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# On a class of Hausdorff compacts and GSG Banach spaces

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**Abstract.** The paper deals with compact Hausdorff spaces homeomorphic to weak-star compact RN-sets in dual Banach spaces. One of the main theorems asserts that if $K$ is such a compact space, then $K$ contains a $G_\delta$ dense metrizable subset of $G_\delta$ points. This is a generalization of the Benyamini–Rudin–Wage result on Eberlein compacts.

**0. Introduction.** A compact Hausdorff space is said to be an *Eberlein compact* if it is homeomorphic to a weakly compact subset of a Banach space. It is well known that every Radon probability on an Eberlein compact has a separable support. Moreover, if $K$ is a weakly compact set in a Banach space and $\mu$ is a Radon probability on $K$, then there is a sequence of strongly compact sets $\{K_n\}_{n=1}^{\infty}$ such that $\mu\big(\bigcup_n K_n\big) = 1$ (the Phillips theorem). Thus the natural generalization of the notion of a weakly compact set in a Banach space is the notion of an RN-set in a dual Banach space (cf. [7], [9], [12]): we say that a weak-star compact set $K \subset X^*$ is an RN-*set* if every Radon probability on $K$ is supported a.e. on a countable union of strongly compact sets (let us mention that a convex weak-star compact set $K$ is an RN-set iff it has the Radon–Nikodym property; see for example [7], [9], [12]. Now we only have to make one step to get a new class of compact Hausdorff spaces which contains the class of Eberlein compacts.

DEFINITION 0.1. A compact Hausdorff space is said to be a *compact of type* RN if it is homeomorphic to a weak-star compact RN-set in a dual Banach space.

It is now convenient to formulate one of the main results of the paper concerning compact spaces of type RN (it follows directly from Corollary 3.1 in Section 3). The author hopes that the introduction of the new notion is partially justified by this result (and by the results obtained in Section 2).

THEOREM 0.1. *If $K$ is a compact space of type* RN, *then there exists a $G_\delta$ dense metrizable subset $S$ of $K$ every point of which is a $G_\delta$ point of $K$.*

COROLLARY 0.1 ([16], [1]). *If $K$ is an Eberlein compact, then there is a $G_\delta$ dense metrizable subset $S$ of $K$ every point of which is a $G_\delta$ point of $K$.*

The class of sets in Banach spaces which is in a sense dual to the class of RN-sets is the class of sets measurable in themselves which was introduced and considered for the first time by the author in [4] (and in detail in [5]): a bounded subset $A$ of a Banach space $X$ is said to be *measurable in itself* if for each finite Radon measure $\mu$ on the unit cell $K$ of $X^*$ with its weak-star topology the set of functions $A = \{\langle x, \cdot\rangle\colon x \in A\}$ on $K$ is equimeasurable. Since every weakly compact set is measurable in itself (see [5]), a natural generalization of the notion of WCG space is the notion of the Banach space generated by a bounded subset measurable in itself:

DEFINITION 0.2. A Banach space $X$ is said to be GSG if there is a bounded subset $A$ of $X$, measurable in itself, whose linear span is dense in $X$.

The following remark explains a discrepancy in terminology:

Remark 0.1. As we have already said, sets measurable in themselves (under this very name) were considered for the first time in papers [4] and [5]. Later C. Stegall's attention was drawn to this class of sets. He named them "sets with the GS property" and the Banach spaces which these sets generate — "GSG spaces" (see [14]). Preserving for these Banach spaces the name suggested by C. Stegall, the author thinks at the same time that it is fair to keep their original name for sets measurable in themselves.

Our paper is devoted to a study of compact Hausdorff spaces of type RN and the GSG Banach spaces defined above. In Section 1 we prove a factorization theorem, which will be useful in the next section. The proof of the theorem is inspired by the Davis–Figiel–Johnson–Pełczyński construction in [2] but differs from it in details. As corollaries of our factorization theorem we get some interesting results on RN-sets and sets measurable in themselves in Banach spaces (see Corollaries 1.1–1.3).

In Section 2 we investigate some general properties of compact spaces of type RN and GSG Banach spaces. In particular, we show that a compact Hausdorff space is of type RN iff $C(K)$ is a GSG space (Proposition 2.5) and, more generally, if $X$ is a Banach space, then the space $C(K, X)$ of all continuous $X$-valued functions on $K$ is GSG iff $X$ is a GSG space and $K$ is a compact space of type RN (Corollary 2.5).

Section 3 is devoted to the study of some properties of the Radon–Nikodym operators with applications to RN-sets in dual Banach spaces. An immediate consequence of our results in this section is Theorem 0.1 above. The central place in Section 3 is occupied by Lemma 3.2, in which some geometrical properties of non-Radon–Nikodym operators are obtained. The proof of this lemma is very simple which makes several of its consequences interesting, some of them being well known. In particular, we give an alternative (and simple) proof of C. Stegall's theorem on dual spaces with the Radon–Nikodym property. Another interesting application of Lemma 3.2 is Corollary 3.1, which says that every weak-star compact RN-set $K \subset X^*$ contains a dense $G_\delta$ subset at every point of which the identity map $(K, \sigma(X^*, X)) \to (K, \|\cdot\|)$ is continuous.



In Section 4 we state some problems concerning GSG spaces and compact spaces of type RN inspired by the known results on Eberlein compacts and WCG Banach spaces.

