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Uniformly smooth partitions of unity on superreflexive Banach spaces

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Abstract. It is proved that a real Banach space X is superreflexive iff X admits partitions of unity formed by functions with uniformly continuous differential.

Partitions of unity of the kind mentioned in abstract on separable superreflexive spaces are constructed in [18]. We will work in real Banach spaces.

Let us recall that a Banach space Y is said to be *finitely representable* in a Banach space X ([10]) if for each finite-dimensional subspace $F \subset Y$ and each $\varepsilon > 0$ there is an isomorphism T of F onto some subspace of X with $\|T\|\cdot\|T^{-1}\| < 1+\varepsilon$.

For $\varepsilon > 0$, an *ε-tree* T in a Banach space X is a set of points $x_{ij} \subset X$, $i, j = 0, 1, 2, \ldots,\ \ j < 2^i$, such that for each such i, j,

$$x_{ij} = \tfrac{1}{2}(x_{i+1,2j} + x_{i+1,2j+1}) \quad \text{and} \quad \|x_{i,2j} - x_{i,2j+1}\| \geqslant \varepsilon$$

([9], [10]). If i is allowed to be only $\leqslant n$, then we speak on an *n-ε tree* $T_{p,\varepsilon}$.

A Banach space X is *superreflexive* ([10]) if only reflexive Banach spaces Y are finitely representable in X. This is the case iff X admits an equivalent norm which is both uniformly convex and uniformly smooth ([6]) and iff for each $\varepsilon > 0$ there is an n such that no n-ε tree $T_{n,\varepsilon}$ lies in the unit ball B_1 of X ([10]).

A bounded subset B of a Banach space X is called *dentable* ([16]) if for each $\varepsilon > 0$ there is an $f \in X^*$ and $\delta > 0$ such that

$$\operatorname{diam}\{x \in B,\ f(x) \geqslant \sup_B f - \delta\} < \varepsilon.$$

A norm $\|\cdot\|$ on X is said to be *rough* ([13], [14]) if there is an $\varepsilon > 0$ such that for every $x \in X$ and every $\delta > 0$ there is a $v \in X$, $\|v\| \leqslant 1$ with $\|x+tv\| \geqslant \|x\| + \varepsilon|t| - \delta$ for each $|t| \leqslant \|x\|$.

The following definition will be suitable in this note.

DEFINITION 1. If $(X, \|\cdot\|)$ is a Banach space, $K, \varepsilon, \delta, \eta > 0$, $\delta < 1$, and $|\cdot| \leqslant K\|\cdot\|$ is a pseudonorm on X, then a *dual tree* $D(K, \varepsilon, |\cdot|, \delta, \eta)$

is a set of points $x_{ij} \in X$, $i, j = 0, 1, \ldots,$ $j < 2^i$, such that for each such i, j,

$$x_{ij} = \tfrac{1}{2}(x_{i+1,2j} + x_{i+1,2j+1}), \qquad \|x_{i,2j} - x_{i,2j+1}\| \leqslant 2\delta$$

and

$$|x_{ij} + t(x_{i+1,2j} - x_{ij})| \geqslant |x_{ij}| + \varepsilon\delta|t| - \eta$$

for any $|t| \leqslant 1$.

If i is allowed to be only $\leqslant n$, we speak on the *dual n-tree* $D_n(K, \varepsilon, |\cdot|, \delta, \eta)$.

Dual trees can easily be constructed e.g. in l_1.

All the differentials of maps: $X \to Y$ are taken in the Fréchet sense and their continuity, uniform continuity and so on, is taken in the sense of $X \to L(X, Y)$ ($L(X, Y)$ is the Banach space of all bounded linear operators of X into Y with its supremum norm).

$\boldsymbol{N}$ ($\boldsymbol{R}$) denote the set of all positive integers (all reals, respectively).

