On domination and separation of ideals in commutative Banach algebras

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Abstract. We introduce and study the concept of domination property and approximate domination property of ideals in commutative Banach algebras, and connections of these concepts with separation of ideals by means of annihilating nets. Among other results we show that domination and separation properties coincide, as well as the approximate domination and bounded separation properties.

1. Separation of ideals. All algebras in this paper are assumed to be commutative, complex Banach algebras with unit, unless otherwise stated. The unit element will be designated by $e$.

The maximal ideal space of an algebra $A$ will be designated by $\mathfrak{M}(A)$, and its Shilov boundary by $\Gamma(A)$. We shall treat elements of $\mathfrak{M}(A)$ both as ideals and as multiplicative-linear functionals. We say that an ideal $I \subset A$ consists of joint topological divisors of zero if there is a net $(\xi_n) \subset A$, $\|\xi_n\| \geq \varepsilon > 0$, such that $\lim (\xi_n x_n) = 0$ for all $x \in I$. In this case we say that the net $(\xi_n)$ annihilates $I$ and write $(\xi_n) \perp I$, or $(\xi_n) \in I^\perp$. The symbol $I(A)$ will designate the set of all (not necessarily closed) proper ideals of $A$, including the zero ideal, consisting of joint topological divisors of zero.

The members of $I(A)$ will be called shortly $l$-ideals. We put also $\mathcal{E}(A) = I(A)/\mathfrak{M}(A)$. It is known that the closure of an $l$-ideal is again an $l$-ideal, $\mathcal{E}(A)$ is a closed subset of $\mathfrak{M}(A)$ containing $\Gamma(A)$ (cf. [9]), and every $l$-ideal is contained in a maximal ideal $\mathcal{M}$ belonging to $\mathfrak{M}(A)$ (cf. [9]).

1.1. Definition. We say that an ideal $I \subset A$ can be separated from an element $x_0 \notin I$ if there is a net $(x_n) \perp I$ and $x_n x_0 \to 0$. We say that two ideals $I_1$ and $I_2$ can be separated if one of them can be separated from an element of the other. We say that an ideal $I \subset A$ has the separation property if it can be separated from each element $a \notin I$. In above definition we replace nets by bounded nets, we obtain the concepts of bounded separation of an ideal $I$ from an element $a$, bounded separation of two

* This paper was written during the authors stay at the University of Kansas in second semester of 1978/79.
ideals and bounded separation property. One can easily see that instead of using all bounded nets, we can limit ourselves to nets \((z_a)\) with \(|z_a| = 1\) for all \(a\).

The above concepts make sense only for \(l\)-ideals, and particularly when all ideals in \(A\) are \(l\)-ideals. The latter holds when \(\mathcal{R}(A) = \mathcal{M}(A)\) and this condition is satisfied for regular algebras, i.e., semi-simple algebras such that for every closed subset \(F \subset \mathcal{R}(A)\) and \(\mathcal{M} \notin F\), there is an element \(v \in A\) whose Gelfand transform \(v^*\) is zero on \(F\) and \(v^*(\mathcal{M}) \neq 0\).

The hull of an ideal \(I \subset A\) is the set \(h(I) = \{M \in \mathcal{R}(A) : I \subset M\}\) and we say that two ideals \(I_1\) and \(I_2\) in \(A\) can be separated by means of spectral synthesis if \(h(I_1) \neq h(I_2)\). An important example of regular algebras are the group algebras \(L_1(G)\) for LCA groups \(G\), or rather these algebras when a unit is adjoined. An important theorem of Mallivain [3] [5] states that for a non-compact LCA group \(G\) the algebra \(L_1(G)\) always contains closed ideals \(I_1 \neq I_2\) with \(h(I_1) \neq h(I_2)\).

The motivation for the study of the concept of separation by means of nets is to provide a tool which works better than spectral synthesis.

The following results are proved in [15]:

1.2. Proposition. If two ideals \(I_1\) and \(I_2\) of a regular Banach algebra can be separated by means of spectral synthesis, then they can also be boundedly separated.

For a regular algebra \(A\) and for any closed set \(F \subset \mathcal{R}(A)\) there always exists a unique smallest ideal \(I_F\) with \(h(I_F) = F\). It is

\[I_F = \{x \in A : x^*(M) = 0\text{ for all }M \in F\text{ in some open neighborhood of }F\}.

