatment of the problem is developed concerning almost everywhere convergences of operator averages with time tending to infinity and to zero for positive linear operators.

In our consideration, the basic setting is the function class $M_p^a(X)$ ($1 \leqslant p < \infty$, $0 \leqslant a < \infty$) consisting of all functions f such that

$$\int_{|f|>t} \left(\frac{|f|}{t}\right)^p \left(\log \frac{|f|}{t}\right)^a d\mu < \infty \tag{1.1}$$

for every $t > 0$. The important one in extending pointwise ergodic theorems on $L_p(X)$ to functions of $M_p^a(X)$ is the indispensable inequalities for quasi-linear operators of weak type (p, p). In Section 2 we introduce two function classes, $L_p(X)+L_\infty(X)$ and $L^p(X)[\log^+ L(X)]^a$, and some properties of $M_p^a(X)$ are investigated in connection with these classes. In Section 3 we shall prove some pointwise ergodic theorems with discrete time for products of positive linear operators on $L_p(X)$, $1 < p < \infty$, which are at the same time contractions of $L_p(X)$ and of $L_\infty(X)$. Moreover, the results obtained on $L_p(X)$ are then extended to functions of $M_p^a(X)$ as generalizations of those due to Fava [9]. In Section 4 we consider the continuous extensions for the case of continuous semigroups. In addition, the continuous analogues of the Akcoglu–Chacon convexity theorem [2] and of the Akcoglu theorem [1] on pointwise convergences for positive operators on $L_p(X)$, $1 < p < \infty$, are also obtained with a few applications. The last section includes two local ergodic theorems pertaining products of continuous semigroups.

2. Quasi-linear operators and function classes M_p^a. Let there be given two measure spaces $(X, \mathscr{B}, \mu)$ and $(Y, \mathscr{F}, \nu)$. Equalities and inequalities are meant in the almost everywhere sense from now on. In addition we shall sometimes suppress the argument of a function, writing f for $f(x)$. An operator T, which maps $\mathscr{B}$-measurable functions defined on X to $\mathscr{F}$-measurable functions defined on Y, is called *quasi-linear* if

$$|T(f+g)| \leqslant \varrho(|Tf|+|Tg|), \qquad |T(af)| = |a|\,|Tf| \tag{2.1}$$

for some constant $\varrho > 0$. In case $\varrho = 1$, T is said to be *sublinear*. Furthermore we say that T is of *weak type* (p, p) if for any $f \in L_p(X) \cap D(T)$, $1 \leqslant p < \infty$, with the domain $D(T)$ of T and for any $\lambda > 0$

$$\nu\{|Tf| > \lambda\} \leqslant \frac{C}{\lambda^p} \int_X |f|^p d\mu, \tag{2.2}$$

where C is a constant independent of f and λ. The space underlying the following exposition will be σ-finite measure space $(X, \mathscr{B}, \mu)$. We denote

by $L_p(X)+L_\infty(X)$ the class of all functions f such that $f = g+h$, $g \in L_p(X)$, $h \in L_\infty(X)$. An operator T defined on $L_p(X)+L_\infty(X)$ is called *L_∞-bounded* provided that with some constant $K \geqslant 1$

$$\|Tf\|_\infty \leqslant K\|f\|_\infty \quad \text{for} \quad f \in L_\infty(X). \tag{2.3}$$

Examples of L_∞-bounded quasi-linear operators of weak type (p, p) may be taken from the works by Dunford and Schwartz [7] and Akcoglu [1] for the ergodic maximal operators concerning linear contractions on $L_p(X)$. Another example is the Hardy–Littlewood maximal operator, which is given by

$$Uf(x) = \sup_{a<x<\beta} \frac{1}{\beta - a} \int_a^\beta |f(y)|\,dy \tag{2.4}$$

for $f \in L_p(-\infty, +\infty)+L_\infty(-\infty, +\infty)$.

Lemma 2.1. *Let T be an L_∞-bounded quasi-linear operator of weak type (p, p) defined on $L_p(X)+L_\infty(X)$. Then for any $f \in L_p(X)+L_\infty(X)$ and every $t > 0$*

$$\mu\{|Tf| > 2K\varrho t\} \leqslant \frac{C}{t^p} \int_{|f|>t} |f|^p d\mu, \tag{2.5}$$

where C is a constant independent of f and t.

Proof. If the right-hand side of (2.5) is infinite, then the lemma holds trivially. We may thus consider the case where $|f|$ is integrable over the set where $|f| > t$. Define

$$f^{(t)} = f\chi_{\{|f|>t\}}, \qquad f_{(t)} = f\chi_{\{|f|\leqslant t\}} \tag{2.6}$$

for $t > 0$, where χ_A denotes the characteristic function of the set A. Then $f = f^{(t)}+f_{(t)}$ and $|Tf| \leqslant \varrho|Tf^{(t)}|+K\varrho t$ since $\|f_{(t)}\|_\infty \leqslant t$. Hence we have

$$\mu\{|Tf| > 2K\varrho t\} \leqslant \mu\{|Tf^{(t)}| > K\varrho t\} \leqslant \frac{C}{t^p} \int_X |f^{(t)}|^p d\mu \tag{2.7}$$

which implies (2.5).

In the sequel we denote by $M_p^a(X)$, $1 \leqslant p < \infty$, $0 \leqslant a < \infty$, the classes defined by (1.1). Let $L^p(X)[\log^+ L(X)]^a$ denote the class of all functions f for which

$$\int_X |f|^p[\log^+|f|]^a d\mu < \infty, \tag{2.8}$$

where $\log^+ u = \log\max(u, 1)$ for $u \geqslant 0$.

Theorem 2.1. *Let $1 \leqslant p$, $q < \infty$ and $0 \leqslant a$, $\beta < \infty$.*

(2.9) *$M_p^a(X)$ is a linear space.*

(2.10) $L_p(X) \subset M_p^0(X) \subset L_p(X)+L_\infty(X)$.

(2.11) $M_p^a(X) \subset M_q^a(X)$ *for* $q < p$.

(2.12) $M_p^a(X) \subset M_p^\beta(X)$ *for* $\beta < a$.

(2.13) $L_q(X) \subset M_p^a(X)$ *for* $q > p$.

(2.14) $M_p^a(X) \subset L^p(X)[\log^+ L(X)]^a \subset L_p(X) + L_\infty(X)$.

(2.15) $M_p^a(X) = L^p(X)[\log^+ L(X)]^a$ *if and only if* $\mu(X) < \infty$.

(2.16) *The linear span of* $\bigcup_{q>p} L_q(X) \subset M_p^a(X)$.

