in 1. That is, \((S^n s_0)\) is a basis for \(l^1\) which, as our proof shows, is similar to \((e_n)\).

Using essentially the same ideas we can prove:

**Theorem 4.2.** If \(s_0 = (1, a_1, a_2, \ldots) \in l^1\) and \(f(s) = 1 + a_1 s + a_2 s^2 + \ldots\) with \(f(s) \neq 0\) for all \(s\) in the unit disc \(|s| < 1\), then \((S^n s_0)\) is a basis for \(l^1\) which is similar to \((e_n)\).

**Proof.** As in the proof of Theorem 4.2, our assumptions imply that \(f^{-1}(s) = 1 + d_s s + \ldots\) for \((e_n)\) in \(l^1\). Therefore the operator \(Q = I + d_s s + \ldots\) is a bounded linear operator on \(l^1\) and clearly \(Q^* = Q = I\). That is, \(Q\) is invertible on \(l^1\) and since \(Q s_0 = s_0\), the theorem is proved.

**Corollary 4.3.** If \(s_0 \in l^1\), the following are equivalent:

(i) \((S^n s_0)\) is a basis for \(l^1\) similar to \((e_n)\).

(ii) \((S^n s_0)\) is a basis for \(l^1\).

(iii) The function \(f(s) = 1 + a_1 s + a_2 s^2 + \ldots\) has no zero on the disc \(|s| < 1\).

Finally we state without proof a more general version of Theorem 3.1 (for the case \(X = l^1\)) which settles the question of similarity of \((S^n s_0)\) and \((e_n)\) in \(l^1\).

**Theorem 4.4.** Let \(s_0 = (1, a_1, a_2, \ldots) \in l^1\) and \(f(s) = 1 + a_1 s + a_2 s^2 + \ldots\) Then \((S^n s_0)\) is a basis sequence in \(l^1\) which is similar to \((e_n)\) if and only if \(f(s) \neq 0\) for all \(s\) on the unit circle \(|s| = 1\).

The proof uses essentially the same ideas outlined above along with certain estimates on the norms of linear combinations of \((S^n s_0)\) and will be given in a subsequent paper devoted to more general problems in this area.

**References**


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**A refinement of the Helson–Szegö theorem and the determination of the extremal measures**

by

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**Abstract.** Let \(E_M\) be the set of measures on the unit circle which satisfy the M. Riesz inequality for the Hilbert transform with constant \(M\). \(E_M\) is determined by an associated class \(H_M\) of analytic functions. We give a geometric characterization of the elements of \(H_M\) and derive a refinement of the Helson–Szegö theorem. The extremal measures in the cone \(E_M\) are determined. Our basic result is the construction of a subfield of extremal measures by means of which every element in \(E_M\) can be naturally obtained.

**I. Introduction.** Let \(T\) denote the unit circle, \(f\) the Fourier transform of \(f \in L^1(\mathbb{T})\) and \(f\) its conjugate function. If \(M > 1\) is a fixed constant and \(\mu \geq 0\) a measure on \(T\), the Hilbert transform, we shall write \(\mu \in E_M\) if:

\[
\int_{\mathbb{T}} |f|^2 d\mu < \int_{\mathbb{T}} |f|\varphi^2 d\mu, \quad \forall f \in L^1(\mu) \cap L^2.
\]

Set \(E = \bigcup_{M>1} E_M\). Helson and Szegö proved [3] that \(\mu \in E_M\) if \(\mu\) is absolutely continuous with respect to Lebesgue measure, \(d\mu = \varphi(x) dx\), and

\[
\omega = \exp(\varphi(x) - \varphi(y)), \quad \varphi(x) \in L^2, \quad \|\mu\|_{L^2} < 2.
\]

Consequently, from now on we shall write \(\mu \in E_M\) if \(\mu \in E_M\). Cotlar and Sadosky proved [2] that \(\mu \in E_M\) iff there exists \(\alpha \in H^2(\mathbb{T})\) such that:

\[
\alpha - \alpha^2 - 2(\alpha^2 - 4\alpha^2) Re(\alpha) \rightarrow 0, \quad \text{a.e.}
\]

In this paper we study first those functions \(h\) that, by (I.3), characterize \(E_M\). Then we state a version of Helson–Szegö theorem for each \(E_M\) and, in particular, a simple proof of (I.2), deduced from (I.3). Finally we determine the extremal maps of the cone \(E_M\). Our basic result is:

**Theorem 1.** Let \(g = g_1 + ig_2 \neq 0\) be a function of \(H^2(\mathbb{T})\) with \(g(0) = 0\) and \(\tau(\varphi) = \langle \varphi, \varphi \rangle\), where

\[
\tau_0 = \arctan \left( \frac{|M-1|}{2M} \right), \quad \tau_0 = \arctan \left( \frac{|M+1|}{2M} \right), \quad \tau = \arctan \left( \frac{4M+1}{2M} \right).
\]

\(E_M = \{ \mu \in E_M : \mu = \mu \varphi \} \), where \(\varphi \in E_M\) and \(\varphi \neq 0\).
Then \( w = \exp(u + \bar{f}) \), where \( C \) is any positive constant, belongs to an extremal ray in \( E_M \). Let \( E_M \) be the set of rays obtained in this way: if \( w \in E_M \), log \( w \) is a weak star limit, in the duality of

\[
H_{\infty} = \{ f \in L^1; \hat{f} \in L^\infty, \hat{f}(0) = 0 \},
\]

of convex combinations of logarithms of elements of \( E_M \).

II. The class \( H_M \) of analytic functions. Let \( H_M \) be the set of all functions \( h \in H^2(T) \) that verify (I.3), that is,

\[
H_M = \{ h \in H^2(T); 3w_0 \leq 0, w \in L^1(T) \text{ and } h \text{ verifies } (I.3) \}.
\]

(II.a) Lemma. We have

\[
H_M = \{ h = h_1 + i h_2 \in H^2(T); h_1 \leq 0, |h_2| \leq (M - 1)|h_1|/2M^{1/2}, \text{ a.e.} \}
\]

Proof. If \( h \in H_M \) and \( h \neq 0 \), (I.3) says that for almost every \( z \) there exists a positive solution of

\[
-4Mw^2 - 2(M + 1)uw + |h(z)|^2 = 0,
\]

so \( h_1(z) = 0 \) and

\[
(M - 1)^2h_2(z)/4M \geq h_2(z).
\]

Reciprocally, if \( h = h_1 + i h_2 \in H^2(T) \) and \( h_1 \leq 0, (M - 1)^2h_2^2/4M \geq h_2^2 \), setting \( w = -(M + 1)h_1/4M \), it is clear that \( h \in H_M \), thus proving (II.a) and consequently that \( H_M \) is a convex cone.

Given \( h \in H_M \), we set:

(II.1) \( w_j = (4M)^{-1/2}((M + 1)h_1 - (M - 1)^2/2M^{1/2}) \),

and \( w_0 \) are the solutions of the second degree equation considered above, so we have

(II.b) Corollary. Given \( h \in H_M \), we and \( h \) verify (I.3) iff \( w_0 \leq w \leq w_2 \). In particular \( -\text{Re} h_M \) in \( H_M \).

So each \( h \in H_M \) defines a "band" of functions belonging to \( E_M \). We denote this band with \( W_{M,a} = \{ w; \text{measurable function}, w_0 \leq w \leq w_2 \}, \) where \( w_1, w_2 \) are given by (II.1). Therefore \( w \in E_M \) if \( 3h \in H_M \), so \( w \in W_{M,a} \). Now we consider the angle:

\[
S_M = \{(x, y) \in E; \pi < 0, |y| < (M - 1)|x|/2M^{1/2} \},
\]

(II.2) and

\[
H_M = \{ h \in H^2(T); h \in H_M \}.
\]

(II.c) Proposition. \( h \in H_M \) and \( h \neq 0 \) iff \( h \) is given by the boundary values of \( f \), an analytic function on the unit disc \( D \), such that \( f(D) \subset S_M \).