1.3. Proposition. If \(I \subset A\) is regular, then for any closed \(F \subset \mathcal{R}(A)\) the ideal \(I_F\) has the separation property.

Since usually the ideal \(I_F\) is non-closed, the above proposition shows that it is possible to separate an ideal from its closure.

1.4. Proposition. There exists a regular algebra \(A\) and two ideals \(I_1, I_2\) in \(A\) which can be separated, but cannot be boundedly separated.

2. Domination property.

2.1. Definition. We say that an element \(x \in A\) is dominated by elements \(x_1, x_2, \ldots, x_n \in A\) if there exists a constant \(C > 0\) such that for all \(s \in A\) the following holds true:

\[|s| \leq C \sum_{i=1}^n |x_i|.|s_n|.

In this case we write \(s < (x_1, \ldots, x_n)\). We say that an element \(x \in A\) is dominated by an ideal \(I \subset A\) if \(x < (x_1, \ldots, x_n)\) for some \(n\)-tuple \((x_1, \ldots, x_n)\) of elements of \(I\). In this case we write \(x < I\). We say that an ideal \(I \subset A\) has the domination property if the relation \(x < I\) implies \(x \in I\). The family of all (not necessarily closed) ideals in \(A\) with domination property will be designated by \(D(A)\).

We say that a family \(F = (I_a)\) of ideals in \(A\) is directed, if for any \(I_1, I_2 \in F\) there exists \(I_3 \in F\) such that \(I_1 \cup I_2 \subset I_3\).

2.2. Proposition. Let \(A\) be a commutative Banach algebra with unit.

(i) \(D(A) \subset D(A)\subset \mathcal{R}(A)\).

(ii) If \(I_\alpha \in D(A)\), then \(\bigcap I_\alpha \in D(A)\).

(iii) If \((I_\alpha)\) is a directed family of ideals and \(I_\alpha \in D(A)\) for all \(\alpha\), then \(\bigcup I_\alpha \in D(A)\).

Proof. (i): Let \(M \in \mathcal{R}(A)\) and \(x_1, x_2, \ldots, x_n \in M\). If \(x < (x_1, \ldots, x_n)\), then any not annihilating \(M\) annihilates \(x\) also. Since \(M\) is maximal, we have \(x \in M\) and so \(M \in D(A)\). If \(I \notin D(A)\), then the unit element \(e\) is dominated by some elements in \(I\). Thus \(I \notin D(A)\).

(ii): If \(I = \bigcap I_\alpha\) and \(x_1, \ldots, x_n \in I\), then any element \(x\) dominated by \((x_1, \ldots, x_n)\) is in \(I\) for all \(\alpha\) and so \(x \in I\). Thus \(I \in D(A)\).

(iii): The proof follows immediately from Definition 2.1.

Remark. Since the ideal \(I^\gamma\) can be expressed as a union of a directed family of ideals which are intersections of maximal ideals, then Proposition 1.3 is a consequence of above proposition and Theorem 2.8 below.

2.3. Corollary. \(D(A) \cap \mathcal{R}(A) = \mathcal{E}(A)\).

2.4. Corollary. Every ideal in \(D(A)\) is contained in a maximal ideal which also belongs to \(D(A)\).

Problem 1. Suppose that \(I \in D(A)\). Does the closure \(\bar{I}\) belong to \(D(A)\) too?

2.5. Corollary. If \(\Omega\) is a compact Hausdorff space, then every closed ideal in \(A = C(\Omega)\) is in \(D(A)\).

The proof follows from the fact that \(A\) is a regular algebra and every closed ideal in \(A\) is an intersection of maximal ideals.

There are, however, non-closed ideals in \(A = C[0, 1]\) which are not in \(D(A)\). If \(\pi(t) = t\) and \(I\) is the principal ideal \(\pi(A)\), then the element \(x(t) = x, t \in I\) is dominated by \(I\) and does not belong to \(I\).