Our terminology is standard. Let us mention only that if $A$ is a subset of a Banach space $X$, then $\overline{\Gamma(A)}$ denotes its closed absolutely convex hull; if $A = \overline{\Gamma(A)}$ is a bounded subset, then $X_A$ is the Banach space whose unit cell is $A$. If $V$ is an absolutely convex neighbourhood in $X$, then $\hat{X}_V$ denotes the completion of a normed linear space $X_V$ generated by $V$. It is known that $(\hat{X}_V)^* = X^*_{V^0}$.

**1. A factorization theorem.** Let $X$ be a Banach space, let $K$ be a weak-star compact subset of $X^*$ and let $B = \overline{\Gamma(K)}^*$ be its weak-star closed absolutely convex hull. For each $n = 1, 2, \ldots$ we consider an equivalent norm $|\cdot|_n$ on $X^*$ determined by the absolutely convex set $B_n = nB + n^{-1}D(X^*)$, where $D(X^*)$ is the unit cell in $X^*$. Since $B_n$ is weak-star compact, the set $B_n^0$ determines an equivalent norm $\|\cdot\|_n$ on $X$ so that $(X, \|\cdot\|_n)^* = (X^*, |\cdot|_n)$. Put $p_0(x) = \inf_n\{\|x\|_n\}$ for $x \in X$ and let $p$ be the "absolutely convex hull" of $p_0$, i.e. the Minkowski functional of the set $\overline{\Gamma(p_0^{-1}[0, 1])}$. Then $p$ is a seminorm on $X$ and $p(x) \leqslant p_0(x) \leqslant c\|x\|$ for each $x \in X$. The Banach space associated with $(X, p)$ will be denoted by $Q(X, K)$ and its natural norm — by $|||\cdot|||$. Let $\Psi_1\colon X \to Q(X, K)$ be the canonical map and let $\tilde{B}_n = (\Psi_1^*)^{-1}(B_n)$. For $z \in Q(X, K)$ it is convenient to denote the number $\sup_{z' \in \tilde{B}_n} |\langle z, z'\rangle|$ by $\|z\|_n$ (if $x \in X$ and $z = \Psi_1(x)$, then $\|z\|_n = \|x\|_n$). Note that it follows from the definition of $p_0$ and $p$ that $\sup_{|||z|||\leqslant 1} |\langle z, z'\rangle| = \sup_{\inf_n\{\|z\|_n\}\leqslant 1} |\langle z, z'\rangle|$ for each $z' \in Q(X, K)^*$. If $x \in X$, then $\sup_{x' \in B} |\langle x, x'\rangle| \leqslant \sup_{x' \in B_n} |\langle x, x'\rangle| = \|x\|_n$ for each $n = 1, 2, \ldots$, i.e. $p_{B^0}(x) \leqslant |||\Psi_1(x)|||$, whence the canonical map $\Psi_2\colon Q(X, K) \to \hat{X}_{B^0}$ is continuous.

THEOREM 1.1. *Let $\mathfrak{B}$ be the category of all Banach spaces, and let $\mathscr{C}$ be a class of all pairs $(X, K)$ where $X \in \mathfrak{B}$ and $K$ is a weak-star compact subset of $X^*$. The above mapping $Q\colon \mathscr{C} \to \mathfrak{B}$ has the following properties:*

*If $(X, K) \in \mathscr{C}$ and $B = \overline{\Gamma(K)}^*$, then*

(1) *the canonical operator $\Psi_{K^0}\colon X \to \hat{X}_{K^0}$ factors through the space*

$Q(X,K)\colon X \xrightarrow{\Psi_1} Q(X,K) \xrightarrow{\Psi_2} \hat{X}_{K^0}$ *and*

(a) *the set* $\Psi_1(X)$ *is dense in* $Q(X,K)$;

(b) $\Psi_2^{**}$ *is one-to-one (or* $\Psi_2^*(X_B^*)$ *is dense in* $Q(X,K)^*$);

(c) *the map* $\Psi_2\colon \big(\Psi_1(D(X)),\, |||\cdot|||\big) \to \big(\Psi_{K^0}(D(X)),\, \|\cdot\|\big)$ *is a homeomorphism* ($D(Z)$ *denotes the unit cell of* $Z$);

(d) *the map* $\Psi_1^*\colon \big(\Psi_2^*(D(X_B^*)),\, \|\cdot\|\big) \to (B, \|\cdot\|)$ *is a homeomorphism*;

(2) $Q(X,K) = Q(X,B)$;

(3) $K$ *is norm separable iff* $Q(X,K)^*$ *is separable*;

(4) $K$ *is an* RN-*set iff* $Q(X,K)^*$ *has the* RN *property*.

Proof. Evidently the mapping $Q$ has properties (1a) and (2). Let us prove that it has also the other properties.