Proof. If \( h \in H_M \), \( h \in H^2(T) \). Set

\[
f(z) = (2\pi)^{-1/2} \int_0^{2\pi} P(r, u - b)h(t)d\tau, \quad r < 1,
\]

where \( P \) is the Poisson kernel; then

\[
h(\delta) = \lim f(\delta) \text{ a.e.}
\]

Being \( P \geq 0, (II.a) \) implies that \( f(D) \subset S_M \).

Reciprocally, let \( f \) be analytic on \( D \) and \( f(D) \subset S_M \). Set:

(II.3) \( \alpha(M) = \arg [(M - 1)/2M^{1/2}] \),

so the measure of the angle \( S_M \) equals \( 2\alpha(M) \), if \( f \in H^2(D) \), \( \forall p \in [0, \pi/2] \). (See (I.1). Since \( 2\alpha(M) < \pi, f \in H^2(D) \), then \( h(\delta) = \lim f(\delta) \) exists a.e. and \( h \in H^2(T) \). Clearly \( h \in H_M \).

In the above proof we have shown that \( h \in H_M \) implies \( h \in H^2 \), and \( \forall p \in [0, \pi/2] \).

Since \( w_0 \geq (\delta_2, \Pi (M)) \) says that

(II.d) Corollary. \( \forall h \in H_M, \exists \in \text{Lo}(T), \forall p \in (0, \pi/2] \), \( \alpha(M) \) given by (II.3).

Also, since \( |h_1| \leq 2M^{1/2} \), it is evident that

\[
W_{M,a} \subset \text{Lo}(T) \Rightarrow h \in \text{Lo}(T)
\]

\[
\Rightarrow \exists w \in W_{M,a} \text{ such that } w \in \text{Lo}(T).
\]

If \( w \in E_M \), there exists \( \varphi \in H^1(D), \forall p \in [0, \pi/2] \), such that \( w = |\varphi| \); in fact, if \( w \in E_M, 3h \in H_M \), such that \( w \in W_{M,a} \Rightarrow w \geq C|h|, C \), a constant, and \( |\arg h| \leq \pi/2 \) because \( h \in H^2 \Rightarrow \log |h| \in L^1 \); then \( \exists \varphi \in H^1 \) such that \( w = |\varphi| \); hence \( w \in E_M \). Finally \( \varphi \in H^1(D), \forall p \in [0, \pi/2] \).

(II.e) Corollary. \( \forall h \in H_M, h^{*} \in H_M \) and \( (W_{M,a})^{*} = 4M W_{M,a-1} \).

Proof. If \( h(D) \subset S_M \), then \( h^{-1}(D) \subset S_M \). It is easy to see that, with obvious notation, \( w_{1,a} = 4Mw_{1,a}/|h| \), \( 4Mw_{2,a-1} \), and \( w_{2,a} = 4Mw_{2,a-1} \), so \( (W_{M,a})^{*} \subset \{ w; w \in W_{M,a} \} = 4M W_{M,a-1} \).

Examples. (i) Let \( k \in [0, 2a(M)/\pi], f(x) = -(1 + e)^{1/2} \). Then \( f(D) \subset S_M \) and \( w_{0}(w) = [\cosh(k/2)]^{1/2} \cos(\cosh(2)/2) \{(M + 1) + (1 - e)^2/4M(2)^{1/2}/M \}^{1/2} \), \( j = 1, 2 \).

(ii) If \( h_1 = -2, h_2 = 0 \), then \( w_0 = M^{-1}, w_2 = 1, \) so it is a measurable function such that \( 0 < h_0 \leq h_2 \), with \( b/a \leq M \), then \( w \in E_M \).

If \( g: D \to D \) is analytic and

\[
P(z) = -(1 + g(z))/(1 - g(z)),
\]

\( a > 0 \), \( 0 < k \leq 2a(M)/\pi \), then \( f(D) \subset S_M \). By subordination, this is the general form of the functions in \( H_M \).

We shall say that \( h \in H_M \) is a "boundary function" in \( H_M \) if \( h(T) \subset \beta S_M \Rightarrow (M - 1)/2M^{1/2} = \bar{h}_2/4M \).
If \( F : D \to \{ \text{Re} z < 0 \} \) is analytic, then

\[
F(z) = \frac{1}{2\pi i} \int_C \frac{F(t)}{t - z} \, dt
\]

with \( |f(t)| \leq t \), \( t \in [0, 2\pi] \), so \( \arg F(z) = \pi(t)/2 \). Then every \( F \in H_M \) has a representation (II.4) with

\[
f \in L_M = \{ g : T \to \mathbb{R}, \quad |g(t)| \leq 2a(M)/|\xi|, \}
\]

convex and compact with the weak star topology of \( L^\infty \). The Krein–Milman theorem states that every \( g \in L_M \) can be approximated, in that topology, by convex combinations of extremal points of \( L_M \). Now, if \( F \) is extremal in \( L_M \), iff \( |f(t)| = 2a(M)/|\xi| \), that is, iff the \( F \) associated to \( f \) by (II.4) is a "boundary function" in \( H_M \). In this case, going from \( F \in H_M \) to the associated \( g \in L_M \), we can say that the "boundary functions" determine \( H_M \).

III. Propositions related to the Helson–Szegő theorem.

(III.a) Lemma. Let \( h \in H_M \) and \( v \in W_M \). Then \( v = e^{\text{exp}(u - \theta)} \), with

(i) \( u = \pi - \text{arg} h \), so \( |v|_{\infty} \leq e(\xi) \);
(ii) \( O = \beta(0)/2M^{12} \);
(iii) \( w = \log(2M^{13}|h|/|\xi|) \).

Let \( U_M(v) = \text{arch} \big( (M + 1)/2M^{12} \big) \). Then \( |u| \leq U_M(v),\ U_M(v) + : w = \log(w_1/\omega),\ U_M(v) - w = \log(w_1/\omega) \).

Proof. Since \( h(D) \in S(M) \), \( \arg h = (\theta - \omega)/(\xi) \) can be defined, with \( |\theta - \omega| = \xi |\omega| \leq a(M) \) and \( \theta = \log(\beta(0)/2M^{13}) \). From the definition (III.1) of \( u \), \( w \), we get

\[
\log(w_1/\omega) = \log \left( \frac{M + 1}{2M^{13}} \cos \left[ \frac{(M + 1)^2}{4M} \cos \theta - 1 \right] \right) - \log 2M^{12},
\]

so

\[
\log(2M^{13}w_1/|\omega|) = -U_M(v), \quad \log(2M^{13}w_1/|\omega|) = U_M(v).
\]

Then, setting \( v = \log(2M^{13}w_1/|\omega|) \), the result follows.

Note. Helson–Szegő's proof of (I.2) uses properties of analytic outer functions. The characterization (I.3) has been proved in a direct and elementary way; moreover, it is immediate that (I.2) implies (I.3) for some \( M [\xi] \). (III.a) states the reciprocity. So (III.a) with (I.3) gives a new proof of Helson–Szegő theorem. (See also (III.e) below.) As (III.a) rests only on (II.b), this proof is a simpler and more elementary one, because it does not use refined properties of \( H \).

(III.b) Lemma. Let \( w = e^{\text{exp}(u + \theta)}, \ C \) a positive constant, \( |u| \leq U_M(v), |\omega|_{\infty} \leq a(M) \).

Set

\[
h(z) = 2M^{13}e^{\text{exp} [(\pi - \theta)(0)] + \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\theta + \theta - \omega}}{e^{\theta - \theta - \omega}} v(t) \, dt}.
\]

Then

(i) \( h \in H_M \) and (on \( T \)) \( \text{arg} h = \pi - \theta \); (ii) \( w_1/w = \text{exp}(U_M(v)+u), \ w_2/w = \text{exp}(U_M(v)-u) \); (iii) \( w \in W_M \).