2.6. Lemma. Let \(R\) be a set of indices and let \((x^*_\gamma)_{\gamma \in R}\) be a family of nets of elements of \(A\) indexed by the same directed system \(S\) of indices \(\gamma\). Then there exists a single net \((x^*_\alpha)\) such that

\[x \in A : x^*_\gamma \to 0 \iff \lim_{\gamma \in R} x^*_\gamma \to 0\text{.}\]

If, moreover, \(|x^*_\alpha| = 1\text{ for all }\gamma \in S \text{ and all } \alpha \in R\), then \(|x^*_\alpha| = 1\text{ for all }\alpha\).
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Proof. Put

\[ J = \bigcap \{ x \in A : x \to 0 \} \]

If \( J = A \), then we can assume \( a = \gamma \) and put \( x_\gamma = x \) for an arbitrarily chosen \( x \in E \). Suppose then that \( J \neq A \) and form a new directed system of indices \( \beta \) in the following way. We put \( \beta = \langle F_1, \beta_1, \beta_2 \rangle \), where \( F_1 = F_1(\beta) \) is a finite non-void subset of \( A \setminus J \), \( F_2 = F_2(\beta) \) is a finite non-void subset of \( J \) and \( \beta = (\beta_1, \beta_2) \) is a positive integer. Write \( \beta_1 \geq \beta_2 \) if for the corresponding sets \( F_1, F_2 \), and integers \( \kappa \) we have \( F_1(\beta_1) = F_1(\beta_2) \), \( F_2(\beta_1) = F_2(\beta_2) \), and \( \kappa(\beta_1) \geq \kappa(\beta_2) \).

By assumption, to any \( x \notin J \) there correspond an index \( \gamma = \gamma(x) \) and a positive constant \( c(\gamma) \) such that

\[ \limsup \sup |x_\gamma(x)| = c(\gamma) > 0. \]

Now, to each index \( \beta \) we associate \( n \) indices \( a_\beta(\beta_1), \ldots, a_\beta(\beta_n) \), where \( n \) is the cardinality of \( F_1(\beta) \), and order them in the following way. If \( \alpha = a_\beta(\beta) \), \( \alpha' = a_\beta(\beta') \), then \( \alpha > \alpha' \) if \( \beta > \beta' \), or in the case when \( \beta = \beta' \) if \( i > i' \).

In this way we obtain a directed system of indices \( a \). Define now the net \( (x_\gamma(x)) \) in the following way. For \( a = a_\beta(\beta) \) write \( F_1(\beta) = \{ y_1, \ldots, y_m \} \) and \( x_\beta = x_{\beta}(\gamma) \) where \( \gamma \in 8 \) is chosen in such a way that

\[ |x_{\gamma}(y_j)| < 1/k(\beta) \quad \text{for} \quad j = 1, 2, \ldots, m \]

and

\[ |x_{\gamma}(x_{\beta}(\gamma))| > 1/c(\gamma). \]

Such an index \( \gamma \) exists because of definition of \( c(\gamma) \) and because the net \( (x_{\beta}(\gamma))_{\beta} \) annihilates \( J \). This is clear that the net \( (x_{\beta}(\gamma))_{\beta} \) annihilates \( J \) and does not annihilate \( A \setminus J \) since for any \( x \in A \setminus J \) we have

\[ \limsup |x_{\beta}(\gamma)| > 1/c(\gamma). \]

2.7. Theorem. Let \( A \) be a commutative Banach algebra with unit \( e \). Then an ideal \( I \subseteq A \) has the domination property if and only if it is of the form

\[ I = \{ x \in A : x \to 0 \}, \]

where \( (x) \) is a set of elements of \( A \) which does not tend to zero.

Proof. Suppose that \( I \) is a subset of \( A \) of the form (3). Since \( x \to 0 \), it is a proper ideal in \( A \). If \( x \notin I \), then relation (1) shows that \( x \mapsto 0 \) and so \( x \in I \). Thus \( I \in D(A) \). Suppose now that \( I \in D(A) \). We shall construct a directed system \( S \) of indices \( \gamma \) such that for each \( \gamma \notin I \) there is a net \( (x_\gamma(x))_{\gamma} \) in \( I^+ \) with \( x_\gamma(x) \to 0 \). Once we have the family of nets \( (x_\gamma(x)) \), \( x \in A \setminus I \), then Lemma 2.6, with \( B = A \setminus I \) and \( x_\gamma = x_\gamma(x) \), provides us with a set \( (x) \) which realizes formula (3). The directed systems of indices \( \gamma \) will consist of pairs \( \gamma = (\beta, k) \), where \( \beta \) is a finite subset of \( I \) and \( k \) is a positive integer. For \( \gamma = (\beta, k_1) \) and \( \gamma' = (\beta', k_2) \) we write \( \gamma > \gamma' \) if \( \beta > \beta' \) and \( k > k_2 \). Fix an element \( x \notin I \). Since \( I \) has the domination property and \( \gamma \notin I \), for each \( \gamma = (\beta, k) \), \( \beta = (y_1, \ldots, y_m) \in I \), there exists by formula (1) an element \( x_\gamma(x) \) such that

\[ \exists \sum_{\gamma \in I} \| [x_\gamma(x)] \| < \infty. \]