(1b): Let $z'' \neq 0$ be in $Q(X,K)^{**}$. For each $k = 1, 2, \ldots$ we have (see the remarks preceding Theorem 1.1):

$$\begin{aligned} 0 &< \sup\{|\langle z', z''\rangle|\colon z' \in Q(X,K)^*,\ \sup_{\substack{\inf_n\{\|z\|_n\}\leqslant 1\\ z\in Q(X,K)}} |\langle z', z\rangle| \leqslant 1\} \\ &\leqslant \sup\{|\langle z', z''\rangle|\colon z' \in Q(X,K)^*,\ \sup_{\|z\|_k \leqslant 1} |\langle z', z\rangle| \leqslant 1\} \\ &= \sup\{|\langle z'', z'\rangle|\colon z' \in \tilde{B}_k\} \\ &\leqslant \sup\{|\langle z'', z_1'\rangle|\colon z_1' \in k(\Psi_1^*)^{-1}(B)\} + \\ &\quad + \sup\{|\langle z'', z_2'\rangle|\colon z_2' \in k^{-1}(\Psi_1^*)^{-1}(D(X^*))\} \\ &\leqslant \sup\{|\langle \Psi_2^{**} z'', y'\rangle|\colon y' \in kD(X_B^*)\} + \\ &\quad + k^{-1}\sup\{|\langle z'', z'\rangle|\colon z' \in (\Psi_1^*)^{-1}(D(X^*))\} \\ &= k\|\Psi_2^{**} z''\|_{(X_B^*)^*} + k^{-1}C, \end{aligned}$$

where $C < +\infty$ is a constant not depending on $k$. This implies that $\Psi_2^{**} z'' \neq 0$.

(1c): Take $\{z_m\}_{m=1}^\infty \subset \Psi_1(D(X))$ and $\delta > 0$. Suppose that for each $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that $z_m \in \varepsilon \Psi_1(B^0)$ if $m > M$. We must show that there is an $N = N(\delta)$ with $|||z_m||| < \delta$ for each $m > N$. Let $k$ be an integer such that $k^{-1} \leqslant \delta$. Put $M = M(\varepsilon)$ for $\varepsilon = (4k^2)^{-1}$ and $N = N(\delta) = M$. For every $n = 1, 2, \ldots$ and every $m > N$ we have:

$$\begin{aligned} \|z_m\|_n &= \sup\{|\langle z_m, z'\rangle|\colon z' \in \tilde{B}_n\} \\ &\leqslant n\|\Psi_2(z_m)\|_{\hat{X}_{B^0}} + n^{-1}\sup\{|\langle z_m, z'\rangle|\colon z' \in (\Psi_1^*)^{-1}(D(X^*))\} \leqslant n\varepsilon + n^{-1}. \end{aligned}$$



Hence

$$|||z_m||| \leqslant \inf_n\{\|z_m\|_n\} \leqslant \inf_n\{n\varepsilon + n^{-1}\} \leqslant 2k\varepsilon + (2k)^{-1} = k^{-1} \leqslant \delta.$$

(1d): Take $\{z_m'\}_{m=1}^\infty \subset \Psi_2^*(D(X_B^*))$ and $\delta > 0$. Suppose that for each $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that $\|\Psi_1^*(z_m')\| \leqslant \varepsilon$ if $m > M$. We must show that there is an $N = N(\delta)$ with

$$\sup\big\{|\langle z_m', z\rangle|\colon \inf_n\{\|z\|_n\} < 1\big\} \leqslant \delta$$

for each $m > N$. Put $N = M(\varepsilon)$ for $\varepsilon = \delta^2$. Let $z$ be an element of $Q(X,K)$ such that $|||z||| < 1$. Then there is an $n$ with $\|z\|_n \leqslant 1$, i.e.

$$z \in \big(n^{-1}(\Psi_1^*)^{-1}(D(X^*)) + n\Psi_2^*(D(X_B^*))\big)^0.$$

Since $z_m' \in \Psi_2^*(D(X_B^*))$, it follows that $|\langle z_m', z\rangle| \leqslant n_{\!1}^{-1}$ for each $m$ and if $m > M(\varepsilon)$, then $|\langle z_m', z\rangle| \leqslant n\varepsilon$. Thus if $m > N$, then

$$|\langle z_m', z\rangle| \leqslant \min\{n^{-1}, n\varepsilon = n\delta^2\}.$$

Now, if $n^{-1} \leqslant \delta$, then $|\langle z_m', z\rangle| \leqslant \delta$, and if $n^{-1} > \delta$, then $n\varepsilon = n\delta^2 < \delta$, so $|\langle z_m', z\rangle| \leqslant \delta$ again. Hence if $m > N$, then $\|z_m'\|_{Q(X,K)^*} \leqslant \delta$.

(3): If $K$ is norm separable, then so is $B$. Since by (1b) the linear span of $(\Psi_1^*)^{-1}(B)$ is dense in $Q(X,K)^*$, it follows from (1d) that $Q(X,K)^*$ is also separable.

(4): Let $K$ be an RN-set. Then $B$ is also an RN-set (see [7], [9], [14]). It follows from (1d) and from the definition of RN-sets (see Introduction) that $\Psi_2^*(D(X_B^*))$ is an RN-set in the space $Q(X,K)^*$. But by (1b) the linear span of $\Psi_2^*(D(X_B^*))$ is dense in $Q(X,K)^*$. Thus (4) follows immediately from the next statement (see [9], Proposition 1.1 or 1.3 for a proof).

LEMMA 1.1. *Let $Z$ be a Banach space. If $Z^*$ is* RNG (*i.e. there exists a weak-star compact* RN-*set with a (norm) dense linear span in* $Z^*$), *then* $Z^*$ *has the* RN *property*.

Remark 1.1. Of course, as in the case of the Davis–Figiel–Johnson–Pełczyński intermediate space [2], $K$ is weakly compact iff $Q(X,K)$ is reflexive, and $\Psi_{K^0}$ "contains $l^1$" iff $Q(X,K)$ does. In view of Theorem 1.1 this implies in particular that every weakly compact (weakly conditionally compact) set embeds with its norm topology into a reflexive Banach space (into a Banach space containing no subspaces isomorphic to $l^1$).