Proof. (i) From the definition of \( h \) it follows that

\[
\text{arg} h = \text{Im} \left\{ [\pi - \theta(0)] + \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\theta + \theta - \omega}}{e^{\theta - \theta - \omega}} v(t) \, dt \right\}
\]

so, on \( T \), \( \text{arg} h = \pi - \theta(0) + \theta = \pi - \theta = h \in H_M \).

(ii) On \( T \), \( |h| = 2M^{13}C \), so \( w = e^{\text{exp}(u)} 2M^{13} \). In the last proof we saw that \( w_j = e^{\text{exp}(U_M(v)+U_M(v))/2M^{13}}, j = 1,2 \).

(iii) Since \( U_M(v) \leq \omega \leq \omega \), (ii) says that \( w_1 \leq w \leq w_2 \) so \( w \in W_M \).

(III.c) Proposition. \( w \in R_M \) iff \( w = e^{\text{exp}(u + \theta)}, \ C \) a positive constant, \( |u| \leq U_M(v) \) and \( \text{Im} \{ \pi - \theta(0) \} = \text{Im} \{ \pi - \theta_0(0) \} \).

Proof. If \( w \in R_M \), \( 3 \in H_M \) such that \( w \in W_M \) \( (\text{III.a}) \) says that \( w \) is as stated. (III.b) proves the reciprocity.

(III.d) Corollary (Helson–Szegő theorem for \( R_M \)). \( w \in R_M \) iff \( w = e^{\text{exp}(u + \theta)}, \ w \leq U_M(v), \ |w|_{\infty} \leq a(M) \).

(III.e) Corollary (Helson–Szegő theorem). We have

\[
w \in R \supseteq \bigcup \{ R_M ; M > 1 \} \iff w = e^{\text{exp}(u + \theta)}, \ u, v \leq \omega, |w|_{\infty} \leq \pi/2.
\]

Proof. If \( w = e^{\text{exp}(u + \theta)}, \ |w|_{\infty} \leq \pi/2, \ u \leq \omega \), then \( 3 \in H_M \) such that \( \text{conv} \subseteq 2M^{13}[(M + 1)/2M^{12}], \text{ and } |u| \leq \text{arch}(2M^{12}) \), so \( \text{arch}(2M^{12}) \leq a(M) \text{ and } |w| \leq U_M(v) \); the result follows by (III.d), which also states the reciprocity.

IV. Characterization of the extremal rays of \( R_M \) in terms of \( H_M \).

\( w \in R_M \) defines an extremal ray of \( R_M \) iff \( w = (w^* + w)/2 \), \( w \) and \( w^* \in R_M \) imply \( w^* = C\omega, \ w^* = C\omega \), \( C \) positive constants. Set

\[
\text{Ext} R_M = \{ w \in R_M ; w \text{ belongs to an extremal ray in } R_M \}.
\]
Proposition. Let \( h \in H_M \), \( w \in W_{MA} \). If there exists \( A \subseteq T \), \( |A| > 0 \), such that in \( A \), \( w_i < w \iff w_j \), then \( w \notin \text{Ext} R_M \).

Proof. Fix \( \varepsilon > 0 \) such that in a set \( A \subseteq T \), \( |A| > 0 \), \( w_i < w \iff w_j \), \( w_i < w - \varepsilon \), \( w_j < w \).

Setting \( w' = w_{X_{T-A}} + (w - \varepsilon)_{X_A} \), \( w'' = w_{X_{T-A}} + (w + \varepsilon)_{X_A} \), the result follows.

Corollary. If \( w \in \text{Ext} R_M \), then \( \exists h = h_1 + h_2 \in H_M \) and \( A \subseteq T \) such that

\[
\omega = w_{X_A} + w_{X_{T-A}} = -((M+1)h_1 - (M - 1)h_2) + 4Mh_2 + 4Mh_1^2/3/4M_M.
\]

Proof. Let \( -f_M(w) \) be the first member of \((1,2,3)\) that is, \( f_M(w) = 4Mw^3 + 2(M+1)w + |h|\). Then \( f_M(w) - f_M(w) = 2M - Mw + 2h + h_1 \).

Now, \( 2Mw + h_1 \leq 0 \); moreover, \( 2Mw + h_1 < 0 \iff h_1 < 0 \). Consequently, \( f_M(w) \leq 0 \), the inequality being strict iff \( h_1 < 0 \). Since \( f_M(w) \) is a second degree polynomial with two real roots, \( w_M1, w_M2, \) it follows that \( w_M1 < w_M2 \), with strict inequality iff \( h_1 < 0 \). In the same way we can prove the result concerning \( w_{X_A} \) and \( w_{X_{T-A}} \).

Corollary. \( \text{Ext} R_M = B_M - \bigcup_{1 < M < M_M} B_M \).

Proof. Let \( h', h'' \in H_M \), \( 0 < h < 1 \), and \( h = h' + (1-t)h'' \), then

\[
w_i < tw'_i + (1-t)w''_i \iff tw'_i + (1-t)w''_i < w_i,
\]
equalities hold iff \( h'/h'' \) is real. So \( W_M = \{w_{X_M} : tw'_i + (1-t)w''_i < w_i\} \).

Proof. We want to compare \( W_M \) with \( W_{MA} = \{w_{X_M} : tw'_i + (1-t)w''_i < w_i\} \).

Set \( A = \{t((M-1)h'_2 - 4Mh''_2) + (1-t)((M-1)h'_2 - 1)2Mh''_2 + 2t((M-1)h'_2 - 4Mh''_2)\} \).

Then

\[
w_i - [tw'_i + (1-t)w''_i] = (A - B)/4M_M,
\]

with \( A, B > 0 \). Set

\[
A = (M-1)h'_2 - 4Mh''_2 > 0,
D = ((M-1)h'_2 - 4Mh''_2)^2 + (M-1)h'_2 - 4Mh''_2 > 0.
\]

Then

\[
A^2 - B^2 = 2t(1-t)(C - D),
\]

and

\[
C - D = 4M((M-1)h'_2 - 4Mh''_2)^2.
\]

Consequently \( w_i \geq tw'_i + (1-t)w''_i \), and equality holds only when \( h'_2 - 4Mh''_2 = 0 \). Analogously,

\[
w_i - [tw'_i + (1-t)w''_i] = -(A - B)/4M_M.
\]

Now suppose \( h'/h'' \) is real a.e. on \( T \). Consider the analytic function \( f = h'/h'' \); then \( f(D) = S = \{x : \text{arg} s < b\} \) for some \( b < \pi \) because \( h', h'' \in H_M \). Set \( F(s) = (1 + s)^{-1} t \) such that \( F(D) = S \); by subordination, \( F = F', g, D \rightarrow D \), analytic. Now \( f(e^{st}) \in E \) a.e. on \( T \rightarrow g(e^{st}) \) is constant. So we have the following

Lemma. If \( h', h'' \in H_M \) and \( h'/h'' \) is real a.e., then \( h'/h'' \) is constant (and positive).

Note. The above lemma implies that if \( h \) is a "boundary function" of \( H_M \) then it belongs to an extremal ray of the cone \( H_M \). In fact, if \( h = h' + (1-t)h'' \), \( h', h'' \in H_M \), it is clear that, on \( T \), \( \text{arg} h = \text{arg} h' \), so \( h'/h'' \) is real a.e. (\( \omega \)).

Proposition. Let \( h', h'' \in H_M \), \( 0 < h < 1 \), and \( h = th' + (1-t)h'' \). If \( h' \neq \text{Ch}' \), \( C \) a positive constant, then

\((i)\) \( W_{MA} = \{w_{X_M} : tw'_i + (1-t)w''_i < w_i\} \), \( w_i \leq tw'_i + (1-t)w''_i \leq w_i \).

(II) \( W_{MA} = \{w_{X_M} : tw'_i + (1-t)w''_i < w_i\} \) is a set of positive measure in \( T \).