It follows that \( (x_\gamma(x)) \notin I \) and \( x_\gamma(x) \to 0 \). The conclusion follows.

2.8. Theorem. An ideal \( I \subseteq A \) has the domination property if and only if it has the separation property.

Proof. If \( I \in D(A) \), then by formula (2) the ideal \( I \) has the separation property. If \( I \notin D(A) \), then there is an element \( x \notin I \) which is dominated by an \( n \)-tuple \( (a_1, \ldots, a_n) \in I \). Formula (1) implies that every net annihilating \( I \) must also annihilate \( x \), and so \( I \) cannot be separated from \( x \).

For an arbitrary ideal \( I \subseteq D(A) \) there always exists a unique smallest ideal in \( D(A) \) which contains \( I \). It will be denoted by \( I^0 \). The proof of the following proposition is left to the reader.

2.9. Proposition. Let \( I \subseteq I(A) \). Then

\[ I^0 = I^{1 \perp} = \bigcap \{ J \in D(A) \mid I \subseteq J \} = \{ x \in A : x \to 0 \}. \]

Here for a family \( N \) of nets of elements of \( A \) we put \( N^+ = \{ x \in A : x \to 0 \} \) for all \( x \in N \). The symbol \( I^{1 \perp} \) denotes \((I^0)^\perp \). We have also \( N^+ = N^{1 \perp} \) and \( I^0 = I^{1 \perp} \) for any family of nets \( N \) containing at least one net which does not tend to zero, and for any ideal in \( I(A) \).

For two ideals \( I, J \subseteq I(A) \) write \( I < J \) if for each \( x \in I \) we have \( x < J \). One easily sees that \( I < J \) is equivalent to each of the following relations

\[ I^0 < J \quad I < J^0 \quad I \subseteq J^0 \quad I^0 \subseteq J \quad I \subseteq J \]

The relations \( I < J \) and \( J < I \) imply \( I^0 = J^0 \), so they imply \( I = J \) in the case where \( I, J \in D(A) \). One has also \( I^0 = I^0 \) for all \( I \subseteq I(A) \). Since two ideals \( I, J \subseteq I(A) \) can be separated if and only if \( I^0 \neq J^0 \), we have the following

2.10. Proposition. Two ideals \( I, J \subseteq I(A) \) can be separated if and only if \( I^0 \neq J^0 \).

Another characterization of \( D(A) \) will be obtained by means of the concept of extension of an algebra.

An extension of a separable algebra of \( A \) is an algebra \( B \) with unit which contains \( A \) under a unitary topological isomorphism. In this situation we write \( A \subseteq B \). By a theorem of Lindberg [2] we can always replace the
norm in \( B \) by an equivalent norm, so that the embedding in question becomes an 

isometry. Thus without loss of generality we can consider only isometric extensions of \( A \). If \( I \) is an ideal in \( A \), then for any extension \( B \supseteq A \) we denote by \( I_B \) the smallest ideal in \( B \) containing \( I \). \( I_B \) will designate the closure of \( I_B \) in \( B \). We have \( I_B = \left\{ \sum x_i b_i : x_i \in I, b_i \in B \right\} \).

