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Packing spheres in C_p spaces

by

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Abstract. Packing numbers for the spaces C_p and their subspaces are studied. The exact value is found for $1 \leqslant \varrho \leqslant 2$ and for the subspaces T_p of all triangular matrices $1 \leqslant \varrho < \infty$. Results are also obtained for other subspaces.

1. Introduction. A collection of balls of radius r is said to be *packed* in the unit ball U of a Banach space X if each pair has disjoint interiors and they are all contained in U. The *packing number* $P(X)$ is defined by

$$P(X) = \sup\{r\colon \text{infinitely many balls of radius } r \text{ can be packed in } U\}.$$

In this paper, the packing numbers of the C_p spaces are studied.

The packing numbers $P(l_p)$ were found by Burlack, Rankin, and Robertson [2] to be $[1+2^{1-1/p}]^{-1}$, $1 \leqslant p < \infty$. In 1970, Kottman [6] showed that $1/3 \leqslant P(X) \leqslant 1/2$ for any infinite dimensional Banach space. Results of Wells and Williams [10], along with slight modifications in [4] show that for $L_p(\mu)$ with μ not purely atomic, $P(L_p)$ is the same as for l_p for $1 \leqslant p \leqslant 2$, but for $p \geqslant 2$, $P(L_p) = [1+2^{1/p}]^{-1}$.

To see that an infinite number of balls of radius $1/2$ can be packed in the unit ball of c_0 and hence in l_∞, choose the center of the kth ball to be $(1/2, 1/2, \ldots, 1/2, -1/2, 0, 0, \ldots)$ where the $-1/2$ occurs in the kth position. For Banach spaces containing subspaces isomorphic to c_0, a theorem of R. C. James ([5], Lemma 2.2) implies X contains a subspace nearly isometric to c_0 and hence $P(X) = 1/2$. This result was extended by Kottman [7] to show that if Y is isomorphic to l_p, then $P(Y) \geqslant P(l_p)$. Thus the study of packing leads to results about the subspace structure.

A Riesz–Thorin theorem and some properties of special matrices are used to study the packing number of the C_p spaces and some subspaces. Although the C_p spaces have many of the properties of both l_p and L_p, they are not isomorphic to either.

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2. Main results. In the following, H will denote a separable Hilbert space, $\mathscr{B}(H)$ the bounded linear operators on H, and A^* the adjoint of A.

If A is a compact operator in $\mathscr{B}(H)$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0$ are the eigenvalues of $(AA^*)^{1/2}$, then for $1 \leqslant p < \infty$ we define

$$\|A\|_p = \Big\{\sum_{n=1}^{\infty} \mu_n^p\Big\}^{1/p} = \{\operatorname{tr}(AA^*)^{p/2}\}^{1/p}.$$

The space C_p consists of all compact operators A such that $\|A\|_p$ is finite. The number $\|A\|_\infty$ will denote the operator norm of A. The space C_1 is the trace class and C_2 the Hilbert–Schmidt operators.

To find upper bounds for the packing number of C_p, we use an interpolation theorem in the same manner as in [4] and [10].

Let $X_1, X_2, \ldots, X_n$ be Hilbert spaces and $P = (p_1, p_2, \ldots, p_n)$ be an n-tuple of real numbers with $1 \leqslant p_k \leqslant \infty$. Define $\oplus C_{p_k}(X_k)$ to be the linear space of all vectors $A = (A_1, A_2, \ldots, A_n)$, $A_k \in C_{p_k}(X_k)$, with usual coordinate addition and scalar multiplication. In this space introduce the norm

$$\|A\|_{P,r} = \Big\{\sum_{k=1}^{n} \|A_k\|_{P_k}^r \lambda_k\Big\}^{1/r}$$

where $1 \leqslant r < \infty$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an n-tuple of positive weights. In case $r = \infty$, write

$$\|A\|_{P,\infty} = \max_{1\leqslant k\leqslant n} \|A_k\|_{P_k}.$$

Denote by $C_{P,r}(\lambda)$ the set of A such that $\|A\|_{P,r}$ is finite.