Corollary. Under the hypothesis of (IV,g), if \( w = w_{X_A} + w_{X_{T-A}} \), then \( w \notin W_{MA} = \{w_{X_M} : tw'_i + (1-t)w''_i < w_i\} \).

Lemma. If \( w \in \text{Ext} R_M \), \( w_0, w \in R_M \), \( w_0, w \in R_M \), not belonging to the same ray and such that \( w = (w_0 + w')/2 \).

Then \( w', w'' \in H_M \) be such that \( w' \in W_{MA} \), \( w'' \in W_{MA} \) such that \( w = (w' + w'')/2 \).

Let \( h', h'' \in H_M \) such that \( w \in W_{MA} \), \( w' \in W_{MA} \), when \( w \in \hat{W}_{MA} \) and \( h' \neq h'' \). If \( w \in W_{MA} \), \( h' \neq h'' \), \( w' = w_{X_{T-A}} + w_{X_{T-A}} \), then \( w' = w_{X_{T-A}} + w_{X_{T-A}} \), \( w'' = w_{X_{T-A}} + w_{X_{T-A}} \), \( w' = w_{X_{T-A}} + w_{X_{T-A}} \), \( w'' = w_{X_{T-A}} + w_{X_{T-A}} \). Then \( w' = w'' \), \( w' = w'' \), \( w'' = w'' \), \( w'' = w'' \), which is not possible. But if \( h'/h'' \) is not a real constant, \( \omega \) states that \( w = w_{X_A} + w_{X_{T-A}} \) is impossible. The result follows.

The last lemma and (IV,a) prove the following

Proposition. Let \( w \in \text{Ext} R_M \). Then \( w \in \text{Ext} R_M \) if \( w \in W_{MA} \) implies \( w = w_{X_A} + w_{X_{T-A}} \).
That is, \( w \) belongs to an extremal ray if and only if it belongs to a "band", it equals the upper function of the band in some subset of \( T \) and, in its complement, it equals the lower function.

**Definition.** For every \( w \in E_M \) let

\[
H_{M,w} = \{ k \in H_M : w \in W_{M,k} \}.
\]

If \( 0 < t < 1 \), then

\[
|k + (1-t)k'|^2 = t(k_1^2 + k_2^2) + (1-t)(k_1'^2 + k_2'^2) + 2t(1-t)(k_1 k_1' + k_2 k_2') - t(k_1 h_1 + k_2 h_2)^2,
\]

so:

\[
|k + (1-t)k'|^2 \leq t|k|^2 + (1-t)|k'|^2,
\]

and equality holds only when \( k = k' \), (IV.2). This relation shows that:

**IV.2 Proposition.** The set \( H_{M,w} \) is convex.

**IV.2.** Proposition. Let \( h \in H_{M,w} \) and \( w = u_{k,x} + w_{k,x_T - A} \). Then \( h \) is an extremal point of the convex set \( H_{M,w} \).

Proof. Suppose \( h = (k + h')/2 \), \( k', h'' \in H_{M,w} \), \( k \neq h'' \). The hypotheses on \( w \) says that \( 0 = -4Mw_3w - 2(M - 1)w - |h|^2, \) a.e. (IV.2) ensures that \( 3A \subset T, |A| > 0 \), such that in \( A \):

\[
-4Mw_3w - 2(M - 1)w - |h|^2 \geq -4Mw_3w - 2(M - 1)w - |h|^2 + |h''|^2/2 > 0 > 4Mw_3w - 2(M - 1)w - |h''|^2/2,
\]

which contradicts \( k', h'' \in H_{M,w} \).

**IV.2.** Corollary. If \( w \in E_{M,w} \), then \( H_{M,w} \) contains only one element.

**Proof.** (IV.2) and (IV.2) state that every element in \( H_{M,w} \) is an extremal point. In this way we get a first characterization of the extremal rays of \( E_M \).

**IV.2.** Theorem 2. The following conditions are equivalent:

(i) \( w \in E_{M,w} \);

(ii) there exists one and only one \( h \in H_M \) such that \( w \in W_{M,k} \) and

\[
w = u_{k,x} + w_{k,x_T - A} = -2(M - 1)h + (k_1^2 + k_2^2)/4M, \text{ a.e.}
\]

Proof. (IV.2) and (IV.2) say that (i) implies (ii); (IV.2) proves the reciprocal.

**V. Characterization of the extremal rays of \( E_M \) by means of Helson–Szegö theorem.**

**V.1.** Lemma. Let \( w \in E_{M,w} \). Then there exists only one \( v \) such that

(i) \( |v|_M \leq a(M) \);

(ii) \( v = Ce^{i\arg h} \), with \( C \) a positive constant and \( |w| \leq U_M(v) \).

**Proof.** There exists at least one such \( v \) because of (III.5). Lemma (III.6) says that \( w \in W_{M,A} \), with \( v = -\arg h \). Since \( w \in E_{M,w} \), (IV.2) shows that \( h \), and consequently \( v \), is well determined.

**V.1.** Lemma. Under the same hypothesis and with the same notation as in (V.1), it must be \( |w| = U_M(v) \).

**Proof.** (IV.2) shows that \( w = u_k \) or \( w = w_3 \) a.e., so (III.5) implies, respectively, \( U_M(v) = u_k \) or \( U_M(v) = w_3 = 0 \).

**V.1.** Lemma. Under the same hypothesis and with the same notation as in (V.1), \( w(v) \) is also unique.

**Proof.** If \( w = C \exp(u_k + \bar{v}) = C \exp(v_3 + \bar{v}) \), then \( u_k = v_3 + \bar{v} \), \( k \) a real constant. Considering (V.1), we can see that \( |w_k| = |w_3| = u_k \) is constant \( = \tau \) equals a.e. a constant \( t \).

Suppose \( t \leq a(M) = \frac{\pi}{\bar{C} \exp(u_k + \bar{v})} \). Then \( w = C \exp(u_k + \bar{v}) = C \exp(v_3 + \bar{v}) \) with \( u_k = v_3 \), \( C \) \( = C \exp(u_k + \bar{v}) \), and \( |v_3| \leq a(M) \). (III.5) says that \( w \in E_{M,w} \) and (IV.2) that \( w \) would not belong to \( E_{M,w} \). Then, it must be \( w = a(M) \) a.e. \( = U_M(v) = 0 \) a.e. \( \Rightarrow w = u_k \).

**V.1.** Definition. A (measurable) function has property \( (E_M) \) if the following conditions are satisfied.

**E.1.** There exists one and only one pair \( (v, w) \) such that

(i) \( |v|_M \leq a(M) \);

(ii) \( |w| \leq U_M(v) \);

(iii) \( v = C e^{i\arg h} \), for a (well determined) positive constant \( C \).

**E.2.** \( |w| = U_M(v) \).

What we have proved up to now is that \( w \in E_{M,w} \Rightarrow w \) has property \( (E_M) \).

Reciprocally:

**V.2.** Lemma. If \( w \) has property \( (E_M) \), then \( w \in E_{M,w} \).

**Proof.** \( w \in E_{M,w} \) because of (III.5). Lemma (III.6) says that \( w \in W_{M,A} \) with \( h \) defined above. Suppose \( k \in H_M \) is such that \( w \in W_{M,k} \); then (III.5) states that \( w = C \exp(v_3 + \bar{v}) \), with \( v_3 = -\arg h' \), \( C' = |C|/2M^{1/2} \), and \( |v_3| \leq a(M) \), \( |w| \leq U_M(v) \). So, by (E.2), \( v = v_3 \), \( w = w_3 \Rightarrow C = C' \). From \( \arg h = \pi - \pi = \pi - \bar{\arg h} \) we see that \( h' \) is real a.e. on \( T \); then (IV.2) shows that \( h' = h \), \( h \) a positive constant. Consequently \( h = |C|/|C'| \Rightarrow C = C' \). So there is only one \( h \in H_{M,w} \) such that \( w \in W_{M,A} \).