If \( I \in \mathcal{I}(A) \), then \( I_B \in \mathcal{I}(B) \), so it is a proper ideal in \( B \). We put \( I_B^0 = (I_B)^0 \) for any \( I \in \mathcal{I}(A) \), \( A \subseteq B \). Under this notation we have

2.11. PROPOSITION. An ideal \( I \in \mathcal{I}(A) \) is in \( D(A) \) if and only if for each extension \( B \supseteq A \) we have

\[
I = I_B^0 \cap A.
\]

Proof. Taking \( B = A \) we see that the above formula implies \( I = I_B^0 \in D(A) \). Suppose now that \( I \in D(A) \), the converse holds clearly. Suppose that \( I \) is not in \( D(A) \), we have to show that \( I_B^0 \cap A \subseteq I \). So let \( x \in I_B^0 \cap A \). If \( (a_i) \perp I \), then \( (a_i) \perp I_B \). But \( I_B^0 = I_B^{1/2} \) and \( I_B = I_B^{1/2} \), thus \( (a_i) \perp I_B \) and so \( x \in I_B \). Thus \( x \in I_B \cap A = I \).

In [10] we posed the following domination conjecture: If \( x < (a_1, \ldots, a_n) \in A \), then there is an extension \( B \supseteq A \) and elements \( b_1, \ldots, b_n \in B \) such that

\[
x = \sum \pi a_i b_i.
\]

(Clearly every element in \( A \) of this form is dominated by \( (a_1, \ldots, a_n) \).) However, as was recently shown by Vladimir Müller [4], this conjecture fails in general. Nevertheless, in some instances the conjecture is true and we have the following domination theorems.

2.12. THEOREM (Arens [1]). If \( x, y \in A \), \( x < y \), then there is an extension \( B \supseteq A \) and element \( b \in B \) such that

\[
x = yb.
\]

2.13. THEOREM ([11]). If \( A \) is a uniform algebra and \( x < (a_1, \ldots, a_n) \) \( \in A \), then there is an extension \( B \supseteq A \), which is also a uniform algebra, and elements \( b_1, \ldots, b_n \in B \) such that

\[
x = \sum \pi a_i b_i.
\]

In order to obtain results on separation of ideals in some algebras without unity we need an extension of Arens theorem. An approximate identity for \( A \) is a net \( (\lambda_n) \subset A \) such that \( \lambda_n x \to x \) for all \( x \in A \). We do not assume that the net \( (\lambda_n) \) is bounded.

2.14. PROPOSITION. Let \( A \) be a commutative Banach algebra with an approximate identity \( (\lambda_n) \). Let \( x, y \in A \) and suppose that there is a positive \( C \)

such that for all \( z \in A \)

\[
\|xz\| \leq C \|yz\|.
\]

Then there exists a subalgebra \( B \supseteq A \) and an element \( b \in B \) such that

\[
x = yb.
\]

Proof. Let \( A_1 = A \oplus C \) be \( A \) with an adjoined identity \( e \) and with the norm given by \( \|x + \lambda e\| = \|x\| + \|\lambda\| \) for \( x \in A, \lambda \in C \). Relation (3) implies \( \|x(z + \lambda e)\| \leq C \|y(z + \lambda e)\| \) for all \( x \in A, \lambda \in C \). By passing to the limit we obtain \( \|x(z + \lambda e)\| \leq \|y(z + \lambda e)\| \) for \( x < y \in A_1 \), and we apply Arens domination theorem.

For \( w \) not necessarily unital commutative Banach algebra \( A \) and \( x_0 \in A \) denote by \( I_0(x_0) \) the smallest ideal in \( A \) containing \( x_0 \), i.e., \( I_0(x_0) = (x_0 + \lambda \delta_0 : \lambda \in A, \delta_0 \in C) \).

We shall use Proposition 2.14 in the proof of the following result.

2.15. PROPOSITION. Let \( A \) be a commutative regular Banach algebra either with unit element or with an approximate identity. If \( x_0 \in A \), \( 0 \in \sigma(x_0) \), and \( 0 \) is not an isolated point of the spectrum \( \sigma(x_0) \), then all ideals \( I_0(x_0) \)

\[
n = 1, 2, \ldots \text{can be mutually separated.}
\]

Proof. It is clear that an element \( x \) is dominated by \( I_0(x_0) \) if and only if \( x < x_0^* \), so by Proposition 2.10 it is sufficient to show that for \( n > m \) the element \( x_0^{*n} \) cannot be dominated by \( x_0^* \). Suppose then that \( x_0^{n} < x_0^* \).

