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On the cohomology of sheaves $\mathcal{S}\,\varepsilon\, L$

by

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Abstract. Some results on the cohomology groups of sheaves $\mathcal{S}\varepsilon L$ are given. Using those results we get the splitness of Dolbeault complexes of holomorphic Banach bundles at only positive dimension over Stein manifolds.

Introduction. Let L be a complex locally convex space and let L' denote the strongly dual space of L. By $\mathcal{U}(L)$ we denote the set of all balanced convex neighbourhoods of zero in L. For every $U \in \mathcal{U}(L)$ let $p(U)$ denote the Minkowski functional of U and let $L(U)$ be the completion of $L(U)/p(U) \overset{\text{def}}{=} L(U)/p(U)^{-1}(0)$ equipped with the norm $p(U)$.

A locally convex space L is said to be *s-nuclear* (see [9]) if for every $U \in \mathcal{U}(L)$ there exists a $V \in \mathcal{U}(L)$ such that $V \subset U$ and the canonical map $\omega(V, U)\colon L(V) \to L(U)$ is s-nuclear, i.e., there exist sequences $\{\lambda_j\} \subset \boldsymbol{R}$, $\{u_j'\} \subset L(V)'$ and $\{v_j\} \subset L(U)$ such that

$$\text{(S)}\qquad \begin{aligned} &\sup\{\|u_j'\| + \|v_j\|\colon j = 1, 2, \ldots\} < \infty, \\ &\lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0, \\ &\sum_{j=1}^{\infty} \lambda_j^p < \infty \quad \text{for} \quad p > 0, \end{aligned}$$

and

$$\omega(V, U)u = \sum_{j=1}^{\infty} \lambda_j u_j'(u) v_j \quad \text{for} \quad u \in L(V).$$

Let $(X, \mathcal{O})$ be an analytic space and $\mathcal{S}$ a coherent analytic sheaf on X. For every quasi-complete locally convex space L consider the sheaf $\mathcal{S}\,\varepsilon\, L$ on X generated by the presheaf

$$U \mapsto \mathcal{S}(U)\,\varepsilon\, L \overset{\text{def}}{=} \mathrm{HOM}_e\big(L_c', \mathcal{S}(U)\big)$$

for every open set $U \subset X$, where L_c' denotes the space L' equipped with the compact-open topology and $\mathrm{HOM}_e\big(L_c', \mathcal{S}(U)\big)$ the space of all continuous linear maps from L_c' into $\mathcal{S}(U)$ equipped with the topology of uniform convergence on all equicontinuous subsets of L'. Since the presheaf $U\mapsto \mathrm{HOM}_s(L'_c, \mathscr{S}(U))$ is a sheaf, we have

$$H^0(U, \mathscr{S}\varepsilon L) = H^0(U, \mathscr{S})\varepsilon L$$

for every open set $U \subset X$, where we write $H^0(U, \mathscr{S}) = \mathscr{S}(U)$.

Let us note that $H^0(U, \mathcal{O}\varepsilon L)$ is the space of holomorphic functions on U with values in L, equipped with the compact-open topology [2].

The aim of this paper is to study the cohomology groups of sheaves $\mathscr{S}\varepsilon L$. In § 1 we prove that if X is an analytic space having a countable topology and $\mathscr{S}$ a coherent analytic sheaf on X and if $H^1(X, \mathscr{S}\varepsilon F') = 0$ for every s-nuclear Fréchet space F, then $\mathscr{S}(X)$ is isomorphic to C^m for some $m \leqslant \infty$. Moreover, we prove also that if X is a Stein space, then $H^q(X, \mathscr{S}\varepsilon L) = 0$ for every $q \geqslant 2$ and for every quasi-complete locally convex space L. These results will be applied to the study of the splitness of Dolbeault complexes of holomorphic Banach bundles over complex manifolds in § 2.

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1. Cohology groups of sheaves $\mathscr{S}\varepsilon L$. In this section we prove the following

THEOREM 1.1. (i) *Let X be an analytic space having a countable topology and $\mathscr{S}$ a coherent analytic sheaf on X. If $H^1(X, \mathscr{S}\varepsilon F') = 0$ for every s-nuclear Fréchet space F, then $H^0(X, \mathscr{S})$ is isomorphic to C^m for some $m \leqslant \infty$.*

(ii) *If X is a compact analytic space and $\mathscr{S}$ a coherent analytic sheaf on X such that $H^q(X, \mathscr{S}) = 0$ for some $q > 0$, then $H^q(X, \mathscr{S}\varepsilon L) = 0$ for every quasi-complete locally convex space L.*

THEOREM 1.2. (i) *If X is an irreducible analytic space having a non-constant holomorphic function and if F is a Fréchet space which does not admit a continuous norm, then $H^1(X, \mathcal{O}\varepsilon F') \neq 0$.*

(ii) *If X is a Stein space and $\mathscr{S}$ a coherent analytic sheaf on X, then*

$$H^q(X, \mathscr{S}\varepsilon L) = 0$$

for every $q \geqslant 2$ and for every quasi-complete locally convex space L.

The proof of Theorem 1.1 is based on the following

PROPOSITION 1.3. *Let E be a subspace of a Fréchet space F. If E is s-nuclear, then there exist an s-nuclear Fréchet space $\tilde{E}$ and continuous linear maps $e\colon E\to\tilde{E}$ and $h\colon F\to\tilde{E}$ such that $h|E = e$ and e is an embedding.*

We need the following.

LEMMA 1.4. *For every s-nuclear map T from a Banach space A into a Banach space B there exist s-nuclear maps P and Q between Banach spaces such that $T = QP$.*

Proof. Since T is s-nuclear, there exist sequences $\{\lambda_j\}\subset \boldsymbol{R}$, $\{u'_j\}\subset A'$ and $\{v_j\}\subset B$ satisfying condition (S) and such that

$$Tu = \sum_{j=1}^{\infty} \lambda_j u'_j(u) v_j \quad \text{for} \quad u\in A. \tag{1.1}$$

For $u \in A$ and $\xi = (\xi_j) \in l^1$, putting

$$Pu = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, u'_j(u) e_j, \quad Q\xi = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \xi_j v_j,$$

where $\{e_j\}$ denotes the canonical basis of l^1, we get s-nuclear maps P and Q such that $T = QP$.

LEMMA 1.5. *Every s-nuclear map T from a subspace of a Banach space into a Banach space can be extended to an s-nuclear map.*

Proof. The lemma follows from the Hahn–Banach theorem.

Proof of Proposition 1.3. Let $\{U_n\}$ be a decreasing sequence of balanced convex neighbourhoods of zero in F such that

(a) $\{U_n\cap E\}$ forms a basis of neighbourhoods of zero in E.

