STUDIA MATHEMATICA, T. LXX. (1981)

Eventual continuity in Banach algebras of differentiable functions

by

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Abstract. Question 1 of recent paper of Badé, Curtis, and Laursen is answered. An eventually continuous homomorphism from $C^{(n)}$ into a Banach algebra which is eventually continuous is proved to be $C^{(2n+\varepsilon)}$-continuous for each $\varepsilon > 0$, but an example is given which shows that it need not be $C^{(2n)}$-continuous.

1. This note is a response to Question 1 raised by Badé, Curtis, and Laursen in [2].

If n is a positive integer, denote by $C^{(n)}$ or $C^{(n)}(I)$ the set of functions having at least n continuous derivatives on the unit interval, $I = [0, 1]$. With pointwise algebraic operations and norm

$$\|f\|_n = \sum_{k=0}^{n} \frac{1}{k!} \sup\{|f^{(k)}(t)|:\ t \in I\} \qquad (f \in C^{(n)}),$$

$C^{(n)}$ is a commutative Banach algebra. Let $\mathfrak{B}$ be a Banach algebra, let $\nu\colon C^{(n)} \to \mathfrak{B}$ be a homomorphism, and let k be a positive integer with $k \geqslant n$. Then ν is *$C^{(k)}$-continuous* if $\nu|(C^{(k)}, \|\cdot\|_k) \to \mathfrak{B}$ is continuous, and ν is *eventually continuous* if ν is $C^{(k)}$-continuous for some $k \geqslant n$. Two recent results proved by Badé, Curtis, and Laursen are the following:

PROPOSITION 1.1. *Let $\nu\colon C^{(n)} \to \mathfrak{B}$ be a homomorphism.*

(i) ([1], Theorem 2.5). *If ν is eventually continuous, then ν is $C^{(2n+1)}$-continuous.*

(ii) (from [2], Theorem 3.11). *If the radical of $\mathfrak{B}$ is finite-dimensional, then ν is $C^{(2n)}$-continuous.*

On the other hand, it follows from [5], Theorem 3.4, that, if $k < 2n$, then $\nu\colon C^{(n)} \to \mathfrak{B}$ need not be $C^{(k)}$-continuous even if $\mathfrak{B}$ is finite-dimensional.

These results leave open the question, raised as Question 1 in [2], whether or not every eventually continuous homomorphism from $C^{(n)}$ into a Banach algebra is necessarily $C^{(2n)}$-continuous. Here, I give an example which shows that this is not true (at least if the continuum hypothesis be assumed), but I prove in Theorem 2.1 that there is a sense in which the number "$2n+1$" of Proposition 1.1(i) can be replaced by "$2n+\varepsilon$ for each $\varepsilon > 0$".

In fact, the positive results can be given in the more general context discussed by Laursen in [7]. Let Y be a Banach space, let $B(Y)$ be the Banach algebra of bounded linear operators on Y, let $\mathfrak{A}$ be a commutative Banach algebra, and let $\varrho\colon \mathfrak{A} \to B(Y)$ be a homomorphism, not necessarily continuous. A linear operator $S\colon \mathfrak{A} \to Y$ is *of class* $\mathscr{I}$ (with respect to ϱ) if the map $S(a\,\cdot) - \varrho(a)S(\cdot)$, $\mathfrak{A} \to Y$, is continuous for each a in $\mathfrak{A}$. The class $\mathscr{I}$ is the class of *intertwining* operators. (A more general case is considered in [7], Definition 2.1.) Homomorphisms, derivations into modules, and centralizers are examples of such maps.

Now let Y be a Banach space, and let $S\colon C^{(n)} \to Y$ be of class $\mathscr{I}$. With definitions analogous to the above, Proposition 1.1 extends to this case. Let $\mathfrak{S}(S)$ be the separating space of S.