Suppose $Y_1, Y_2, \ldots, Y_m$ is another collection of Hilbert spaces, $\eta = (\eta_1, \ldots, \eta_m)$ with $\eta_i \geqslant 0$ and $Q = (q_1, q_2, \ldots, q_m)$, $1 \leqslant q_i \leqslant \infty$ and define $C_{Q,s}(\eta)$. Consider linear maps taking finite rank operators in $C_{P,r}(\lambda)$ into finite rank operators in $C_{Q,s}(\eta)$. Then the following interpolation theorem is established in [3].

THEOREM 1. *Suppose* $1 \leqslant P_i, Q_i \leqslant \infty, 1 \leqslant r_i, s_i \leqslant \infty$, $i = 1, 2$ *and*

$$1/P = (1-t)/P_1 + t/P_2, \qquad 1/Q = (1-t)/Q_1 + t/Q_2,$$

$$1/r = (1-t)/r_1 + t/r_2, \qquad 1/s = (1-t)/s_1 + t/s_2.$$

Let L be a bounded linear operator taking finite rank operators in C_{P_i,r_i}, $i = 1, 2$ into finite rank operators in C_{Q_i,s_i}, $i = 1, 2$ with bounds M_1 and M_2, respectively. Then L takes $C_{P,r}$ into $C_{Q,s}$ and

$$\|L(A)\|_{Q,s} \leqslant M_1^{1-t} M_2^t \|A\|_{P,r} \quad \textit{for} \quad A \in C_{P,r}.$$

This theorem is used to establish the following inequalities.

COROLLARY 2. *Let $1 \leqslant p \leqslant \infty$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ any collection of positive numbers such that $\sum_{i=1}^{n} \lambda_i = 1$. Then for any $A_1, A_2, \ldots, A_n$ in C_p,*

(i) $$\sum_{i,j=1}^{n} \lambda_i\lambda_j \|A_i - A_j\|_p^p \leqslant 2\gamma^{2-p} \sum_{i=1}^{n} \lambda_i \|A_i\|_p^p, \qquad 1 \leqslant p \leqslant 2$$

and

(ii) $$\sum_{i,j=1}^{n} \lambda_i\lambda_j \|A_i - A_j\|_p^{p'} \leqslant \big(2\gamma\,(p-2)/(p-1)\big) \sum_{i=1}^{n} \lambda_i \|A_i\|_p^{p'}, \qquad 2 < p < \infty$$

where $p' = p/(p-1)$ and $\gamma = \max_{1\leqslant i\leqslant n}(1-\lambda_i)$.

Proof. Let P_i, $i = 1, 2$ be the constant n-tuple with each component 1 and Q_i, $i = 1, 2$ be the constant n^2-tuple with each component 1. Setting $t_1 = r_1 = 1$ and $t_2 = r_2 = 2$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\eta = (\lambda_i\lambda_j)_{i,j=1}^n$, define L from $C_{P_i,t_i}(\lambda)$ into $C_{Q_i,r_i}(\eta)$ by

$$L(A_1, \ldots, A_n) = (A_i - A_j)_{i,j=1}^n.$$

Now

$$\begin{aligned}\|LA\|_{1,1} &= \sum_{i,j=1}^{n} \lambda_i\lambda_j \|A_i - A_j\|_1 \\ &\leqslant \sum_{i,j=1}^{n} \lambda_i\lambda_j (\|A_i\|_1 + \|A_j\|_1) - 2\sum_{i=1}^{n} \lambda_i^2 \|A_i\|_1 \\ &= 2\sum_{i=1}^{n} (1-\lambda_i)\lambda_i \|A_i\|_1 \leqslant 2\gamma \sum_{i=1}^{n} \lambda_i \|A_i\|_1 \\ &= 2\gamma \|A\|_{1,1}.\end{aligned}$$

It follows from properties of Hilbert space that

$$\|LA\|_{2,2} \leqslant \sqrt{2}\,\|A\|_{2,2}.$$

For $1 \leqslant p \leqslant 2$, choose $s = t$ so that $1/p = (1-t) + t/2 = 1 + t/2$ and apply the theorem to obtain (i). Statement (ii) can be obtained in a similar way.