Remark 1.2. Let $X$ and $Y$ be Banach spaces and let $T$ be an operator from $X$ to $Y$. Recall that $T$ is said to be an RN *operator* if it takes each $X$-valued measure $m$ of finite variation into a measure having a derivative with respect to the variation of $m$. It is known that $T^*$ is an RN operator iff $T^*$ maps bounded subsets of $Y^*$ into RN-sets in $X^*$ and iff $T$ maps bounded subsets of $X$ into sets measurable in themselves in $Y$ (see [7], [9], [14]). Thus Theorem 1.1 implies that every dual RN operator factors through a Banach space with the RN property. This factorization theorem for dual RN operators was obtained independently in 1976–1977 by S. Heinrich, C. Stegall and the present author (see [7], [9], [14], [3]).

COROLLARY 1.1. *Let $K$ be a weak-star compact* RN*-set in a dual Banach space $X^*$ and let $B = \overline{\Gamma(K)}^*$. There exist a Banach space $Z$ and an operator $U: X \to Z$ such that*

(1) $\overline{U(X)} = Z$;

(2) *$Z^*$ has the* RN *property;*

(3) *the set $(U^*)^{-1}(B)$ is weak-star compact;*

(4) *the map $U^*: \big((U^*)^{-1}(B), \|\cdot\|\big) \to (B, \|\cdot\|)$ is a homeomorphism.*

Proof. Just apply Theorem 1.1, (1d) and (4).

COROLLARY 1.2. *Let $A$ be a bounded set measurable in itself in a Banach space $Y$. There exist a Banach space $Z$ and an operator $U: Z \to Y$ such that*

(1) $U(D(Z)) \supset A$;

(2) *$Z^*_{**}$ has the* RN *property;*

(3) *the set $(U^{-1})(A)$ is bounded in $Z$;*

(4) *the map $U: \big(U^{-1}(A), \|\cdot\|\big) \to (A, \|\cdot\|)$ is a homeomorphism.*

Proof. Let us consider the natural operator $\Phi: Y_{\overline{\Gamma(A)}} \to Y$. The set $\overline{\Gamma(A)}$ is measurable in itself and thus $\Phi^*$ is an RN operator (see [7], [14], [4]). Now it is enough to put $K = \Phi^*(D(Y^*))$ and to apply Theorem 1.1, (1c).

We conclude this section with an interesting result which is a direct consequence of Theorem 1.1 ((1d) and (3)) above and Corollary 8 in [2].

COROLLARY 1.3. *Norm separable weak-star compact subsets of dual Banach spaces embed (with their norm and weak-star topologies) into dual spaces with boundedly complete bases. More precisely, if $K$ is a norm separable weak-star compact subset of a dual Banach space $X^*$, then there exist a Banach space $Z$ with shrinking basis and an affine map $\varphi: \overline{\Gamma(K)}^* \to Z^*$ which is one-to-one and weak-star-to-weak-star continuous so that $\varphi$ is a homeomorphism in the norm topologies.*

**2. Some properties of compact spaces of type RN and of GSG Banach spaces.** In this section we obtain some general results on compact spaces of type RN and GSG spaces, establishing in particular a connection between these two notions. Most of the facts are inspired by the well-known theorems concerning the properties of Eberlein compacts and WCG spaces. Some questions remaining open in this paper are listed in Section 4.

Let us begin with the following obvious statement:

PROPOSITION 2.1. (a) *Every Eberlein compact is a compact space of type* RN.



(b) *Every finite Radon measure on a compact space of type* RN *has a separable support.*

*Moreover, if $K$ is a compact of type* RN *and $\mu \in C^*_+(K)$, then there exists a sequence $\{K_n\}$ of metrizable compact subsets of $K$ with $\mu\big(\bigcup_n K_n\big) = \mu(K)$.*

It is shown in [14] that every weak-star compact RN-set in a dual Banach space is sequentially compact. A direct consequence of this fact is

PROPOSITION 2.2. *If $K$ is a compact space of type* RN, *then every sequence of elements of $K$ contains a converging subsequence.*

An immediate consequence of Corollary 1.1 is

PROPOSITION 2.3. *Every compact space of type* RN *is homeomorphic to a weak-star compact subset of a dual Banach space with the* RN *property.*

In the next two propositions a connection between compact spaces of type RN and GSG Banach spaces is established.

PROPOSITION. 2.4 *If $X$ is* GSG, *then the unit cell $D(X^*)$ of $X^*$ is a compact space of type* RN *in its weak-star topology.*

Proof. If $A$ is a bounded subset of $X$ which is measurable in itself, then its closed absolutely convex hull is also measurable in itself (see [5]). Now, if $X$ is GSG, then there is a bounded set $A$ measurable in itself, $A = \overline{\Gamma(A)} \subset X$, such that the canonical operator $\Phi: X_A \to X$ has a dense range. It follows that $\Phi^*$ is an RN operator (see [14], [5]) and hence $\Phi^*(D(X^*))$ is an RN-set in $(X_A)^*$ (see [4], Theorem 7 or [7], Corollary 5). Since $\Phi^*$ is one-to-one, $D(X^*)$ is homeomorphic to an RN-set $\Phi^*(D(X^*))$ in its weak-star topology.