PROPOSITION 1.2. *Let $S\colon C^{(n)} \to Y$ be of class $\mathscr{I}$.*

(i) ([7], Theorem 5.5). *If S is eventually continuous, then S is $C^{(2n+1)}$-continuous.*

(ii) ([7], Theorem 5.26). *If $\mathfrak{S}(S)$ is finite-dimensional, then S is $C^{(2n)}$-continuous.*

2. We now introduce the algebras with which we shall be concerned.

If $\alpha \in (0,1]$, let $\mathrm{Lip}_\alpha I$ or Lip_α be the Lipschitz algebra (of order α) on I, so that $\mathrm{Lip}_\alpha I$ is the set of functions f on I with $p_\alpha(f) < \infty$, where

$$p_\alpha(f) = \sup\{|f(x)-f(y)|/|x-y|^\alpha\colon x, y \in I,\ x \neq y\}.$$

With pointwise operations, $\mathrm{Lip}_\alpha I$ is a Banach algebra on I with respect to the norm $\|\cdot\|_\alpha$, where

$$\|f\|_\alpha = |f|_I + p_\alpha(f) \qquad (f \in \mathrm{Lip}_\alpha I)$$

and $|f|_X = \sup\{|f(x)|\colon x \in X\}$ defines the uniform norm on a compact space X. (Note that our two definitions of $\|\cdot\|_1$ coincide on $C^{(1)}$, and that $C^{(1)}$ is a proper closed subalgebra of Lip_1.)

If $\alpha \in (0,1)$, then

$$\mathrm{lip}_\alpha I = \{f \in \mathrm{Lip}_\alpha I\colon |f(x)-f(y)|/|x-y|^\alpha \to 0 \text{ as } |x-y| \to 0\}.$$

It is noted in [9], 2.1, where the basic theory of these algebras is developed, that lip_α is a closed subalgebra of Lip_α (for $\alpha \in (0,1)$) and that each is a regular algebra on I.

For $\beta > 1$, write $\beta = n+\alpha$ with $n \in \mathbf{N}$ and $\alpha \in (0,1]$. Define $\mathrm{Lip}_\beta I$ or Lip_β by

$$\mathrm{Lip}_\beta I = \{f \in C^{(n)}(I)\colon f^{(n)} \in \mathrm{Lip}_\alpha I\}.$$

Then Lip_β is a Banach algebra on I with respect to the norm $\|\cdot\|_\beta$, where

$$\|f\|_\beta = \sum_{k=0}^{n} |f^{(k)}|_I + p_\alpha(f^{(n)}) \qquad (f \in \mathrm{Lip}_\beta I).$$

See [8], §4, for example. In fact $\|fg\|_\beta \leqslant C\|f\|_\beta\|g\|_\beta$ $(f, g \in \mathrm{Lip}_\beta)$, where C is a constant depending only on β. If $\alpha \in (0,1)$, $\mathrm{lip}_\beta I = \{f \in C^{(n)}(I)\colon f^{(n)} \in \mathrm{lip}_\alpha I\}$. Both Lip_β and lip_β are regular Banach algebras on I in these cases.

If $\beta = n+\alpha$, if $k \in \{0, 1, \ldots, n\}$, and if $t_0 \in I$, we set

$$M_{\beta,k}(t_0) = \{f \in \mathrm{Lip}_\beta I\colon f(t_0) = \ldots = f^{(k)}(t_0) = 0\}.$$

Write $M_{\beta,k}$ for $M_{\beta,k}(0)$.

Let Y be a Banach space, and let S be a linear operator from $C^{(n)}$ or Lip_β into Y such that S is of class $\mathscr{I}$. Then S is Lip_γ*-continuous* if $S|(\mathrm{Lip}_\gamma, \|\cdot\|_\gamma) \to Y$ is continuous for some γ with $\gamma > n$ or $\gamma \geqslant \beta$, and S is *eventually continuous* if it is Lip_γ-continuous for some γ.

The first result extends Proposition 1.2(i).

THEOREM 2.1. *Let $S\colon C^{(n)} \to Y$ be of class $\mathscr{I}$. If S is eventually continuous, then S is $\mathrm{Lip}_{2n+\varepsilon}$-continuous for each $\varepsilon > 0$.*

Proof. As in [7], Theorem 5.5, we can suppose that S has a singleton singularity set, say $\{t_0\}$. Let $I(S)$ be the continuity ideal of S. Since S is eventually continuous, $|z-t_0|^q$ belongs to $I(S)$ for some $q \in \mathbf{N}$, and so, by [7], 5.3, $|z-t_0|^{n+\frac{1}{2}\varepsilon}$ belongs to $I(S)$ for each $\varepsilon > 0$.