To apply this to packing, assume n disjoint balls of radius r with centers $A_1, A_2, \ldots, A_n$ are contained in the unit ball of C_p. If $1 \leqslant p \leqslant 2$, the A_i's must satisfy (i) for any collection of λ_i's. In particular, choose the $\lambda_i = i/n$, $i = 1, 2, \ldots, n$, then

$$\sum_{i,j=1}^{n} (1/n^2)\|A_i - A_j\|_p^p \leqslant 2(1 - 1/n)^{2-p} \sum_{i=1}^{n} (1/n)\|A_i\|_p^p$$

which implies

$$(*)\qquad (1/n^2)n(n-1)(2r)^p \leqslant 2(1-1/n)^{2-p}n(1-r)^p$$

since the centers must be at least $2r$ apart and $\|A_i\|_p \leqslant 1-r$. Solving this for r, we have

$$r \leqslant \{1+[2(1-1/n)]^{(p-1)/p}\}^{-1}.$$

Letting $n\to\infty$, $r \leqslant [1+2^{1-1/p}]^{-1}$.

Using (ii) and performing same operations as above we find that for $2\leqslant p\leqslant\infty$, infinite packing is possible only if $r\leqslant[1+2^{1/p}]^{-1}$.

THEOREM 3. *If $1\leqslant p<\infty$, the packing numbers satisfy*

$$(1)\qquad P(C_p) = [1+2^{1-1/p}]^{-1},\quad 1\leqslant p\leqslant 2,$$

$$(2)\qquad [1+2^{1/2}]^{-1}\leqslant P(C_p)\leqslant[1+2^{1/p}]^{-1},\quad 2\leqslant p<\infty.$$

Furthermore, if $r>[1+2^{1-1/p}]^{-1}$, $1\leqslant p\leqslant 2$ or $r>[1+2^{1/p}]^{-1}$, $2\leqslant p<\infty$, then the number of spheres of radius r which can be packed in the unit ball of C_p does not exceed

$$(3)\qquad C_p(r) = [1-1/2((1-r)/r)^{p/(p-1)}]^{-1},\quad 1\leqslant p\leqslant 2,$$

$$(4)\qquad C_p(r) = [1-1/2((1-r)/r)^p]^{-1},\quad 2\leqslant p<\infty.$$

Proof. Equality in (1) follows from the fact that l_p is isometric to a subspace of C_p. The left hand side of (2) is obtained since l_2 is isometric to a subspace of C_p. Inequality (3) is obtained by solving $(*)$ for n.

It was shown by Spence [9] that the numbers $C_p(r)$ obtained above are best possible in complex l_p-spaces and hence the same is true in C_p. The results of Kottman [7] and Theorem 3 lead to structural results for C_p.

COROLLARY 4. *If X is a Banach space and Y is an infinite dimensional subspace isomorphic to C_p, then $P(X)\geqslant[1+2^{1-1/p}]^{-1}$, $1\leqslant p\leqslant 2$, and $P(X)\geqslant[1+2^{1/2}]^{-1}$, $2\leqslant p<\infty$.*

COROLLARY 5. *If $1\leqslant p_1\leqslant p_2\leqslant 2$, then C_{p_1} is not isomorphic to a subspace of C_{p_2}.*

The remainder of the paper is concerned with finding packing numbers of certain subspaces of C_p. The following lemma yields a useful relationship among the C_p norms.

LEMMA 6. *If $A\in C_p$, $A\geqslant 0$, $1\leqslant p<\infty$, and $p\geqslant r>0$, then $\|A^r\|_{p/r} = \|A\|_p^r$.*

Proof.

$$\|A^r\|_{p/r} = \{\operatorname{tr}[A^r(A^r)^*]^{p/2r}\}^{r/p} = \{\operatorname{tr}[A^{2r}]^{p/2r}\}^{r/p} = \{\operatorname{tr}A^p\}^{r/p} = \|A\|_p^r.$$

Relative to a fixed orthonormal basis $\{e_i\}_{i=1}^\infty$ of H, associate the matrix $A(i,j)$ with the operator A by $A(i,j) = (Ae_i, e_j)$. Let n be a positive integer and R, S be subsets of the positive integers. Define the operators $P_n, P_{S,R}$ and E_n on C_p by:

$$P_nA(i,j) = \begin{cases}A(i,j), & \max\{i,j\}\leqslant n,\\ 0 & \text{otherwise,}\end{cases}$$

$$P_{S,R}A(i,j) = \begin{cases}A(i,j), & (i,j)\in S\times R,\\ 0 & \text{otherwise,}\end{cases}$$

$$E_nA(i,j) = \begin{cases}A(i,j), & \min\{i,j\}\leqslant n,\\ 0 & \text{otherwise.}\end{cases}$$

Denote by T_p the subspace of C_p consisting of those A in C_p for which $A(i,j) = 0$ for $j>1$. Let $E_{n,p}$ be the subspace of C_p consisting of those A in C_p having the property $A = E_nA$.