We summarize the known results and the results of this note in

THEOREM 1. *The following properties of a real Banach space X are equivalent:*

(i) *X is superreflexive.*

(ii) *X admits a real-valued function with bounded nonempty support and uniformly continuous differential.*

(iii) *For any open cover $\mathscr{U}$ of X, there is a locally finite partition of unity on X subordinated to $\mathscr{U}$ and formed by functions with uniformly continuous differential.*

(iv) *Negation of : There is an $\varepsilon > 0$ and $K > 0$ such that for any $n \in \boldsymbol{N}$ and any $\delta \in (0, 1)$ and $\eta > 0$ there is a pseudonorm $|\cdot| \leqslant K\|\cdot\|$ on X and a dual tree $D_n(K, \varepsilon, |\cdot|, \delta, \eta) \subset X$.*

Proof. (i) ⇒ (ii) easily follows from the Enflö renorming theorem ([6]) of superreflexive Banach spaces mentioned above. If $\|\cdot\|$ is a uniformly Fréchet differentiable norm on X and $\varphi \in C^\infty(\boldsymbol{R})$ with $\varphi(0) > 0$, $\varphi(t) = 0$ for every $|t| > 1$, then $\varphi(\|x\|^2)$ is the desired function.

(ii) ⇒ (iv). We use a variant of an argument of E. B. Leach and J. H. M. Whitfield ([14]). Assume that non(iv) holds for some $K > 0$ and $\varepsilon \in (0, 1)$ and that X admits a real-valued function with uniformly continuous differential and such that $f(0) = 0$, $f(x) = 2$ for $\|x\| > K^{-1}$.

Choose $\delta \in (0, \varepsilon)$ from the uniform differentiability of f to ε, so such that

$$f(x+h) - f(x) \leqslant f'(x)h + \varepsilon\|h\| \quad \text{for each } x \in X,\ h \in X,\ \|h\| \leqslant \delta.$$

Now choose a positive integer n and a real number $\eta > 0$ such that $n\varepsilon\delta < 2$, $n(\varepsilon\delta - \eta) > 1$. Let $D_n(K, \varepsilon, |\cdot|, \delta, \eta) \equiv \{x_{ij}\} \subset X$ be a dual tree. Consider the function $f_1(x) = f(x - x_{00})$ on X. We have, for $i, j = 0, 1, 2, \ldots,$

$j < 2^i$, $i \leqslant n-1$,

$$f_1\big(x_{ij} + t(x_{i+1,2j} - x_{ij})\big) \leqslant f_1(x_{ij}) + f_1'(x_{ij})\big(t(x_{i+1,2j} - x_{ij})\big) + \varepsilon\delta$$

for any $|t| \leqslant 1$. So, choosing $t = \pm 1$ dependent on the sign of $f_1'(x_{ij})(x_{i+1,2j} - x_{ij})$, we have

$$f_1(x_{i+1,2j}) \leqslant f_1(x_{ij}) + \varepsilon\delta \quad \text{ or } \quad f_1'(x_{i+1,2j+1}) \leqslant f_1(x_{ij}) + \varepsilon\delta.$$

So, since $f_1(x_{00}) = f(0) = 0$, by induction on i, we have that there is a $j < 2^n$ such that $f_1(x_{nj}) \leqslant n\varepsilon\delta < 2$. On the other hand, since for each allowed i, j

$$|x_{ij} + t(x_{i+1,2j} - x_{ij})| \geqslant |x_{ij}| + |t|\varepsilon\delta - \eta,$$

we have similarly that

$$|x_{nj}| \geqslant |x_{00}| + n(\varepsilon\delta - \eta) > |x_{00}| + 1.$$

Thus, $\|x_{nj} - x_{00}\| \geqslant (1/K)|x_{nj} - x_{00}| > K^{-1}$ and $f(x_{nj} - x_{00}) = f_1(x_{nj}) < 2$, a contradiction.

(iv) ⇒ (i). If X is not superreflexive, neither is X^* ([10]), so, by another result of R. C. James ([9]), there is a Banach space $(Y, \|\cdot\|)$ with some ε-tree T_ε in its unit ball B_1, which is finitely representable in X. The ε-tree T_ε is a nondentable set, so (see e.g. [4]) neither is $\overline{B_1 + \overline{\operatorname{conv}}\, T_\varepsilon \cup (-T_\varepsilon)}$ which is a unit ball of some norm $|||\cdot|||$ on Y for which $\frac{1}{2}\|\cdot\| \leqslant |||\cdot||| \leqslant \|\cdot\|$. Thus $|||\cdot|||^*$, the dual norm on Y^*, is rough for some $\varepsilon < 1$ ([12]).