Proof. The proofs of (2.9)–(2.11) are straightforward. To see (2.12) it is enough to note that $\mu\{t < |f| \leqslant 2t\} < \infty$ for $f \in M_p^a(X)$ and $t > 0$. Then

$$(2.17)\quad \int_{|f|>t} \left(\frac{|f|}{t}\right)^p \left[\log \frac{|f|}{t}\right]^\beta d\mu \leqslant (2t)^{p-1}(\log 2)^\beta \mu\{t < |f| \leqslant 2t\} + \\ +(\log 2)^{-(a-\beta)} \int_{|f|>t} \left(\frac{|f|}{t}\right)^p \left[\log \frac{|f|}{t}\right]^a d\mu < \infty$$

which implies (2.12). The inclusion (2.13) follows from (2.12) and the following:

$$(2.18)\quad \int_{|f|>t} \left(\frac{|f|}{t}\right)^p \left[\log \frac{|f|}{t}\right]^k d\mu \leqslant C_1 \int_{|f|>t} \left(\frac{|f|}{t}\right)^p \left(\frac{|f|}{t}\right)^{q-p} d\mu \\ \leqslant C_2 \int_X |f|^q d\mu < \infty$$

for $f \in L_q(X)$ and $k = [a]+1$ with some constants C_1 and C_2 since $[\log^+ u]^k \leqslant \text{const.}(u)^{q-p}$ for $u \geqslant 0$. The first part of (2.14) is clear. Using (2.6), we have $f = f^{(3)} + f_{(3)}$ and $f_{(3)} \in L_\infty(X)$ and

$$(2.19)\quad \int_X |f^{(3)}|^p d\mu \leqslant (\log 3)^{-a} \int_X |f|^p [\log^+ |f|]^a d\mu$$

proving the second part of (2.14). Evidently, $M_p^a(X) = L^p(X)[\log^+ L(X)]^a$ implies $\mu(X) < \infty$. To prove the converse, suppose that $\mu(X) < \infty$ and $f \in L^p(X)[\log^+ L(X)]^a$. With $f^{(3)}$ and $f_{(3)}$ given by (2.6) we have

$$(2.20)\quad \int_{|f|>t} \left(\frac{|f|}{t}\right)^p \left[\log \frac{|f|}{t}\right]^a d\mu \leqslant C_1 \int_{|f|>t} |f|^p [\log^+ |f|]^a d\mu + C_2 \int_{|f|>t} |f|^p d\mu \\ \leqslant C_1 \int_{|f|>t} |f|^p [\log^+ |f|]^a d\mu + C_3 \int_X (|f^{(3)}|^p + |f_{(3)}|^p) d\mu < \infty$$

for some constants C_1, C_2 and C_3. The relation (2.20), combined with (2.14), shows that $M_p^a(X) = L^p(X)[\log^+ L(X)]^a$, and so (2.15). The inclusion (2.16) follows immediately from (2.9) and (2.13). The proof of Theorem 2.1 has hereby been completed.

Remark. (i): The class $M_p^0(X)$ contains properly $L_p(X)$. We show this fact for the Lebesgue measure μ on the real line. In fact, let $f(x) \geqslant 0$ be a continuous even function defined on the real line with $f(x) \to 0$ as $x \to \infty$, which is strictly decreasing in $x \geqslant 0$ and which is such that $\int_{-\infty}^{+\infty} [f(x)]^p dx = \infty$. For example, take $f(x) = (|x|+1)^{-(1/p)}$. Obviously the function f belongs to $M_p^0(X)$ but does not belong to $L_p(X)$.

(ii): The class $M_p^a(X)$ contains properly the linear span of $\bigcup_{q>p} L_q(X)$. In fact, let $g(x) \geqslant 0$ be a continuous even function defined on the real line with $g(x) \to 0$ as $x \to \infty$, which is strictly decreasing in $x \geqslant 0$ and which is such that $x^q / g_+^{-1}(1/x) \to 0$ as $x \to \infty$ for any fixed $q > p$, where $g_+^{-1}(t)$ is the non-negative part of the inverse function of $g(x)$. For example, we may take $g(x) = 1/\log(|x|+2)$. Then g belongs to $M_p^a(X)$ but does not belong to the linear span of $\bigcup_{q>p} L_q(X)$, where $X = (-\infty, +\infty)$. It is an easy exercise to show this.

3. Pointwise ergodic theorems. In this section some generalizations of the ergodic theorems of Dunford and Schwartz [7] are proved for positive operators without any hypotheses on the $L_1(X)$-norm. In what follows the symbol $A(n)$ will denote the average $(1/n)\sum_{k=0}^{n-1} T^k$ and sometimes, when it is desirable to show the dependence of $A(n)$ upon T, the symbol $A(T, n)$ will be used instead of $A(n)$.

THEOREM 3.1. *Let $T_1, \ldots, T_k$ be k positive linear operators on $L_p(X)$ with $1 < p < \infty$ such that $\|T_i\|_p \leqslant 1$, $\sup\{\|T_i^n\|_\infty : n \geqslant 0\} \leqslant K$, $i = 1, 2, \ldots, k$, for some constant $K \geqslant 1$. Then for every $f \in L_p(X)$ the multiple sequence*

$$(3.1)\quad A(n_1, \ldots, n_k)f = \frac{1}{n_1 \ldots n_k} \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_k=0}^{n_k-1} T_1^{i_1} \ldots T_k^{i_k} f$$

is convergent (as $n_1 \to \infty, \ldots, n_k \to \infty$ independently) almost everywhere on X, as well as in the norm of $L_p(X)$. Furthermore, the functions (3.1) are, for $n_1, \ldots, n_k > 0$, all dominated by a function in $L_p(X)$.

Proof. The proof is by induction. The case $k = 1$ has already been discussed by Akcoglu [1] without the $L_\infty(X)$-norm condition. We now assume $k > 1$ and the almost everywhere convergence has been proved for the case of $k-1$ operators $T_1, \ldots, T_{k-1}$. Since the sequence $A(n_k)$ is bounded and $L_p(X)$ is reflexive for $1 < p < \infty$, the decomposition theorem in [7], p. 133, shows that the linear manifold $\mathfrak{M}_k$ generated by functions of the form

$$(3.2)\quad f = g + (h - T_k h), \quad h \in L_p(X) \cap L_\infty(X),\ g \in L_p(X),\ T_k g = g$$

is dense in $L_p(X)$. For a function f of the form (3.2) we have

$$(3.3)\quad A(n_1, \ldots, n_k)f = A(n_1, \ldots, n_{k-1})g + \frac{1}{n_k} A(n_1, \ldots, n_{k-1})(h - T_k^{n_k}h)$$

and

$$(3.4)\quad \left\| \frac{1}{n_k} A(n_1, \ldots, n_{k-1})(h - T_k^{n_k}h) \right\|_\infty \leqslant \frac{2K^k}{n_k} \|h\|_\infty.$$

Adapting the induction hypothesis to (3.3), together with (3.4), shows that the functions (3.3) converge almost everywhere as $n_1 \to \infty, \ldots, n_k \to \infty$ independently. This implies the almost everywhere convergence of $A(n_1, \ldots, n_k)f$ for any $f \in \mathfrak{M}_k$. However, applying the theorem of Akcoglu [1] repeatedly proves that the sequence $A(n_1, \ldots, n_k)f$, for $f \in L_p(X)$, is dominated by a function in $L_p(X)$. Accordingly we can deduce the almost everywhere convergence of $A(n_1, \ldots, n_k)f$ for $f \in L_p(X)$ from the Banach convergence theorem. The convergence of (3.1) in the norm of $L_p(X)$ may be proved by induction as in the proof of Dunford and Schwartz [7]. Theorem 3.1 has herewith been proved.