By Theorem 2.11 we have \( x_0^* (e - be_0)= 0 \)

and there is a sequence \( x_0 \) in \( \mathcal{R}(A) \) with \( x_0 \neq I_0(x_0) \). Since \( A \) is regular, we have \( \mathcal{R}(A) = \mathcal{I}(A) \) and all functionals \( f \) can be extended to functionals \( F \) in \( \mathcal{R}(B) \). Since \( F_0(x_0) \neq 0 \), relation (4) implies \( F_0(b)F_0(x_0^{*n}) = 1 \) for all \( b \), and so \( F_0(b) \to \infty \). A contradiction, since \( \mathcal{I}(B) < \mathcal{I}(A) \), and the conclusion follows.

We close this section with the following questions.

PROBLEM 2. For which Banach algebras is the domination conjecture true?

PROBLEM 3. Let \( A \) be a regular Banach algebra with unit element. Do all closed ideals in \( A \) belong to \( D(A) \)?

PROBLEM 4.(?) Suppose that \( x < (a_1, \ldots, a_n) \in A \). Does there exist an extension \( B \supseteq A \) and elements \( b_1, \ldots, b_n \in B \) such that \( x = \sum \pi a_i b_i \) ? Here we do not assume that \( B \) is a Banach algebra, but any topological algebra containing \( A \) under a homeomorphic imbedding preserving the unit element.

(?) A negative answer to this problem has been given in [12] [added in proof].
3. Approximate domination.

3.1. Definition. Let $A$ be a commutative Banach algebra with unit element $e$ and let $I$ be a (proper) ideal in $A$. An element $x \in A$ is said to be **approximately dominated by $I$ if for each $\varepsilon > 0$ there exist elements $x_1, \ldots, x_n \in I$ such that for all $x \in A$ we have**

\[
||x|| \leq \sum ||x_i|| + \varepsilon ||x||.
\]

In this case we write $x \prec I$. We say that an ideal $I \subset A$ possesses the **approximate domination property** if the relation $x \prec I$ implies $x \in I$. The family of all ideals in $A$ possessing the approximate domination property will be designated by $D_A$. 

3.2. Proposition. If $A$ is as above, then $D_A = D(A)$.

Proof. The relation $x \prec I$ clearly implies $x \prec J$. Thus if $I \in D_A$ and $x \prec I$, then $x \prec J$. So $x \in I$ and $I \in D(A)$.

3.3. Corollary. $D_A \subset I(A)$.

3.4. Proposition. Every ideal in $D_A$ is closed.

Proof. Let $I \in D_A$ and $y \in I$. For any given $\varepsilon > 0$ we find an element $x \in I$ with $||x - y|| < \varepsilon$. Since $||x|| = ||(y - x) + x|| \leq ||y - x|| + \varepsilon ||x||$ for all $x \in A$, we have $y \in I$ and so $y \in I$. Thus $I \in D_A$.

3.5. Proposition. Let $A$ be as above. Then

(i) $I(A) \subset D_A$,

(ii) If $I \in D_A$, then $I = \bigcap \{N \in A : N \subset D_A\}$.

Proof. (i): Let $M \in I(A)$ and choose a net $(a_n) \subset M$, $||a_n|| = 1$. If $x \prec a_n$, then relation (5) implies that for each $\varepsilon > 0$ we have $\lim_{n \to \infty} ||a_n|| \leq \varepsilon$, and by the maximality of $M$ we have $x \in M$. So $M \in D_A$.

(ii): If $x \prec I$, then $x \prec a_n$ for all $a_n$, so $x \in I$ for all $a_n$ and $x \in I$.

3.6. Theorem. Let $A$ be a commutative Banach algebra with unit $e$. Then an ideal $I \subset A$ has the approximate domination property if and only if it is of the form

\[I = \{x \in A : x \cdot a \to 0\},\]

where $(a_n)$ is a net of elements of $A$ such that $||a_n|| = 1$ for all $a_n$.