(b) The canonical maps $\eta^n_{n+1} = \omega(U_{n+1}\cap E, U_n\cap E)\colon E_{n+1}\to E_n$ are s-nuclear, where $E_n = E(U_n\cap E)$. By Lemma 1.5 for each $n\geqslant 1$ there exists an s-nuclear map $\tilde{\eta}^n_{n+1}\colon F_{n+1}\to E_n$ such that $\tilde{\eta}^n_{n+1}|E_{n+1} = \eta^n_{n+1}$, where $F_n = F(U_n)$. Applying Lemma 1.4 to $\tilde{\eta}^n_{n+1}$ we get s-nuclear maps P^n_j and Q^n_j such that

$$\tilde{\eta}^n_{n+1} = Q^n_1 P^n_1, \ldots; P^n_j = Q^n_{j+1}P^n_{j+1}, \ldots \tag{1.2}$$

Let $\tilde{E}_n$ denote the completion of F_n/p_n equipped with the norm

$$p_n(\bar{u}) = \|\tilde{\eta}^{n-1}_n u\| + \|P^1_{n-2}\omega^2_n u\| + \ldots + \|P^{n-2}_1\omega^{n-1}_n u\|, \tag{1.3}$$

where $\omega^n_{n+1} = \omega(U_{n+1}, U_n)$.

By (1.3) the map ω^n_{n+1} induces naturally a continuous linear map $\tilde{\omega}^n_{n+1}\colon \tilde{E}_{n+1}\to\tilde{E}_n$. Putting

$$\tilde{E} = \lim_{\leftarrow}\{E_n, \tilde{\omega}^m_n\}, \quad h = \lim_{\leftarrow} h_n,$$

where $h_n\colon F_n\to\tilde{E}_n$ denotes the canonical map, we get a Fréchet space $\tilde{E}$ and a continuous linear map $h\colon F\to\tilde{E}$ such that $e = h|E$ is an embedding.

To finish the proof it suffices to check that $\tilde{\omega}^{n-1}_{n+1}$ is s-nuclear for $n\geqslant 2$. For every $n\geqslant 2$, by (1.2) and (1.3) we can define a continuous linear map

$$\alpha_{n+1}\colon \tilde{E}_{n+1}\to \operatorname{Im}\tilde{\eta}_n^{n-1}\oplus \operatorname{Im}P_{n-2}^1\oplus \ldots \oplus \operatorname{Im}P_1^{n-2}$$

by the formula

$$\alpha_{n+1}(\overline{u}) = \{\tilde{\eta}_n^{n-1}\,\omega_{n+1}^n u,\ Q_{n-1}^1 P_{n-1}^1\,\omega_{n+1}^2 u,\ \ldots,\ Q_2^{n-2}P_2^{n-2}\,\omega_{n+1}^{n-1} u\}.$$

Since $\alpha_{n+1} = \gamma_n \tilde{\omega}_{n+1}^n$, where γ_n denotes the embedding of $\tilde{E}_n$ into $\operatorname{Im}\tilde{\eta}_n^{n-1}\oplus \operatorname{Im}P_{n-2}^1\oplus \ldots \oplus \operatorname{Im}P_1^{n-2}$ given by the formula

$$\gamma_n(\overline{u}) = \{\tilde{\eta}_n^{n-1} u,\ P_{n-2}^1\,\omega_n^2 u,\ \ldots,\ P_1^{n-2}\,\omega_1^{n-2} u\}$$

and since Q_j^n are s-nuclear, it follows that α_{n+1} is s-nuclear. Thus, by the following lemma, we infer that $\tilde{\omega}_{n+1}^{n-1}$ is s-nuclear.

LEMMA 1.6. *Let $\alpha\colon A\to B$ and $\beta\colon B\to C$ be continuous linear maps such that $e\alpha$ is s-nuclear for some embedding $e\colon B \hookrightarrow \tilde{B}$ and β is quasi-nuclear. Then $\beta\alpha$ is s-nuclear.*

Proof. Since β is quasi-nuclear, β is represented in the form $\beta = \eta\gamma$, where $\gamma\colon B\to l^\infty$ and $\eta\colon l^\infty\to C$ are continuous linear maps [9]. By the Hahn–Banach theorem there exists a continuous linear map $\tilde{\gamma}\colon \tilde{B}\to l^\infty$ such that $\tilde{\gamma}e = \gamma$. Since $e\alpha$ is s-nuclear, so is $\tilde{\gamma}e\alpha$. Hence $\beta\alpha = \eta\tilde{\gamma}e\alpha$ is s-nuclear.

LEMMA 1.7. *Let F be a Fréchet–Montel space and let there exist a continuous linear map P from $E = \prod_{j=1}^{\infty} B_j$, where B_j are Banach spaces, onto F. Then $F \cong C^m$ for some $m \leqslant \infty$.*

Proof. Let $G_n = \prod_{j=1}^{n} U_j \times \prod_{j>n} B_j$, where $U_j = \{u\in B_j\colon \|u\| < 1\}$. Since P is open, it follows that for every n, the formula

$$P_n(\{u_1,\ldots,u_n\}) = \pi(PG_n)P(\{u_1,\ldots,u_n,0,\ldots\})$$

defines a continuous linear open map from $E/p(G_n) = \prod_{j=1}^{n} B_j$ onto $F/p(PG_n)$, where $\pi(PG_n)$ denotes the canonical map from F onto $F/p(PG_n)$. By hypothesis, F is Montel, it follows that P_n is compact. Hence $\dim F(PG_n) < \infty$. On the other hand, since $\{PG_n/m\}_{n,m=1}^{\infty}$ forms a basis of neighbourhoods of zero in F, it follows that F is isomorphic to C^m for some $m \leqslant \infty$.

LEMMA 1.8. *Let X be an analytic space and F a Fréchet space. Then for every $f\in(\mathcal{O}\varepsilon F')_z$ there exist $U\in\mathcal{U}(F)$ and $\tilde{f}\in(\mathcal{O}\varepsilon F(U)')_z$ such that $f = \pi(U)'\tilde{f}$.*

Proof. Since a map f from an analytic subset V of an open set G in C^n into a quasi-complete locally convex space L is holomorphic if and only if f can be extended to a holomorphic map on a neighbourhood of every point $z\in V$ in G [2], we can assume without loss of generality that X is an open set in C^n for some n. Let $f\in(\mathcal{O}\varepsilon F')_z$, $z\in X$. Take a holomorphic map $\tilde{f}$ from a neighbourhood G of z in X into L such that $\tilde{f}_z = f$ and $\tilde{f}(G)$ is bounded. Since F is Fréchet, $\tilde{f}(G)$ is contained and bounded in $F(U)'$ for some $U\in\mathcal{U}(F)$. Then, by the Cauchy integral formula, we infer that $\tilde{f}\colon G\to F(U)'$ is holomorphic. Obviously, $f = [\pi(U)'\tilde{f}]_z$.

LEMMA 1.9. *Let X be a paracompact analytic space and $\mathcal{S}$ a coherent analytic sheaf on X. Then $H^0(X,\mathcal{S})$ is s-nuclear.*

Proof. (a) First we observe that subspaces and quotient spaces of s-nuclear spaces are s-nuclear.