Now fix $\varepsilon > 0$ and take $f \in M_{2n+\varepsilon,2n}(t_0)$. Set $g = |z-t_0|^{-n-\frac{1}{2}\varepsilon}f$ (with $g(t_0) = 0$). We claim that $g \in C^{(n)}$ and that $\|g\|_n \leqslant K\|f\|_{2n+\varepsilon}$ for some constant K. In fact, if $t \in I\setminus\{t_0\}$, $g^{(n)}(t)$ is a finite sum of terms $|f^{(n-j)}(t)|/|t-t_0|^{n+j+\frac{1}{2}\varepsilon}$ for $j = 0, 1, \ldots, n$. But

$$\frac{|f^{(n-j)}(t)|}{|t-t_0|^{n+j}} \leqslant \sup\{|f^{(2n)}(s)|\colon s \in [t_0-|t-t_0|,\ t_0+|t-t_0|]\},$$

and so $|g^{(n)}(t)| \leqslant K_1 p_\varepsilon(f^{(2n)})|t-t_0|^{\frac{1}{2}\varepsilon}$ for some constant K_1. Thus, $g^{(n)}(t) \to 0$ as $t \to t_0$, so that $g \in C^{(n)}$. The claim then follows.

The remainder of the proof is the same as that in [7], 5.5, noting that $M_{2n+\varepsilon,2n}(t_0)$ is closed and of finite codimension in $\mathrm{Lip}_{2n+\varepsilon}$.

A similar proof establishes the next closely related result. It is only necessary to change the technical lemmas.

THEOREM 2.2. *Let $S\colon \mathrm{Lip}_\beta \to Y$ be of class $\mathscr{I}$. If S is eventually continuous, then S is $\mathrm{Lip}_{2\beta+\varepsilon}$-continuous for each $\varepsilon > 0$.*

LEMMA 1. *If $f \in M_{n+\alpha,n}$, where $n \in \mathbf{Z}^+$ and $\alpha \in (0,1)$, and if $\beta \in [0,\alpha)$, then $f/z^{n+\beta} \in \mathrm{Lip}_{\alpha-\beta}$ and $\|f/z^{n+\beta}\|_{\alpha-\beta} \leqslant Kp_\alpha(f^{(n)})$ for some constant K depending only on $n+\alpha$ and β. If, further, $f \in \mathrm{lip}_{n+\alpha}$, then $f/z^{n+\beta} \in \mathrm{lip}_{\alpha-\beta}$.*

Proof. Let $g = f/z^{n+\beta}$ with $g(0) = \lim_{t\to 0} g(t) = 0$. We shall estimate $|g(x)-g(y)|/|x-y|^{\alpha-\beta}$ for $x, y \in I$ with $x \neq y$. We can suppose that $y > x$. For $h \in \mathrm{Lip}_\alpha$ and $[u, v] \subset I$, set

$$p_{\alpha,[u,v]}(h) = \sup\{|h(s)-h(t)|/|s-t|^\alpha \colon s, t \in [u, v], s \neq t\}.$$

First suppose that $y \in (x, 2x]$. Then

$$\begin{aligned} &|f(x)-f(y)| \\ &\leqslant \sum_{k=1}^{n} \frac{1}{k!}|x-y|^k |f^{(k)}(x)| + \frac{1}{n!}|x-y|^{n+\alpha} p_{\alpha,[x,y]}(f^{(n)}) \\ &\leqslant \sum_{k=1}^{n} \frac{1}{k!(n-k)!}|x-y|^k x^{n-k+\alpha} p_{\alpha,[0,x]}(f^{(n)}) + \frac{1}{n!}|x-y|^{n+\alpha} p_{\alpha,[x,y]}(f^{(n)}) \\ &\leqslant K_1 |x-y| x^{n-1+\alpha}\big(p_{\alpha,[0,x]}(f^{(n)}) + p_{\alpha,[x,y]}(f^{(n)})\big) \end{aligned}$$

for some K_1, noting that $|x-y| \leqslant x$. Hence

$$\begin{aligned} \frac{|f(x)-f(y)|}{x^{n+\beta}|x-y|^{\alpha-\beta}} &\leqslant K_1 \left|\frac{x-y}{x}\right|^{1-\alpha+\beta} \big(p_{\alpha,[0,x]}(f^{(n)}) + p_{\alpha,[x,y]}(f^{(n)})\big) \\ &\leqslant 2K_1 p_\alpha(f^{(n)}). \end{aligned}$$

