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The Markov process determined by a weighted composition operator

by

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Abstract. The ergodic theory of transformations of the form $Tf = \varphi f \circ \tau$ on $L'(X, \Sigma, m)$, where τ is a measurable mapping from X to X and φ is a measurable function from X to R, is studied. Particular attention is paid to determining the conservative and dissipative parts of such operators.

1. Introduction. Transformations of the form $Tf = f\circ\tau$ on $L'(X, \Sigma, m)$, where τ is a measurable mapping from X to X, have been, and continue to be, widely studied. Such mappings form the basis for a large part of the modern development of ergodic theory and Markov processes. In this paper the authors examine the broader class of weighted composition operators on $L'(X, \Sigma, m)$, that is, operators of the form $Tf = \varphi\cdot f\circ\tau$. Particular attention is paid to determining the conservative and dissipative parts of such operators.

In Section 2 the basic properties of weighted composition operators are established, such as evaluation of norm and adjoint. Most of the notation employed in the paper is formalized in this section.

Section 3 is concerned with the development of the structure of the σ-ring of invariant sets for a weighted composition operator. Theorem (3.2) establishes several equivalent characterizations of invariance and is the major result of this section.

The results of Section 3 are, in Section 4, focused on the examination of the conservative and dissipative sets for such processes. In the case where τ is measure preserving a complete analysis of these parts is accomplished ((4.2) and (4.3)).

Section 5 concludes the paper with two examples of such operators. In the first example it is shown that even in the measure preserving τ, isometric operator, case conservatism is not a metric equivalence invariant. The second example indicates how complicated the non-measure preserving case can be.

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2. Preliminaries. Throughout this paper (X, Σ, m) is a probability space and $L' = L'(X, \Sigma, m)$ is the Banach space of equivalence classes of absolutely integrable real valued functions on X. For S a Σ-measurable subset of X, $L'(S) = L'(S, \Sigma|_S, m|_S)$.

If Σ_0 is a subring of Σ, then to each non-negative Σ-measurable (or L') function f there is associated a Σ_0-measurable function $E(f|\Sigma_0)$ so that for each Σ_0 set S

$$\int_S f\,dm = \int_S E(f|\Sigma_0)\,dm.$$

$E(f|\Sigma_0)$ is the *expected value* of f given Σ_0. See [3] for a general discussion of expected values.

Let τ be a mapping from X to X such that $\tau^{-1}(S)$ is in Σ for each S in Σ, and such that $m\circ\tau^{-1}$ is absolutely continuous with respect to m. The symbol τ^n stands for the n-fold composition of τ with itself. Further, let φ be a measurable mapping of X into $(0, \infty)$. We define the *weighted composition operator* $T_{\varphi,\tau}$ on L' by

$$T_{\varphi,\tau}(f) = \varphi\cdot f\circ\tau$$

provided the resulting function is also in L'. If $\|T_{\varphi,\tau}\|_1 \leqslant 1$, we will call $T_{\varphi,\tau}$ a *weighted composition process* since in this case $(T_{\varphi,\tau}, X, \Sigma, m)$ is a Markov process, in the sense of Foquel [4].

We wish to establish formulas for the norms and adjoints of such operators. We will use the following well-known facts frequently throughout this paper.

1. If g is Σ_0-measurable, then $E(fg|\Sigma_0) = E(f|\Sigma_0)g$.

2. f is $\tau^{-1}\Sigma$-measurable if and only if $f = g\circ\tau$ for some Σ-measurable function g.

In view of (2) above it makes sense to refer to the function $E(f|\tau^{-1}\Sigma)\circ\circ\tau^{-1}$.

