Integral inequalities with weights for
the Hardy maximal function*

by

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Abstract. Necessary and sufficient conditions are obtained in order that inequalities of the form
\[ \int_{\mathbb{R}^n} \Phi(|Mf(x)|)w(x) \, dx \leq C \int_{\mathbb{R}^n} \Phi(|f(x)|)w(x) \, dx \]
hold, where \( Mf \) is the Hardy maximal function of \( f \) and \( \Phi \) is an appropriate Young's function. This result gives similar inequalities for the usual singular integral operators.

1. Our aim is to study weighted integral inequalities involving the maximal function operator \( M \) defined for Lebesgue-measurable \( f \) on \( \mathbb{R}^n \) by
\[ (Mf)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n; \]
as is always the case below, \( Q \) is a nondegenerate cube with sides parallel to the axes. More specifically, we extend to the context of Orlicz classes the result of B. Muckenhoupt, [4], for Lebesgue classes:
\[ \int_{\mathbb{R}^n} \left( |(Mf)(x)|^p w(x) \right)^{\frac{1}{p}} \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx, \]
\( p \) fixed, \( 1 < p < \infty \), and \( C \) independent of Lebesgue-measurable \( f \), if and only if \( w \) is in the class \( A_p \) of these weight functions for which
\[ \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{\frac{1}{p}} \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p(p-1)}} \, dx \right)^{p-1} \leq K, \]
for all cubes \( Q \).

The integral inequalities of interest to us are of the form
\[ \int_{\mathbb{R}^n} \Phi(|(Mf)(x)|)w(x) \, dx \leq C \int_{\mathbb{R}^n} \Phi(|f(x)|)w(x) \, dx. \]

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The Young's functions $\Phi(t)$ involved in is given by

$$\Phi(t) = \int_0^t \varphi(s) \, ds, \quad t > 0,$$

where $\varphi(u)$ is a non-decreasing function defined for $u > 0$ with $\varphi(0^+) = 0$. We require that $\Phi(t)$ satisfies the $\Delta_2$ condition

$$\Phi(2t) \leq B \Phi(t), \quad t > 0,$$

which is equivalent to the more general property

$$\Phi \bigl( t \Phi(t) \bigr) \leq B \Phi(t), \quad t > 0,$$

(with possibly different $B$). It is also important that the Young's function $\Psi(t) = \int_0^t \varphi^{-1}(s) \, ds$, complementary to $\Phi(t)$, obey the $\Delta_2$ condition. (Here $\varphi^{-1}(u) = \sup\{s: \varphi(s) \leq u\}$.) These restrictions ensure that $\lim \varphi(t) = \lim \varphi^{-1}(t) = 0$ and $\lim \varphi(t) = \lim \varphi^{-1}(t) = \infty$, and hence that these functions are equivalent to strictly monotonic ones. We will use the following properties of $\Phi(t)$ without explicit reference:

1. $\Phi(t)$ is essentially equal to $t \Psi(t)$,
2. $t \leq \Phi^{-1}(t) \Psi^{-1}(t) \leq 2t$.

The Orlicz space $L_\Phi = L_\Phi(w)$, $w(s)$ positive and locally-integrable on $\mathbb{R}^n$, consists of all Lebesgue-measurable functions on $\mathbb{R}^n$ for which there is a $K > 0$ such that

$$\int_{\mathbb{R}^n} \Phi \left( \frac{f(x)}{K} w(x) \right) dx \leq 1.$$

The norm of $f$ in $L_\Phi$ is the infimum over all such $K$. Under our restrictions on $\Phi(t)$ and $\Psi(t)$, the spaces $L_{\Phi}$ and $L_{\Psi}$ are mutually dual and, in particular, are reflexive.

Muszynszka and Orlicz, [3], have associated a pair of indices with a given $L_\Phi$. A generalization of these, or rather their reciprocals, has been given in the more general context of rearrangement invariant spaces in Boyd [1]. There, the upper and lower indices $\alpha$ and $\beta$ are defined by

$$\alpha = \inf_{s > 0} \frac{\ln \frac{h(s)}{s}}{\ln s}, \quad \beta = \sup_{s > 0} \frac{\ln \frac{h(s)}{s}}{\ln s},$$

where, for Orlicz spaces,

$$h(s) = \sup_{t > 0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(st)},$$

We refer to [1] for a complete discussion of their properties, some of which will be introduced below as needed. We just mention that for the $\Phi(t)$ we consider, $0 < \beta \leq \alpha < 1$; that in the case of Lebesgue spaces, $L_p$, when $\Phi(t) = t^p$, one has $\alpha = \beta = p^{-1}$.