If $f \in \mathrm{lip}_{n+\alpha}$ and $\varepsilon > 0$, take $\eta > 0$ so that $p_{\alpha,[0,\eta]}(f^{(n)}) < \varepsilon/2K_1$, and then take $\delta > 0$ so that $(\delta/\eta)^{1-\alpha+\beta} < \varepsilon/2K_1 p_\alpha(f^{(n)})$ and so that $p_{\alpha,[x,y]}(f^{(n)}) < \varepsilon/2K_1$ if $|x-y| < \delta$. Then $|f(x)-f(y)|/x^{n+\beta}|x-y|^{\alpha-\beta} < \varepsilon$ if $|x-y| < \delta$.

Also, $|f(y)| \leqslant y^{n+\alpha} p_{\alpha,[0,y]}(f^{(n)})/n!$, so that

$$\frac{|f(y)|}{|x-y|^{\alpha-\beta}}\left|\frac{1}{x^{n+\beta}} - \frac{1}{y^{n+\beta}}\right| \leqslant \frac{1}{n!} p_{\alpha,[0,y]}(f^{(n)}) \left(\frac{y}{x}\right)^{\alpha-\beta} \frac{|(y/x)^{n+\beta}-1|}{|1-(y/x)|^{\alpha-\beta}}.$$

Let $\varphi(s) = s^{\beta-\alpha}\{(1+s)^{n+\beta}-1\}$ for $s > 0$. Since $\alpha-\beta < 1$, $\varphi(s) \to 0$ as $s \to 0+$, and so

$$\begin{aligned} \frac{|f(y)|}{|x-y|^{\alpha-\beta}}\left|\frac{1}{x^{n+\beta}} - \frac{1}{y^{n+\beta}}\right| &\leqslant \frac{2^{\alpha-\beta}}{n!} p_{\alpha,[0,y]}(f^{(n)}) |\varphi|_{[0,y/x-1]} \\ &\leqslant \frac{2^{\alpha-\beta}}{n!} p_\alpha(f^{(n)}) |\varphi|_{[0,1]}. \end{aligned}$$

Again, if $f \in \mathrm{lip}_{n+\alpha}$ and $\varepsilon > 0$, we can take $\delta > 0$ so that

$$\frac{|f(y)|}{|x-y|^{\alpha-\beta}}\left|\frac{1}{x^{n+\beta}} - \frac{1}{y^{n+\beta}}\right| < \varepsilon \quad (|x-y| < \delta).$$

Thus, since

$$\frac{|g(x)-g(y)|}{|x-y|^{\alpha-\beta}} \leqslant \frac{1}{x^{n+\beta}} \frac{|f(x)-f(y)|}{|x-y|^{\alpha-\beta}} + \frac{|f(y)|}{|x-y|^{\alpha-\beta}}\left|\frac{1}{x^{n+\beta}} - \frac{1}{y^{n+\beta}}\right|,$$

the result follows in this case.

Secondly, suppose that $y > 2x$. Then

$$\begin{aligned} \frac{|g(x)-g(y)|}{|x-y|^{\alpha-\beta}} &\leqslant \frac{|f(x)|}{x^{n+\beta}|x-y|^{\alpha-\beta}} + \frac{|f(y)|}{y^{n+\beta}|x-y|^{\alpha-\beta}} \\ &\leqslant \frac{1}{n!}\left(\left|\frac{x}{x-y}\right|^{\alpha-\beta} p_{\alpha,[0,x]}(f^{(n)}) + \left|\frac{y}{x-y}\right|^{\alpha-\beta} p_{\alpha,[0,y]}(f^{(n)})\right) \\ &\leqslant \frac{1}{n!}(1+2^{\alpha-\beta}) p_\alpha(f^{(n)}). \end{aligned}$$

If $|x-y| \to 0$, then, in this case, $x, y \to 0$. Thus, if $f \in \mathrm{lip}_{n+\alpha}$, then $|g(x)-g(y)|/|x-y|^{\alpha-\beta} \to 0$ as $|x-y| \to 0$.