(2.1) Proposition. *Let $T = T_{\varphi,\tau}$. Then*

(a) $\|T\|_1 = \left\| [E(\varphi|\tau^{-1}\Sigma)\circ\tau^{-1}]\dfrac{dm\circ\tau^{-1}}{dm}\right\|_\infty$;

(b) $T^*g = [E(\varphi g|\tau^{-1}\Sigma)\circ\tau^{-1}]\dfrac{dm\circ\tau^{-1}}{dm}$,

equivalently

(b′) $(T^*g)\circ\tau = [E(\varphi g|\,\tau^{-1}\Sigma)]\left(\dfrac{dm\circ\tau^{-1}}{dm}\right)\circ\tau.$

Proof. Let f be in L'. Then

$$\|Tf\|_1 = \int_X \varphi|f\circ\tau|\,dm = \int_X E(\varphi|\,f\circ\tau|\,|\tau^{-1}\Sigma)\,dm = \int_X E(\varphi|\tau^{-1}\Sigma)|\,f\circ\tau|dm$$
$$= \int_X [E(\varphi|\tau^{-1}\Sigma)\circ\tau^{-1}]|f|\,dm\circ\tau^{-1}$$
$$= \int_X [E(\varphi|\tau^{-1}\Sigma)\circ\tau^{-1}]\frac{dm\circ\tau^{-1}}{dm}\,|f|\,dm.$$

In general,

$$\sup\left\{\int_X |hg|dm:\ \|h\|_1 = 1\right\} = \|g\|_\infty,$$

so the validity of (a) is established. In order to prove (b) and (b′), let f be in L' and g be in L^∞. Then

$$(Tf, g) = \int_X (\varphi)(f\circ\tau)(g)\,dm = \int_X E(\varphi g|\,\tau^{-1}\Sigma)f\circ\tau\,dm$$
$$= \int_X [E(\varphi g|\tau^{-1}\Sigma)\circ\tau^{-1}]\frac{dm\circ\tau^{-1}}{dm}f\,dm,$$

so that (b) and (b′) hold.

We will especially be interested in the class of weighted composition operators $T_{\varphi,\tau}$ for which τ is *measure preserving*, that is $m(\tau^{-1}S) = m(S)$ for every S in Σ. In this case we have $dm\circ\tau^{-1}/dm = 1$.

We will also be concerned with *isometric* operators, $\|Tf\|_1 = \|f\|_1$ for each f in L^1. It follows immediately from the derivation of 2.1 (a) that $T = T_{\varphi,\tau}$ is isometric if and only if $[E(\varphi|\tau^{-1}\Sigma)\circ\tau^{-1}]\dfrac{dm\circ\tau^{-1}}{dm} = 1$ a.e. dm.

A Σ-set S is said to be *invariant* for $T = T_{\varphi,\tau}$ if $T^*1_S = 1_S$, the characteristic function of S. See [4] for a full discussion of invariant sets. Let $\mathcal{I} = \mathcal{I}(\varphi, \tau)$ be the σ-ring of invariant sets for $T_{\varphi,\tau}$.

The *conservative set* for T is $C = \{x\in X:\ \sum_{n=0}^{\infty}(T^n1)(x) = \infty\}$ and the *dissipative set* for $T_{\varphi,\tau}$ is $D = X - C$ ([4]). From [4], Ch. II, and (3.2), it follows that C (but not in general D) is an invariant set for T.

Throughout this paper all set references are to be interpreted as valid up to sets of measure 0. For example, the statement $A\subseteq B$ is to be interpreted as $m(B - A) = 0$.

3. Characterization of invariant sets. Let $T = T_{\varphi,\tau}$ be a weighted composition process and let $\Sigma' = \{A\in\Sigma:\ \tau^{-1}A = A\}$.

We say that τ is *full* on the set $S\in\Sigma$ if whenever $E\subseteq S$, $E\in\Sigma$, and $m(E) > 0$, then $m(\tau^{-1}E) > 0$.

(3.1) Lemma. *Let $S \in \Sigma'$. Then the following are equivalent:*
(a) *T is an isometry on $L'(S)$;*
(b) $T^*1_S \geqslant 1_S$;
(c) (i) *τ is full on S, and*
(ii) $(T^*1_S)\circ\tau \geqslant 1_S$.