So, for each $n \in \boldsymbol{N}$ and each $\delta \in (0, 1)$, $\eta > 0$, we can construct a dual tree $D_n(1, \varepsilon, |||\cdot|||, \delta, \eta) \equiv \{x'_{ij}\} \subset Y^*$. Namely, choose $x'_{00} = 0$, x'_{10} — such a point that $|||x'_{10}||| = \delta$. Then, having chosen x'_{ij} for $i < k$, choose for $x'_{k-1,j}$, $j < 2^{k-1}$, by the roughness property of $|||\cdot|||^*$, a $v \in Y^*$, $|||v|||^* \leqslant 1$ such that

$$|||x'_{k-1,j} + tv|||^* \geqslant |||x'_{k-1,j}|||^* + |t|\varepsilon - \eta \quad \text{ for each } |t| \leqslant |||x'_{k-1,j}|||^* \geqslant \delta.$$

Now put $x'_{k,2j} = x'_{k-1,j} + \delta v$, $x'_{k,2j+1} = x_{k-1,j} - \delta v$. We have

$$|||x'_{k-1,j} + t(x'_{k,2j} - x'_{k-1,j})|||^* = |||x'_{k-1,j} + t\delta v|||^* \geqslant |||x'_{k-1,j}|||^* + |t|\delta\varepsilon - \eta$$

for any $|t| \leqslant 1$.

Let $E_n \subset Y$ be such a finite-dimensional subspace of Y that for the restriction map Re: $Y^* \to E_n^*$ we have

$$|||\mathrm{Re} f_{(E_n, |||\cdot|||)^*}|||^* \geqslant (1 - \varepsilon')|||f_{(Y, |||\cdot|||)^*}|||^*$$

for each $f \in \operatorname{sp} D_n(1, \varepsilon, |||\cdot|||, \delta, \eta)$, where $\varepsilon' < \min\big(1, \eta(\max|||x'_{ij}|||^*)^{-1}\big)$. Let T: $(E_n, \|\cdot\|) \to X^*$ be an isomorphism into X^* such that $\|T\| = 1$, $\|T^{-1}\| \leqslant 1 + \varepsilon' < 2$ and define for $x \in X$,

$$|x| = |||T^* x|||^*_{(E_n, |||\cdot|||)^*}.$$

Then $|x| \leqslant 2\|x\|$ for $x \in X$. Let us put $x_{ij} = \frac{1}{2}(T^*)^{-1}\mathrm{Re}\, x'_{ij}$, for $i \leqslant n$, $j < 2^i$. Then

$$\|x_{i,2j} - x_{i,2j+1}\| \leqslant \delta$$

and

$$\begin{aligned} |x_{ij} + t(x_{i+1,2j} - x_{ij})| &= \tfrac{1}{2}\big|\big|\big|\mathrm{Re}\big(x'_{ij} + t(x'_{i+1,2j} - x'_{ij})\big)\big|\big|\big|^*_{(E_n, |||\cdot|||)^*} \\ &\geqslant (1-\varepsilon')\,|||x'_{ij} + t(x'_{i+1,2j} - x'_{ij})|||^*_{(Y, |||\cdot|||)^*} \\ &\geqslant \tfrac{1}{2}|||x'_{ij} + t(x'_{i+1,2j} - x'_{ij})|||^* - \tfrac{1}{2}\varepsilon' \max |||x'_{ij}|||^* \geqslant \tfrac{1}{2}|||x'_{ij}|||^* + |t|\,\delta\varepsilon/2 - \eta \\ &\geqslant \tfrac{1}{2}|||\mathrm{Re}\, x'_{ij}|||^* + |t|\,\delta\varepsilon/2 - \eta = |x_{ij}| + |t|\,\delta\varepsilon/2 - \eta . \end{aligned}$$

So, $\{x_{ij}\} \equiv D_n(2, \varepsilon/2, |\cdot|, \delta, \eta)$ is a dual tree in X; non (iv) holds. Therefore (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv). Obviously, (iii) $\Rightarrow$ (ii). So, to complete the proof of Theorem 1, it remains to show (i) $\Rightarrow$ (iii). This will be done in three lemmas.