The result follows.

LEMMA 2. *If $f \in M_{n+\alpha,k}$ where $n \in \mathbf{Z}^+$, $\alpha \in (0, 1)$, and $k \in \{0, 1, \ldots, n\}$, and if $\beta \in (0, \alpha)$, then $f/z^{k+\beta} \in \mathrm{Lip}_{n+\alpha-k-\beta}$ and $\|f/z^{k+\beta}\|_{n+\alpha-k-\beta} \leqslant K\|f\|_{n+\alpha}$ for some constant K. If, further, $f \in \mathrm{lip}_{n+\alpha}$, then $f/z^{k+\beta} \in \mathrm{lip}_{n+\alpha-k-\beta}$.*

Proof. If $g = f/z^{k+\beta}$, then $g^{(n-k)}$ is a finite sum of terms of the form $f^{(n-k-j)}/z^{k+j+\beta}$ for $j = 0, 1, \ldots, n-k$. The result then follows from Lemma 1.

LEMMA 3. *If $f \in M_{n+\alpha,n}$, where $n \in \mathbf{Z}^+$ and $\alpha \in (0, 1)$, and if $\gamma > 0$, then $z^\gamma f \in M_{n+\alpha,n}$ and $\|z^\gamma f\|_{n+\alpha} \leqslant K\|f\|_{n+\alpha}$ for some constant K. If, further, $f \in \mathrm{lip}_{n+\alpha}$, then $z^\gamma f \in \mathrm{lip}_{n+\alpha}$.*

Proof. Let $g = z^\gamma f$. Then $g^{(n)}$ is a finite sum of terms of the form $z^{\gamma-j} f^{(n-j)}$ for $j = 0, 1, \ldots, n$. By Lemma 1, $f^{(n-j)}/z^j \in \mathrm{Lip}_\alpha$ and $\|f^{(n-j)}/z^j\|_\alpha \leqslant K_j p_\alpha(f^{(n)})$ for some K_j $(j = 0, 1, \ldots, n)$. Also, $f^{(n-j)}/z^j \in \mathrm{lip}_\alpha$ if $f \in \mathrm{lip}_{n+\alpha}$. So it suffices to prove the result in the case $n = 0$. We can suppose that $\gamma < \alpha$. Then $|x^\gamma - y^\gamma|/|x-y|^\alpha \leqslant K_1 y^{\gamma-\alpha}$ for some K_1 and $y^{\alpha-\gamma}|x^\gamma - y^\gamma|/|x-y|^\alpha \to 0$ as $x/y \to 1$. So, if $g = z^\gamma f$, then

$$\frac{|g(x)-g(y)|}{|x-y|^\alpha} \leqslant x^\gamma \frac{|f(x)-f(y)|}{|x-y|^\alpha} + |f(y)| \frac{|x^\gamma - y^\gamma|}{|x-y|^\alpha},$$

and hence $p_\alpha(g) \leqslant p_\alpha(f) + K_1 p_\alpha(f)$. Also, if $f \in \mathrm{lip}_\alpha$ and $|x-y| \to 0$, then $|g(x)-g(y)|/|x-y|^\alpha \to 0$. The result follows.

LEMMA 4. *If $n \in \mathbf{N}$ and $\alpha \in (0, 1)$, and if J is a closed primary ideal of $\mathrm{lip}_{n+\alpha}$, then $J \supset \mathrm{lip}_{n+\alpha} \cap M_{n+\alpha,n}(t_0)$ for some $t_0 \in I$.*

Proof. This follows easily from the result of Sherbert ([9], 4.2) that, if $\alpha \in (0, 1)$, then each closed primary ideal of lip_α is a maximal ideal.