Proof. Assume (a) holds and let $A \subseteq S$, $m(A) > 0$. Then

$$m(A) = \|1_A\|_1 = \|T1_A\|_1 = (\varphi 1_A\circ\tau, 1) = (T1, 1_A\circ\tau) = \big(1, T^*(1_A\circ\tau)\big).$$

But $T^*(1_A\circ\tau) = [E(\varphi 1_A\circ\tau|\,\tau^{-1}\Sigma)\circ\tau^{-1}]\dfrac{dm\circ\tau^{-1}}{dm}$. Now, $1_A\circ\tau = 1_{\tau^{-1}(A)} = 1_{\tau^{-1}(A)}\cdot 1_S$ since $A \subseteq S$ and $\tau^{-1}S = S$, so that

$$T^*(1_A\circ\tau) = \big[E(\varphi 1_S(1_A\circ\tau)|\,\tau^{-1}\Sigma)\circ\tau^{-1}\big]\frac{dm\circ\tau^{-1}}{dm} = [E(\varphi 1_S|\,\tau^{-1}\Sigma)\circ\tau^{-1}]\cdot\frac{dm\circ\tau^{-1}}{dm}\cdot 1_A = (T^*1_S)1_A.$$

Thus

$$m(A) = \|T1_A\|_1 = \|T^*(1_A\circ\tau)\|_1 = \int_A (T^*1_S)\,dm \leqslant m(A).$$

It follows that $T^*1_S = 1$ a.e. on S, i.e. $T^*1_S \geqslant 1_S$. The converse clearly holds, so that (a) and (b) are equivalent.

Suppose now that (b) holds. If $A \subseteq S$ with $m(\tau^{-1}A) = 0$, then as above, we have $m(A) = \|T^*(1_A\circ\tau)\|_1 = \|T^*1_{\tau^{-1}(A)}\|_1 = 0$, so τ is full on S. Also, since $T^*1_S \geqslant 1_S$ and $\tau^{-1}S = S$, $(T^*1_S)\circ\tau \geqslant 1_S\circ\tau = 1_S$, so (b) implies (c).

Finally, suppose that (c) is valid. Let $A = \{x \in S: (T^*1_S)(x) < 1\}$. Then $\tau^{-1}A = \{x: \tau(x) \in S \text{ and } (T^*1_S)(\tau(x)) < 1\} = \{x \in S: (T^*1_S)(\tau(x)) < 1\} = \emptyset$. (The second equality follows from the fact that $\tau^{-1}S = S$.) But τ is full on S, so $m(A) = 0$, that is to say, (b) holds and the proof is complete.

We are now ready to characterize the invariant ring $\mathscr{I}(\varphi, \tau)$.

(3.2) Theorem. *Let $S \in \Sigma$. Then S is in $\mathscr{I}(\varphi, \tau)$ if and only if $S \in \Sigma'$ and any of the (equivalent) conditions (a), (b), or (c) from (3.1) holds.*

Proof. Since $\tau^{-1}(S) = S$, $\tau^{-1}(X-S) = X-S$. Therefore,

$$\int_{X-S} T^*1_S dm = \int_X (1_{X-S})(T^*1_S)\,dm = \int_X (1_S)(T1_{X-S})\,dm = \int_X \varphi 1_S 1_{\tau^{-1}(X-S)}dm = \int_X (\varphi 1_S)(1_{X-S})\,dm = 0,$$

so that $T^*1_S = 0$ off S. Suppose that (3.1b) holds, i.e. $T^*1_S \geqslant 1_S$. Since $T^*1_S \leqslant 1$ and vanishes off S, $T^*1_S = 1_S$, so that $S \in \mathscr{I}(\varphi, \tau)$.

Now suppose that S is in $\mathscr{I}(\varphi, \tau)$. We need only show that $\tau^{-1}S = S$. Note that

$$0 = \int_{X-S} T^*1_S\,dm = \int_S T1_{X-S}\,dm = \int_S \varphi 1_{\tau^{-1}(X-S)}\,dm.$$

Thus $\tau^{-1}(X-S) \subset X-S$. Moreover, $T^*1_{X-S} = T^*1 - T^*1_S = T^*1 - 1_S$, so $0 = \int_S T^*1_{X-S}dm = \int_{X-S}\varphi 1_{\tau^{-1}S}\,dm$. It follows that $\tau^{-1}S \subseteq S$ and so $\tau^{-1}S = S$.