(b) From (a) by the Cartan Theorem B [6] and by the paracompactness of X we can assume that X is an analytic set in an open polydisc

$$\Delta^n = \Delta_1^n = \{z = (z_1,\ldots,z_n)\in C^n\colon \max|z_j| < 1\}$$

in C^n for some n and $\mathcal{S} = \mathcal{O}$.

(c) Since the restriction map $\mathcal{O}(\Delta^n)\to\mathcal{O}(X)$ is surjective [6], it suffices to check that $\mathcal{O}(\Delta^n)$ is s-nuclear.

(d) Let $0<\varepsilon<1$ and let $0<\delta<\varepsilon$. By the Taylor expansion at zero of every element $f\in\mathcal{O}(\Delta^n)$, the canonical map

$$\omega(\varepsilon,\delta)\colon \mathcal{O}(\Delta^n)(U_\varepsilon)\to\mathcal{O}(\Delta^n)(U_\delta)$$

is represented in the form

$$\omega(\varepsilon,\delta)f = \sum_\alpha (\delta/\varepsilon)^{|\alpha|}u'_\alpha(f)v_\alpha,$$

where

$$U_\varepsilon = \{f\in\mathcal{O}(\Delta^n)\colon \sup\{|f(z)|\colon z\in\bar{\Delta}_\varepsilon^n\}\leqslant 1\},$$

$$\alpha = (\alpha_1,\ldots,\alpha_n),\qquad |\alpha| = \alpha_1+\ldots+\alpha_n,$$

$$u'_\alpha\in[\mathcal{O}(\Delta^n)(U_\varepsilon)]',$$

$$u'_\alpha(f) = \varepsilon^{|\alpha|}/(2\pi i)^n\int\limits_{|\xi_j|=\varepsilon}\frac{f(\xi)\,d\xi_1\ldots d\xi_n}{\xi_1^{\alpha_1+1}\ldots\xi_n^{\alpha_n+1}},$$

$$v_\alpha\in\mathcal{O}(\Delta^n)(U_\delta),\qquad v_\alpha(z) = 1/\delta^{|\alpha|}z_1^{\alpha_1}\ldots z_n^{\alpha_n},\qquad z\in\bar{\Delta}_\delta^n.$$

Since $\sum_\alpha(\delta/\varepsilon)^{p|\alpha|}<\infty$ for every $p>0$ and $\sup\{\|u'_\alpha\|+\|v_\alpha\|\}<\infty$, it follows that $\omega(\varepsilon,\delta)$ is s-nuclear.

Proof of Theorem 1.1. (i). By Lemma 1.9 there exists a decreasing basis of balanced convex neighbourhoods of zero in $E = H^0(X,\mathcal{S})$, $\{U_n\}$, such that the maps $\eta_{n+1}^n = \omega(U_{n+1},U_n)\colon E_{n+1}\to E_n$ are s-nuclear. Put $F = \prod_{j=1}^{\infty}E_j$. Applying Proposition 1.3 to the canonical embedding

$e\colon E\to F$, $e(u)=\{\pi(U_n)u\}$, we get an s-nuclear Fréchet space $\tilde{E}$ and continuous linear maps $\tilde{e}\colon E\to\tilde{E}$, $h\colon F\to\tilde{E}$ such that $\tilde{e}$ is an embedding and $he=\tilde{e}$. Since $\tilde{E}/\mathrm{Im}\,\tilde{e}$ is Fréchet–Montel, the sequence

$$0\to(\tilde{E}/\mathrm{Im}\,\tilde{e})'\to\tilde{E}'\xrightarrow{\tilde{e}'}E'\to 0$$

is topologically exact. Hence the sequence

$$0\to\mathscr{S}\varepsilon(E/\mathrm{Im}\,\tilde{e})'\to\mathscr{S}\varepsilon\tilde{E}'\xrightarrow{\mathrm{id}\,\varepsilon\tilde{e}'}\mathscr{S}\varepsilon E'$$

is exact. We prove that the map $\mathrm{id}\,\varepsilon\tilde{e}'$ is surjective. Let $z\in X$ and $f\in(\mathscr{S}\varepsilon E')_z$. Take a surjective morphism $\mu\colon\mathcal{O}^p|G\to\mathscr{S}|G$ and $\tilde{f}\in H^0(G,\mathscr{S}\varepsilon E')$ inducing f, where G is a Stein neighbourhood of z in X. Let V be a relatively compact neighbourhood of z in G. Then there exists $W\in\mathscr{U}(\mathcal{O}^p(G))$ such that $\mathcal{O}^p(G)(W)$ is isomorphic to a Hilbert space and $g|V=\pi(W)g|V$ for all $g\in\mathcal{O}^p(G)$. Since the map $\hat{\mu}_G\colon H^0(G,\mathcal{O}^p)\to H^0(G,\mathscr{S})$ induced by μ is surjective [6], $\hat{\mu}_G$ induces a continuous linear map γ from $\mathcal{O}^p(G)(W)$ onto $H^0(G,\mathscr{S})(\tilde{W})$, where $\tilde{W}=\hat{\mu}_G(W)$. Thus there exists a continuous linear map $\beta\colon E\to\mathcal{O}^p(V)$ such that $\hat{\mu}_V\beta=\tilde{f}|V$. By Lemma 1.8 we can assume that $\beta\in\mathcal{O}^p(V)\varepsilon E(U_0)'$ for some $U\in\mathscr{U}(\tilde{E})$, where $U_0=\tilde{e}^{-1}(U)$. On the other hand, since the restriction map $\tilde{E}(U)'\to[\mathrm{Im}\,\tilde{e}(U\cap\mathrm{Im}\,\tilde{e})]'$ is surjective, there exists $\eta\in\mathcal{O}^p(V)\varepsilon\tilde{E}'$ such that $(\mathrm{id}\,\varepsilon\tilde{e}')\eta=\beta$ [4]. Obviously

$$\alpha=(\hat{\mu}_V\varepsilon\mathrm{id})\eta\in H^0(V,\mathscr{S}\varepsilon\tilde{E}')\quad\text{and}\quad[(\mathrm{id}\,\varepsilon e')\alpha]_z=f.$$

Since $\tilde{E}/\mathrm{Im}\,\tilde{e}$ is s-nuclear, $H^1(X,\mathscr{S}\varepsilon(\tilde{E}/\mathrm{Im}\,\tilde{e})')=0$. Thus the map

$$\mathrm{id}\,\varepsilon\tilde{e}'\colon H^0(X,\mathscr{S}\varepsilon\tilde{E}')\equiv\mathrm{HOM}_s(\tilde{E},\mathscr{S}(X))\to H^0(X,\mathscr{S}\varepsilon E')\equiv\mathrm{HOM}_s(E,\mathscr{S}(X))$$

is surjective. Hence there exists an element $\theta\in\mathrm{HOM}_s(\tilde{E},\mathscr{S}(X))$ such that $\theta\tilde{e}\sigma=\sigma$ for all $\sigma\in H^0(X,\mathscr{S})$. Setting $P=\theta h$, we get a continuous linear map from F onto $H^0(X,\mathscr{S})$. Since $H^0(X,\mathscr{S})$ is Fréchet–Montel, by Lemma 1.7 $H^0(X,\mathscr{S})$ is isomorphic to C^m for some $m\leqslant\infty$. This completes the proof of (i).