LEMMA 1. *Let S be a ring of continuous real-valued functions on a Banach space X satisfying the following conditions.*

(i) *For each $S_0 \subset S$ with $\{\mathrm{supp} f,\ f \in S_0\}$ discrete in X and* supp$f$ *bounded for each $f \in S_0$, there is a $g \in S$ with* $\mathrm{supp}\, g = \bigcup_{f \in S_0} \mathrm{supp} f$ *(where* $\mathrm{supp} f = f^{-1}(\boldsymbol{R} \setminus \{0\})$).

(ii) *For each nonnegative $f \in S$ and $\varepsilon > 0$ there is a $g \in S$ with $0 \leqslant g \leqslant 1$ and $g^{-1}(0) = f^{-1}(0)$ and $g^{-1}(1) = f^{-1}(\langle \varepsilon, \infty))$.*

(iii) *If U_1, U_2 are open subsets of X with disjoint closures and $f \in S$ satisfies $f(x) = 0$ for $x \notin U_1 \cup U_2$, then the function $f_1 \in S$, where*

$$f_1(x) = \begin{cases} f(x) & \text{for } x \notin U_2, \\ 0 & \text{for } x \in U_2. \end{cases}$$

Then X admits S-partitions of unity (locally finite, subordinated to any open cover) on X iff $\{\mathrm{supp} f,\ f \in S\}$ contains a σ-discrete basis of the topology of X.

Proof. Let $\{\mathrm{supp} f,\ f \in S\}$ contain a σ-discrete basis of the topology of X and let $\mathscr{U}$ be a cover of X by open bounded sets. We will construct a locally finite partition of unity $S_0 \subset S$ with supp$f$ refining for each $f \in S_0$.

Under our assumption, there are subsets S_i, $i \in \boldsymbol{N}$ of S such that $\mathscr{V}_i = \{\mathrm{supp} f,\ f \in S_i\}$ is a discrete refinement of $\mathscr{U}$ and $\bigcup_i \mathscr{V}_i$ covers X. By (i), there are functions $g_i \in S$ with $\mathrm{supp}\, g_i = \bigcup_{f \in S_i} \mathrm{supp} f$ for $i \in \boldsymbol{N}$. We can assume without loss of generality that $g_i \geqslant 0$ (otherwise replace g_i by g_i^2). Let $g_{ij} \in S$ be such that $0 \leqslant g_{ij} \leqslant 1$ and $g_{ij}^{-1}(0) = g_i^{-1}(0)$, $g_{ij}^{-1}(1) = g_i^{-1}(\langle 1/j, \infty))$ ((ii)). Let $n \to (i_n, j_n)$ be a bijection of $\boldsymbol{N}$ onto $\boldsymbol{N} \times \boldsymbol{N}$. We put $h_0 \equiv 0$ and $h_n = g_{i_n, j_n}$, $\lambda_n = h_n(1 - h_{n-1}) \ldots (1 - h_0)$ (similarly as in [18]) and $\lambda_{n,f}(x) = \lambda_n(x)$ if $x \in \mathrm{supp} f$ and $\lambda_{n,f}(x) = 0$ if $x \notin \mathrm{supp} f$, for $n \in \boldsymbol{N}$ and $f \in S_{i_n}$.

Since S is a ring, $\lambda_n \in S$ for all $n \in \boldsymbol{N}$; moreover, it follows from (iii) and the discreteness of $\{\mathrm{supp} f,\ f \in S_{i_n}\}$ that $\lambda_{n,f} \in S$ for all $n \in \boldsymbol{N}$ and $f \in S_{i_n}$. Furthermore, $\mathrm{supp}\, \lambda_{n,f} \subset \mathrm{supp} f$ and therefore $\{\mathrm{supp}\, \lambda_{n,f},\ n \in \boldsymbol{N},\ f \in S_{i_n}\}$ refines $\mathscr{U}$. So it remains to be checked that $\{\lambda_{n,f},\ n \in \boldsymbol{N},\ f \in S_{i_n}\}$ is a locally finite partition of unity.

Given $x \in X$, there is a neighborhood U of x such that $h_n | U = 1$ for some n. To see this, it suffices to take $i, j \in \boldsymbol{N}$ so that $g_i(x) > 1/j$ and n so that $(i_n, j_n) = (i, j)$; we may then let $U = \{x' \in X,\ g_i(x') > 1/j\}$.