The following result holds for any Markov process on a probability measure space with the property that $\|Tf\|_1 = \|T|f|\|_1$.

(3.3) Corollary. *X is invariant if and only if T is an isometry.*

Since $C = C(T)$ is an invariant set, T is an isometry on $L'(C)$. Let $I(T)$ be the largest (modulo sets of measure 0) invariant set in $\mathscr{I}(\varphi, \tau)$. Set $DI(T) = I(T) - C(T)$, and $SD(T) = X - I(T)$. Then T is an isometry on $L'(I(T))$. Furthermore, $\tau^{-1}(DI(T)) = DI(T)$ and $\tau^{-1}(SD(T)) = SD(T)$. It follows that $T = T_C \oplus T_{DI} \oplus T_{SD}$ where T_M is the restriction of T to the space $L'(M)$. In the case where τ is measure preserving we will succinctly and completely characterize these "parts" of T.

Note that if τ is measure preserving, then τ is full on every Σ-set. It follows from (3.2c) that $\mathscr{I}(1, \tau) = \Sigma'$, In particular, the dissipative set for $T_{\varphi,\tau}$ is in Σ'.

4. The conservative and dissipative sets. In this section we characterize the conservative and dissipative parts of $T_{\varphi,\tau}$ in terms of φ and τ when τ is measure preserving. For notational convenience let τ^k be the k-fold composition of τ with itself, set $\varphi_0 = 1$ and $\varphi_{n+1} = \prod_{k=0}^{n}\varphi\circ\tau^k$. Thus $T^nf = \varphi_n f\circ\tau^n$ or equivalently $T^n = T_{\varphi_n,\tau^n}$. Let $E_n(f) = E(f|\,\tau^{-n}\Sigma)$, so that $T^{*n}f = E_n(\varphi_n f)\circ\tau^{-n}$. The following lemma is related to [3], p. 732, Ex. 27. However, in this setting a more detailed analysis is possible.

(4.1) Lemma. *Let $T = T_{\varphi,\tau}$ be a weighted composition operator (not necessarily contractive) with τ being measure preserving. Further suppose that $\log\varphi$ is in L'. Then*

$$\lim_{n\to\infty}\varphi_n^{1/n} = \exp[E(\log\varphi|\Sigma')] \text{ a.e. } dm.$$

Proof. We have

$$\log\varphi_n^{1/n}(x) = \frac{1}{n}\log\varphi_n(x) = \frac{1}{n}\log\Big(\prod_{k=0}^{n-1}\varphi(\tau^k(x)\Big) = \frac{1}{n}\sum_{k=0}^{n-1}[(\log\varphi)\circ\tau^k](x)$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}(T^k_{1,\tau}\log\varphi)(x).$$

The result follows from the Birkhoff pointwise ergodic theorem [1].

(4.2) THEOREM. *Let $T = T_{\varphi,\tau}$ be a weighted composition operator such that τ is measure preserving and $\log\varphi$ is summable. Then $D = \{E(\log\varphi|\Sigma') < 0\}$.*

Proof. If x is in $\{E(\log\varphi|\Sigma') < 0\}$, then from (4.1) we have $\overline{\lim}\,[\varphi_n(x)]^{1/n} < 1$ so that $\sum\varphi_n(x) < \infty$. Thus $\{E(\log\varphi|\Sigma') < 0\} \subseteq D$.

Set $f = 1/(\sum_{n=0}^{\infty}\varphi_n)$. Then $f = 0$ off D and $1 > f > 0$ on D. Direct computation shows, for $n \geq 1$, that

$$(f\circ\tau^n)\Big[1-\Big(\sum_{k=0}^{n-1}\varphi_k\Big)f\Big] = \varphi_n f.$$

It follows from this that

$$\overline{\lim}\,(\varphi_n)^{1/n} \leq \overline{\lim}\,(f\circ\tau^n)^{1/n}$$

and the proof of (4.1) can be modified to show that

$$\overline{\lim}\,(f\circ\tau^n)^{1/n} = \lim\,(f\circ\tau^n)^{1/n} < 1$$

if $0 < f < 1$ Using (4.1) it can now he seen that $E(\log\varphi|\Sigma') < 0$ on D.