(ii). By $\mathscr{B}(L)$ we denote the set of all bounded balanced convex subsets of L. Let $K\in\mathscr{B}(L)$ and let $L(K)$ denote the Banach space spanned by K. This space is equipped with the norm ϱ_K, where ϱ_K is the Minkowski functional of K on $L(K)$. It is known [2] that

$$(1.4)\qquad(\mathscr{S}\varepsilon L)_z=\bigcup\{(\mathscr{S}\varepsilon L(K))_z\colon K\in\mathscr{B}(L)\}\quad\text{for}\quad z\in X.$$

Let $\omega\in H^q(X,\mathscr{S}\varepsilon L)$. Since $H^q(X,\mathscr{S}\varepsilon L)=\varinjlim H^q(\mathscr{U},\mathscr{S}\varepsilon L)$, by the compactness of X and by (1.4) we find an $K\in\mathscr{B}(L)$ such that ω is induced by some element $\omega(K)\in Z^q(\mathscr{U},\mathscr{S}\varepsilon L(K))$, where $\mathscr{U}$ is a Stein open covering of X and $Z^q(\mathscr{U},\mathscr{S}\varepsilon L(K))=\mathrm{Ker}\,\delta^q$, $\delta^q=\delta^q(\mathscr{U},\mathscr{S}\varepsilon L(K))\colon C^q(\mathscr{U},\mathscr{S}\varepsilon L(K))\to C^{q-1}(\mathscr{U},\mathscr{S}\varepsilon L(K))$ are coboundary maps and $C^q(\mathscr{U},\mathscr{S}\varepsilon L(K))$ are equipped with the product topology. Since $\mathscr{U}$ is a Leray covering of X, by hypothesis the map $\delta^{q-1}\colon C^{q-1}(\mathscr{U},\mathscr{S})\to Z^q(\mathscr{U},\mathscr{S})$ is surjective. Then, by the nuclearity of $C^{q-1}(\mathscr{U},\mathscr{S})$ it follows that the map

$$\delta^{q-1}(\mathscr{U},\mathscr{S}\varepsilon L(K))\colon C^{q-1}(\mathscr{U},\mathscr{S}\varepsilon L(K))\to Z^q(\mathscr{U},\mathscr{S}\varepsilon L(K))$$

is surjective. Hence $\omega(K)=\delta^{q-1}\tilde{\omega}$ for some $\tilde{\omega}\in C^{q-1}(\mathscr{U},\mathscr{S}\varepsilon L(K))$. Thus $\omega=0$ and (ii) is proved.

Proof of Theorem 1.2. (i): Let $H^1(X,\mathcal{O}\varepsilon F')=0$, where F is some Fréchet space which does not admit a continuous norm. By a theorem of Bessaga and Pełczyński [1], there exists a complemented subspace of F which is isomorphic to C^∞. Hence $H^1(X,\mathcal{O}\varepsilon C^{\infty\prime})=0$.

Let f be a non-constant holomorphic function on X. Put $a=\sup\{|f(z)|\colon z\in X\}$. By the irreducibility of X we can assume that $a=\infty$ since if $a<\infty$, we replace f by the function $(r-f(z))^{-1}$, where $|r|=a$ and $r=\lim f(x_n)$ for a sequence $\{x_n\}\subset X$. Take a sequence $\{z_n\}\subset R(X)$, where $R(X)$ denotes the regular part of X such that $|f(z_n)|\to\infty$ and $f(z_k)\neq f(z_j)$ for $k\neq j$. Select $\varphi\in\mathcal{O}(C)$ such that $\varphi^{-1}(0)=\{f(z_n)\}_{n=1}^\infty$. Put $V=(\varphi f)^{-1}(0)=\bigcup_{j=1}^\infty V_j$ where $V_j=f^{-1}(f(z_j))$. Since $\{f(z_j)\}_{j=1}^\infty$ is discrete and since f is continuous, there exists an open covering $\mathscr{U}=\{U_j\}_{j=0}^\infty$ of X such that

$$V_j\subset U_j\quad\text{for}\quad j\geqslant 1,$$

$$U_i\cap U_j\neq\emptyset\quad\text{for}\quad i\neq j,\ i,j\geqslant 1\ \text{and}\ V\cap U_0=\emptyset.$$

Put $f_{ij}=0$ for $i,j\geqslant 1$ and $f_{0j}=-f_{j0}=e_j/\varphi_j$ for $j\geqslant 0$, where $e_0=0$, $e_j=(\underbrace{0,\ldots,0,1}_{j})$, $j\geqslant 1$. Obviously $\{f_{ij}\}\in Z^1(\mathscr{U},\mathcal{O}\varepsilon C^{\infty\prime})$. Since $H^1(X,\mathcal{O}\varepsilon C^{\infty\prime})=0$, we can assume that $f_{ij}=f_i-f_j$, where $f_i\in H^0(U_i,\mathcal{O}\varepsilon C^{\infty\prime})$ and thus we have

$$e_i+\varphi f f_i=e_j+\varphi f f_j\quad\text{on}\quad U_i\cap U_j\quad\text{for}\quad i,j\geqslant 0.$$

Thus the formula

$$\tilde{f}(z)=e_j+\varphi f f_j(z)\quad\text{for}\quad z\in U_j$$

define an element $\tilde{f}\in H^0(X,\mathcal{O}\varepsilon C^{\infty\prime})$ such that $\tilde{f}(V_j)=e_j$ for $j\geqslant 1$. Take $z_0\in R(X)$ and a neighbourhood G of z_0 such that $\tilde{f}(G)$ is bounded. Then $\tilde{f}(G)\subset\bigoplus_{j=1}^{n_0}Ce_j$ for some n_0. Put

$$\tilde{G}=\{z\in R(X)\colon\exists G_z\ni z\colon\tilde{f}(G_z)\subset\bigoplus_{j=1}^{n_0}Ce_j\}.$$

Note that $\tilde{G}$ is non-empty and open. Let $z\in\bar{\tilde{G}}\cap R(X)$ and let G_z be a connected neighbourhood of z in $R(X)$ such that $\tilde{f}(G_z)\subset\bigoplus_{j=1}^{n_0+p}Ce_j$ for some p.