COROLLARY 3.1. *On the hypothesis of Theorem 3.1 without positivity of the operators, if there exist the linear moduli of the operators then Theorem 3.1 is also true for non-positive operators.*

A linear contraction T on $L_1(X)$ is said to be *regular* if for some $f \in L_1(X)$ with $f \neq 0$,

$$(3.5)\quad \sum_{k=0}^{\infty} |T^k f| = +\infty$$

almost everywhere on X (cf. [2]). The regularity of T on $L_1(X)$ implies the regularity of the linear modulus of T. For the existence of linear modulus on $L_1(X)$, see Chacon and Krengel [6].

COROLLARY 3.2. *Let $T_1, \ldots, T_k$ be k positive linear operators on $L_p(X)$, with $1 \leqslant p \leqslant p_0$ for some $p_0 > 1$,* such that *$\|T_i\|_p \leqslant 1$, $i = 1, \ldots, k$, for $1 \leqslant p \leqslant p_0$ and such that they are regular on $L_1(X)$. Then Theorem 3.1 remains true for $f \in L_p(X)$ with $1 < p < \infty$.*

Proof. According to the convexity theorem of Akcoglu and Chacon [2], the operators $T_1, \ldots, T_k$ may be extended in such a way that $\|T_i\|_p \leqslant 1$, $i = 1, \ldots, k$, for $1 \leqslant p \leqslant \infty$. These extensions coincide with $T_1, \ldots, T_k$, respectively, on $L_1(X)$, and so applying of the Riesz convexity theorem proves the corollary through Theorem 3.1.

THEOREM 3.2. *Let $T_1, \ldots, T_k$ be k positive linear operators on $L_p(X) + L_\infty(X)$, $1 \leqslant p < \infty$, such that $\|T_i\|_p \leqslant 1$, $\sup\{\|T_i^n\|_\infty : n \geqslant 0\} \leqslant K$, $i = 1, \ldots, k$, for some constant $K \geqslant 1$. Then for every $f \in M_p^{\delta_1(p)(k-1)}(X)$ (where $\delta_1(p)$ means the Kronecker delta) the sequence $A(n_1, \ldots, n_k)f$ converges almost everywhere on X as $n_1 \to \infty, \ldots, n_k \to \infty$ independently.*

Proof. The case $p = 1$ has already been discussed by the author in [11]. (Cf. [9].) Now let $p > 1$ and $f \in M_p^0(X)$. Define

$$(3.6)\quad U_i f = \sup_{n>0} |A(T_i, n)f|, \quad i = 1, \ldots, k,$$

$$(3.7)\quad f^*(x) = \sup_{n_1, \ldots, n_k > 0} |A(n_1, \ldots, n_k)f(x)|.$$

From (3.1) and (3.6) we get $|A(n_1, \ldots, n_k)f| \leqslant U_1 \ldots U_k f$ and thus by (3.7) $f^* \leqslant U_1 \ldots U_k f$. The product operator $U_1 \ldots U_k$ is an L_∞-bounded positive sublinear operator, because so is each U_i. Furthermore, a repeated use of the Akcoglu theorem [1] will show that the product operator is of weak type (p, p). Therefore, with $f^{(t)}$ given by (2.6) for $t > 0$, one gets by Lemma 2.1

$$(3.8)\quad \mu\{f^* > 2K^k t\} \leqslant \mu\{U_1 \ldots U_k f > 2K^k t\} \leqslant \mu\{U_1 \ldots U_k f^{(t)} > K^k t\} \leqslant \frac{C}{t^p} \int_{|f|>t} |f|^p d\mu$$

for some constant C independent of f and t. Let D_r denote the set of all k-tuples $a = (n_1, \ldots, n_k)$ for integers n_i with $n_i \geqslant r$ $(i = 1, \ldots, k)$. Let us define

$$(3.9)\quad \omega(f)(x) = \limsup_{\substack{a,b \in D_r \\ r \to \infty}} |V(k: a)f(x) - V(k: b)f(x)|$$

by putting $V(k: a) = A(n_1, \ldots, n_k)$ for $a = (n_1, \ldots, n_k)$. Clearly, ω is subadditive and $\omega(f) \leqslant 2f^*$. Now choose a sequence $\{f_n\}$ of simple functions having support of finite measure, such that $\lim_n f_n = f$ pointwise and $|f - f_n| \leqslant 2|f|$ for all $n \geqslant 1$. Since $\omega(f) \leqslant \omega(f - f_n) + \omega(f_n)$ and $\omega(f_n) = 0$ by Theorem 3.1, we have $\omega(f) \leqslant \omega(f - f_n) \leqslant 2(f - f_n)^*$, so that by (3.8)

$$(3.10)\quad \mu\{\omega(f) > 8K^k t\} \leqslant \mu\{(f - f_n)^* > 4K^k t\} \leqslant \frac{C}{t^p} \int_{|f-f_n|>2t} |f - f_n|^p d\mu \leqslant \frac{C}{t^p} \int_{|f|>t} |f - f_n|^p d\mu$$

for each $t > 0$. But the last integral of (3.10) tends to zero as $n \to \infty$ by virtue of the Lebesgue dominated convergence theorem. Hence

$$(3.11)\quad \mu\{\omega(f) > 8K^k t\} = 0 \quad \text{for any } t > 0$$

which implies that $\omega(f) = 0$ and completes the proof of Theorem 3.2.

COROLLARY 3.3. *On the hypothesis of Theorem 3.2 without positivity of operators, if there exist the linear moduli of the operators in question, then Theorem 3.2 is also true for non-positive operators.*

The following corollary is an extension of Fava's theorem [9].

COROLLARY 3.4. *Let $T_1, \ldots, T_k$ be positive linear operators on $L_p(X)+L_\infty(X)$, with $1 \leqslant p \leqslant p_0$ for some $p_0 > 1$, such that $\|T_i\|_p \leqslant 1$, $i = 1, \ldots, k$, for $1 \leqslant p \leqslant p_0$ and such that they are regular on $L_1(X)$. Then the same conclusion as in Theorem 3.2 holds.*

The following theorem is an extension of the Akcoglu and Chacon theorem [2].

THEOREM 3.3. *Let $T_1, \ldots, T_k$ be k commuting positive linear operators on $L_1(X)+L_\infty(X)$ such that $\|T_i\|_p \leqslant 1$, $i = 1, \ldots, k$, $1 \leqslant p \leqslant p_0$, for some $p_0 > 1$ and such that they are regular on $L_1(X)$. Then for every $f \in M_1^0(X)$ the limit*

$$\lim_{n\to\infty} \frac{1}{n^k} \sum_{i_1=0}^{n-1} \ldots \sum_{i_k=0}^{n-1} T_1^{i_1} \ldots T_k^{i_k} f \tag{3.12}$$

exists almost everywhere on X.