If τ is measure preserving and $T = T_{\varphi,\tau}$ is a process, then somewhat more can be said about the pieces of T in terms of φ. It turns out that we need not assume that $\log\varphi$ is in L'. We will use the following notation. If $A \in \Sigma$, then A_* is the largest (as usual up to sets of measure 0) $\mathscr{I}(\varphi,\tau)$ set in A ([4], Ch. III).

(4.3) THEOREM. *Let τ be measure preserving, $A_0 = \{\varphi = 1\}$ and $A_1 = \{E_1(\varphi) = 1\}$. Then*

(a) $I(T) = A_{1*}$,

(b) $C(T) = A_{0*}$,

(c) $DI(T) = A_{1*} - A_{0*}$, *and*

(d) $SD(T) = X - A_{1*}$.

Proof. (a): Since $\dfrac{dm\circ\tau^{-1}}{dm}\circ\tau \equiv 1$, a set S is invariant if and only if τ is full on S and $E_1(\varphi 1_S) \geq 1_S$. Thus, if S is invariant, $1 \geq E_1(\varphi 1_S) = E_1(\varphi)1_S = 1$ on S, so $S \subset A_1$ and consequently $S \subset A_{1*}$. If (b) holds,

then (c) and (d) are valid, as they would be the definitions of the respective parts. The proof will be complete upon the verification of (b).

(b): Suppose first that $\log\varphi$ is summable. Certainly, $C(T) \supseteq A_{0*}$ since $\sum_{n=0}^{\infty}\varphi_n = \infty$ on A_{0*}. Let $A \in \Sigma'$ with $A \subseteq X - A_{0*}$. On A we have, via Taylor's theorem,

$$\log\varphi = (\varphi-1) - \frac{(\varphi-1)^2}{[Z(\varphi)]^2}$$

for some function Z on the range of φ. Then

$$\int_A \log\varphi = \int_A(\varphi-1) - \int_A\frac{(\varphi-1)^2}{[Z(\varphi)]^2}.$$

Since $A \in \Sigma' \subseteq \tau^{-1}\Sigma$, and since $E_1(\varphi) \leq 1$ a.e., $\int_A(\varphi-1) = \int_A E_1(\varphi) - m(A) \leq 0$. Thus $\int_A\log\varphi \leq \int_A\big((\varphi-1)^2/[Z(\varphi)]^2\big)$. But $A \subseteq X - A_{0*}$ so φ cannot be identically 1 on A. Consequently $\int_A\big((\varphi-1)^2/[Z(\varphi)]^2\big) > 0$, so that $\int_A\log\varphi < 0$. Thus $E(\log\varphi|\Sigma') < 0$ a.e. on $X - A_{0*}$. Now, from (4.1), we have that $X - A_{0*} \subseteq D(T)$.

This shows that $A_{0*} = C(T)$. Dropping the assumption that $\log\varphi$ is in L', let $\psi = \frac{1}{2}(1+\varphi)$. Then $\log\psi$ is summable and $T_{\psi,\tau}$ is a weighted composition process. Moreover, $\{\psi = 1\} = \{\varphi = 1\} = A_0$. Hence $C(T_{\psi,\tau}) = A_{0*}$. Now

$$\log[\psi_n^{1/n}] = \frac{1}{n}\sum_{k=0}^{n-1}\log\psi\circ\tau^k = \frac{1}{n}\sum_{k=0}^{n-1}\log(\tfrac{1}{2}+\tfrac{1}{2}\varphi\circ\tau^k)$$

$$\geq \frac{1}{n}\sum_{k=0}^{n-1}\tfrac{1}{2}\log\varphi\circ\tau^k(\text{concavity of }\log) = \frac{1}{2}\cdot\frac{1}{n}\log\varphi_n = \log\sqrt{\varphi_n^{1/n}}.$$