PROPOSITION 2.5. *Let $K$ be a compact Hausdorff space. The following four assertions are equivalent:*

(1) *$K$ is a compact space of type* RN;

(2) *The unit cell in $C^*(K)$ is a compact space of type* RN *in its weak-star topology;*

(3) *The space $C(K)$ is* GSG;

(4) *There is a bounded subset $A$ of $C(K)$ measurable in itself separating the points of $K$.*

Proof. That (3) implies (2) follows from Proposition 2.4; that (2) implies (1) is obvious.

(1) $\Rightarrow$ (4): If $K$ is a compact space of type RN, then by Proposition 2.3 there is a Banach space $X$ such that $X^*$ has the RN property and $K$ is homeomorphic to a weak-star compact subset $S$ of $X^*$. If $j: X \to C(S)$ is a natural map $(jx)(\cdot) = \langle x, \cdot\rangle$, then the set $j\big(D(X)\big)$ is measurable in itself (see [14], [5]) and separates the points of $S$. Clearly this proves that (1) implies (4).

(4) $\Rightarrow$ (3): A straightforward argument shows that if $A$ and $B$ are

two bounded sets measurable in themselves in $C(K)$, then the set $AB = \{fg: f \in A, g \in B\}$ is also measurable in itself and hence each set $A^n = AA \dots A$ is measurable in itself. Furthermore, another straightforward argument shows that if $\{B_n\}_{n=1}^{\infty}$ is a sequence of uniformly bounded subsets measurable in themselves of $C(K)$, then the set $B = \bigcup_{n=1}^{\infty} 2^{-n} B_n$ is measurable in itself. Now if $A$ is a bounded set measurable in itself in $C(K)$ separating the points of $K$, then the set $\{1\} \cup (\bigcup_{n=1}^{\infty} 2^{-n} A^n)$ is measurable in itself and its linear span is dense in $C(K)$ by the Stone–Weierstrass theorem.

COROLLARY 2.1. *If $K$ is a compact space of type* RN, *then the compact space of all Radon probabilities on $K$ has the same property.*

Proof. Apply Proposition 2.5, (1) ⇒ (2).

COROLLARY 2.2. *If $K$ is a compact space of type* RN, *then every weakly compact operator from $C(K)$ has a separable range.*

Proof. If $K$ is a compact space of type RN, then by Proposition 2.5 there is a closed absolutely convex bounded set $A$ measurable in itself, $A \subset C(K)$, generating the space $C(K)$. Let $\Phi$ be the canonical operator from $(C(K))_A$ into $C(K)$. Since $\Phi^*$ is an RN operator for each weakly compact operator $U$ from $C(K)$, the map $U\Phi$ is compact (see [6], Proposition 1), whence $U\Phi$ has a separable range. Thus $U$ also has a separable range since the range of $\Phi$ is dense in $C(K)$.

COROLLARY 2.3. *Let $X$ be a Banach space and let $K$ be a weak-star compact set in $X^*$ whose weak-star closed absolutely convex hull coincides with the unit cell of $X^*$. If $K$ is a compact space of type* RN *in its weak-star topology, then*

(1) *$X$ is a subspace of a* GSG *space, and*

(2) *every absolutely $p$-summing operator from $X$ has a separable range.*

Proof. This is a consequence of Proposition 2.5 and Corollary 2.2 since $X$ is a subspace of $C(K)$ and every $p$-integral operator from $X$ factors through a weakly compact operator from $C(K)$.

An easy consequence of Propositions 2.2 and 2.4 is

PROPOSITION 2.6. *If $X$ is a Grothendieck space, then $X$ is a subspace of a* GSG *space iff $X$ is reflexive.*

COROLLARY 2.4. *A space $L^{\infty}(\mu)$ is a subspace of a* GSG *space iff it is finite-dimensional.*

In the next proposition $E(X)$ denotes the Banach space of all $X$-valued measurable functions $f: \Omega \to X$ such that $\|f(\cdot)\|_X \in E$ with the natural norm $\|f\|_{E(X)} = \| \, \|f(\cdot)\|_X \|_E$.

PROPOSITION 2.7. (1) *Let $X$ be a Banach space, let $(\Omega, \Sigma, \mu)$ be a finite measure space and let $E$ be a Banach ideal in the space $L^0(\mu)$ of all measurable functions on $\Omega$. If $X$ and $E$ are* GSG, *then $E(X)$ is also* GSG.

(2) *If $X$ and $Y$ are two* GSG *Banach spaces, then their injective tensor product $X \hat{\otimes} Y$ is also* GSG.

Proof. (1): If $X$ is GSG, then it follows from Corollary 1.2 (or from [14]) that there is a Banach space $Z$ and an operator $U: Z \to X$ such that $Z^*$ has the RN property and $\overline{U(Z)} = X$. On the other hand, if $E$ is GSG, then $l^{\infty}$ does not embed into $E$ by Corollary 2.4; hence each order interval in $E$ is weakly compact. Thus $E$ satisfies condition (A): if $e_n \downarrow 0$, then $\|e_n\| \to 0$. Now, since $\mu$ is a finite measure, there is an essentially positive element $e$ in $E$, and from condition (A) it follows that the weakly compact interval $[-e, e]$ generates the space $E$. By using the Davis–Figiel–Johnson–Pełczyński construction [2], it is easy to see that the canonical map $E_{[-e,e]} \to E$ factors through a reflexive Banach space $F$ which is an ideal in $L^0(\mu)$, whence the natural inclusion $j: F \to E$ has a dense range. Now, the map $j \otimes U: F(Z) \to E(X)$ has a dense range and the space $F(Z)$ is GSG (apply the following easy consequence of the main theorem in [13]: if $Z^*$ and $F^*$ have the RN property, then so does $F(Z)^*$). Thus the space $E(X)$ is also GSG.