Note that the above result does not hold with lip_{n+a} replaced by Lip_{n+a}: see [9], 6.2.

Proof of Theorem 2.2. Let $S\colon \mathrm{Lip}_\beta \to Y$ be of class $\mathscr{I}$, and suppose that S is Lip_γ-continuous. By slightly increasing β if necessary, we may suppose that $\beta \notin N$. Suppose that $\beta = n+a$ with $n \in \mathbf{Z}^+$ and $a \in (0,1)$. Now $S\colon \mathrm{lip}_\beta \to Y$ is of class $\mathscr{I}$, and we define the continuity ideal $I(S)$ with respect to this map:

$$I(S) = \{f \in \mathrm{lip}_\beta\colon\ g \mapsto S(fg),\ \mathrm{lip}_\beta \to Y, \text{ is continuous}\}.$$

Since lip_β is a regular algebra, [7], 3.1, shows that S has a finite singularity set, which we take to be the singleton $\{0\}$. Since each closed primary ideal of lip_β has finite codimension in lip_β, we can apply [7], 3.2, to carry through the argument of [7], 5.5, and conclude that $I(S) \cap \mathrm{lip}_\gamma$ is closed in lip_γ. Hence, using Lemma 4 again, $z^q \in I(S)$ for some $q \in N$.

Lemma 3 shows that the map $f \mapsto z^\gamma f$ is a bounded linear operator on $\{f \in \mathrm{lip}_\beta\colon f(0) = \ldots = f^{(n)}(0) = 0\}$, and this is what is required to ensure that the analogue of [7], 5.2, holds. As in [7], 5.3, it follows that $z^{\beta+\varepsilon} \in I(S)$ for each $\varepsilon > 0$.

If $f \in M_{2\beta+\varepsilon,n}$ and $g = z^{-\beta-\frac{1}{2}\varepsilon} f$, then it follows from Lemma 2 that $g \in \mathrm{Lip}_{\beta+\frac{1}{2}\varepsilon}$. But $\mathrm{Lip}_{\beta+\frac{1}{2}\varepsilon} \subset \mathrm{lip}_\beta$, so that $g \in \mathrm{lip}_\beta$ and this suffices for us to conclude the proof as before.

3. In this section, we give the example which answers Question 1 of [2] in the negative.

THEOREM 3.1. *Let n belong to N. If the continuum hypothesis holds, then there is a Banach algebra $\mathfrak{B}$ and a unital homomorphism $\nu\colon C^{(n)}(I) \to \mathfrak{B}$ such that ν is $C^{(2n+1)}$-continuous, but such that ν is not $C^{(2n)}$-continuous.*

Proof. Let $M = \{f \in C(I)\colon f(0) = 0\}$, and let

$$L = \{f \in C(I)\colon |f(t)| = O(t^\varepsilon) \text{ as } t \to 0+ \text{ for each } \varepsilon > 0\}.$$

Then L is an ideal in $C(I)$ and $L \subsetneq M$. Take

$$f_0(t) = \begin{cases} (-\log t)^{-1} & (t \in (0,1]),\\ 0 & (t = 0),\end{cases}$$

so that $f_0 \in M$, but $f_0^n \notin L$ for any $n \in N$. Then there is a prime ideal, say P, with $L \subset P \subset M$ and $f_0 \notin P$.

Assuming the continuum hypothesis, there is a radical Banach algebra R and a unital homomorphism $\lambda\colon C(I) \to R^\#$ with $\ker\lambda = P$: here, $R^\#$ is the Banach algebra formed by adjoining an identity, say e, to R. (See [4] and [6]: the result is discussed in [3], Theorem 9.6.) In particular, $\lambda|L = 0$ and $\lambda(f_0) \neq 0$.