**E.1.** \( (E_M) \) says that \( |w| = U_M(v) \); by (III.6) \( w = u_k \) or \( w = w_3 \) a.e. Considering (IV.2), the result follows.

Summing up we have

**V.2.** Theorem 3. \( w \in E_{M,w} \Rightarrow w \) has property \( (E_M) \).
VI. The set $P_M$.

(VI.a) Definitions. Set $G(v, u) = u + \varepsilon$, $P_M = \{(v, u) \in L^0 \times L^0 : ||u||_\infty \leq \alpha(M), |u| \leq U_M(v)\}$, $L_M = G(P_M)$.

(VI.b) Lemma. $L_M$ is a strictly concave function.

From the lemma, considering that $U_M$ is a decreasing function in $[0, \alpha(M)]$, the following result follows.

(VI.c) Proposition. $P_M$ is convex, symmetric and absorbent. In the norm topology of $L^0 \times L^0$, $P_M$ is the same as the closure of its interior.

(VI.d) Definitions. Let $L_M$ and $\hat{L}_M$ be the "cylinders" given by

$$\hat{P}_M = \{(v, u) + (0, \varepsilon) : (v, u) \in L_M, \varepsilon \geq R\},$$

$$\hat{L}_M = G(\hat{P}_M) = L_M + R.$$

Let us consider the sets $\text{Ext}\hat{P}_M$ and $\text{Ext}\hat{L}_M$ of extremal generators of these cylinders and the space $L = \{v = u + \varepsilon, \varepsilon \geq R, v, u \in L^0\}$, where functions that differ in a constant are identified. Then $\text{Ext}\hat{P}_M$ is just the set of extreme points of $L_M$, a convex subset of $L$.

(VI.e) Proposition. $\hat{L}_M$ is a weak star compact and convex subset of $L$.

Proof. $L$ is the dual of $H_{\text{rep}}$ with a norm equivalent to the one given by

$$||\varphi||_{L^0} = \inf ||\varphi||_{L^0} = \{\varphi \in L^0 : \varphi = \varepsilon + u + \varepsilon, \varepsilon \geq R, v + \varepsilon \in L^0\}.$$  

In this norm $\hat{L}_M$ is bounded, because $\varphi \in \hat{L}_M$ implies $||\varphi||_{L^0} \leq U_M(0) + \alpha(M)$. Let us now prove that $\hat{L}_M$ is closed. If $\varphi$ belongs to the norm closure of $L_M$, for each natural number $n$ may be written as $\varphi = \varepsilon_n + u_n + \varepsilon_n$, with $\varepsilon_n \geq R$, $(v_n, u_n) \in L_M$, $\lim ||\varepsilon_n||_\infty = 0$. Considering eventually subsequences, we may assume that $\varepsilon_n \rightarrow v$ in the weak star topology of $L_0$ to $v, u$, respectively. Set $p_n = u_n + \varepsilon_n$ clearly, $\varphi_n = \varphi_n(u_n, v_n), v_n = 0$, so $\varphi$ and $\varphi_n$ represent the same element of $L$.

Obviously, $||\varphi||_{L^0} \leq \alpha(M)$; moreover, in a set $A \subset L$ such that $|A| = 0$ the following inequalities hold: $|v| \leq \lim ||u_n||_\infty, |u| \leq \lim ||u_n||_\infty$. Since $U_M$ is a decreasing function in $[0, \alpha(M)]$, in every point of $A$ we have: $U_M(v) \geq U_M(\lim ||u_n||_\infty) = \lim U_M(v_n) \geq \lim ||u_n||_\infty \geq |u|$. Consequently, $(v, u) \in L_M$ and so $\varphi \in \hat{L}_M$. Considering the theorem of Bourbaki–Alaoglu, the proof is over.

(VI.f) Corollary. Let $(v, u) \in P_M$; then $(v, u)$ is an extremal point of the convex set $P_M$ iff $|(v, u)| = U_M(\varepsilon, |v|), a.e.$

Proof. If $|(v, u)| < U_M(v)$ in a set of positive measure it is easy to construct $u_1, u_2$ such that $u = (u_1 + u_2)/\beta, (u_1 - u)$ and $(u_2 - u)$ are not constant and a fortiori not zero and $|u_1|, |u_2| \leq U_M(v)$. Then $(v, u) = \frac{1}{\beta}(v, u_1) + \frac{1}{\beta}(v, u_2)$ and $v \neq U_M(v)$ in $P_M$, so $(v, u)$ is not an extremal point of $P_M$. This proves that $(v, u) \notin \text{Ext}\hat{P}_M$.

Reversely, suppose $|(v, u)| = U_M(v) \ a.e.$ and $(v, u) = \frac{1}{\beta}(v', u') + \frac{1}{\beta}(v'', u'')$ in $P_M$. Then $|(v, u)| = U_M(v) \geq \frac{1}{\beta}U_M(v') + \frac{1}{\beta}U_M(v'') \geq \max(|v'|, |v''|)/\beta = |v'| \ a.e.$ Since $U_M$ is strictly concave, we must have $v = v' = u''$ to get equality in the first inequality, hence $(v, u) = (v', u') \ a.e.$, and we must have $u = u' = u'' \ a.e.$ to get equality in all the inequalities.

(VI.g) Proposition. Let $(v, u) \in P_M$; then the following conditions (a) and (b) are equivalent:

(a) $(v, u) \in \text{Ext}\hat{P}_M$.

(b) $|(v, u)| = U_M(v)$ and at least one of the following conditions is satisfied:

(i) $u$ changes its sign,

(ii) $v = 0$.

Proof. (a) $\Rightarrow$ (b): We saw at the beginning of the proof of (VI.f) that (b) implies $|(v, u)| = U_M(v)$. Suppose that (i) and (ii) are both false; then we may assume that $u \geq 0$ and that $|v'| > 0, 1 < |v| < 2$.

Set $u_1 = u = U_M[a(M')]$ then $u_1 \geq 0$ and $u_1 < u$, so $(v, u_1) \notin P_M$ and $(v, u_1)$ is not an extremal point of $P_M = (v, u_1) \notin \text{Ext}\hat{P}_M$. The result follows.

(b) $\Rightarrow$ (a): Let $(v', u'), (v'', u'') \in P_M$ and $\gamma$ a constant such that $v = v' + v''$, $u = u' + u''/2 + \gamma$. In the set $A = \{u \geq 0\}$ we have:

$$u = U_M(v) \geq \frac{1}{2}U_M(v') + \frac{1}{2}U_M(v'') \geq \frac{1}{2}|v'| + \frac{1}{2}|v''| \geq \frac{1}{2}(u' + u'')/2;$$

so, if $|A| > 0$, $\gamma \geq 0$. Analogously, if $|(u < 0)| > 0$, $\gamma \leq 0$. Then, if $(b_1)$ holds, $0 = v = v' + v'', u = u' - u''$, because $|u| = U_M(v)$ ensures that $(v, u)$ is an extremal point of $P_M$. But $(b_1)$ does not hold, then $||u||_\infty = \alpha(M)$; for each natural $n$, $\exists \beta_n \in T$ such that $|\beta_n| > 0$ and, in $\beta_n$,

$$|v| \geq \alpha(M) - 1/n \Rightarrow |v'|, |v''| \geq \alpha(M) - 2/n = |u''| - |u'| \geq 2U_M(a(M')) - 2/n;$$

consequently, $|v| = 2U_M(a(M)) - 2/n \rightarrow 0$, and the result follows.

(VI.h) Corollary. Let $(v, u) \in P_M$ be such that $\text{exp}(u + \varepsilon) \in \text{Ext}\hat{P}_M$.

Then $(v, u) \in \text{Ext}\hat{P}_M$.