Since $\tilde{f}|G_z\colon G_z \to \bigoplus_{j=1}^{n_0+p} Ce_j$ is holomorphic, $\tilde{f}(G_z \cap \tilde{G}) \subset \bigoplus_{j=1}^{n_0} Ce_j$ and $G_z \cap \tilde{G}$ is non-empty open in G_z, by the connectedness of G_z it follows that $\tilde{f}(G_z) \subset \bigoplus_{j=1}^{n_0} Ce_j$. Thus $z \in \tilde{G}$ and hence $\tilde{G}$ is closed in $R(X)$. Since X is irreducible, $R(X)$ is connected. Hence $\tilde{G} = R(X)$ and thus $\tilde{f}\,(R(X)) \subset \bigoplus_{j=1}^{n_0} Ce_j$. This is impossible since $z_{n_0+1} \in R(X)$ and $\tilde{f}(z_{n_0+1}) \notin \bigoplus_{j=1}^{n_0} Ce_j$. Thus (i) is proved.

(ii): Let X, $\mathcal{S}$ and L be as in (ii) and $q \geqslant 2$. We prove that $H^q(X, \mathcal{S}\varepsilon L) = 0$. We write $X = \bigcup_{n=1}^{\infty} W_n$, where W_n are relatively compact sets in X such that

$$W_n \subset \hat{W}_n \subset W_{n+1} \quad \text{for} \quad n \geqslant 1.$$

It is known [2] (Theorem B*, p. 338) that

$$H^p(\hat{W}_n, \mathcal{S}\varepsilon L) = 0 \quad \text{for} \quad n, p \geqslant 1. \tag{1.5}$$

Let

$$0 \to \mathcal{S}\varepsilon L \to J_0 \xrightarrow{d_0} J_1 \xrightarrow{d_1} \ldots$$

be a flabby resolution of $\mathcal{S}\varepsilon L$. Since

$$0 \to \mathcal{S}\varepsilon L|\hat{W}_n \to J_0|\hat{W}_n \to J_1|\hat{W}_n \to \ldots$$

is a flabby resolution of $\mathcal{S}\varepsilon L|\hat{W}_n$, by (1.5) we have

$$\widehat{\operatorname{Ker} d_p}|\hat{W}_n = \widehat{\operatorname{Im} d_{p-1}}|\hat{W}_n \quad \text{for} \quad n, p \geqslant 1. \tag{1.6}$$

Let $\sigma \in \widehat{\operatorname{Ker} d_q}$. By (1.6) for every $n \geqslant 1$ there exists an $\alpha^{(n)} \in H^0(\hat{W}_n, J_{q-1})$ such that $\widehat{d_{q-1}}\,\alpha(n) = \sigma|\hat{W}_n$. Since $\widehat{d_{q-1}}(\alpha^{(n)} - \alpha^{(n-1)})|\hat{W}_{n-1} = 0$, by (1.6) there exists a $\beta^{(n-1)} \in H^0(\hat{W}_{n-1}, J_{q-2})$ such that

$$\widehat{d_{q-2}}\,\beta^{(n-1)} = (\alpha^{(n)} - \alpha^{(n-1)})|\hat{W}_{n-1}.$$

Since J_p are flabby for every n, there exists an $\tilde{\beta}^{(n-1)} \in J_{q-2}(X)$ such that $\tilde{\beta}^{(n-1)}|\hat{W}_{n-1} = \beta^{(n-1)}$. Define an $\alpha \in J_{q-1}(X)$ by

$$\alpha|W_n = \Big(\alpha^{(n)} - \widehat{d_{q-2}}\Big(\sum_{k<n} \tilde{\beta}^{(k)}\Big)\Big)\Big|W_n.$$

This is well defined since for $n \geqslant m$ we have

$$\alpha^{(n)} - \widehat{d_{q-2}}\Big(\sum_{k<n} \tilde{\beta}^{(k)}\Big) - \Big(\alpha^{(m)} - \widehat{d_{q-2}}\Big(\sum_{k<m} \tilde{\beta}^{(k)}\Big)\Big)$$
$$= \alpha^{(n)} - \alpha^{(m)} - \widehat{d_{q-2}}\Big(\sum_{m \leqslant k<n} \tilde{\beta}^{(k)}\Big)$$
$$= \alpha^{(n)} - \alpha^{(m)} - \sum_{m \leqslant k<n} (\alpha^{(k+1)} - \alpha^{(k)}) = 0 \quad \text{on} \quad W_n.$$

Finally on W_n, $\widehat{d_{q-1}}\,\alpha = \sigma|W_n$. Thus $\widehat{d_{q-1}}\,\alpha = \sigma$.

COROLLARY 1.10. Cousin's First Problem for holomorphic functions on a Stein space with values in F', where F is a Fréchet space which does not admit a continuous norm, may not admit a solution.

It is known [2] that Cousin's First Problem for holomorphic functions on a Stein space with values in a Fréchet space has a solution.

Now we apply Theorems 1.1 and 1.2 to get the splitness of Dolbeault complexes of holomorphic vector bundles over Stein manifolds at only positive dimension. This result has been by Palamodov [8]. We need the following

LEMMA 1.11. *Let Δ be a compact polydisc in C^n and ω a C^∞-form of bidegree (p, q) on a neighbourhood of Δ with values in a quasi-complete locally convex space L. If $q > 0$ and $\bar{\partial}_L^q \omega = 0$, where $\bar{\partial}_L^q = \bar{\partial}^q \varepsilon \mathrm{id}$, then there exists a C^∞-form η of bidegree $(p, q-1)$ on a neighbourhood of Δ with values in L such that $\bar{\partial}_L^{q-1}\eta = \omega$.*

Proof. For $L = C$ the lemma has been established in [6]. Let us notice that the proof in [6] gives also a proof of Lemma 1.10 since we have the following

Remark 1.12. Let D be a simple connected domain in C and g a C^∞-function on D with values in a quasi-complete locally convex space L such that $\operatorname{supp} g$ is compact. Then setting

$$f(z) = 1/2\pi i \iint_D \frac{g(\xi)\, d\xi \wedge d\bar{\xi}}{\xi - z}$$

for every $z \in D$ we get a C^∞-function f on D with values in L so that $\bar{\partial}_L^0 f = g$. Moreover, if g is C^∞ or holomorphic in some additional parameters, so is f.

Let ξ be a holomorphic Banach bundle over a complex manifold X. By $\mathcal{O}_\xi$ we denote the sheaf of germs of holomorphic sections of ξ on X and by $\Omega_\xi^{(p,q)}$ the sheaf of germs of C^∞-forms of bidegree (p, q) on X with values in ξ. Lemma 1.11 implies that the sequence

$$0 \to \mathcal{O}_\xi \to \Omega_\xi^0 \xrightarrow{\bar{\partial}_\xi^0} \Omega_\xi^1 \xrightarrow{\bar{\partial}_\xi^1} \ldots \tag{1.7}$$

is exact, where $\Omega^q_\xi = \Omega^{(0,q)}_\xi$. The complex

$$D(\xi):\ 0\to\mathcal{O}_\xi(X)\to\Omega^0_\xi(X)\overset{\hat{\bar\partial}^0_\xi}{\to}\Omega^1_\xi(X)\overset{\hat{\bar\partial}^1_\xi}{\to}\dots$$

of global sections of (1.7) on X is called the *Dolbeault complex of* ξ. We say that the complex $D(\xi)$ splits at q if there exists a continuous linear map γ_q: $\operatorname{Im}\hat{\bar\partial}^q_\xi\to\Omega^q_\xi(X)$ such that $\hat{\bar\partial}^q_\xi\gamma_q = \mathrm{id}$.