(2): Let $Z$ and $W$ be Banach spaces such that $Z^*$ ans $W^*$ have the RN property and there are operators $U: Z \to X$ and $V: W \to Y$ with dense ranges. Then the space $(Z \hat{\otimes} W)^*$ has the RN property (this follows from the fact that if $E$ and $F$ are two separable spaces with separable duals then the space $(E \hat{\otimes} F)^*$ is a quotient of the separable space $E^* \otimes F^*$). Since the natural map from $Z \hat{\otimes} W$ to $X \hat{\otimes} Y$ is an operator with a dense range, we conclude that the space $X \hat{\otimes} Y$ is GSG.

An easy consequence of Propositions 2.5 and 2.7, (2) is

COROLLARY 2.5. *Let $K$ be a compact Hausdorff space and let $X$ be a Banach space. The space $C(K, X)$ is* GSG *iff $K$ is a compact space of type* RN *and $X$ is a* GSG *space.*

To prove this we only need to note that $C(K, X) = C(K) \hat{\otimes} X$.

**3. Points of weak-star-to-norm continuity of dual** RN **operators.** In this section we show that each dual RN operator $T^*$ has many points of weak-star-to-norm continuity on every weak-star compact set (in a sense almost all points of every weak-star compact set are points of continuity of $T^*$).

Let $M$ be a topological Hausdorff space and let $f$ be a map from $M$ into a metric space $R$. We shall write $f \in D(M, R)$ if for each $\varepsilon > 0$ and each closed subset $K$ of $M$ there is an open set $V \subset M$ such that the oscillation $\omega(f; K \cap V)$ of $f$ on $V \cap K$ is less than $\varepsilon$.

The following almost obvious statement will be useful in the proof of the main result of the section.

Lemma 3.1. *Let $M$ be a Baire space and let $f$ be a map from $M$ into a metric space $R$. If $f \in D(M, R)$, then there is a $G_\delta$ dense subset $S$ of $M$ such that the map $f\colon M \to R$ is continuous at each point of $S$.*

Proof. Let $A_\varepsilon$ be the union of all open subsets $V$ of $M$ with the property that the oscilation of $f$ on $V$ is less than $\varepsilon$. Then $A_\varepsilon$ is a dense open subset of $M$ (since otherwise the open subset $M \setminus \bar{A}_\varepsilon$ contains an open subset on which the oscilation of $f$ is less than $\varepsilon$). Now, it is sufficient to put $S = \bigcap_{n=1}^{\infty} A_{n^{-1}}$.

The basis for all the results of this section is the following lemma, which has a very simple proof:

Lemma 3.2. *Let $T$ be an operator from a Banach space $X$ into a Banach space $Y$. Let $K$ be a weak-star compact subset of $Y^*$ and let $\varepsilon > 0$ be a number such that $\omega(T^*, V) > \varepsilon$ for each (weak-star) open subset $V$ of $K$. Then there is a separable subspace $X_0$ of $X$ such that the subset $(T|X_0)^*(K)$ of $X_0^*$ contains a weak-star compact set $Q$ homeomorphic to the Cantor set such that $\|p-q\| > \varepsilon$ if $p, q \in Q$, $p \neq q$. In particular, $Q$ is not an RN-set in $X_0^*$.*

Proof. Let $W$ be an open subset of the compact space $T^*(K)$ and let $V = (T^*)^{-1}(W) \cap K$. Since $\omega(T^*, V) > \varepsilon$, we can find $w_1$ and $w_2$ in $W$ and $x \in X$, $\|x\| \leqslant 1$, such that $|\langle x, w_1 - w_2\rangle| > \varepsilon$. Now, there are two open subsets $W_1$ and $W_2$ of $W$ with the properties that $\bar{W}_1 \cup \bar{W}_2 \subset W$ and $\inf\{|\langle x, \tilde{w}_1 - \tilde{w}_2\rangle|\colon \tilde{w}_1 \in W_1,\ \tilde{w}_2 \in W_2\} > \varepsilon$.

Using this argument, we can find a sequence $\{x_j\}_{j=1}^{\infty} \subset X$, $\|x_j\| \leqslant 1$, and a system of open sets $\{W_j\}_{j=1}^{\infty}$, $W_j \subset T^*(K)$, such that $\bar{W}_{2n} \cup \bar{W}_{2n+1} \subset W_n$ and $\inf\{|\langle x_n, w_1 - w_2\rangle|\colon w_1 \in W_{2n},\ w_2 \in W_{2n+1}\} > \varepsilon$.

Let $X_0$ be the closed linear span of $\{x_j\}_{j=1}^{\infty}$ and let $Q_n = W_n|X$, $\tilde{Q} = \bigcup\{\bigcap_{k=1}^{\infty} Q_{n_k}\colon n_1 < n_2 < \ldots\}$. By standard arguments we can see that there exists a continuous map $\varphi$ from $Q$ onto the Cantor set $\Delta$ such that if $n_1 < n_2 < \ldots$ and $n_1' < n_2' < \ldots$, then $\varphi(\bigcap_k Q_{n_k})$ consists of one point of $\Delta$ (if not empty) and either $\bigcap_k Q_{n_k} = \bigcap_k Q_{n_k'}$ or $\varphi(\bigcap_k Q_{n_k'}) \neq \varphi(\bigcap_k Q_{n_k'})$. Since $\tilde{Q}$ is a metrizable compact space, there is (by a topological result of Kuratowski) a subset $Q$ of $\tilde{Q}$ homeomorphic to $\Delta$ and such that $\varphi|Q$ is a homeomorphism. It follows from the definition of $\varphi$ and $X_0$ that $Q$ is the compact space we wished to find.