Proof. Suppose that an ideal $I \subset A$ is of the form (6). If $x \prec I$, then relation (5) implies that for each $\varepsilon > 0$ we have $\lim_{n \to \infty} ||a_n|| \leq \varepsilon$ and so $x \in I$. Thus $I \subset D_A$. Suppose now that $I \in D_A$. We should have a directed system $S$ of indices $\gamma$ such that for each $x \in I$ there is a net $(s_\gamma(a)) \subset I$, $||s_\gamma(a)|| = 1$ for all $a \in S$, and $s_\gamma(a) \cdot x \to 0$. Once we have the family of nets $(s_\gamma(a)) \in I^I$, $a \in A \setminus I$, with $||s_\gamma(a)|| = 1$ for all $\gamma \in S$, then, by Lemma 2.6 with $R = A \setminus I$ and $s_\gamma = s_\gamma(a)$, we obtain the desired net $s_\gamma$. The system $S$ is exactly the same as in the proof of Theorem 2.7. Fix an element $x \in I$ and $\varepsilon > 0$. Since $I$ has the approximate domination property, then for each $\gamma = (\varepsilon_1, \varepsilon_2)$, $F_\gamma = (s, \ldots, a) \in I$, there exists by formula (5) an element $s_\gamma(a)$ with $||s_\gamma(a)|| = 1$ such that

\[||s_\gamma(a)|| \geq \varepsilon \sum ||x_i|| + \varepsilon \geq \varepsilon\]

(we use the notation of (5), $h_i$ instead of $e_i$). Relation (7) shows that $||s_\gamma(a)|| \leq ||x||/h_i$ for all $i > \gamma_1$, where $r_\gamma = (S_\gamma, h_\gamma)$ and $c_\gamma \in F_\gamma$. Since $h_i \to \infty$, this implies that $s_\gamma(a) \cdot x_i \to 0$. Since we can take for $a_i$ an arbitrary element of $I$, this implies $s_\gamma(a) \cdot x_i \to 0$. On the other hand, formula (7) shows that $ax_{\gamma_1} \to 0$. The conclusion follows.

3.7. Theorem. An ideal $I \subset A$ has the approximate domination property if and only if it has the bounded separation property.

Proof. If $I \in D_A$, then by formula (6) the ideal $I$ possesses the bounded separation property. If $I \notin D_A$, then there is an element $x \notin I$ such that for each $\varepsilon > 0$ there are elements $x_1, x_2, \ldots, x_n \in I$ with

\[||x|| \leq \sum ||x_i|| + \varepsilon \]

for all $x \in A$. Let $(x_\gamma) \subset I$ and $||x_\gamma|| = 1$ for all $\gamma$. Formula (8) implies that $\limsup ||x_\gamma|| \leq \varepsilon$, and since it holds for all $\varepsilon > 0$, we have $\limsup ||x_\gamma|| = 0$. Thus every bounded net $(x_\gamma)$ annihilating $I$ annihilates $x$ also and so $I$ does not possess the bounded separation property.

For an ideal $I \in I(A)$ denote by $I^B$ the set of all nets $(a_n)$ in $I^I$ with $||a_n|| = 1$ for all $a_n$ and denote by $I^{B_0}$ the smallest ideal in $D_A$ which contains $I$. The proof of the following proposition is left to the reader.

3.8. Proposition. Let $I \in I(A)$. Then

\[I^{B_0} = I^{B_1} = I^{B_2} = I \cap D(A): I \subset D(A) \subset A \subset A \subset I \subset I.
\]

If in the relations following Proposition 2.8 we replace $D$ by $D_A$, we obtain again valid relations. In particular we obtain

3.9. Proposition. If $I, J \in I(A)$, then $I$ can be boundary separated from $J$ if and only if $I^{B_0} \neq J^{B_0}$.

If for $I \in I(A)$ and for an extension $B \supset A$ we put $I^{B_0} = (I^{B_0})^B$ and $I^{B_0} = (I^{B_0})^A$, we obtain the following proposition whose proof is similar to that of Proposition 2.11.

3.10. Proposition. Let $I \in I(A)$. Then $I \in D_A(A)$ if and only if for each extension $B \supset A$ we have

\[I = I^{B_0} \cap A = I^{B_0} \cap A.
\]
We do not know any example of an ideal \( I \in D_0(A) \) for a semi-simple Banach algebra \( A \) which is not of the form
\[
I = \bigcap \{ M \in \mathcal{E}(A) : I \subseteq M \}.
\]
(9)
By Proposition 3.5 every ideal of this form is in \( D_0(A) \). We shall now show that for uniform algebras \( D_0(A) \) coincides with the ideals of the form (9).

3.11. Proposition. Let \( A \) be a uniform algebra with unit \( e \). Then every ideal in \( D_0(A) \) is of the form (9).