Proof. Suppose the statement is false. Since (V.h) says that $|(v, u)| = U_M(v^2)$, both $(b_1)$ and $(b_2)$, in (VI.g), must be false. So we may assume that $u \geq 0$ and $||u||_\infty < \alpha(M) = 3M' = 1, M' < M$, such that $||u||_\infty < \alpha(M')$. Set $u_1 = u - U_M[a(M')]$ and we shall see that $|u_1| \leq U_M(v)$ which is equivalent to

$$U_M(v) - U_M(a(M')) \leq U_M(v^2) \Rightarrow \text{arch}(M + 1) \leq \text{arch}(M + 1) + \text{arch}(2M'/2M'^2);$$

setting

$$\theta = (M + 1)(2M/2M'^2) \Rightarrow \text{arch}(M + 1) + \text{arch}(2M'/2M'^2) \leq \theta.$$
the last inequality is true because \( \operatorname{arch}(a) - \operatorname{arch}(t) \leq \operatorname{arch} a, \forall t \geq 1. \) So \( |v| \leq U_{M} (v) \) and

\[
\begin{align*}
|v_{w} &= c(M) \Rightarrow (v, u_{1}) \in E_{M} \Rightarrow e^{\nu}_{w} u_{1} = e^{\nu}_{w} u_{1} e^{\nu}_{w} R_{M} \\
&= e^{\nu}_{w} \notin \mathbb{R} M.
\end{align*}
\]

Now we can state the relation between the extremal generators of these cylinders and the extremal rays of \( E_{M} \).

(VI.1) Proposition. The following conditions are equivalent:

(a) \( f \in \operatorname{Ext} L_{M} \);

(b) \( e^{\nu} \notin \operatorname{Ext} E_{M} \);

(c) \( e^{-1}(f) \cap \mathbb{R} M \) contains only one element that belongs to \( \operatorname{Ext} E_{M} \).

Proof. (a) \( \Rightarrow \) (b): Let \( f = \psi (v_{1}, u_{2} + \nu), (v_{1}, u_{2}) \in E_{M} \), \( C \) a constant; (III.b) says that \( w = e^{\nu} \notin \operatorname{Ext} E_{M} \). Suppose \( w \notin \operatorname{Ext} E_{M} \); then (IV.j) ensures that \( z \in H_{M} \) and \( A \subset T \) such that \( w \in W_{M, A}, |A| > 0 \) and, in \( A \), \( w_{1} < w < w_{2} \); so \( B \subset T \) and \( a > 1 \) such that \( |B| > 0 \) and, in \( B \), \( w_{1} \leq w_{2} / a \). Set \( w' = w_{x_{2} - B} + \frac{1}{a} w_{y_{2} B} \), \( w'' = w_{x_{2} - B} + \frac{1}{a} w_{y_{2} B} \); then \( w', w'' \in W_{M, A} \) and \( w_{2} = w'r'' \). Then (III.a) shows that

\[
w = \exp (u_{1} + \nu), \quad w' = \exp (u_{1} + \nu), \quad w'' = \exp (w'' + \nu),
\]

\[
(v, u_{1}), (v', u_{1}) \in E_{M} \text{ from } w = w' = w'' \text{ we get } u_{1} = (u_{1} + \nu)2 + + (w'' + \nu)2 \text{ since } w'/w \text{ is not a constant, } (u_{1} + \nu) - (u_{1} + \nu) = 0 \text{ not one either, so } (u_{1} + \nu) \notin \operatorname{Ext} E_{M} \Rightarrow f \notin \operatorname{Ext} E_{M} \).
\]

(b) \( \Rightarrow \) (c): Suppose \( (v, u_{1}) \in E_{M} \) and \( f = k + u_{1} = k' + u_{1} + e^{\nu}, k' \) and \( h' \) real constants; then \( w = e^{\nu} = e^{\nu} + e^{h'} = e^{h'} \), \( C \) and \( C' \) positive constants.

Let \( h \) be defined in terms of \( e \) and \( C \) as in (III.b) and, in the same way, \( k \) by means of \( v' \) and \( C' \); then \( w \in W_{M, W_{M, A}} \). Since \( w \in \operatorname{Ext} E_{M} \) (IV.n) states that \( k = k' \), so \( v = v' \), \( C = C' \) and, consequently, \( w = w' \). Thus \( e^{\nu}(f) \cap \mathbb{R} M \) contains only one element, \( (v, u_{1} + \nu) \); since \( e^{\nu} = e^{\nu} \cap \mathbb{R} M \) shows that \( (v, u_{1} + \nu) \in \operatorname{Ext} E_{M} \).

(c) \( \Rightarrow \) (a): Let \( f = f(u' + \nu) / 2 \), \( (v', u') \in E_{M} \), \( k = (v' + \nu)2 \); since \( ((v' + \nu)2 / 2, (v' + \nu)2 / 2) \in E_{M} \), \( (k + (v' + \nu)2 / 2, (v' + \nu)2 / 2) \) is the only element in \( e^{\nu}(f) \cap \mathbb{R} M \), in order that it belong to an extremal generator, it is necessary that \( v' = v', u' = u' \); \( u' = u' \), so \( f \in \operatorname{Ext} E_{M} \), and the proof is over.

Let \( N \) be the kernel of \( f \). Then following is the basic result of this section.

(VI.1) Theorem 4. Let \( w \in \mathbb{R} M \); then (a) and (b) are equivalent:

(a) \( w \in \operatorname{Ext} E_{M} \);

(b) \( w = \exp (u_{1} + \nu), \quad \nu \) a positive constant, \( (v_{1}, u_{2}) \in E_{M}, |u_{2}| = U_{M}(u_{1}), (v_{1}, u_{2} + \nu) \cap \mathbb{R} M = (v_{1}, u_{2}) \) and at least one of the following conditions is satisfied: (I) \( \nu \) changes its sign, (II) \( |u_{2}| = a(M) \).

Proof. (a) \( \Rightarrow \) (b): (I) says that \( w = \exp (u_{1} + \nu) \), with \( (v_{1}, u_{2}) \in E_{M} \). Let \( f = u_{1} + \nu \); (II) states that \( (v_{1}, u_{2} + \nu) \cap \mathbb{R} M \) contains only one element, evidently \( (v_{1}, u_{2}) \). Since \( (v_{1}, u_{2}) \in \operatorname{Ext} E_{M}, \) (VI.1) finishes this part of the proof.

(b) \( \Rightarrow \) (a): (VII.1) Shows that \( (v_{1}, u_{2}) \in \operatorname{Ext} E_{M} \), so the result follows from (VII.1).

(VII.1) Proof of the main theorem. The preceding characterizations of the extremal rays are not constructive now we shall construct explicitly a subset \( E_{M} \subset \operatorname{Ext} E_{M} \) such that, in the sense specified in Theorem 1, every \( v \in E_{M} \) can be obtained by means of elements of \( E_{M} \).

Let \( K_{1} \) be the interior of \( E_{M} \) in the norm topology of \( L_{x} \times L_{y} \) and \( K_{2} = (v_{1}, u_{2}) + N \), where \( (v_{1}, u_{2}) \) is as in (VII.1) and \( N = \{ v' - \nu; v' \in L_{x} \} \). Considering (VII.1), a well known corollary of the Hahn--Banach theorem ensures the existence of an hyperplane \( H \) that separates \( K_{1} \) and \( K_{2} \). Since \( K_{1} \) is open, \( K_{1} \cap H = \emptyset \), so \( H \) is not dense; consequently, it is not difficult to prove the following result.

(VII.1.1) Proposition. Let \( (v_{1}, u_{2}) \) be as in (VII.1). Then there exists \( \nu, \mu, \) belonging to the topological dual of \( L_{x} \) such that:

(i) \[ \int_{v_{1}} (d_{1}u + \nu d_{1}) = 0, \quad \forall u \geq 0, \]

(ii) \[ 1 - \int_{v_{1}} (d_{1}u + \nu d_{1}) \geq \int_{v_{1}} (d_{1}u + \nu d_{1}), \quad \forall (v, u) \in E_{M}. \]

It seems reasonable to suppose that \( (v_{1}, u_{2}) \) will have some special properties when the functional can be represented by functions. So we set the following.