Now we are ready to prove the main result of this section.

Theorem 3.1. *Let $T$ be an operator from $X$ to $Y$. If $T^*$ is an RN operator, then for each weak-star compact subset $K$ of $Y^*$ there exists a subset $S$ of $K$ having the following properties:*

(1) *$S$ is a $G_\delta$ dense subset of $K$;*

(2) *the map $T^*|K\colon K \to (X^*, \|\cdot\|)$ is continuous at every point of $S$.*



Proof. According to Lemma 3.1 we only have to show that if there is a weak-star compact set $K$ in $Y^*$ such that $T^*|K \notin D(K, X^*)$, then $T^*$ is not an RN operator. Let $K$ be such a set. Without loss of generality we may assume that there is an $\varepsilon > 0$ such that the oscilation of $T^*$ on every open subset $V$ of $K$ is more than $\varepsilon$. Now we can apply Lemma 3.2: if $T^*$ is an RN operator, then for each subspace $X_0$ of $X$ $(T|X_0)^*(K)$ is an RN-set in $X^*$, and so it is clear that Lemma 3.2 implies Theorem 3.1.

Remark 3.1. The converse of the theorem is also valid. In fact, it is easy to see that for each topological Hausdorff space $M$ and each metric space $R$ every map $f \in D(M, R)$ is universally (Lusin-) measurable (see [10]). Thus if $T^*$ is in $D(K, X^*)$, where $K$ is the unit cell in $Y^*$ with its weak-star topology, then the map $T^*$ is universally measurable as the map from $K$ into $X^*$, whence $T^*$ is an RN operator in virtue of Theorem 7 in [4].

Corollary 3.1. *Let $K$ be a weak-star compact subset of $X^*$. If $K$ is an RN-set, then there exists a subset $S$ of $K$ having the following properties:*

(1) *$S$ is a $G_\delta$ dense metrizable subset of $K$;*

(2) *the identity map $K \to (K, \|\cdot\|)$ is continuous at every point of $S$;*

(3) *each point of $S$ is a $G_\delta$ point of $K$.*

Proof. This follows immediately from Theorem 3.1 since $K$ is an RN-set iff $\overline{\Gamma(K)}^*$ is and iff $\Phi_{\overline{\Gamma(K)}^*}$ is an RN operator (see [14], [5]).

Corollary 3.2. *If $K$ is a weakly compact subset of a Banach space, then there is a subset $S$ of $K$ having properties (1)–(3) of the previous corollary when $K$ is equipped with the weak topology.*

With the help of Corollary 3.1 we get a simple proof of the main result in [13]:

Corollary 3.3. *If $X$ is a separable Banach space such that $X^*$ has the RN property, then $X^*$ is separable.*

Proof. If $X$ is separable and $X^*$ has the RN property, then the unit cell $K = D(X^*)$ is a metrizable weak-star compact RN-set in $X^*$. By Corollary 3.1 for every weak-star closed subset $K_1$ of $K$ the identity map $K_1 \to (K_1, \|\cdot\|)$ has a point of continuity. It follows (from the Baire theorem for Banach space valued functions) that the identity map $K \to (K, \|\cdot\|)$ is a Baire-1 function on $K$ with values in $X^*$. Therefore the set $K$ is norm separable.

We conclude this section with a corollary of the above results, proving only a part of it (a complete proof is based on Corollary 3.1 and some properties of universally measurable mappings). This statement gives an answer to a problem posed in [15] by M. Talagrand in a particular case.

Corollary 3.4. *Let $Y$ be a subspace of a dual Banach space $X^*$ and let $K$ be a subset of $Y$ which is weak-star compact in $X^*$. If the space $Y$ is*

*$K$-analytic in its weak topology, then the $\sigma(X^*, X)$ closed absolutely convex hull $B$ of $K$ belongs to $Y$ and there exists a $G_\delta$ dense set $S$ of $G_\delta$ points in $K$ (for the topology $\sigma(X^*, X)$) such that the identity map $(K, \sigma(X^*, X)) \to (K, \|\cdot\|)$ is continuous at each point of $S$.*

Proof (in the case where $K = B$). Let $\Phi$ be the canonical operator from $X_B^*$ into $X^*$. Note that $\Phi$ is a dual of the canonical map from $X$ to $X_{B^0}$ and $X_B^* = Y_B$. Hence $\Phi$ is a dual operator and factors through the weakly $K$-analytic Banach space $Y$. Now Corollary 3.4 follows from Corollary 3.1 and from

LEMMA 3.3. *Let $U$ be an operator from a Banach space $Z$ into a Banach space $W$. If $U^*$ factors through a weakly $K$-analytic Banach space, then $U^*$ is an* RN *operator.*

This is a simple consequence of Theorem 1 in [8].

Remark 3.2. A complete proof of Corollary 3.4 with some other close results will appear elsewhere.