Therefore it follows that $\lambda_k | U = 0$ for all but finitely many k's and since

$$(1 - h_1)(1 - h_2) \ldots (1 - h_k) = 1 - \lambda_1 - \ldots - \lambda_k,$$

we infer that $\{\lambda_k,\ k \in \boldsymbol{N}\}$ is a locally finite partition of unity. Moreover, $\{\mathrm{supp} f,\ f \in S_{i_k}\}$ is discrete in X and $\sum \lambda_{k,f} = \lambda_k$ for each $k \in \boldsymbol{N}$, and thus $\{\lambda_{k,f},\ k \in \boldsymbol{N}, f \in S_{i_k}\}$ is a locally finite partition of unity of X subordinated to $\mathscr{U}$.

The converse implication in Lemma 1 is clear. As any metrizable space, X has σ-discrete coverings $\mathscr{U}_n$, $n \in \boldsymbol{N}$ such that $\bigcup \mathscr{U}_n$ is a basis of the topology of X (see e.g. [7]). If $S_n \subset S$, $n \in \boldsymbol{N}$ are partitions of unity subordinated to $\mathscr{U}_n$, then $\bigcup \{\mathrm{supp} f,\ f \in S_n\}$ is a σ-discrete basis of the topology of X.

To use Lemma 1, we will need the following lemma.

LEMMA 2. *For any superreflexive Banach space X there is a homeomorphic embedding H of X into $l_2(\Gamma)$ for some Γ which is a differentiable map with the differential uniformly continuous on bounded sets of X.*

Proof. We use some arguments of [2]. First, for any superreflexive Banach space X there is a $p \in (1, \infty)$ and a one-to-one, norm 1 linear operator T of X into $l_p(\Gamma)$ for some Γ. It follows by the use of the result of J. Lindenstrauss on the existence a projectional resolution of identity in reflexive spaces ([1]) and the result of R. C. James ([11]) that for each superreflexive Banach space X there is a $p \in (1, \infty)$ such that for each such projectional resolution $\{P_a\}$ of identity on a subspace Y of X, we have $\left(\sum \|(P_{a+1} - P_a)x\|^p\right)^{1/p} = 2\|x\|$ for any $x \in Y$. Now, working with the class B_p of all superreflexive spaces which have p as this index, we can easily show the existence of T by use of the natural injection $u\colon X \to \left(\sum \oplus (P_{a+1} - P_a)X\right)_p$ and the induction on the density of X.

So, let $T\colon X \to l_p(\Gamma)$ be a one-to-one, norm 1 linear operator. Further we follow the argument of S. Mazur ([15]). Let $s = (r-2)/2 > p + 1$ be an even integer. Consider the one-to-one map $\Phi\colon l_p(\Gamma) \to l_2(\Gamma)$ defined by

$$\Phi x(\gamma) = x^{r/2}(\gamma).$$

We will show that Φ is differentiable with differential uniformly continuous on bounded sets of $l_p(\Gamma)$. For this, first observe that for $k, h \in l_p(\Gamma)$,

$$\lim_{t\to 0} t^{-1}\big(\Phi(k+th)(\gamma)-\Phi(k)(\gamma)\big)$$
$$= \lim_{t\to 0} t^{-1}\big((k+th)^{r/2}(\gamma)-k^{r/2}(\gamma)\big) = \tfrac{1}{2}rk(\gamma)^{(r-2)/2}h(\gamma) \in l_2(\Gamma).$$

Furthermore, whenever $k, h \in l_p(\Gamma)$, $\|k\| \leqslant c$, $\|h\| \leqslant \delta < 1$, then $\|k\|_{l_s(\Gamma)} \leqslant c$, $\|h\|_{l_s(\Gamma)} \leqslant \delta$ and by the use of the Hölder inequality and the inequality $|a^s - b^s| \leqslant s\ |a-b|(|a|+|b|)^{s-1}$ for $a, b \in \boldsymbol{R}$, $s \in \boldsymbol{N}$, we can estimate