PROPOSITION 1.13. *Let ξ be a holomorphic vector bundle on a Stein manifold X. Then $D(\xi)$ splits at q if and only if $q>0$.*

Proof. Lemma 1.11 implies that for every quasi-complete locally convex space L the sequence

$$(1.8)\qquad 0\to\mathcal{O}_\xi\varepsilon L\to\Omega^0_\xi\varepsilon L\to\Omega^1_\xi\varepsilon L\to\dots$$

which is obtained by tensoring the sequence (1.7) with L is exact. Hence

$$(1.9)\qquad H^q(X,\mathcal{O}_\xi\varepsilon L) = \operatorname{Ker}(\bar\partial^q_\xi\varepsilon\mathrm{id})^\wedge/(\operatorname{Im}\bar\partial^q_\xi\varepsilon\mathrm{id})^\wedge = \operatorname{Ker}\hat{\bar\partial}^q_\xi\varepsilon L/\operatorname{Im}(\bar\partial^{q-1}_\xi\varepsilon\mathrm{id})^\wedge.$$

Let $q>0$. Applying Theorem 1.2 (ii) to $L=(\operatorname{Im}\hat{\bar\partial}^q_\xi)'$, by (1.9) it follows that the map $(\bar\partial^q_\xi\varepsilon\mathrm{id})^\wedge$: $\Omega^q_\xi(X)\varepsilon(\operatorname{Im}\hat{\bar\partial}^q_\xi)'\to(\operatorname{Im}\hat{\bar\partial}^q_\xi)\varepsilon(\operatorname{Im}\hat{\bar\partial}^q_\xi)'$ is surjective. This implies that $D(\xi)$ splits at q.

Now we prove that $D(\xi)$ does not split at 0. For a contradiction by (1.9) it follows that

$$H^1(Z,\mathcal{O}_\xi\varepsilon L) = 0$$

for every quasi-complete locally convex space L, where Z is some component of X. By Theorem 1.1 (i) the space $\mathcal{O}_\xi(Z)$ is isomorphic to C^m for $m\leqslant\infty$. This is imposible, since $\mathcal{O}_\xi(Z)$ is an infinite-dimensional Fréchet space having a continuous norm. The proposition is proved.

2. Splitness of Dolbeault complexes of holomorphic Banach bundles.

Let X be an analytic space. We say that X has a *Stein morphism* if there exists a holomorphic map π from X into a Stein space W such that $\pi^{-1}(U)$ is Stein for every Stein open set U belonging to a Stein open covering $\mathcal{U}$ of W. Since $\pi(\partial\pi^{-1}(V))\subseteq\partial V$ for every open set $V\subset W$ it is easy to see that $\pi^{-1}(V)$ is Stein for every Stein open set V in X contained in some $U\in\mathcal{U}$. Thus by a lemma of Stehlé [11] there exists a Stein open covering $\{V_j\}_{j=1}^\infty$ of W such that

(S_1) $\Omega_j=\bigcup_{i\leqslant j}V_i$ are Stein.

(S_2) $\Omega_j\cap V_{j+1}$ are Runger domains in V_{j+1}.

(S_3) $\tilde{\mathcal{U}}=\{X'_j=\pi^{-1}(V_j)\}$ is a Stein covering of X.

Since X'_{j+1} is Stein, by (S_2) it is easy to check that $X_j\cap X'_{j+1}$ is Runger in X'_{j+1}.

In this section we prove the following

THEOREM 2.1. *Let ξ be a holomorphic Banach bundle over a Stein manifold X. Then $D(\xi)$ splits at q if and only if $q>0$.*

THEOREM 2.2. *Let ξ be a holomorphic Banach bundle over a complex manifold X having a Stein morphism. Then*

(i) *$D(\xi)$ splits at $q\geqslant 2$.*

(ii) *$D(\xi)$ does not split at 0.*

Proof of Theorem 2.1. We can assume that X is connected. First we prove that $D(\xi)$ does not split at 0. For a contradiction by (1.9) we have

$$(2.1)\qquad H^1(X,\mathcal{O}_\xi\varepsilon C^{\infty\prime}) = 0.$$

Select $\sigma\in\mathcal{O}_\xi(X)$, $\sigma\neq 0$. Since X is Stein, there exists a sequence $\{z_n\}\subset X$ and $f\in\mathcal{O}(X)$ such that

$$\sigma(z_j)\neq 0\quad\text{for}\quad j\geqslant 1,$$
$$|f(z_j)|\to\infty\quad\text{and}\quad f(z_j)\neq f(z_k)\quad\text{for}\quad j\neq k.$$

Consider the function $\varphi\in\mathcal{O}(C)$ and the covering $\mathcal{U}$ as in the proof of Theorem 1.2 (i). Put

$$\sigma_{ij}=0\quad\text{for}\quad i,j\geqslant 1\quad\text{and}\quad \sigma_{0j}=-\sigma_{j0}=\sigma/\varphi f\otimes e_j\quad\text{for } j\geqslant 0.$$

Obviously $\{\sigma_{ij}\}\in Z^1(\mathcal{U},\mathcal{O}_\xi\varepsilon C^{\infty\prime})$. Hence by (2.1) we can assume that

$$\sigma_{ij}=\sigma_i-\sigma_j,\quad\text{where}\quad \sigma_j\in H^0(U_j,\mathcal{O}_\xi\varepsilon C^{\infty\prime}),$$

whence

$$\varphi f\sigma_i+\sigma\otimes e_i=\varphi f\sigma_j+\sigma\otimes e_j\quad\text{on}\quad U_i\cap U_j.$$

Thus there exists an element $\beta\in H^0(X,\mathcal{O}_\xi\varepsilon C^{\infty\prime})$ such that $\beta(z_j)=\sigma(z_j)\otimes e_j$ for $j\geqslant 1$. By the connectedness of X and since $H^0(X,\mathcal{O}_\xi\varepsilon C^{\infty\prime})=H^0(X,\mathcal{O}_\xi)\varepsilon C^{\infty\prime}$ it follows that

$$\beta\in H^0(H,\mathcal{O}_\xi)e_1\oplus\dots\oplus H^0(X,\mathcal{O}_\xi)e_{n_0}\quad\text{for some } n_0.$$

This contradicts the relation

$$\beta(z_{n_0+1})=\sigma(z_{n_0+1})e_{n_0+1}\notin H^0(X,\mathcal{O}_\xi)e_1\oplus\dots\oplus H^0(X,\mathcal{O}_\xi)e_{n_0}$$

since $\sigma(z_{n_0+1})\neq 0$.

Now let $q>0$. We prove that $D(\xi)$ splits at q.

(a) First we prove that

(2.2) $H^p(X,\mathcal{O}_\xi\varepsilon L)=0$ for every $p\geqslant 2$ and for every quasi-complete locally convex space L.