Proof. It is shown in [8] that for a uniform algebra \( A \) we have \( \Gamma(A) = \mathcal{E}(A) \). Let \( I \in D_0(A) \) and consider the embedding \( A \subseteq B = C(\Gamma(A)) \). Since every closed ideal of \( B \) is an intersection of maximal ideals (c.f. e.g. [7]), we have
\[
I_B = \bigcap \{ M \in \mathcal{B}(B) : I \subseteq M \} \subseteq \bigcap \{ M \in \mathcal{B}(B) : I \subseteq M \}.
\]
(10)
Thus
\[
I_B \cap A = \bigcap \{ \mathcal{M} \cap A : \mathcal{M} \in \mathcal{B}(B), I \subseteq \mathcal{M} \cap A \}.
\]
But the intersections \( \mathcal{M} \cap A, \mathcal{M} \in \mathcal{B}(B) \) are precisely the elements of \( \Gamma(A) = \mathcal{E}(A) \).

So
\[
I_B \cap A = \bigcap \{ \mathcal{M} \in \mathcal{E}(A) : I \subseteq \mathcal{M} \}.
\]
The conclusion will follow if we show \( I = I_B \cap A \). But this follows immediately from Proposition 3.10, since by formula (10) and Propositions 3.4, 3.5 and 3.8 we have \( I_B = I_B \).

3.12. Corollary. For uniform algebras the spectral synthesis separate better than bounded nets. So by Proposition 1.3 separation by spectral synthesis is for uniform regular algebras equivalent to separation by bounded nets.

Another type of ideal in \( D_0(A) \) which must be of the form (9) is given by the ideal \( I = a^+ \) of the form (9).

3.13. Proposition. If \( A \) is a semi-simple algebra and \( a \in A \), then the ideal \( I = a^+ \) is of the form (9).

Proof. Let \( S = \{ M \in \mathcal{E}(A) : a^+(M) \neq 0 \} \). If \( x \in a^+ \), then \( ax = 0 \) and so \( a^+(M) = 0 \) for all \( M \in S \). Thus \( a^+ \subseteq M \) for all \( M \in S \). Put
\[
J = \bigcap \{ M : M \in S \}.
\]
So \( a^+ \subseteq J \). If \( x \in J \), then \( a^+(M) = 0 \) for all \( M \in S \) and so \( a^+(M) a^+(M) = 0 \) for all \( M \in \mathcal{E}(A) \). Since \( \Gamma(A) = \mathcal{E}(A) \) and \( A \) is semi-simple it follows that \( ax = 0 \) or \( x \in a^+ \). So \( a^+ \subseteq J \) and the conclusion follows.

If \( A \) is not a semi-simple algebra, then there are ideals in \( D_0(A) \) which are not of the form (9). For example, if \( \alpha \in A \) is an element such that \( \alpha M = 0 \) and \( x \alpha \ldots \alpha x \ldots \alpha \) is not of the form (9), such an element \( \alpha \) can be found in the convolution algebra \( L^1(0, 1) \) with unit \( \alpha \).

Problem 5. Is every ideal in \( D_0(A) \) of the form (9), if \( A \) is a regular algebra?

The following problem is an analog of the domination conjecture, it is not disproved by the Müller example.

Problem 6. Let \( x < A \subseteq A \). Does there exist an extension \( B \supseteq A \) such that \( x \in D_0(B) \)?

We give a proof of this conjecture for uniform algebras.

3.14. Proposition. If \( A \) is a uniform algebra and \( x < A \), then \( x \) is an ideal in \( A \), then there is an extension \( B \supseteq A \) such that \( x \in D_0(B) \).

Proof. Let \( B = C(\Gamma(A)) \). If \( x \) is not contained in an ideal \( M \in \mathcal{E}(A) \), then \( I_B = B \) and so \( x \in D_0(B) \). Otherwise \( I_B = \bigcap \{ M \in \mathcal{B}(B) : I \subseteq M \} \) and \( I_B \cap A = \bigcap \{ M \in \mathcal{E}(A) : I \subseteq M \} \). By Proposition 3.5 we have \( I_B \cap A \subseteq D_0(A) \). Since \( I \subseteq I_B \cap A \), the relation \( x < A \) implies \( x \subseteq I_B \cap A \) and so \( x \in D_0(A) \).

References


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Received May 21, 1979
Revised version September 4, 1979