We now state our main result.

**Theorem 1.** Let $w(s)$ be a positive, locally-integrable function on $\mathbb{R}^n$ and let $\Phi(t) = \int_0^t \varphi(u) \, du$ be a Young's function which, together with its complementary function $\Psi(t)$, satisfies the $\Delta_2$ condition. Then, in order that the inequality

$$(1) \quad \int_{\mathbb{R}^n} \Phi \left( \frac{f(x)}{1 + \varphi(x)} \right) w(x) dx \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{f(x)}{1 + \varphi(x)} \right) w(x) dx$$

be valid for $C$ independent of $f$, it is necessary and sufficient that either one of the following holds:

2. $w(s)$ is in the class $A_{\Phi}$; that is,

$$\int_{\mathbb{R}^n} \frac{w(s)}{|Q|^s} ds \leq K$$

for all cubes $Q$ and all $s > 0$;

3. $w(s)$ is in the class $A_{\Phi}$, where $p^1$ is the upper index of $L_\Phi$; that is

$$p^1 = \inf_{s > 0} \frac{\ln \frac{h(s)}{s}}{\ln s}, \quad h(s) = \sup_{t > 0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(st)}.$$

In §2 we show that (2) is necessary for (1), in §3 that (2) implies (3), which in turn is sufficient for (1).

Finally, arguments similar to those of [2], Theorem III, show that, given $w(s) \in A_{\Phi}$, integral inequalities of the form (1) hold for the usual singular integral operators. Indeed, for the Hilbert transformation, the condition $w(s) \in A_{\Phi}$ is also necessary.

2. It will be enough to obtain the condition of (2) with $e = 1$, provided that $K$ is seen to depend only on $C$. To begin, we claim there is a constant $C_1$ so that for all cubes $Q$ and all $e > 0$

$$(4) \quad \|w\|_{L_\Phi} \leq C_1,$$

Here $\| \cdot \|$ denotes the norm in $L_\Phi(\phi w)$; $\| \cdot \|$ the norm in $L_{\phi w}(\phi w)$. Firstly, our assumptions on $w(s)$ ensure that $0 < B < \infty$, where $B = \|w\|_{L_\Phi}$.
For $B = 0$ implies that the $L_1$ norm of $\mathcal{Q}^*$ is zero, which means, in turn, that $|Q| = 0$, making $Q$ a degenerate cube. Again, $B = \infty$ requires the existence of a nonnegative function $f$ in $L_1(Q)$ on $Q$ with $\int f(x)dx = \infty$. This forces $\mathcal{M}_f = \infty$ on $Q$, which isn't consistent with (1) if $Q$ is nondegenerate.

Next, the converse of Hölder's inequality allows us to choose a nonnegative function $f$ supported on $Q$, so that $|f| = 1$ and $\int f(x)dx = |\mathcal{Q}|$. Then, for $x \in Q$,

$$ \int \Phi(|\mathcal{Q}|^{|w(x)|} f(x))dx = C; $$

and so

$$ \int \Phi(|\mathcal{Q}|^{|w(x)|} f(x))dx = O \int \Phi(|\mathcal{Q}|^{|w(x)|} f(x)) = C; $$

that is,

$$ \Phi(|\mathcal{Q}|^{|w(x)|} f(x)) = C. $$

On taking $C_m = h(C^{-1})$, (4) follows.

Now, from the definition of $|\mathcal{Q}|^{|w(x)|}$ and (4),

$$ (C_m|\mathcal{Q}|^{|w(x)|} f(x)) = C \int \Phi(|\mathcal{Q}|^{|w(x)|} f(x))dx = C, $$

where $t \to C_m|\mathcal{Q}|^{|w(x)|} f(x)$ for all $t > 0$ and $C_m = C_m^{-1}$. Let $x > 0$ satisfy $C_m|\mathcal{Q}|^{|w(x)|}$.