By the proof of Theorem 1.2 (ii) it suffices to prove that

$$H^p(\hat{W}, \mathcal{O}_\xi \varepsilon L) = 0$$

for every compact subset W of X and every $p \geqslant 1$. Since the relation

$$\bigcup \{B\varepsilon L(K) \colon K \in \mathscr{B}(L)\} = \bigcup \{[B\varepsilon L](K) \colon K \in \mathscr{B}(B\varepsilon L)\}$$

holds for all Banach spaces B, it follows that

$$(\mathcal{O}_\xi \varepsilon L)_z = \bigcup \{(\mathcal{O}_\xi \varepsilon L(K))_z \colon K \in \mathscr{B}(L)\} \quad \text{for} \quad z \in X.$$

Thus

$$H^p(\hat{W}, \mathcal{O}_\xi \varepsilon L) = \varinjlim_{K \in \mathscr{B}(L)} H^p(\hat{W}, \mathcal{O}_\xi \varepsilon L(K)) = \varinjlim_{K \in \mathscr{B}(L)} H^p(\hat{W}, \mathcal{O}_\xi \varepsilon L(K)).$$

Since $\hat{W}$ has a basis of Stein neighbourhoods, we infer that

$$H^p(\hat{W}, \mathcal{O}_\xi \varepsilon L) = \varinjlim_{\substack{G \supset \hat{W} \\ G \text{ is Stein}}} \varinjlim_{K \in \mathscr{B}(L)} H^p(G, \mathcal{O}_\xi \varepsilon L(K)) = 0.$$

(b) Using (2.2) to $L = (\operatorname{Im} \hat{\hat{\partial}}{}^q_\xi)'_c$, by the relation $((\operatorname{Im} \hat{\hat{\partial}}{}^q_\xi)'_c)'_c = \operatorname{Im} \hat{\hat{\partial}}{}^q_\xi$, we infer that the map

$$\mathrm{HOM}_s(\operatorname{Im} \hat{\hat{\partial}}{}^q_\xi, \Omega^q_\xi(X)) \to \mathrm{HOM}_s(\operatorname{Im} \hat{\hat{\partial}}{}^q_\xi, \operatorname{Im} \hat{\hat{\partial}}{}^q_\xi)$$

induced by

$$\hat{\hat{\partial}}{}^q_\xi \colon \Omega^q_\xi(X) \to \operatorname{Im} \hat{\hat{\partial}}{}^q_\xi$$

is surjective. This implies that $D(\xi)$ splits at q. The theorem is proved.

The proof of Theorem 2.2 is based on the following

PROPOSITION 2.3. *Let ξ and X be as in Theorem 2.2 and let $z_0 \in X$. Let $J_{\xi z_0}$ denote the subsheaf of the sheaf $\mathcal{O}_\xi$ consisting of germs which vanish at z_0. Then*

(i) there exists a Stein open covering $\tilde{\mathscr{U}}_1$ of X such that the map δ^0: $C^0(\tilde{\mathscr{U}}_1, J_{\xi z_0}) \to Z^1(\tilde{\mathscr{U}}_1, J_{\xi z_0})$ has dense image.

(ii) $H^q(X, \mathcal{O}_\xi \varepsilon L) = 0$ for every $q \geqslant 3$ and for every quasi-complete locally convex space L.

Proof. (i): We write $\mathscr{S} = J_{\xi z_0}$. Take a Stein open covering $\tilde{\mathscr{U}}_1$ of X such that $\tilde{\mathscr{U}}_1 < \tilde{\mathscr{U}}$, $\tilde{\mathscr{U}}_1$ forms a basis of open sets in X and

$$\tilde{\mathscr{U}}_1 | X_{j+1} = \tilde{\mathscr{U}}_1 | X_j \cup \tilde{\mathscr{U}}_1 | X'_{j+1} \tag{2.3}$$

for $j \geqslant 1$, where $\tilde{\mathscr{U}}_1 | G = \{U \in \tilde{\mathscr{U}}_1 \colon U \subset G\}$. We prove that $\operatorname{Im} \delta^0(\tilde{\mathscr{U}}_1, \mathscr{S})$ is dense in $Z^1(\tilde{\mathscr{U}}_1, \mathscr{S})$. It suffices to show that

$$\overline{\operatorname{Im} \delta^0(\tilde{\mathscr{U}}_1 | X_j, \mathscr{S})} = Z^1(\tilde{\mathscr{U}}_1 | X_j, \mathscr{S}) \quad \text{for} \quad j \geqslant 1. \tag{2.4}$$

Since $\tilde{\mathscr{U}}_1 | X_1$ is a Leray covering of the Stein manifold X_1, (2.4) holds for $j = 1$.

Now given $\varphi \in Z^1(\tilde{\mathscr{U}}_1 | X_{j+1}, \mathscr{S})$. Take a sequence $\{\alpha^j_n\} \subset C^0(\tilde{\mathscr{U}}_1 | X_j, \mathscr{S})$ such that $\delta^0 \alpha^j_n \to \varphi | \tilde{\mathscr{U}}_1 | X_j$. Since $(\delta^0 \alpha^j_n - \varphi) | \tilde{\mathscr{U}}_1 | X_j \cap X'_{j+1} \to 0$ and the map

$$\varrho \colon A = C^0(\tilde{\mathscr{U}}_1 | X_j, \mathscr{S}) \oplus C^0(\tilde{\mathscr{U}}_1 | X_j \cap X'_{j+1}, \mathscr{S}) \to Z^1(\tilde{\mathscr{U}}_1 | X_j \cap X'_{j+1}, \mathscr{S})$$

defined by

$$\varrho(\sigma, \beta) = \delta^0(\sigma - \beta) | \tilde{\mathscr{U}}_1 | X_j \cap X'_{j+1}$$

is surjective, there exists a sequence $\{(\gamma^1_n, \gamma^2_n)\} \subset A$ converging to zero such that

$$\begin{aligned} \varrho((\gamma^1_n, \gamma^2_n)) &= \delta^0 \alpha^j_n - \varphi | \tilde{\mathscr{U}}_1 | X_j \cap X'_{j+1} \\ &= \delta^0 \alpha^j_n | \tilde{\mathscr{U}}_1 | X_j \cap X'_{j+1} - \delta^0 \beta | \tilde{\mathscr{U}}_1 | X_j \cap X'_{j+1}, \end{aligned}$$

where $\beta \in C^0(\tilde{\mathscr{U}}_1 | X'_{j+1}, \mathscr{S})$, $\delta^0 \beta = \varphi | X'_{j+1}$.