LEMMA 7. *Let R_1, R_2, S_1 and S_2 be nonempty subsets of the positive integers and let $\max R_1<\min R_2$. Suppose $2\leqslant p\leqslant\infty$ and A, B are elements of C_p such that $P_{R_1,S_1}A = A$ and $P_{R_2,S_2}B = B$. Then*

$$\{\|A\|_p^p+\|B\|_p^p\}^{1/p}\leqslant\|A-B\|_p\leqslant\{\|A\|_p^2+\|B\|_p^2\}^{1/2}.$$

Proof.

$$\begin{aligned}\|A-B\|_p^p &= \operatorname{tr}[(A-B)(A-B)^*]^{p/2}\\ &= \operatorname{tr}[AA^*+BB^*]^{p/2}\\ &= \|(AA^*+BB^*)^{1/2}\|_p^p.\end{aligned}$$

One can similarly show $\|A+B\|_p = \|(AA^*+BB^*)^{1/2}\|_p$. The first inequality now follows from Clarkson's inequality [8] and $\|A+B\|_p = \|A-B\|_p$.

The second inequality follows from:

$$\begin{aligned}\|A-B\|_p &= \|(AA^*+BB^*)^{1/2}\|_p\\ &= \|AA^*+BB^*\|_{p/2}^{1/2}\quad\text{(by Lemma 6)}\\ &\leqslant\{\|AA^*\|_{p/2}+\|BB^*\|_{p/2}\}^{1/2} = \{\|A\|_p^2+\|B\|_p^2\}^{1/2}.\end{aligned}$$

The mapping T on C_p defined by $T(A) = A^*$ is an isometry onto C_p. Hence Lemma 7 remains valid when $\max R_1<\min R_2$ is replaced with $\max S_1<\min S_2$.

THEOREM 8. *$P(T_p) = 1/(1+2^{1-1/p})$ for $1\leqslant p\leqslant 2$. $P(T_p) = 1/(1+2^{1/2})$ for $2\leqslant p<\infty$.*

Proof. Notice l_p is embedded isometrically in T_p, and T_p is embedded isometrically in C_p. Thus $P(l_p)\leqslant P(T_p)\leqslant P(C_p)$. Hence $P(T_p) = 1/(1+2^{1-1/p})$ for $1\leqslant p\leqslant 2$. For $2\leqslant p<\infty$, suppose infinitely many balls of radius r are packed in $U(T_p)$. Let $\{A_i\}_{i=1}^{\infty}$ be their centers and let A be a weak limit (without loss of generality assume $\{A_i\}_{i=1}^{\infty}$ converges weakly, since C_p is reflexive for $1 < p < \infty$ [8]). Thus A is an element of T_p and $\|A\|_p \leqslant 1-r$.

Let $\varepsilon > 0$ and fix a positive integer n. There exists a positive integer N such that $\|(I-P_N)A_n\|_p < \varepsilon$, as the operators of finite rank are dense in C_p [8]. By Lemma 7,

$$\begin{aligned}(2r)^2 &\leqslant \|A_m - A_n\|_p^2 = \|P_N(A_m-A_n)+(I-P_N)(A_m-A_n)\|_p^2\\ &\leqslant \|P_N(A_m-A_n)\|_p^2+\|(I-P_N)(A_m-A_n)\|_p^2\\ &\leqslant \|P_N(A_m-A_n)\|_p^2+(1-r+\varepsilon)^2.\end{aligned}$$