$$\|\tfrac{1}{2}r(k+h)^s(\gamma)h(\gamma)-\tfrac{1}{2}rk^s(\gamma)h(\gamma)\|_{l_2(\Gamma)} \leqslant \|\tfrac{1}{2}r(k+h)^s(\gamma)-k^s(\gamma)\|_{l_1(\Gamma)}$$
$$\leqslant \frac{r}{2}\cdot s\cdot \sum |h(\gamma)|\big(|(k+h)(\gamma)|+|k(\gamma)|\big)^{s-1}$$
$$\leqslant \frac{rs}{2}\Big(\sum |h(\gamma)|^s\Big)^{s-1}\Big(\sum \big(|(k+h)(\gamma)|+|k(\gamma)|\big)^s\Big)^{(s-1)/s}$$
$$\leqslant \frac{rs\,\delta}{2}\,(2^s\|k+h\|^s_{l_s(\Gamma)}+\|k\|^s_{l_s(\Gamma)})^{(s-1)/s} \leqslant \delta\ rs\cdot 2^{s-1}\big((c+1)^s+c^s\big)^{1-1/s}.$$

Now, similarly as in [17], define

$$H\colon X \to l_2(\Gamma\cup 1) \qquad (1 \notin \Gamma)$$

by

$$Hx = (\Phi Tx, \|x\|^2),$$

where $\|\cdot\|$ is a uniformly Fréchet differentiable and uniformly convex norm on X ([6]) and Φ, T as above in this proof.

Then H is one-to-one with differential uniformly continuous on bounded sets of X. So, to complete the proof of Lemma 2 it remains to show that H is a homeomorphism into $l_2(\Gamma\cup 1)$. If $\lim H(x_n) = H(x)$ in $l_2(\Gamma\cup 1)$, $x_n, x \in X$, then by the form of Φ, $\lim Tx_n(\gamma) = Tx(\gamma)$ for each $\gamma \in \Gamma$ and $\lim \|x_n\| = \|x\|$. Since, moreover, X is reflexive and T one-to-one, we have $\lim x_n = x$ in the weak topology of X. Thus

$$2\|x\| \geqslant \limsup \|x_n + x\| \geqslant \liminf \|x_n + x\| \geqslant 2\|x\|$$

by the weak-lower semicontinuity of $\|\cdot\|$ By the uniform convexity of $\|\cdot\|$ we have $\lim x_n = x$ in X. Thus, to finish the proof of Theorem 1, it suffices to show

LEMMA 3. *Let X be a superreflexive Banach space and let $H\colon X \to l_2(\Gamma\cup 1)$ be a homeomorphic embedding constructed in Lemma 2. Let $\omega_1^i(\delta)$, $i = 1, 2$, $\delta > 0$ be moduli of continuity of H, H' on the unit ball of X, respectively. Let S be a ring of all real-valued functions f on X which are Fréchet differentiable and have the following property:*

For each $f \in S$ and $n \in \boldsymbol{N}$ there is a constant $c_n(f) > 0$ such that the moduli of continuity $\omega_n^i(\delta)$ of f, f', respectively, on the n-ball $B_n(0) \subset X$ satisfy

$$\omega_n^i(\delta) \leqslant c_n(f)\max \omega_1^i(\delta) \quad \text{for} \quad \delta > 0.$$

Then S satisfies (i)–(iii) *of Lemma* 1 *and* $\{\operatorname{supp} f, f \in S\}$ *contains a σ-discrete basis of the topology of X.*

Proof. Using the fact that uniformly continuous map on a ball in a Banach space is bounded, we easily see that S is actually a ring. Also, let us observe that H, H' have the property defining the ring S, because of their $r/2$ $(r/2-1)$ positive homogeneity, respectively.

(i) If $S_0 \subset S$ and $\{\operatorname{supp} f_\alpha, f_\alpha \in S_0\}$ discrete in X, and $\operatorname{supp} f_\alpha$ bounded for $f_\alpha \in S_0$, then by multiplying f_α's by some constants $c_\alpha > 0$ we ensure that all $c_\alpha f_\alpha$, $c_\alpha f'_\alpha$ have moduli of continuity $\leqslant \omega_1(\delta) \equiv \max_i \omega_1^i(\delta)$ for $\delta > 0$. Then define

$$f(x) = \begin{cases} f_\alpha(x) & \text{for} \quad x \in \operatorname{supp} f_\alpha, \\ 0 & \text{for} \quad x \in X \setminus \bigcup_{f_\alpha \in S_0} \operatorname{supp} f_\alpha. \end{cases}$$