Thus a $\epsilon$ exists such that the left hand side of the equation is a continuous function of $\epsilon$ which tends to infinity as $\epsilon \to 0$, and to zero as $\epsilon \to \infty$. Indeed, since $|\mathcal{Q}|^{|w(x)|} = 1/\Phi^{-1}(1/|\mathcal{Q}|)$,

$$ C_m|\mathcal{Q}|^{|w(x)|} = C_m|\mathcal{Q}|^{|w(x)|} f(x) = C_m|\mathcal{Q}|^{|w(x)|} f(x); $$

which means the desired $\epsilon$ is essentially equal to $|\mathcal{Q}|^{|w(x)|} f(x) = C_m|\mathcal{Q}|^{|w(x)|} f(x) = C_m|\mathcal{Q}|^{|w(x)|} f(x)$. We thus have, for some $B_1$ comparable to $B_1$,

$$ \int \Phi(|\mathcal{Q}|^{|w(x)|} f(x))dx = B_1|\mathcal{Q}|^{|w(x)|} f(x); $$

yielding (2) with $K = B_2B_1^{-1}$, where $B_1$ corresponds to $\Lambda = 2B_2C_1$ in the generalized $\Lambda$ condition for $\Phi(t)$.

3. In this section we prove that (2) implies (3) and that (3) suffices for (1). The former is a consequence of the three results proved below and the following interpolation criterion due to Stein and Weiss.

Suppose $T$ is a sublinear operator defined on functions $L^2$, $E$ a subset of $\mathcal{L}^2$ of finite Lebesgue measure, and that $w(x)$ is a nonnegative, locally-integrable function on $E$.

Suppose, further, $T$ is simultaneously of restric-
bounded above by
\[ \|x_{\lambda} \|_{L^p(\mathbb{R})} \leq C \Phi^{-1}(\|w(\mathbb{R})\|_p) \Phi^{-1}(\|E(\mathbb{R})\|_p). \]

The latter term, however, is less than 2C(w(E)\|w(Q))^{1/p}. For \( h_\delta(s) \geq s^{-1/p} \) when \( 0 < s < 1 \), this means that for fixed \( s < 1 \) there is a \( t > 0 \) with \( \Phi^{-1}(t) |\Phi^{-1}(st) > s^{-1/p} \) and so \( \Phi^{-1}(st) |\Phi^{-1}(t) < 2s^{1/p} \). Taking \( s = w(E)\|w(Q) \) and \( \epsilon = 1/\|w\|_p \) yields (6).

**Lemma 2.** Let \( \Phi \) and \( p \) be as in Theorem 1. For \( \delta > 0 \) define the Young's function \( \Phi_{\delta} \) by the equation
\[ \Phi^{-1}_{\delta}(t) = (\Phi^{-1}(t))^{1+\delta}. \]

Then, the upper index of \( L_{\Phi_{\delta}} \) is greater than \( p^{-1} \). Moreover, if \( w(s) \in A_\delta \), then \( w(s) \in A_{\delta} \) for all sufficiently small \( \delta \).

**Proof.** To prove the second assertion it will be enough to establish the condition of (2) for \( \Phi_{\delta} \), where \( s = 1 \), provided it is seen the \( C \) only on \( K \).

Set \( v(s) = \Phi^{-1}(1/\|w(s)\|_p) \). Then \( w(s) = 1/\|v\|_p \) and \( w(s) \in A_\delta \) implies
\[ \left( \frac{1}{\delta} \right) \left( \frac{1}{\|v\|_p} \right) \|w\|_p \leq K. \]

We show there exist \( \alpha, \beta, > 0 \), independent of \( Q \), so that for \( s \in Q \), \( v(s) > C\|\phi(Q)\|_p \) we have \( |E| > \beta\|Q\| \). On the complement of \( E \) in \( Q \), \( E' \), \( \psi(s) \leq C\|\phi(Q)\|_p \) and so, using (7),
\[ |E'|/\|\psi\|\phi\|Q\|_p \leq K/\|v\|_p \|Q\|_p. \]

Therefore,
\[ |E'|/\|Q\|_p \leq K \|\phi\|_p \|\phi\|_Q \|v\|_p \|Q\|_p \|v\|_Q \|Q\|_p. \]

As established below, given a fixed \( r < p \) there is an \( s_r \), \( 0 < s_r < 1 \), with
\[ \Phi(st) > s_r^{1-\delta} \Phi(t), \quad t > 0, \quad 0 < s < 1. \]

This will mean \( |E'|/\|Q\|_p \leq Ks_r^{1-\delta} \|Q\|_p \|Q\|_p \leq 1/2 \) for small \( a \).