Thus

$$(2r)^2-(1-r+\varepsilon)^2 \leqslant \|P_N(A_m-A_n)\|_p^2.$$

This inequality is independent of m, and $\{A_i\}_{i=1}^{\infty}$ conve ges to A weakly. Hence

$$(2r)^2-(1-r+\varepsilon)^2 \leqslant \|P_N(A-A_n)\|_p^2.$$

Letting N tend to infinity and ε to zero,

$$(2r)^2-(1-r)^2 \leqslant \|A-A_n\|_p^2 \quad \text{for every } n.$$

Again let $\varepsilon > 0$ and choose M so that $\|P_M A\|_p < \varepsilon$. From above,

$$\begin{aligned}(2r)^2-(1-r)^2 &\leqslant \|A-A_n\|_p^2\\ &= \|P_M(A-A_n)+(I-P_M)(A-A_n)\|_p^2\\ &\leqslant \|P_M(A-A_n)\|_n^2+\|(I-P_M)(A-A_n)\|_p^2\\ &\leqslant \|P_M(A-A_n)\|_p^2+(1-r+\varepsilon)^2.\end{aligned}$$

Letting n tend to infinity and ε to zero yields

$$(2r)^2-(1-r)^2 \leqslant (1-r)^2$$

or

$$r \leqslant 1/(1+2^{1/2}).$$

The above proof is a modification of the proof of Theorem 16.4 in [10].

THEOREM 9. *$P(E_{k,p}) = 1/(1+2^{1/2})$, for $2 \leqslant p < \infty$ and k a fixed positive integer.*

Proof. Suppose infinitely many balls of radius r are packed in $\bigcup(E_{k,p})$. Without loss of generality, assume the $\{A_n\}$ converge weakly to A, and the A_i have the property $P_{m_i}A_i = A_i$. Let $\varepsilon > 0$, and choose $n_0 \geqslant k$ such that $\|(I-P_{n_0})A\|_p < \varepsilon/8$. Pick B_1 in $\{A_n\}$ such that $\|P_{n_0}(A-B_1)\|_p < \varepsilon/8$ and choose $n_1 > n_0$ such that $P_{n_1}B_1 = B_1$. Choose B_2 in $\{A_n\}$ such that $\|P_{n_1}(A-B_2)\|_p < \varepsilon/8$ and pick $n_2 > n_1$ such that $P_{n_2}B_2 = B_2$. Continuing, obtain a sequence of integers $\{n_i\}_{i=1}^{\infty}$ such that $k \leqslant n_0 < n_1 < n_2 < \ldots$, and also a sequence of operators $\{B_i\}_{i=1}^{\infty}$ in $E_{k,p}$ such that $\|B_i\|_p \leqslant 1-r$, $P_{n_i}B_i = B_i$, and $\|P_{n_i}(A-B_{i+1})\| < \varepsilon/8$. Let $R = \{1, 2, \ldots, k\}$, $S_i = \{n_{i-1}+1, n_{i-1}+2, \ldots, n_i\}$, and $\alpha_i = \|P_{R,S_i}B_i\|_p$, $\beta_i = \|P_{S_i,R}B_i\|_p$. Notice that $\alpha_i, \beta_i \in [0,1]$ for $i = 1, 2, \ldots$ By passing to a subsequence, if necessary, there exist $\alpha, \beta \in [0,1]$ such that $|\alpha-\alpha_i| < \varepsilon/8$, $|\beta-\beta_i| < \varepsilon/8$, and $(\alpha_i^p+\beta_i^p)^{1/p} \leqslant 1-r$. Now consider for $i > j$,