Proof. According to the Akcoglu and Chacon convexity theorem, $T_1, \ldots, T_k$ are extended in such a way that $\|T_i\|_p \leqslant 1$, $i = 1, \ldots, k$, for $1 \leqslant p \leqslant \infty$. Observe that the ergodic maximal operator U given by

$$Uf(x) = \sup_{n>0} |A(n_1, \ldots, n_k) f(x)| \tag{3.13}$$

for $f \in L_p(X)$ is of weak type (p, p). Then the theorem may be proved by the same argument as that in Theorem 3.2.

4. The case of continuous semigroups. Let $\{T(t)\colon t \geqslant 0\}$ be a strongly continuous semigroup of bounded linear operators on $L_p(X)$ with $1 \leqslant p \leqslant \infty$. Then it is well known ([12]) that there exist $K > 0$ and $a \geqslant 0$ such that $\|T(t)\|_p \leqslant Ke^{at}$ for $t \geqslant 0$. If we take $K = 1$ and $a = 0$, then $T(t)$ turns out to be a linear contraction on $L_p(X)$. Now for $f \in L_p(X)$ there exists a scalar representation $\xi(t, x)$ of $T(t)f$ (which is often denoted by $T(t)f(x)$) such that

(4.1) *$\xi(t, x)$ is measurable in (t, x) on the product space $[0, \infty) \times X$;*

(4.2) *$\xi(t, x)$ is uniquely determined up to a set of points (t, x) whose product measure is zero;*

(4.3) *$\xi(t, \cdot) = T(t)f(\cdot)$ in $L_p(X)$ for almost all $t \geqslant 0$;*

(4.4) *there exists a null set $N(f)$ which may depend on f but which is independent of t and is such that for any $x \in X - N(f)$, $\xi(\cdot, x)$ is integrable over every finite t-interval and*

$$\int_0^\alpha \xi(t, \cdot)\,dt = \int_0^\alpha T(t)f(\cdot)\,dt \quad \text{in } L_p(X) \quad \text{for} \quad 0 \leqslant \alpha < \infty. \tag{4.5}$$

With the function $\xi(t, x)$ we define

$$\int_0^\alpha T(t)f(x)\,dt = \int_0^\alpha \xi(t, x)\,dt \tag{4.6}$$

for any $\alpha \geqslant 0$. The point of this definition is that the left-hand integral of (4.6) is a continuous function of α, for almost every x. Given a semigroup $\{T(t)\colon t \geqslant 0\}$ of linear operators on $L_p(X)+L_\infty(X)$, we shall say that the semigroup is *L_p-continuous* and *L_∞-integrable* if $T(t)$ is strongly continuous when restricted to $L_p(X)$ and strongly integrable over every finite t-interval when restricted to $L_\infty(X)$. We also say that the semigroup is *restrictedly strongly continuous* if $T(t)$ is L_p- and L_∞-continuous when restricted to $L_p(X)$ and $L_\infty(X)$, respectively. Let $\{T(t)\colon t \geqslant 0\}$ be an L_p-continuous and L_∞-integrable semigroup of bounded linear operators on $L_p(X)+L_\infty(X)$. For $f = g+h$ with $g \in L_p(X)$, $h \in L_\infty(X)$, we define

$$\int_0^\alpha T(t)f\,dt = (L_p)\int_0^\alpha T(t)g\,dt + (L_\infty)\int_0^\alpha T(t)h\,dt, \tag{4.7}$$

where the signs preceding the integrals indicate the norms with respect to which each integral is defined. It is easy to verify that definition (4.7) is consistent. To see this it is enough to observe that for any $f \in L_p(X) \cap L_\infty(X)$

$$(L_p)\int_0^\alpha T(t)f\,dt = (L_\infty)\int_0^\alpha T(t)f\,dt. \tag{4.8}$$

Using scalar representations $T(t)g(x)$ and $T(t)h(x)$ of $T(t)g$ and $T(t)h$, respectively, we obtain a scalar representation $T(t)f(x)$ of $T(t)f$ by taking $T(t)f(x) = T(t)g(x)+T(t)h(x)$. Then the ordinary Lebesgue integral

$$\int_0^\alpha T(t)f(x)\,dt = \int_0^\alpha T(t)g(x)\,dt + \int_0^\alpha T(t)h(x)\,dt, \tag{4.9}$$

as a function of x, is a scalar representation of $\int_0^\alpha T(t)f\,dt$. In what follows we consider the following average

$$A(\alpha)f = \frac{1}{\alpha}\int_0^\alpha T(t)f\,dt, \qquad \alpha > 0, \tag{4.10}$$

for $f \in L_p(X)+L_\infty(X)$.

We begin by proving the following theorem which is a continuous version of Akcoglu's theorem [1].

THEOREM 4.1. *Let $\{T(t)\colon t \geqslant 0\}$ be a strongly continuous semigroup of positive linear contractions on $L_p(X)$ with $1 < p < \infty$. Then for any $f \in L_p(X)$ the average (4.10) converges (as $\alpha \to \infty$) almost everywhere on X,* *as well as in the norm of* $L_p(X)$. *Moreover, there holds the following strong type inequality*

$$\|f^*\|_p \leqslant \frac{p}{p-1}\|f\|_p, \tag{4.11}$$

where $f^*(x) = \sup_{a>0}|A(a)f(x)|$.

Proof. Since $L_p(X)$ is reflexive for $1<p<\infty$, the convergence in the norm of $L_p(X)$ follows from the fact that the averages $A(a)$ are bounded and $T(n)/n \to 0$ strongly as $n\to\infty$. The proof of almost everywhere convergence is based upon the following identity which will serve to reduce the present object to the discrete case. Let $a \geqslant 1$ and $f\in L_p(X)$. Then

$$\frac{1}{a}\int_0^a T(t)f\,dt = \frac{[a]}{a}\left\{\frac{1}{[a]}\sum_{k=0}^{[a]-1} T(k)\Big(\int_0^1 T(t)f\,dt\Big)+\frac{T([a])}{[a]}\Big(\int_0^r T(t)f\,dt\Big)\right\}, \tag{4.12}$$

where $a = [a]+r$, $0\leqslant r<1$. However, since

$$\Big\|\int_0^1 T(t)f\,dt\Big\|_p \leqslant \|f\|_p, \quad \Big\|\int_0^r T(t)f\,dt\Big\|_p \leqslant \|f\|_p, \tag{4.13}$$

$$\lim_{a\to\infty}\frac{T([a])}{[a]}\int_0^r T(t)f\,dt = 0 \text{ a.e.}, \tag{4.14}$$

we may apply the Akcoglu theorem to the right-hand side of (4.12) to conclude that $A(a)f$ converges almost everywhere as $a\to\infty$. Using an approximation argument as in Dunford and Schwartz [7], the estimate (4.11) for f^* can be obtained by reduction to a result in [1]. Hence the theorem follows.