Arguing as in [3], Theorem IV, we have the “reverse Hölder inequality”
\[ \left( \frac{1}{\|Q\|} \right) \int_\|Q\| \|v\|_{1/\|\phi\|_p} \|w\|_p \|Q\|_p \|Q\|_p \|Q\|_p \leq C_1 \left( \frac{1}{\|Q\|} \right) \int_\|Q\| \|v\|_{1/\|\phi\|_p} \|w\|_p \|Q\|_p \|Q\|_p \]
for all sufficiently small \( \delta \). Thus,
\[ \psi(1/\|Q\|) \int_\|Q\| \|v\|_{1/\|\phi\|_p} \|w\|_p \|Q\|_p \|Q\|_p \|Q\|_p \leq C_1 \left( \frac{1}{\|Q\|} \right) \int_\|Q\| \|v\|_{1/\|\phi\|_p} \|w\|_p \|Q\|_p \|Q\|_p \]
and the latter, by the generalized \( \Delta \) condition for \( \Phi \) and by (7), is no bigger than
\[ C_2 \|\phi\|_p \left( \frac{1}{\|Q\|} \right) \int_\|Q\| \|v\|_{1/\|\phi\|_p} \|w\|_p \|Q\|_p \|Q\|_p \|Q\|_p \leq C_1 \left( \frac{1}{\|Q\|} \right) \|w\|_p \|Q\|_p \|Q\|_p \]
with \( C = C_1 K \). Hence \( v(s) \in A_\delta \) for all small \( \delta \).

To see (8), observe that there exists \( s_1, 0 < s_1 < 1 \), with \( h_\delta(s) \leq s^{-1/p} \) when \( 0 < s < s_1 \). Thus
\[ \Phi^{-1}(t) |\Phi^{-1}(st) < s^{-1/p}, \quad t > 0, \quad 0 < s < s_1. \]

that is,
\[ \Phi(st) \leq s^\delta \Phi(t), \quad t > 0, \quad 0 < s < s_1. \]

For \( s_1 < s < 1 \), (8) follows from the fact that \( \Phi \) increases.

Finally, we show the upper index of \( L_{\Phi_{\delta}} \) is greater than \( p^{-1} \). By duality, it will be sufficient to prove the associate space, \( L_{\Phi_{\delta}} \), has lower index less than \( q^{-1} \), \( (p^{-1} + q^{-1} = 1) \), the lower index of \( L_q \). Given \( \epsilon > 0 \), we have, for fixed, sufficiently large \( s > 1 \),
\[ s^{-1/q} \leq h_{\Phi}(s) \leq s^{-1/q+\epsilon}, \]
where \( h_{\Phi}(s) = \sup_{t>0} \left( t^{-1/q} \right) t^{1/q+\epsilon} \). Thus,
\[ s^{-1/q} \leq \Phi^{-1}(t)/\|Q\|_p \leq \Phi^{-1}(st) \]
for some \( t > 0 \). With \( v = \Psi \) the former gives
\[ \Psi(s) = \Psi(t) \leq s \]
for \( s \geq 1 \) and \( \Psi(t) = 0 \) for \( t > s \). Indeed,
\[ \Psi(s) = \Psi(t) \leq \Psi(s) \leq \Psi(t) \]
for \( s < 1 \). Similarly,
\[ \Psi(s) = \Psi(t) \leq \Psi(s) \leq \Psi(t) \]
for \( s > 1 \).