$$\begin{aligned}2r \leqslant \|B_i-B_j\|_p &= \|P_{n_0}(B_i-B_j)+(P_{n_j}-P_{n_0})(B_i-B_j)+\\ &\qquad +(P_{n_i}-P_{n_j})(B_i-B_j)\|_p\\ &\leqslant \|(P_{n_i}-P_{n_j})B_i-(P_{n_j}-P_{n_0})B_j\|_p+\|P_{n_0}(B_i-B_j)\|_p+\\ &\qquad +\|(P_{n_j}-P_{n_0})B_i\|_p+\|(P_{n_i}-P_{n_j})B_j\|_p\\ &\leqslant \|(P_{n_i}-P_{n_j})B_i-(P_{n_j}-P_{n_0})B_j\|_p+\varepsilon/4+\varepsilon/4\\ &= \|[P_{R,S_i}+P_{S_i,R}+P_{R,S_j}+P_{S_j,R}][(P_{n_i}-P_{n_j})B_i-(P_{n_j}-P_{n_0})B_j]\|_p+\\ &\qquad +\varepsilon/2\\ &= \{\|P_{R,S_i}(P_{n_i}-P_{n_j})B_i-P_{R,S_j}(P_{n_j}-P_{n_0})B_j\|_p^p+\\ &\qquad +\|P_{S_i,R}(P_{n_i}-P_{n_j})B_i-P_{S_j,R}(P_{n_j}-P_{n_0})B_j\|_p^p\}^{1/p}+\varepsilon/2\\ &\leqslant \{[\|P_{R,S_i}(P_{n_i}-P_{n_j})B_i\|_p^2+\|P_{R,S_j}(P_{n_j}-P_{n_0})B_j\|_p^2]^{p/2}+\\ &\qquad +[\|P_{S_i,R}(P_{n_i}-P_{n_j})B_i\|_p^2+\|P_{S_j,R}(P_{n_j}-P_{n_0})B_j\|_p^2]^{p/2}\}^{1/p}+\varepsilon/2\\ &\leqslant \{[(\alpha+\varepsilon/8)^2+(\alpha+\varepsilon/8)^2]^{p/2}+[(\beta+\varepsilon/8)^2+(\beta+\varepsilon/8)^2]^{p/2}\}^{1/p}+\varepsilon/2\\ &= 2^{1/2}\{(\alpha+\varepsilon/8)^p+(\beta+\varepsilon/8)^p\}^{1/p}+\varepsilon/2.\end{aligned}$$

Thus,

$$2r \leqslant 2^{1/2}\{(\alpha+\varepsilon/8)^p+(\beta+\varepsilon/8)^p\}^{1/p}+\varepsilon/2 \quad \text{for} \quad \varepsilon > 0.$$

Letting ε tend to zero yields

$$2r \leqslant 2^{1/2}(\alpha^p+\beta^p)^{1/p} \leqslant 2^{1/2}(1-r)$$

or

$$r \leqslant 1/(1+2^{1/2}).$$

It was shown by Arazy and Lindenstrauss [1] that C_p is isomorphic to T_p. This combined with other properties of C_p seem to indicate that $P(C_p) = P(T_p)$ but the techniques above do not yield this result.

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Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions

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Abstract. The pairs of nonnegative weight functions (U, V) for which the modified Hardy operator $P_\eta f(x) = x^{-\eta}\int_0^x f(t)\,dt$, η real, is of weak type (p, q) are characterized. Dual results for the operator $Q_\eta f(x) = x^{-\eta}\int_x^\infty f(t)\,dt$ are given. These results complement the classical (strong) Hardy inequalities and their generalizations considered by Artola, Talenti, Tomaselli and Muckenhoupt. New weighted weak type inequalities for Hilbert transforms and maximal functions are derived as applications of these results.

1. Introduction. Let $1 \leqslant p, q < \infty$ and suppose $U(x)$, $V(x)$ are nonnegative extended real valued functions on $(0, \infty)$. We say that (U, V) is a *strong type* (p, q) *weight pair* for the linear operator T if there is a finite constant C independent of f such that

$$\Big(\int_0^\infty |Tf(x)|^q U(x)\,dx\Big)^{1/q} \leqslant C\Big(\int_0^\infty |f(x)|^p V(x)\,dx\Big)^{1/p}, \tag{1.1}$$

and we say that (U, V) is a *weak type* (p, q) *weight pair* for T if there is a finite constant C independent of f such that for all $y > 0$

$$\Big(\int_{\{x:\,|Tf(x)|>y\}} U(x)\,dx\Big)^{1/q} \leqslant Cy^{-1}\Big(\int_0^\infty |f(x)|^p V(x)\,dx\Big)^{1/p}. \tag{1.2}$$

The smallest choice of constants C in (1.1) and (1.2), called the strong and weak norms of T, are denoted $\|T\|_s$, $\|T\|_w$, respectively. It is well known that (1.1) implies (1.2); moreover, $\|T\|_w \leqslant \|T\|_s$.

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