THEOREM 4.2. *Let* $\{T(t)\colon t\geqslant 0\}$ *be an* L_p*-continuous and* L_∞*-integrable semigroup of positive linear operators on* $L_p(X)+L_\infty(X)$, $1\leqslant p<\infty$, *such that* $\|T(t)\|_p\leqslant 1$, $\sup\{\|T(t)\|_\infty\colon t\geqslant 0\}\leqslant K$ *for some constant* $K\geqslant 1$. *Then for any* $f\in M_p^0(X)$ *the average* $A(a)f$ *converges almost everywhere as* $a\to\infty$.

Proof. The case of $p=1$ is known to hold for $f\in M_1^0(X)$ ([14]). If $p>1$, then by Lemma 2.1 we have

$$\mu\{f^*>2Kt\}\leqslant \frac{C}{t^p}\int_{|f|>t}|f|^p\,d\mu \tag{4.15}$$

for $f\in L_p(X)+L_\infty(X)$ and every $t>0$ since the ergodic maximal operator U given by $Uf=f^*$ is an L_∞-bounded quasi-linear operator of weak type (p,p) by (4.11). Now let us put

$$\omega(f)(x) = \limsup_{a,\beta\to\infty}|A(a)f(x)-A(\beta)f(x)| \tag{4.16}$$

for $f\in M_p^0(X)$. Select a sequence $\{f_n\}$ of simple functions having support of finite measure, such that $f_n\to f$ everywhere as $n\to\infty$ and $|f_n-f|\leqslant 2|f|$ for all $n\geqslant 1$. Then $\omega(f)\leqslant\omega(f-f_n)+\omega(f_n)$ and $\omega(f_n)=0$ by Theorem 4.1. Therefore, $\omega(f)\leqslant\omega(f-f_n)\leqslant 2(f-f_n)^*$ and so

$$\begin{aligned}\mu\{\omega(f)>8Kt\} &\leqslant \mu\{(f-f_n)^*>4Kt\}\\ &\leqslant \frac{C}{t^p}\int_{|f-f_n|>2t}|f-f_n|^p\,d\mu\\ &\leqslant \frac{C}{t^p}\int_{|f|>t}|f-f_n|^p\,d\mu \to 0\end{aligned} \tag{4.17}$$

as $n\to\infty$ on account of the Lebesgue dominated convergence theorem, which implies $\omega(f)=0$, and hence completes the proof of Theorem 4.2.

A strongly continuous semigroup $\{T(t)\colon t\geqslant 0\}$ of linear contractions on $L_1(X)$ is called *regular* if

$$\int_0^{+\infty}|T(t)f|\,dt = +\infty \tag{4.18}$$

almost everywhere for some $f\in L_1(X)$ with $f\neq 0$. We further say that $\{T(t)\colon t\geqslant 0\}$ is *totally regular* if $T(t)$ is regular for each $t\geqslant 0$. Recall that the strong measurability of $\{T(t)\}$ implies the strong continuity of $\{T(t)\}$.

THEOREM 4.3. *Let* $\{T(t)\colon t\geqslant 0\}$ *be a strongly measurable semigroup of positive linear operators on* $L_p(X)$ *with* $1\leqslant p\leqslant p_0$ *for some* $p_0>1$ *such that* $\|T(t)\|_p\leqslant 1$, $t\geqslant 0$, *for* $1\leqslant p\leqslant p_0$ *and such that the semigroup is totally regular on* $L_1(X)$. *Then* $\{T(t)\}$ *can be extended in such a way that* $\|T(t)\|_p\leqslant 1$ *for* $1\leqslant p\leqslant\infty$.

Proof. For each $t\geqslant 0$, $T(t)$ is regular as an operator on $L_1(X)$ by assumption. Thus, applying the convexity theorem of Akcoglu and Chacon [2], $T(t)$ can be extended to a positive linear operator $\tilde{T}(t)$ on $L_p(X)$ with $1\leqslant p\leqslant\infty$ in such a way that $\|\tilde{T}(t)\|_p\leqslant 1$ for $1\leqslant p\leqslant\infty$ and $\tilde{T}(t) = T(t)$ on $L_p(X)$ with $1\leqslant p\leqslant p_0$. Considering that the basic measure space is σ-finite, let $X_1\subset X_2\subset\ldots$ be an increasing sequence of sets of finite measure such that $X=\bigcup_{i=1}^{\infty}X_i$, and let χ_i be the characteristic function of X_i. Then for any $f\in L_p(X)$ with $1\leqslant p\leqslant\infty$, $f\geqslant 0$ we have

$f(x) = \lim_{i\to\infty} (\chi_i f)(x)$ for almost all $x \in X$. And $\tilde{T}(t)$ is given by

(4.19) $$\tilde{T}(t) f(x) = \lim_{i\to\infty} T(t)(\chi_i f)(x)$$

which is defined for almost all $x \in X$. If f is not positive, then

(4.20) $$\tilde{T}(t) f = \tilde{T}(t) f^{+} - \tilde{T}(t) f^{-}.$$

Now let $\{d_i\}$ be any decreasing sequence of positive numbers with $\lim_{i\to\infty} d_i = 0$, and put for $f \in L_p(X)$ with $1 \leqslant p \leqslant \infty$ and $t, s \geqslant 0$

(4.21) $$E_f(t, s) = \{x\colon |\tilde{T}(t+s) f(x) - \tilde{T}(t)\tilde{T}(s) f(x)| > 0\},$$

(4.22) $$E_f^i(t, s) = \{x\colon |\tilde{T}(t+s)(\chi_i f)(x) - \tilde{T}(t)\tilde{T}(s)(\chi_i f)(x)| > d_i\}.$$

From (4.21) and (4.22) we obtain

(4.23) $$E_f(t, s) \subset \bigcup_{i\geqslant 1} E_f^i(t, s)$$

and $\mu(E_f^i(t, s)) = 0$ for all $i \geqslant 1$. Therefore by (4.23), $\mu(E_f(t, s)) = 0$, from which follows the semigroup property of $\{\tilde{T}(t)\colon t \geqslant 0\}$ on $L_p(X)$ with $1 \leqslant p \leqslant \infty$. The strong measurability of $\{\tilde{T}(t)\}$ is ensured by (4.19), the assumption on $\{T(t)\}$ and a theorem in [8], Ch. III. Hence the proof of the theorem is completed.

THEOREM 4.4. *Let $\{T(t)\colon t \geqslant 0\}$ be a strongly continuous semigroup of positive linear contractions on $L_1(X)$ such that $\|T(t)\|_p \leqslant 1$ on $L_1(X) \cap L_p(X)$ with $1 \leqslant p \leqslant p_0$ for some $p_0 > 1$. Then for any $f \in L_1(X)$*

(4.24) $$\lim_{a\to\infty} \frac{1}{a} \int_0^a T(t) f(x)\, dt$$

exists and is finite almost everywhere on X.

Remark. This theorem is a continuous analogue of a result by Akcoglu and Chacon [2]. If $p_0 = \infty$ then Theorem 4.4 is known as a special case of the Dunford and Schwartz theorem [7]. But, unfortunately, if $p_0 = 1$ then Theorem 4.4 is not true. This fact may be seen from the work by Chacon [4].