Since $X_j \cap X'_{j+1}$ is a Runger domain in X'_{j+1}, there exists a sequence $\{h_n\} \subset H^0(X'_{j+1}, \mathscr{S})$ such that $u_n - h_n \to 0$ on $X_j \cap X'_{j+1}$, where $u_n = (\alpha^j_n - \gamma^1_n) - (\beta - \gamma^2_n) \in H^0(X_j \cap X'_{j+1}, \mathscr{S})$. Let $\tilde{u}_n$ and $\tilde{h}_n$ be trivial extensions of u_n and h_n, repectively, on X_j. Put

$$\alpha^{j+1}_n = \begin{cases} \alpha^j_n - \gamma^1_n - \tilde{u}_n - \tilde{h}_n & \text{on} \quad \tilde{\mathscr{U}}_1 | X_j, \\ \beta - \gamma^2_n - h_n & \text{on} \quad \tilde{\mathscr{U}}_1 | X'_{j+1}. \end{cases}$$

Then $\{\alpha^{j+1}_n\} \subset C^0(\tilde{\mathscr{U}}_1 | X_{j+1}, \mathscr{S})$ and $\delta^0 \alpha^{j+1}_n \to \varphi | \tilde{\mathscr{U}}_1 | X_{j+1}$.

(ii): (a) Since $H^q(Z, \mathcal{O}_\xi \varepsilon L) = 0$ for every $q \geqslant 2$ and for every Stein open set Z in X, by induction on j and considering the Mayer–Vietoris sequence of pairs (X_j, X'_{j+1}) we infer that

$$H^q(X_j, \mathcal{O}_\xi \varepsilon L) = 0 \quad \text{for} \quad q \geqslant 3 \text{ and for } j \geqslant 1. \tag{2.5}$$

(b) Take a flabby resolution

$$0 \to \mathcal{O}_\xi \varepsilon L \to J_0 \xrightarrow{d_0} J_1 \xrightarrow{d_1} \ldots$$

of the sheaf $\mathcal{O}_\xi \varepsilon L$. We prove that for every $j \geqslant 1$ and for every $q \geqslant 3$ the restriction map $\widehat{\operatorname{Ker} d_{q-1}} | X_{j+1} \to \widehat{\operatorname{Ker} d_{q-1}} | X_j$ is surjective. Given $\alpha \in \widehat{\operatorname{Ker} d_{q-1}} | X_j$. Since $X_j \cap X'_{j+1}$ is Stein and $q - 1 \geqslant 2$, by (2.2) we find $\eta \in J_{q-1}(X_j \cap X'_{j+1})$ such that $\hat{d}_{q-1} | X_j \cap X'_{j+1} \eta = \alpha | X_j \cap X'_{j+1}$. Let $\tilde{\eta}$ be an extension of η on X'_{j+1}. Setting

$$\tilde{\alpha} | X_j = \alpha \quad \text{and} \quad \tilde{\alpha} | X'_{j+1} = \hat{d}_{q-2} | X'_{j+1} \tilde{\eta},$$

we get an element $\tilde{\alpha} \in \operatorname{Ker} d_{q-1} | X'_{j+1}$ extending α.

(c) Let $\alpha \in \operatorname{Ker} \hat{d}_q$, $q \geqslant 3$. Take $\alpha_1 \in J_{q-1}(X_1)$ such that $\hat{d}_{q-1} | X_1 \alpha_1$

$= \alpha|X_1$. By (2.5) there exists $\alpha_2' \in J_{q-1}(X_2)$ such that $\widehat{d_{q-1}}\alpha_2' = \alpha|X_2$. Since $(\alpha_2'-\alpha_1)|X_1 \in \operatorname{Ker}\widehat{d_{q-1}}|X_1$, by (b) we find $\alpha_2'' \in \operatorname{Ker}\widehat{d_{q-1}}|X_2$ extending $(\alpha_2'-\alpha_1)|X_1$. Put $\alpha_2 = \alpha_2'-\alpha_2''$. Then $\alpha_2 \in J_{q-1}(X_2)$ and

$$\alpha_2|X_1 = \alpha_1, \qquad \widehat{d_{q-1}}|X_2\alpha_2 = \alpha|X_2.$$

Continuing this process we get elements $\alpha_n \in J_{q-1}(X_n)$ such that

$$\alpha_n|X_{n-1} = \alpha_{n-1} \quad \text{and} \quad \widehat{d_{q-1}}|X_n\alpha_n = \alpha|X_n \quad \text{for} \quad n \geqslant 2.$$

Thus the formula $\beta|X_n = \alpha_n$ for $n \geqslant 1$ defines an element $\beta \in J_{q-1}(X)$ such that $\widehat{d_{q-1}}\beta = \alpha$. Hence (ii) is proved.

The proof of (i) is similar to a proof of Jennane [7].

Proof of Theorem 2.2. The splitness of $D(\xi)$ at $q \geqslant 2$ follows from Proposition 2.3 (ii) and from the proof of Theorem 2.1.

Now we prove that $D(\xi)$ does not split at 0. For a contradiction by Proposition 2.3 (i) we have $H^1(X, \mathcal{O}_\xi) = 0$. Hence, by the splitness of $D(\xi)$, we infer that (2.1) holds. By (2.1) and by the proof of Theorem 2.1 it suffices to show that $\mathcal{O}_\xi(X) \neq 0$ and $\dim \mathcal{O}(X) > 1$.

Obviously $\dim \mathcal{O}(X) > 1$ since X has a Stein morphism.

Assume that $\mathcal{O}_\xi(X) = 0$. Take $z_0 \in X$ such that $\xi_{z_0} \neq 0$. Let $\tilde{\mathcal{U}}_1$ and $\mathcal{S}' = J_{\xi z_0}$ be as in Proposition 2.3. Then $H^1(X, \mathcal{S}') = Z^1(\tilde{\mathcal{U}}_1, \mathcal{S}')/\operatorname{Im}\delta^0(\tilde{\mathcal{U}}_1, \mathcal{S}')$ and $\operatorname{Im}\delta^0(\tilde{\mathcal{U}}_1, \mathcal{S}')$ is dense in $Z^1(\tilde{\mathcal{U}}_1, \mathcal{S}')$. Then, by the exactness of the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & 0 & \longrightarrow & \xi_{z_0} & \xrightarrow{\delta} & H^1(X, \mathcal{S}') & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 \to & C^0(\tilde{\mathcal{U}}_1, \mathcal{S}') & \to & C^0(\tilde{\mathcal{U}}_1, \mathcal{O}_\xi) & \to & C^0(\tilde{\mathcal{U}}_1, \mathcal{S}'') & \to 0 & & \\
 & \downarrow & & \downarrow & & & & & \\
0 \to & Z^1(\tilde{\mathcal{U}}_1, \mathcal{S}') & \to & Z^1(\tilde{\mathcal{U}}_1, \mathcal{O}_\xi) & \to & Z^1(\tilde{\mathcal{U}}_1, \mathcal{S}'') & & & \\
 & & & \downarrow & & & & & \\
 & & & 0 & & & & &
\end{array}$$

we infer that $\operatorname{Im}\delta^0(\tilde{\mathcal{U}}_1, \mathcal{S})$ is closed and hence $H^1(X, \mathcal{S}) = 0$. This implies that $\xi_{z_0} = 0$ which contradicts choice of z_0. The theorem is proved.

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