Since $\{\operatorname{supp} f_\alpha, f_\alpha \in S_0\}$ is discrete, f is well defined, locally depends on one f_α and thus is differentiable. Moreover, if $x, y \in X$, $\|x-y\| \leqslant \delta$, $x \in \operatorname{supp} f_\alpha$, $y \in \operatorname{supp} f_\beta$, $\beta \neq \alpha$, then

$$|f(x)-f(y)| = |f_\alpha(x)-f_\beta(y)| \leqslant |f_\alpha(x)|+|f_\beta(y)|$$
$$= |f_\alpha(x)-f_\alpha(y)|+|f_\beta(y)-f_\beta(x)| \leqslant 2\omega_1(\delta);$$

similarly for f' and the other choice of $x, y \in X$. Thus $f \in S$ and $\operatorname{supp} f = \bigcup_{f_\alpha \in S_0} \operatorname{supp} f_\alpha$.

(ii) If $f \in S$, $\varepsilon > 0$, take a function $g\colon \boldsymbol{R} \to \boldsymbol{R}$, $g \geqslant 0$, which is Lipschitz together with its derivative and $g(\langle -\infty, 0\rangle) = 0$, $g(\langle \varepsilon, \infty)) = 1$. Then $g(f) \in S$ is the desired function for (ii).

(iii) If $f \in S$ and U_1, U_2, f_1 are as in (iii) of Lemma 2, and $x \in X$, then if $x \notin \overline{U}_2$, there is a neighborhood $O_1 \subset X$ of x such that $f_1 = f$ on O_1, so f_1 is differentiable at x. If $x \in \overline{U}_2$, then there is a neighborhood $O_2 \subset X$ of x with $f_1 = O$ on O_2, so f_1 is differentiable at x. Moreover, if $x, y \in B_n(0) \subset X$, $\|x-y\| \leqslant \delta$, then whenever $x, y \notin \overline{U}_2$,

$$|f_1(x)-f_1(y)| = |f(x)-f(y)| \leqslant c_n(f)\max \omega_1^i(\delta).$$

Similarly for f'. If $x \in \overline{U}_1$, $y \in \overline{U}_2$, then $f_1'(y) = 0$ and, by the simple connectedness argument, there is a point u on the line segment $\langle x, y\rangle \subset X$ such that $u \notin \overline{U}_1 \cup \overline{U}_2$. So, then

$$|f_1'(x)-f_1'(y)| = |f_1'(x)-f_1'(u)| = |f'(x)-f'(u)| \leqslant c_n(f)\max \omega_1^i(\delta).$$

Similarly for f. If $x \notin \overline{U}_1 \cup \overline{U}_2$, $y \in \overline{U}_2$, then

$$f_1(x) = f_1(y) = f_1'(x) = f_1'(y) = 0.$$

The same happens if $x, y \in \overline{U}_2$.

So, it remains to show that $\{\operatorname{supp} f, f \in S\}$ contains a σ-discrete basis of the topology of X. By the use of the Stone theorem on the existence of a σ-discrete basis of the topology of X formed by open bounded sets (cf. e.g. [7]), it suffices to show that for any open bounded set $O \subset X$ there is an $f \in S$ with $O = \operatorname{supp} f$. For it take $H(O) \subset l_2(\Gamma \cup 1)$ and an open bounded set $O_1 \subset l_2(\Gamma \cup 1)$ with $O_1 \cap H(X) = H(0)$. By a result of J. Wells ([18], Th. 2, Cor. 2) there is a function $f_1\colon l_2(\Gamma \cup 1) \to \boldsymbol{R}$ with Lipschitz derivative f_1' with $O_1 = \operatorname{supp} f_1$. Take $f = f_1(H)\colon X \to \boldsymbol{R}$. Then $O = \operatorname{supp} f$ and considering the estimation

$$\|f'(H(x))H'(x)h - f'(H(y))H'(y)h\| \leqslant \|f'(H(x))\| \, \|H'(x) - H'(y)\| + \|f'(H(x)) - f'(H(y))\| \cdot \|H(y)\|$$

we may easily derive that $f \in S$. This completes the proof of Lemma 3 and Theorem 1.

We end the paper with a remark that it was proved in [3] that there is no real-valued function with bounded nonempty support and Lipschitz differential on $l_p(\boldsymbol{N})$ for $p < 2$.

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