Proof of Theorem 4.4. Let the conservative and the dissipative parts of $T(1)$ be C and D, respectively. Putting $p(t, \cdot) = \chi_C(\cdot)$ for all $t \geqslant 0$ with the characteristic function χ_C of C, we define

(4.25) $$p_n(\cdot) = \int_n^{n+1} p(t, \cdot)\, dt, \qquad n = 0, 1, 2, \ldots$$

The convexity theorem of Akcoglu and Chacon, together with the fact that C is a $T(1)$-invariant set [3], show that $\{p_n\colon n \geqslant 0\}$ is a $T(1)$-admissible sequence. Assume that $f \in L_1(X)$. Observe that for almost all $x \in X$

(4.26) $$\lim_{a\to\infty} \frac{T([a])}{[a]} \Big(\int_0^1 T(t)|f|(x)\, dt\Big) = 0$$

and that for almost all $x \in C$

(4.27) $$\lim_{a\to\infty} \frac{\int_{[a]}^{a} p(t, x)\, dt}{\sum_{k=0}^{[a]-1} p_k(x)} = 0,$$

where $a = [a] + r$, $0 \leqslant r < 1$. From (4.26) we obtain

(4.28) $$\lim_{a\to\infty} \frac{\int_{[a]}^{a} T(t) f(x)\, dt}{\sum_{k=0}^{[a]-1} p_k(x)} = 0$$

for almost all $x \in C$. Therefore by (4.27), (4.28) and Chacon's theorem [5], for any $f \in L_1(X)$ and for almost all $x \in C$,

(4.29) $$\lim_{a\to\infty} \frac{1}{a}\int_0^a T(t) f(x)\, dt = \lim_{a\to\infty} \frac{\int_0^a T(t) f(x)\, dt}{\int_0^a p(t, x)\, dt}$$

$$= \lim_{a\to\infty} \frac{\left[\dfrac{\sum_{k=0}^{[a]-1} T(k) F(x)}{\sum_{k=0}^{[a]-1} p_k(x)} + \dfrac{\int_{[a]}^{a} T(t) f(x)\, dt}{\sum_{k=0}^{[a]-1} p_k(x)}\right]}{\left[1 + \dfrac{\int_{[a]}^{a} p(t, x)\, dt}{\sum_{k=0}^{[a]-1} p_k(x)}\right]}$$

$$= \lim_{a\to\infty} \frac{\sum_{k=0}^{[a]-1} T(k) F(x)}{\sum_{k=0}^{[a]-1} p_k(x)}$$

exists and is finite, where $F = \int_0^1 T(t) f\, dt \in L_1(X)$. On the other hand, since for almost all $x \in D$

(4.30) $$\Big|\sum_{k=0}^{\infty} T(k) F(x)\Big| < \infty,$$

we have

$$\left|\frac{1}{a}\int_0^a T(t)f(x)\,dt\right| \leqslant \left|\frac{1}{a}\sum_{k=0}^{\infty} T(k)F(x)\right| + \frac{T([a])}{a}\Big(\int_0^1 T(t)|f|(x)\,dt\Big), \tag{4.31}$$

and hence by (4.26), (4.30) and (4.31)

$$\lim_{a\to\infty}\frac{1}{a}\int_0^a T(t)f(x)\,dt = 0 \tag{4.32}$$

almost everywhere on D. Consequently, combining (4.29) and (4.32) proves Theorem 4.4.

THEOREM 4.5. *Let $\{T_i(t)\colon t\geqslant 0\}$, $i = 1, 2, \ldots, k$, be k strongly continuous semigroups of positive linear operators on $L_p(X)$ with $1 < p < \infty$, such that $\|T_i(t)\|_p \leqslant 1$, $\sup\{\|T_i(t)\|_\infty\colon t\geqslant 0\} \leqslant K$, $i = 1, 2, \ldots, k$, for some constant $K\geqslant 1$. Then for every $f\in L_p(X)$ the averages*

$$A(a_1, \ldots, a_k)f = \frac{1}{a_1\ldots a_k}\int_0^{a_1}\cdots\int_0^{a_k} T_1(t_1)\ldots T_k(t_k)f\,dt_1\ldots dt_k \tag{4.33}$$

are convergent (as $a_1\to\infty, \ldots, a_k\to\infty$ independently) almost everywhere on X, as well as in the norm of $L_p(X)$. Moreover, the functions (4.33) are, for $a_1 > 0, \ldots, a_k > 0$, all dominated by a function in $L_p(X)$.

Remark. The interest here lies in the fact that these semigroups need not commute. The case of $p = 1$ and $K = 1$ is known as a special case of the Dunford and Schwartz theorem [7].

Proof of Theorem 4.5. Since the theorem is true for the case of $k = 1$ (Theorem 4.1), we shall apply induction and assume that it has been proved for the case of $k-1$ semigroups $\{T_1(t)\}, \ldots, \{T_{k-1}(t)\}$. The mean convergence follows then easily from the induction hypothesis. Let $A_i(a)$ be the average of the semigroup $\{T_i(t)\}$ on the interval $(0, a)$. Then $A(a_1, \ldots, a_k)f = A_1(a_1)\ldots A_k(a_k)f$. To prove that the function (4.33) is dominated by a function in $L_p(X)$, we may and will assume that $f\geqslant 0$. In view of Theorem 4.1 there are functions $f_1, \ldots, f_k$ in $L_p(X)$ such that

$$\begin{aligned} &A_k(a_k)f \leqslant f_k, \quad a_k > 0,\\ &A_{k-1}(a_{k-1})A_k(a_k)f \leqslant A_{k-1}(a_{k-1})f_k \leqslant f_{k-1}, \quad a_{k-1}, a_k > 0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &A_1(a_1)\ldots A_k(a_k)f \leqslant f_1, \quad a_1, \ldots, a_k > 0, \end{aligned} \tag{4.34}$$

that is, $A(a_1, \ldots, a_k)f$ is dominated by a function in $L_p(X)$. Now $A_k(a)$ is bounded, $T_k(n)/n\to 0$ strongly as $n\to\infty$ and $L_p(X)$ is reflexive for $1 < p < \infty$, and thus the linear manifold $\mathfrak{M}_k$ generated by functions of the form

$$f = g + \sum_{j=1}^{n}\big(h_j - T_k(t_j)h_j\big) \tag{4.35}$$

with $g\in L_p(X)$, $T_k(t)g = g$, $t\geqslant 0$, and with $h_j\in L_p(X)\cap L_\infty(X)$, is dense in $L_p(X)$. For every f of the form (4.35) we have

$$\begin{aligned} A(a_1, \ldots, a_k)f = {}& A(a_1, \ldots, a_{k-1})g + \\ &+ \sum_{j=1}^{n}\frac{1}{a_k}A_1(a_1)\ldots A_{k-1}(a_{k-1})\Big[\int_0^{t_j} T_k(t)h_j\,dt - \int_{a_k}^{a_k+t_j} T_k(t)h_j\,dt\Big], \end{aligned} \tag{4.36}$$

$$\Big\|A(a_1, \ldots, a_k)\Big[\sum_{j=1}^{n}\big(h_j - T_k(t_j)h_j\big)\Big]\Big\|_\infty \leqslant \frac{2K^k}{a_k}\sum_{j=1}^{n}\|h_j\|_\infty t_j. \tag{4.37}$$

The induction hypothesis, combined with (4.36) and (4.37), shows that $\lim A(a_1, \ldots, a_k)f$ exists almost everywhere. Therefore it ensues from this that the almost everywhere convergence of the function (4.33) holds for every $f\in\mathfrak{M}_k$. Considering the situation described above, since $\sup|A(a_1, \ldots, a_k)f| < \infty$ almost everywhere for $f\in L_p(X)$, we may apply the Banach convergence theorem to complete the proof. Hence the theorem follows.