From \( \Psi \) satisfying the \( \Delta \) condition we infer that \( \Psi \) essentially equals \( \Psi(t)^{-1} t^\epsilon \), and so the last inequality reads
\[ \Psi(s) = \Psi(t) \leq \Psi(t) \]
for fixed large \( s \) and all \( T > 0 \). Similarly,
\[ \Psi(s) = \Psi(t) \leq \Psi(t) \]
for fixed large \( s \) and all \( T > 0 \). This shows the lower index of \( L_{\Phi_{\delta}} \) must equal \( q^{-1}(1 + \delta p^{-1}) < q^{-1} \).
Finally, we establish the

Sufficiency of (3), from the well-known "openness" of the condition

for membership in \( A_\alpha \), \( w \) belongs to \( A_\alpha \), for some \( k \), with \( 1 < k < p \), where \( p^{-1} \) is the upper index of \( L_\alpha \). Then, for Lebesgue-measurable \( f \),

\[
\int \Phi\left( \frac{|mf(x)|w(x)}{\alpha} \right) \, dx \leq C \int \Phi\left( \frac{|f(x)|w(x)}{\alpha} \right) \, dx.
\]

Interchanging the order of integration gives

\[
\int \left( \int |f(x)|^\alpha w(x) \, dx \right)^{\frac{1}{\alpha}} \left( \int \Phi\left( \frac{|f(x)|w(x)}{\alpha} \right) \, dx \right)^{\frac{1}{\alpha}} \, dx.
\]

But, using (8) with, say, \( r = (\alpha + p)/2 \), means this is dominated by a constant multiple of \( \int \Phi\left( |f(x)|w(x) \right) \, dx \). This completes our proof.

References


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On the convergence of bilinear and quadratic forms in independent random variables

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Abstract. We consider bilinear and quadratic forms \( \sum a_{ij} X_i Y_j \) and \( \sum a_{ij} X_i X_j \) in independent random variables with expectations 0 and variances 1. Necessary and sufficient conditions for these forms to converge a.s. are given. When the \( X_i \) and \( Y_j \) are normal, we consider \( X - (X_i) \) and \( Y - (Y_j) \) as vectors in \( \mathbb{R}^N \) and ask when \( \sum a_{ij} X_i Y_j \) converges for \( X \) and \( Y \) in a subspace of \( \mathbb{R}^N \) of measure 1 for the distribution law. This is proved to happen precisely when the \( a_{ij} \) define a nuclear operator on \( \mathbb{R}^N \). The natural extension of this theorem to trilinear forms is shown to be false. An analogous result for stochastic integrals is also given.

1. Introduction and statements of results. In this paper all random variables and coefficients will be real-valued. However, our results extend to the complex-valued case with only small modifications. The probability measure is denoted by \( P \).

We shall say that a set of random variables \( X \) stays away from 0 if it contains no sequence tending to 0 in probability, i.e., if there is an \( \varepsilon > 0 \) such that \( P(|X| < \varepsilon) > \varepsilon \) for all \( X \) in the set.

Linear forms \( \sum a_i X_i \) have been considered by Hoffmann-Jorgensen [2], Th. 4.10. If the \( X_i \) are independent, stay away from 0, and satisfy \( E X_i^2 = 1 \) and \( E X_i^3 = 1 \), then the condition \( \sum a_i^2 < \infty \) is necessary and sufficient for \( \sum a_i X_i \) to converge a.s. Notice that this conclusion holds if and only if the \( X_i \) stay away from 0, when the other assumptions are satisfied.

For bilinear forms, several kinds of convergence exist. Call \( (M_k, N_k) \) an admissible sequence if each \( M_k \) and \( N_k \) is a natural number or \( \infty \), and \( M_k \) and \( N_k \) increase to \( \infty \) with \( k \), and both \( M_k \) and \( N_k \) are not \( \infty \). A bilinear form \( \sum a_{ij} X_i Y_j \) is said to converge for such a sequence if \( \sum a_{ij} X_i Y_j \) converges as \( k \to \infty \). Hoffmann-Jorgensen's techniques [2], pp. 153-156, are easily modified to give the following result.

Theorem 1. Let \( X_i \) and \( Y_j, i, j = 1, 2, \ldots, \) be independent, have expectation 0 and variance 1, and stay away from 0. Then the following are equivalent:

(a) \( \sum a_{ij} X_i Y_j \) converges;