Thus $\psi_n^{1/n} \geq \sqrt{\varphi_n^{1/n}}$. Let $a = \lim_{n\to\infty}\psi_n^{1/n}$; then $\overline{\lim}\,\varphi_n^{1/n} \leq a^2$, and so $\{a < 1\} \subset D(T_{\varphi,\tau})$. This implies that $D(T_{\varphi,\tau}) \supseteq D(T_{\psi,\tau}) = X - A_{0*}$, and hence $C(T_{\varphi,\tau}) \subseteq A_{0*}$. Since the reverse inclusion holds regardless of the summability of $\log\varphi$, the proof is complete.

We terminate this section with some results concerning the dissipative part of a conjugate of $T_{\varphi,\tau}$. This material will be used in the subsequent section. The proofs of the following two results are both immediate and omitted.

(4.4) LEMMA. *Let ϱ be a measure isomorphism on (X,Σ) and let $\tau_1 = \varrho^{-1}\circ\tau\circ\varrho$. Then $C(T_{\varphi,\tau_1}) = \varrho^{-1}\big(C(T_{\varphi\circ\varrho^{-1},\tau})\big)$ and $D(T_{\varphi,\tau_1}) = \varrho^{-1}\big(D(T_{\varphi\circ\varrho^{-1},\tau})\big)$.*

(4.5) Corollary. *If* $\tau_1 = \varrho^{-1}\circ\tau\circ\varrho$ *as above and* $\log\varphi\circ\varrho^{-1}$ *is in* L^1, *then*

$$D(T_{\varphi,\tau_1}) = \varrho^{-1}\big(\{E(\log\varphi\circ\varrho^{-1}|\Sigma'(\tau))<0\}\big).$$

5. Two examples. Two operators A and B on the normed set N are said to be *metrically equivalent* if $\|Ax\| = \|Bx\|$ for every x in N. It is easy to see that the weighted composition operators $T_{\varphi,\tau}$ and $T_{\varrho,\tau}$ (same τ) are metrically equivalent if and only if $E_1(\varphi) = E_1(\varrho)$. At first glance it would seem reasonable that the property of being conservative is a metric equivalence invariant, but (4.3) suggests that this is not the case. This first example illustrates that, even in the measure preserving case, most weighted composition processes, including isometric ones, are dissipative.

(5.1) Example. Let $X = [0,1]$ and let m be Lebesgue measure. Define the measure preserving mapping τ from X to X by

$$\tau(x) = \begin{cases} 2x, & 0\leqslant x\leqslant 1/2,\\ 2(1-x), & 1/2\leqslant x\leqslant 1.\end{cases}$$

It is easily verified that $\tau^{-1}\Sigma$ is the σ-ring of all measurable sets in X symmetric about 1/2. Moreover, $T(1,\tau)$ is ergodic, i.e. $\mathscr{I}(1,\varphi) = \{\emptyset, X\}$. Consider $A\in\mathscr{I}(1,\tau)$ with $m(A)>0$. For each α in $(0,1)$ there is a diadic interval $J = [m/2^j, (m+1)/2^j]$ such that $m(A\cap J)\geqslant \alpha/2^j$. For $k\leqslant j$ we have $m(\tau^k J) = 1/2^{j-k}$ and using the fact that $\tau^{-1}A = A$ and τ is measure preserving, we have $m(A\cap\tau^k J)\geqslant \alpha/2^{j-k}$. In particular, for $k=j$ we have $m(A)\geqslant m(A\cap\tau^j J)\geqslant\alpha$. Since α was chosen arbitrarily in $(0,1)$, we are forced to conclude that $m(A)=1$.

In light of (4.3) we may make the following observations:

(a) If φ is not identically 1, then $D(T_{\varphi,\tau}) = X$.

(b) If $E_1(\varphi) = 1$ a.e., then $I(T_{\varphi,\tau}) = X$.