THEOREM 4.6. *Let $\{T_i(t)\colon t\geqslant 0\}$, $i = 1, 2, \ldots, k$, be L_p-continuous and L_∞-integrable semigroups of positive linear operators on $L_p(X)+L_\infty(X)$ with $1\leqslant p < \infty$ such that $\|T_i(t)\|_p \leqslant 1$, $\sup\{T_i(t)\|_\infty\colon t\geqslant 0\}\leqslant K$, $i = 1, 2, \ldots, k$, for some constant $K\geqslant 1$. Then for any $f\in M_p^{\delta_1(p)(k-1)}(X)$ the function (4.33) converges almost everywhere on X as $a_1\to\infty, \ldots, a_k\to\infty$ independently.*

We omit the proof of this theorem since the argument of proof is the same as that in Theorem 3.2 using Theorem 4.1 and Theorem 4.5.

COROLLARY 4.1. *Let $\{T_i(t)\colon t\geqslant 0\}$, $i = 1, 2, \ldots, k$, be strongly continuous, totally regular semigroups of positive linear contractions on $L_1(X)$ such that $\|T_i(t)\|_p\leqslant 1$, $i = 1, 2, \ldots, k$, on $L_1(X)\cap L_p(X)$ with $1\leqslant p\leqslant p_0$ for some $p_0 > 1$. Then Theorem 4.5 holds for every $f\in L_p(X)$ with $1 < p < \infty$.*

Proof. Use Theorem 4.3 and Theorem 4.5.

COROLLARY 4.2. *Let $\{T_i(t)\colon t\geqslant 0\}$, $i = 1, 2, \ldots, k$, be L_1-continuous, L_∞-integrable and totally regular semigroups of positive linear operators on $L_1(X)+L_\infty(X)$ such that $\|T_i(t)\|_p\leqslant 1$, $i = 1, 2, \ldots, k$, on $L_1(X)\cap L_p(X)$ with $1\leqslant p\leqslant p_0$ for some $p_0 > 1$. Then the same assertion as that of Theorem 4.6 is true for the case of $p = 1$.*

Proof. Use Theorem 4.3 and Theorem 4.6.

COROLLARY 4.3. *Under the hypothesis of Corollary 4.2, assume the additional condition that the semigroups are commutative. Then for any* $f \in M_1^0(X)$ *the limit*

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^k} \int_0^\alpha \cdots \int_0^\alpha T_1(t_1) \ldots T_k(t_k) f\, dt_1 \ldots dt_k \tag{4.38}$$

exists and is finite almost everywhere on X.

Proof. The proof is similar to that of Theorem 3.3 using Theorem 4.3 and the Dunford–Schwartz theorem ([7], Theorem 17, p. 169).

5. Local ergodic theorems. In this section we prove

THEOREM 5.1. *Let* $\{T(t_1, \ldots, t_k): t_i \geqslant 0,\ i = 1, 2, \ldots, k\}$ *be an* L_p*-continuous,* L_∞*-integrable* k*-parameter semigroup of positive linear operators on* $L_p(X)+L_\infty(X)$ *with* $1 \leqslant p < \infty$, *such that* $\|T(t_1, \ldots, t_k)\|_p \leqslant \exp[a(t_1+ \ldots +t_k)]$, $\|T(t_1, \ldots, t_k)\|_\infty \leqslant K\exp[b(t_1+ \ldots +t_k)]$ *for some constants* $a \geqslant 0$, $b \geqslant 0$ *and* $K > 0$. *Then for any* $f \in M_p^0(X)$ *with* $1 \leqslant p < \infty$

$$\lim_{\alpha\to 0+} \frac{1}{\alpha^k} \int_0^\alpha \cdots \int_0^\alpha T(t_1, \ldots, t_k) f\, dt_1 \ldots dt_k = f \tag{5.1}$$

almost everywhere on X.

THEOREM 5.2. *Let* $\{T_i(t): t \geqslant 0\}$, $i = 1, 2, \ldots, k$, *be restrictedly strongly continuous semigroups of positive linear operators on* $L_p(X)+L_\infty(X)$ *with* $1 \leqslant p < \infty$, *such that* $\|T_i(t)\|_p \leqslant \exp(a_i t)$, $\|T_i(t)\|_\infty \leqslant K\exp(b_i t)$ *for some constants* $K > 0$, $a_i \geqslant 0$ *and* $b_i \geqslant 0$, $i = 1, 2, \ldots, k$. *Then for every* $f \in M_p^{\delta_1(p)(k-1)}(X)$

$$\frac{1}{\alpha_1 \ldots \alpha_k} \int_0^{\alpha_1} \cdots \int_0^{\alpha_k} T_1(t_1) \ldots T_k(t_k) f\, dt_1 \ldots dt_k \to f \tag{5.2}$$

almost everywhere as $\alpha_1 \to 0+, \ldots, \alpha_k \to 0+$ *independently.*

In what follows the symbol $A_T(\alpha_1, \ldots, \alpha_k)$ will be used instead of $A(\alpha_1, \ldots, \alpha_k)$ when it is desirable to show the dependence of $A(\alpha_1, \ldots, \alpha_k)$ upon $T_i(t)$, $i = 1, 2, \ldots, k$. To prove the above theorems we need the following

LEMMA 5.1. *On the hypothesis of Theorem 5.2, let* $S_i(t) = \exp[-\max(a_i, b_i)t]\ T_i(t)$, $i = 1, 2, \ldots, k$. *Then for every* $f \in L_p(X)+L_\infty(X)$, $1 \leqslant p < \infty$, *the following two statements are equivalent*:

$$\lim_{\alpha_1, \ldots, \alpha_k \to 0+} A_T(\alpha_1, \ldots, \alpha_k) f = f \text{ a.e.} \tag{5.3}$$

$$\lim_{\alpha_1, \ldots, \alpha_k \to 0+} A_S(\alpha_1, \ldots, \alpha_k) f = f \text{ a.e.} \tag{5.4}$$

Proof. Straightforward.