(VII.1.2) Definition. \( (v_{1}, u_{2}) \) as in (VII.1) belongs to the set \( E_{M} \) if there exists \( f_{1}, f_{2} \in L_{x} \) such that \( d_{1}u = f_{1}d_{1}, d_{2}u = f_{2}d_{2} \), satisfy the assertion of Proposition (VII.1). In the way to prove the first part of the main theorem, it will be shown that \( (v_{1}, u_{2}) \in E_{M} \) iff \( (v_{1}, u_{2}) \in (g \in H_{M}, g_{1} > 0), \) where \( g \) as defined in Section I. (VII.1.a) and (VII.1.b) say that:

(VII.1.3) Proposition. Let \( (v_{1}, u_{2}) \) be as in (VII.1). Then \( f_{1} = (f_{1}, f_{2}) \) such that \( -f_{1} - f_{1} = H_{M}, f_{1} = f_{1} = 0 \), \( f_{1} > 0 \), \( f_{2} = 0 \).

(VII.1.4) \[ \int f_{1}d_{1}u + \int f_{2}d_{2}u \leq \int f_{1}v_{1} + \int f_{2}u_{2}, \quad \forall (v, u) \in E_{M}. \]

Proof. Since \( E_{M} \) is convex and absorbs, \( N \) is a closed subspace and by (VII.1) \( \{ (v_{1}, u_{2}) \cap E_{M} = (v_{1}, u_{2}) \), the Hahn--Banach theorem states that \( f \in (L_{x} \times L_{y})' \) such that \( f(M) = 0 \) and \( f((v_{1}, u_{2})) = f_{1} \).
\( \forall s \in \tilde{P}_M. \) If \((v, u) \in \tilde{P}_M\) and \(C\) is any real constant \((v, u + C) \in \tilde{P}_M, \) so it is necessary that \(f_s(0) = 0; \) the result follows. From now on \((v_u, u_u)\) and \(f = (f_1, f_2)\) shall both be as in (VII.c). Consequently \(|u_u| = U_M(v_u), \) so \(u_u = U_M(v_u)(\lambda_{\delta} - 2\delta - \epsilon \rho_u).\) (VII.d) Proposition. With the above notation, the following relations are true, except for the sets of measure 0:

(VII.2) \( (f_1 > 0) \cap \{[v_u] < a(M)\} \subseteq A_{\delta}, \)

\((f_1 < 0) \cap \{[v_u] < a(M)\} \subseteq A_{\delta}. \)

Proof. Set \(v = v_u, \) \(u = U_M(v)(\lambda_{\delta} - 2\delta - \epsilon \rho_u); \) then \((v, u) \in \tilde{P}_M \) and \(\int f_u v_u = \int f_u v_u \) so (VII.1) says that

\[ \int f_u U_M(v_u) - \int f_u U_M(v_u) \leq \int f_u U_M(v_u) - \int f_u U_M(v_u). \]

If the first statement of (VII.2) is false, setting \(A = (f_1 > 0) \cap \{[v_u] < a(M)\} \subseteq (f_1 > 0) \cap \{U_M(v_u) > 0\} \)

it follows that \(|A - A_{\delta}| > 0 \) and \(A_{\delta} - A \subseteq \{U_M(v_u) < 0\}. \) Then

\[ \int f_u U_M(v_u) - \int f_u U_M(v_u) \leq \int f_u U_M(v_u) + \int f_u U_M(v_u) \]

\[ - 2 \int f_u U_M(v_u) \leq \int f_u U_M(v_u) > 0, \] which contradicts (\(\#\)).

The second statement of (VII.3) can be proved in the same way.

(VII.e) Proposition. With the same notation, the following relation is true, except for a set of measure zero:

(VII.3) \( \exists u_u = \text{sgn} f_1. \)

Proof. Let \(|v| = |v_x|, \) \(u = u_u, \) so \((v, u) \in \tilde{P}_M \) and (VII.1) says that

\[ \int f_u < \int f_u v_u. \] Set \(B = (f_1 < 0), v = v_x + \epsilon u \lambda_x; \) then it is clear that:

(i) \(|f_1| < 0) \subseteq C. \)

Now set \(C = (f_1 = 0), v = v_x + \epsilon u \lambda_x; \) then it is clear that:

\[ \int f_u U_M(v_u) = \int f_u U_M(v_u) \subseteq U_M(v_u) \]

and \(\{f_1 = 0) \cap \{v_x < a(M)\} \subseteq \{v_x < a(M)\} \) because \((-f_1 + \epsilon u \lambda_x) \in \tilde{P}_M; \) so \(|u_u| = U_M(v_u) \) a.e. in \(C = v_x = 0 \) a.e. in \(C. \) Consequently

(ii) \(|f_1 = 0) \subseteq \{v_x = 0\} \subseteq \{v_x = 0\}. \)

Let \(D = \{v_x = 0\} - (f_1 = 0) \); suppose \(|D| > 0; \) then \(|v_x = 0 \) if \(|v_x| = a(M) \) and \(v_x = \text{sgn} f_1 \leq a(M) \).

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Then, for every natural $n$, $v_n$ verifies (iii) and $|v_n| < a(M)$ in $\{0 < |v_n| < a(M)\}$. Given any interval $J$, set $v = v_{2x-J} + v_{2x}$, so $v$ verifies (iii)\

Then (ii) states that\

$$\int [f(x)] |v_n| + |f(x)| U_M(v_n) dx < \int [f(x)] |v_n| + |f(x)| U_M(v_n) dx$$

$$\Rightarrow |f(x)| |v_n| + |f(x)| U_M(v_n) < |f(x)| |v_n| + |f(x)| U_M(v_n)$$

Consequently,\

$$\frac{U_M(v_n) - U_M(v_n)}{|v_n| - |v_n|} < \frac{f(x)}{f(x)}$$

holds a.e. in $\{0 < |v_n| < a(M)\}$. So we have

(v) $J_M(|v_n|) < [f(x)|v_n|$ a.e. in $\{0 < |v_n| < a(M)\}$.

Now set $v'_n = v_n$ if $v_n = 0$ or $|v_n| = a(M)$,

$$v'_n = \left\{ \begin{array}{ll} v_n + \frac{a(M) - |v_n|}{\omega} \langle \omega \rangle v_n & \text{in } \{0 < |v_n| < a(M)\}. \end{array} \right.$$

Then $v'_n$ verifies (iii) and $|v'_n| > |v_n|$ in $\{0 < |v_n| < a(M)\}$ so, in the same way as above, we see that

(v') $|f(x)| |v'_n| \leq J_M(|v_n|) a.e.$ in $\{0 < |v_n| < a(M)\}$.

Since $(-f_1 + f_1) \in H^1$, (VII.3) shows that, except for a set of measure 0, $J_M(|v_n|) = 0$ and $v_n = 0$, so $f_1 = f_1 = 0$ and $J_M(|v_n|) = 0$ (VII.4) is proved in $\{0 < |v_n| < a(M)\}$.

In order to finish the proof it is thus enough to show that $f_2 = 0$ a.e. in $B = \{v_n = a(M)\}$. Set

$$v_n = v_n \langle x - \frac{1}{n} \rangle \langle x \rangle, \quad u_n = u_{2x - 2a} + \hat{U}_M(v_n), \quad (sg f_1) \langle x \rangle$$

then $(v_n, u_n) \in F_M$ and $sg v_n, sg u_n = sg v_n$, so (VII.1) and (VII.3) show that

$$\int [f(x)] |v_n| + |f(x)| U_M(v_n) dx < \int [f(x)] |v_n| + |f(x)| U_M(v_n) dx$$

$$\Rightarrow \int [f(x)] \frac{U_M(a) (1 - 1/n)}{(1/n)a a(M)} \to \infty \quad \text{as } n \to \infty.$$  

The result follows.