(c) For $\varphi = \frac{3}{2}1_{[0,1/2]}+\frac{1}{2}1_{[1/2,1]}$, we have $E_1\varphi = 1$ a.e. and so (a) and (b) yield $I(T_{\varphi,\tau}) = D(T_{\varphi,\tau}) = X$.

In particular, $T_{1,\tau}$ and $T_{\varphi,\tau}$ are metrically equivalent while the former is conservative and the latter is dissipative.

In general when τ is not necessarily measure preserving we know that in order for $T_{\varphi,\tau}$ to be conservative it is necessary that $E_1\varphi = \left[\frac{dm\circ\tau^{-1}}{dm}\circ\tau\right]^{-1}$. A natural conjecture would be that $T_{\varphi,\tau}$ is conservative if $\varphi(x) = \left[\frac{dm\circ\tau^{-1}}{dm}\circ\tau(x)\right]^{-1}$. The next example indicates that the situation is not nearly so simple.

(5.2) Example. Let

$$\varrho(x) = \begin{cases} 3x/2, & 0\leqslant x<1/3,\\ (3x/4)+1/4, & 1/3\leqslant x\leqslant 1,\end{cases}$$

and let τ be as in Example (5.1).

Note that

$$\varrho^{-1}(x) = \begin{cases} 2x/3, & 0\leqslant x<1/2,\\ (4x/3)-1/3, & 1/2\leqslant x\leqslant 1.\end{cases}$$

Let $\tau_1 = \varrho^{-1}\circ\tau\circ\varrho$, so that

$$\tau_1(x) = \begin{cases} 2x, & 0\leqslant x\leqslant 1/6,\\ 4x-1/3, & 1/6\leqslant x\leqslant 1/3,\\ 5/3-2x, & 1/3\leqslant x\leqslant 2/3,\\ 1-x, & 2/3\leqslant x\leqslant 1.\end{cases}$$

It follows routinely that

$$\frac{dm\circ\tau_1^{-1}}{dm}\circ\tau_1 = (3/2)1_{(0,1/6)\cup(2/3,1)}+(3/4)1_{(1/6,2/3)}.$$

Since $\mathscr{I}(1,\tau)$ is trivial, we have, according to (4.5), that T_{φ,τ_1} is dissipative if and only if $\int_X \log(\varphi\circ\varrho^{-1})\,dm<0$. We make two observations:

(1) If $\varphi = \left[\frac{dm\circ\tau_1^{-1}}{dm}\circ\tau\right]^{-1}$, then

$$\log\varphi\circ\varrho^{-1} = (\log 2/3)1_{[0,1/4]\cup[3/4,1]}+(\log 4/3)1_{[1/4,3/4]}.$$

Thus $\int_X \log\varphi\circ\varrho^{-1} = -\log 3+(3/2)\log 2<0$;

(2) If

$$\varphi^*(x) = \begin{cases} 1, & 0<x<1/6,\\ 2, & 1/6<x<1/3,\\ 1, & 1/3<x<2/3,\\ 1/2, & 2/3<x<1,\end{cases}$$

then $E(\varphi^*|\tau_1^{-1}\Sigma) = \left[\frac{dm\circ\tau_1^{-1}}{dm}\circ\tau_1\right]^{-1}$, and calculation shows that $\int_X \log\varphi^*\circ\varrho^{-1} = 0$. It follows that T_{φ^*,τ_1} is conservative.

We will show that if φ satisfies $E(\varphi|\tau_1^{-1}\Sigma) = \left[\frac{dm\circ\tau_1^{-1}}{dm}\circ\tau_1\right]^{-1}$ but is not identically φ^*, then T_{φ,τ_1} is dissipative. In order to ascertain this property we develop a method for explicitly computing $E(\varphi|\tau^{-1}\Sigma)$. This technique's development is aided by examination of the following process. In what follows φ is an arbitrary strictly positive L' function.