Remark 3.3. It is easy to see that the proof of Lemma 3.2 is valid in the case where $K$ is a Polish subset of $Y^*$ in its weak-star topology. Thus we conclude in particular that if $K$ is a weak-star Polish subset of a dual Banach space $X^*$ such that the identity map is $j: (K, \sigma(X^*, X)) \to (K, \|\cdot\|)$ is universally measurable then this map is in class $D$ (the converse is also true and follows from Remark 3.1); by Lemma 3.1 this implies that there is a subset $S$ of $K$ having properties (1)–(3) of Corollary 3.1 when $K$ is equipped with its weak-star topology.

Remark 3.2. The proof of Lemma 3.2 is inspired by I. Namioka's proof of a result of C. Stegall on Asplund spaces. However I. Namioka's proof is based on the main result (a difficult one) of paper [13] while our proof is not; moreover, our proof makes it possible to get the result in [13] as a simple consequence.

**4. Some remarks and questions.** H. P. Rosenthal (see [11]) has shown that there exist a probability measure $\mu$ and a subspace $X$ of $Y = L^1(\mu)$ which is not WCG (and hence is not GSG). Furthermore, since a continuous image of an Eberlein compact is an Eberlein compact [1], the unit cell in $X^*$ is an Eberlein compact in its weak-star topology. Thus we conclude that there exist a Banach space $Y$ and its subspace $X$ with the following properties: (1) the space $Y$ is GSG, but $X$ is not GSG; (2) the unit cell of $X^*$ is a compact space of type RN in its weak-star topology. In particular, the converse of Proposition 2.4 is not valid. On the other hand, we know (Corollary 2.3) that if the unit cell in $X^*$ is a compact space of type RN in the topology $\sigma(X^*, X)$, then $X$ is a subspace of a GSG space. This leads to

PROBLEM 4.1. Let $X$ be a subspace of a GSG Banach space. Is the unit cell in $X^*$ a compact of type RN in its weak-star topology?



The Benyamini–Rudin–Wage theorem [1] on Eberlein compacts mentioned above leads to the following question:

PROBLEM 4.2. Let $K$ be a compact space of type RN and let $S$ be a continuous image of $K$. Is $S$ a compact space of type RN?

The next question we state here is inspired by Corollary 2.3 and by the following property of Eberlein compacts: if $K$ is a weak-star compact set in a dual space $X^*$ which is an Eberlein compact in its weak-star topology, then $\overline{\Gamma(K)}^*$ is also an Eberlein compact (indeed, then $\overline{\Gamma(K)}^*$ is contained in the continuous image of the Eberlein compact $(D(C^*(K)), \sigma(C^*(K), C(K)))$; hence $\overline{\Gamma(K)}^*$ is an Eberlein compact by [1]).

PROBLEM 4.3. Let $K$ be a weak-star compact set in a dual Banach space $X^*$ which is a compact space of type RN in its weak-star topology. Is $\overline{\Gamma(K)}^*$ a compact of type RN in the topology $\sigma(X^*, X)$?

We close the paper with the following remark. It is known that every separable Eberlein compact is metrizable. However, this fact is not true for compact spaces of type RN. In fact, in [13] it is shown that there exists a separable non-metrizable Hausdorff compact space $K$ such that $C^*(K) = l^1(K)$. Thus there exists a separable compact space of type RN which is not metrizable.

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# Ein Beitrag zur Lipschitz-Saturation im unendlichdimensionalen Fall

von

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**Abstract.** We study the problem of Lipschitz-saturation in the complex Banach analytic case. In codimension 1 relative Lipschitz-saturation and absolute Lipschitz-saturation are equal; the results of Stutz on equisaturation are also valid in the Banach case. We use only geometric methods and a simple power series calculus.

Von Ramis [4] wurde die Theorie der $C$-analytischen Teilmengen endlicher Definition von $C$-Banachmannigfaltigkeiten systematisch entwickelt. Schickhoff [6] hat die Geometrie dieser Ramisschen Mengen untersucht. Wir setzen in dieser Arbeit wie in [5] die Untersuchungen von Schickhoff fort; insbesondere benutzen wir die Bezeichnungen aus [5], [6].

Gegenstand der vorliegenden Arbeit ist die von Zariski [9], [10], sowie von Pham, Teissier [3] begründete Theorie der Saturation. Da wir die endlichdimensionalen algebraischen Methoden nicht verwenden können, folgen wir dem geometrischen Ansatz von Stutz [8].

In § 1 treffen wir Vorbereitungen; insbesondere führen wir die Begriffe der Lipschitz-Holomorphie, der Puiseux-Regularität und der Lipschitz-Regularität ein. Wie bei Stutz gehen wir dabei von der Situation aus, daß die Singularitätenmenge die Codimension 1 hat.

In § 2 führen wir aus, daß in geeigneten Fällenr elative und absolute Lipschitz-Holomorphie übereinstimmen. Daraus leiten wir Sätze über Lipschitz-Regularität ab, d.h. über die Existenz von Produktdarstellungen im Sinne der Lipschitz-Biholomorphie.

In § 3 beweisen wir, daß die in § 2 angegebenen hinreichenden Bedingungen für Lipschitz-Regularität auch notwendig sind. Unsere Ergebnisse sind also scharf.

In § 4 zeigen wir, daß in der von uns betrachteten Situation "fast immer" Lipschitz-Regularität vorliegt. Ferner erhalten wie eine Verallgemeinerung einer von Stutz bewiesenen Aussage über den $C_5$-Tangentenkegel.

Insgesamt stellen unsere Ergebnisse Verallgemeinerungen Stutz'scher Resultate für den unendlich-dimensionalen Fall dar. Wir können