(VII.5) COROLLARY. The following relations hold except for sets of measure 0:

$$f_1 > 0 = A \cap \{v_n < a(M)\},$$

$$f_2 < 0 = (T - A) \cap \{v_n < a(M)\}.$$  

Proof. (VII.5) says that $|f(x)| \neq 0 \Rightarrow \{v_n < a(M)\} = 0$. So, except for sets of measure 0, (VII.2) shows that

$$f_1 > 0 = \{f_1 > 0\} \cap \{v_n < a(M)\} = A \cap \{v_n < a(M)\},$$

$$f_2 < 0 = \{f_2 < 0\} \cap \{v_n < a(M)\} = (T - A) \cap \{v_n < a(M)\}.$$  

Since $f_2 > 0 \cup f_2 < 0 = \{v_n < a(M)\}$, the result follows.

Now we easily see that $f$ determines $(v_n, u_n)$, let $J_M^+$ be the inverse function to $J_M$, and remember that $u_n = U_M(v_n)$ for $x \geq x_0 - A_0$. Then (VII.2), (VII.4) and (VII.5) state that:

(vii) (vii)

$$u_n = U_M(v_n) = \langle \langle x \rangle - h \rangle f_1 = f_1$$

and $\cos v_n = \sqrt{(b + y^2)/(b + b^2)}$.

Consequently, (VII.6) can be written in the following way:

$$v_n = \arg \sqrt{((M - 1) f_1 (M + 1) f_2 + M f_2)}.$$

(VII.7) $u_n = \arg \sqrt{((M + 1) f_1 + 4 M f_2)}/4 M f_2).$$

So we have proved that

(VII.1) PROPOSITION. Let $(v_n, u_n)$ be as in (VII.1). Then $3 f = (f_1, f_2)$

$\neq \emptyset$ such that $(-f_1 + f_2) \in H^1, f_1(0) = f_2(0) = 0$, and $(v_n, u_n)$ are given by (VII.7).

(VII.1) LEMMA. If $f_1, f_2 \in L^1(T)$ and $(v_n, u_n)$ are defined by (VII.7), then $(v_n, u_n) \in F_M$.

Proof. Clearly $|v_n| L^1(a - (1/a)) = a(M)$.

$$\cos v_n = \sqrt{((M - 1) f_1 f_2 + M f_2)}/4 M f_2).$$

So

$|v_n| = \arg \sqrt{((M + 1) f_1 + 4 M f_2)}/4 M f_2) = U_M(v_n).$

(VII.1) LEMMA. In the same hypothesis of (VII.1), $f(v, u) < f(v, u)$

holds for every $(v, u)$ in $F_M = \{v_n, u_n\}$.
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Proof. Since \(|u| \leq U_M(v), u = hU_M(v)|, with |h| \leq 1. Then
\[
\int f(v, u) = \int [f_1(v, u) + f_2 U_M(v)].
\]
(VII.7) shows that
\[
sg v_0 = sg f_1,
\]
and also that
\[
tg((v_0)) = (M + 1)^{f_1} / [(M + 1)^{f_2} + 4Mf_1]^{1/2} = f_1 / f_2 = J_M((v_0)).
\]
Also
\[
u_0 = U_M(v_0)-(A_{f_2 < v} - A_{f_2 < v}).
\]
(iii) and (iv) show
\[
\begin{align*}
\int f(v_0, u_0) &= \int [f_1(v_0, u_0) + f_2 U_M(v_0)] \\
&= \int f_1 U_M(v_0) - \int f_2 U_M(v_0) > 0
\end{align*}
\]
except \(h = sg f_4 \text{ a.e. in } (v_0 < \alpha(M)), which would imply \(u = u_0). Suppose that \(\mathcal{A} = \{v_0 \neq v_0\} \text{ has positive measure. Consider, in } \mathcal{A}, C = \{U_M(v_0) - U_M(v_0) \leq -U_M(v_0) \leq v_0 - v_0\}; if \(\|v_0 - v_0\| > C > -U_M(v_0) \geq J_M(|v_0|), and, if \(\|v_0 - v_0\| < |v_0|, C < -U_M(v_0), \) so, in \(\mathcal{A}, C \leq -U_M(v_0) = J_M(|v_0|) \leq |v_0| - |v_0|).
\]
Then (iii) shows that, in \(\mathcal{A}, f_1 U_M(v_0) - f_2 U_M(v_0) > J_M(|v_0|) \) a.e. Since \(|\mathcal{A}| > 0,\)
\[
\int f_1 U_M(v_0) - \int f_2 U_M(v_0) > \int f_1 U_M(v_0) + f_2 U_M(v_0)
\]
considering (i) and (v), the results follow.

(VII.k) Proposition. In the same hypothesis of (VII.j), \(\exp(G \circ r(g)) = \exp(u_0 + \hat{v}_0) \in \text{Ext } E_M\) and \(r(g) \in \text{Ext } E_M,\)
\[
\mathcal{P} = \{v, u + \hat{v}_0 \in U_M(v_0) : \langle u_0 + \hat{v}_0 + N \rangle \cap \mathcal{P}_M\}
\]

since \(N \subset \text{Ker } f, it is clear that \(G^{-1}(u_0 + \hat{v}_0) \cap \mathcal{P}_M = \langle u_0 + \hat{v}_0 + N \rangle \cap \mathcal{P}_M\),

contains only one element and that it belongs to \(\text{Ext } \mathcal{P}_M\). So \(\exp(u_0 + \hat{v}_0) \in \text{Ext } E_M\) because of (VII.l). Consequently \(v_0, u_0) \in \text{Ext } E_M\) as in (VII.j); set \(d \mu = g d \tilde{t}, d \nu = \tilde{g} d \tilde{t}.\) Then, the result follows from (VII.j) and Definition (VII.b).

Remark that the first part of Theorem 1 has been proved. In order to prove the second we shall give a geometrical interpretation of \(E_M\).

We say that \(\varphi \in L_M\) belongs to the set \(\text{Ext } \mathcal{P}_M\) of exposed points of \(L_M\) if there exists a linear functional \(F\), continuous in the weak star topology of \(L\), such that \(\mathcal{F}(\varphi) > \mathcal{F}(\varphi), \forall \varphi \in L_M - \{\varphi\}\).

Proposition (VII.3) shows that there exists a bijection between \(\text{Ext } E_M\) and \(\text{Ext } \mathcal{P}_M\), given by \(f \mapsto \varphi\). We shall show that the same relation holds between \(E_M\) and \(\text{Ext } \mathcal{P}_M\).

(VII.k) Proposition. \(\varphi \in \text{Ext } \mathcal{P}_M \iff \varphi \in E_M, \)
The proof. If \(\varphi \in \text{Ext } \mathcal{P}_M\), clearly \(\varphi \in \text{Ext } E_M\), so \(\varphi \in \text{Ext } E_M\). Then \(\varphi = c + u_0 + \hat{v}_0, with (v_0, u_0) \) as in (VII.l). Moreover, there exists \(g \in H, c\) such that \(\int (g v_0 + g \hat{v}_0) > \int (g v_0 + \hat{g} v_0)\) for every \((v, u) \in E_M\).

Thus \((v_0, u_0) \in \text{Ext } E_M,\) Proposition (VII.l) and the definition of \(E_M\) finish this part of the proof. Reciprocally, if \(\varphi \in E_M, \varphi = c + u_0 + \hat{v}_0, with (v_0, u_0)\) as in (VII.l). By (VII.l), \(\varphi \in \text{Ext } \mathcal{P}_M\).

We know that \(L_M\) is a weak star compact and convex subset of \(L, so we may refer to the following theorem of Y. Klee (\(\text{Rem.}\)); Let \(E\) be a separable Banach space and \(E^*\) its dual topological dual. Let \(C\) be a weak star compact and convex subset of \(E^*\). Then \(C\) is the weak star closure of the convex hull of the set of exposed points of \(C\).

This theorem, with \(E = H, C = L_M, says that the proof of Theorem 1 is over.

References

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