Proof of Theorem 5.1. Define a new semigroup $\{S(t_1, \ldots, t_k)\}$ by setting

$$S(t_1, \ldots, t_k) = \exp[-\max(a, b)(t_1+ \ldots +t_k)]T(t_1, \ldots, t_k); \tag{5.5.}$$

then $\|S(t_1, \ldots, t_k)\|_p \leqslant 1$, $\|S(t_1, \ldots, t_k)\|_\infty \leqslant K$. In proving the theorem we may and will assume $0 \leqslant f \in M_p^0(X)$. Let us put

$$V(\alpha)f = \frac{1}{\alpha^k} \int_0^\alpha \cdots \int_0^\alpha S(t_1, \ldots, t_k) f\, dt_1 \ldots dt_k, \tag{5.6}$$

$$f_S^*(x) = \sup_{\alpha>0} V(\alpha)f(x). \tag{5.7}$$

If we define

$$\omega(f)(x) = \limsup_{\alpha\to 0+} V(\alpha)f(x) - \liminf_{\alpha\to 0+} V(\alpha)f(x), \tag{5.8}$$

then ω is subadditive and $\omega(f) \leqslant 2f_S^*$. Now choose a sequence $\{f_n\}$ of simple functions having support of finite measure such that $f_n \to f$ pointwise as $n \to \infty$ and $|f-f_n| \leqslant 2|f|$ for each $n \geqslant 1$. We have $\omega(f) \leqslant \omega(f-f_n)+ +\omega(f_n) = \omega(f-f_n) \leqslant 2(f-f_n)_S^*$ since $\omega(f_n) = 0$ by the McGrath theorem [10]. Upon applying Theorem 4.5 and a slight modification of the Dunford and Schwartz lemma ([7], Lemma 11, p. 159) we see that the ergodic maximal operator U_S given by $U_S f = f_S^*$ is an L_∞-bounded sublinear operator of weak type (p, p). Thus from Lemma 2.1 we get with the decomposition (2.6)

$$\begin{aligned} \mu\{\omega(f) > 8Kt\} &\leqslant \mu\{(f-f_n)_S^* > 4Kt\} \\ &\leqslant \frac{C}{t^p} \int_{|f-f_n|>2t} |f-f_n|^p\, d\mu \\ &\leqslant \frac{C}{t^p} \int_{|f|>t} |f-f_n|^p\, d\mu \to 0 \end{aligned} \tag{5.9}$$

as $n \to \infty$ for every $t > 0$ on account of the Lebesgue dominated convergence theorem, which implies that $\omega(f) = 0$ and that $\lim_{\alpha\to 0+} V(\alpha)f = f$ a.e., because we have

$$|V(\alpha)f-f| \leqslant |V(\alpha)(f-f_n)| + |V(\alpha)f_n - f_n| + |f-f_n| \tag{5.10}$$

and

$$\begin{aligned} \mu\{\limsup_{\alpha\to 0+} |V(\alpha)(f-f_n)| > 4Kt\} &\leqslant \mu\{\sup_{\alpha>0} |V(\alpha)(f-f_n)| > 4Kt\} \\ &\leqslant \mu\{(f-f_n)_S^* > 4Kt\}. \end{aligned}$$

This, together with Lemma 5.1, completes the proof of Theorem 5.1.

Proof of Theorem 5.2. Let $\{S_i(t)\}$, $i=1,2,\ldots,k$, be the semigroups defined in Lemma 5.1 for which $\|S_i(t)\|_p\leqslant 1$, $\|S_i(t)\|_\infty\leqslant K$. For these semigroups the convergence (5.2) in the case of $p=1$ has already been discussed in [11]. We assume $p>1$ and apply induction to prove the theorem. The convergence (5.2) is true for the semigroup $S_1(t)$ (Theorem 5.1). Next suppose that the theorem has been established in the case of $k-1$ semigroups $S_1(t),\ldots,S_{k-1}(t)$. Denoting $B_i(a)=(1/a)\int_0^a S_i(t)\,dt$, we have for $h\in L_p(X)\cap L_\infty(X)$

$$\|B_k(a)h-h\|_\infty\leqslant\frac{1}{a}\int_0^a\|S_k(t)h-h\|_\infty dt \tag{5.11}$$

tending to zero as $a\to 0+$, so that

$$\begin{aligned}|A_S(a_1,\ldots,a_k)h-h|&\leqslant|B_1(a_1)\ldots B_{k-1}(a_{k-1})[B_k(a_k)-I]h|+\\&\quad+|B_1(a_1)\ldots B_{k-1}(a_{k-1})h-h|\\&\leqslant K^{k-1}\|B_k(a_k)h-h\|_\infty+|B_1(a_1)\ldots B_{k-1}(a_{k-1})h-h|\end{aligned} \tag{5.12}$$

which approaches zero as $a_1,\ldots,a_k\to 0+$ by the induction hypothesis. This shows that the limit (5.2) holds good for any simple function with support of finite measure. Since by Theorem 4.5

$$f_S^{**}=\sup_{a_1,\ldots,a_k>0}|A_S(a_1,\ldots,a_k)f|<\infty \tag{5.13}$$

almost everywhere for any $f\in L_p(X)$, the Banach convergence theorem guarantees the existence of the limit on the left-hand side of (5.2) for $S_1(t),\ldots,S_k(t)$ and for any $f\in L_p(X)$. Now, for a general function $f\in M_p^0(X)$, we choose, as in the proof of Theorem 5.1, a sequence $\{f_n\}$ of simple functions having support of finite measure such that $f_n\to f$ pointwise and $|f-f_n|\leqslant 2|f|$ for each n. Let D_δ denote the set of all k-tuples $a=(a_1,\ldots,a_k)$ of real numbers with $1/a_i\geqslant\delta$, $1\leqslant i\leqslant k$, and put $V_S(a)=A_S(a_1,\ldots,a_k)$ for $a=(a_1,\ldots,a_k)$. Let us define

$$\omega(f)(x)=\lim_{\delta\to\infty}\sup_{a,\beta\in D_\delta}|V_S(a)f(x)-V_S(\beta)f(x)|. \tag{5.14}$$

From what we have already observed above we obtain

$$\omega(f)\leqslant 2(f-f_n)_S^*+\omega(f_n),\qquad \omega(f_n)=0 \tag{5.15}$$

and

$$\begin{aligned}\mu\{\lim_{\delta\to\infty}\sup_{a\in D_\delta}|V_S(a)f-f|>8K^kt\}\\ \leqslant\mu\{(f-f_n)_S^*>4K^kt\}+\mu\{|f-f_n|>4K^kt\}\end{aligned} \tag{5.16}$$

for each $t>0$. Moreover, using Theorem 4.1 we have, as in the proof of Theorem 3.2,

$$\mu\{(f-f_n)_S^*>4K^kt\}+\mu\{|f-f_n|>4K^kt\}\leqslant\frac{C}{t^p}\int_{|f|>t}|f-f_n|^p d\mu\to 0 \tag{5.17}$$

as $n\to\infty$. Consequently, (5.15), (5.16) and (5.17) imply $\omega(f)=0$ and

$$\mu\{\lim_{\delta\to\infty}\sup_{a\in D_\delta}|V_S(a)f-f|>8K^kt\}=0. \tag{5.18}$$

Hence Theorem 5.2 follows immediately from this and Lemma 5.1.

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Received October 29, 1979 (1585)