Let $\tau_1(x) = 1-x$ and $\tau_0 = \varrho^{-1}\circ\tau_1\circ\varrho$. Then

$$\gamma(x) = \left[\frac{dm\circ\tau_0^{-1}}{dm}\big(\tau_0(x)\big)\right]^{-1} = \begin{cases}2, & 0<x<1/3,\\ 1/2, & 1/3<x<1.\end{cases}$$

It follows that T_{γ,τ_0} is a conservative process. In fact, $\gamma_k = \gamma$ for k odd, and $\gamma_k = 1$ for k even, hence $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\gamma_k = (3/2)1_{[0,1/3]}+(3/4)1_{[1/3,1]}$ a.e. But then, by the Chacon Identification Theorem ([2]),

$$\lim_{n\to\infty}\left[\left(\frac{1}{n}\sum_{k=0}^{n-1}T^k_{\gamma,\tau_0}\varphi\right)\Big/\left(\frac{1}{n}\sum_{k=0}^{n-1}\gamma_k\right)\right] = E\big(\varphi|\mathscr{I}(\gamma,\tau_0)\big)/E\big(1|\mathscr{I}(\gamma,\tau_0)\big) = E(\varphi|\tau_1^{-1}\Sigma).$$

Thus

$$E(\varphi|\tau_1^{-1}\Sigma)(x) = \begin{cases}(1/3)[\varphi(x)+2\varphi(1-2x)], & 0<x<1/3,\\ (2/3)\big[\varphi(x)+(1/2)\varphi\big((1-x)/2\big)\big], & 1/3<x<1.\end{cases}$$

A straightforward calculation yields

$$\int_{[0,\frac14]\cup[3/4,1]}\log\varphi\circ\varrho^{-1} = \int_0^{\frac14}\log\varphi\big(\varrho^{-1}(x)\big)\,dx + \int_0^{\frac14}\log\varphi\big(1-2\varrho^{-1}(x)\big)\,dx.$$

But $\varrho^{-1}[0,1/4]\subseteq(0,1/3)$, so on $[0,1/4]$, $\frac13\big[\varphi\big(\varrho^{-1}(x)\big)+2\varphi\big(1-2\varrho^{-1}(x)\big)\big] = \big(E(\varphi|\tau_1^{-1}\Sigma)\big)\big(\varrho^{-1}(x)\big) = 2/3$. Hence

$$\varphi\big(\varrho^{-1}(1-x)\big) = \varphi\big(1-2\varrho^{-1}(x)\big) = \big(2-\varphi/\varrho^{-1}(x)\big)/2.$$

Consequently,

$$(*)\qquad \int_{[0,1/4]\cup[3/4,1]}\log\varphi\circ\varrho^{-1} = \int_0^{\frac14}\log\big[(\varphi\circ\varrho^{-1})(2-\varphi\circ\varrho^{-1})/2\big].$$

It is easily verified that such a quantity as the right side of $(*)$ is maximized when $\varphi\circ\varrho^{-1}=1$ on $(0,1/4)$ and $\varphi\circ\varrho^{-1}=1/2$ on $[3/4,1]$. Similarly, $\int_{1/4}^{3/4}\log\varphi\circ\varrho^{-1}$ is strictly maximized when $\varphi\circ\varrho^{-1}$ is 2 on $(1/4, 1/2)$ and is 1 on $(1/2, 3/4)$. However, these conditions imply that $\int_X\log\varphi\circ\varrho^{-1}$ is maximized precisely when $\varphi=\varphi^*$.

In the above example, it turned out that there was a unique φ with $E(\varphi|\tau^{-1}\Sigma) = \left(\frac{dm\circ\tau^{-1}}{dm}\circ\tau\right)^{-1}$ such that $T_{\varphi,\tau}$ was conservative. The φ that works is not $\left[\frac{dm\circ\tau^{-1}}{dm}\circ\tau\right]^{-1}$ as might be supposed from the measure preserving case; rather, φ was that function among those with the appropriate $\tau^{-1}\Sigma$ expected value which maximized $E(\log\varphi|\Sigma')$. We ask whether, for τ conjugate to a measure preserving map, this is always the case.

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