Let $\mathfrak{B} = C^n \times (R^\#)^{n+1}$ with coordinatewise addition and scalar multiplication, so that $\mathfrak{B}$ is a vector space. Identify C with the subfield Ce of $R^\#$, and let $\pi_k\colon \mathfrak{B} \to R^\#$ be the projection map of $\mathfrak{B}$ onto the $(k+1)$-st coordinate for $k \in \{0, 1, \ldots, 2n\}$. If $a, b \in \mathfrak{B}$, define the product ab in $\mathfrak{B}$ by

$$\pi_k(ab) = \sum_{i=0}^{k} \pi_i(a)\pi_{k-i}(b) \qquad (k = 0, 1, \ldots, 2n).$$

Then $\mathfrak{B}$ is a commutative algebra with identity $(1, 0, \ldots, 0)$. If $a \in \mathfrak{B}$, set

$$\|a\| = \sum_{k=0}^{2n} \|\pi_k(a)\|.$$

It is clear that $(\mathfrak{B}, \|\cdot\|)$ is a Banach algebra.

We next define linear maps $S_k\colon M_{n,n-1} \to C(I)$ for $k \in \{0, 1, \ldots, n\}$:

$$S_0(h) = \frac{h}{z^n} \qquad (h \in M_{n,n-1}),$$

and, if $k \geq 1$ and $S_0, \ldots, S_{k-1}$ have been defined, S_k is any linear map such that

$$S_k(h) = S_{k-1}(h/z) \qquad (h \in zM_{n,n-1}),$$
$$S_k(z^n) = 0.$$

Note that, if $h \in M_{n,n-1}$, then $h/z^n \in C(I)$: its value at 0 is $h^{(n)}(0)/n!$. Note also that $z^n \in M_{n,n-1} \setminus zM_{n,n-1}$, so that each S_k is welldefined.

For $f \in C^{(n)}$, define $\varrho(f) \in M_{n,n-1}$ by the formula

$$f = f(0) + zf'(0) + \ldots + \frac{z^{n-1}}{(n-1)!} f^{(n-1)}(0) + \varrho(f),$$

so that $z^{-n}\varrho(f) - f^{(n)}(0)/n!$ belongs to M. Define $T_k\colon C^{(n)} \to R^\#$ by

$$T_k = \lambda \circ S_k \circ \varrho \qquad (k = 0, 1, \ldots, n),$$

and define $\nu\colon C^{(n)} \to \mathfrak{B}$ by

$$\nu(f) = \left(f(0), f'(0), \ldots, \frac{1}{(n-1)!} f^{(n-1)}(0), T_0(f), \ldots, T_n(f)\right) \qquad (f \in C^{(n)}).$$

We claim that ν is an algebra homomorphism. It is clearly linear. To show that ν is multiplicative, we first make some preliminary calculations. Take $h \in M_{n,n-1}$ and $k, l \in \{0, 1, \ldots, n\}$. If $l < k$, then $S_k(z^l h) = S_{k-l}(h)$, and in particular $S_k(z^{n+l}) = 0$, so that $T_k(z^{n+l}) = 0$. If $l = k$, $S_k(z^k h) = S_0(h) = h/z^n$, so that $T_k(z^{n+k}) = e$. If $l > k$, $S_k(z^l h) = S_0(z^{l-k} h) = z^{l-k-n} h \in L$, so that $T_k(z^l h) = 0$. Also, if $h_1, h_2 \in M_{n,n-1}$, then $h_1 h_2 \in M^2_{n,n-1}$. But $M^2_{n,n-1} = z^n M_{n,n-1}$ ([5], 3.1(ii)), so that $T_k(h_1 h_2) = 0$ for $k = 0, 1, \ldots, n-1$, whereas $S_n(h_1 h_2) = S_0(z^{-n} h_1 h_2) = z^{-2n} h_1 h_2 = S_0(h_1) S_0(h_2)$, so that $T_n(h_1 h_2) = T_0(h_1) T_0(h_2)$ because λ is a homomorphism.

Now take $f, g \in C^{(n)}$. For $k \in \{0, 1, \ldots, n-1\}$,

$$\pi_k(\nu(fg)) = \frac{1}{k!}(fg)^{(k)}(0) = \pi_k(\nu(f)\nu(g)).$$

Also,

$$\varrho(fg) = \sum_{l=0}^{n-1} \frac{z^l}{l!}[f^{(l)}(0)\varrho(g) + g^{(l)}(0)\varrho(f)] + \\ + \sum_{l=0}^{n-1}\Big[\sum_{\substack{i,j=0\\ i+j=n+l}}^{n-1} \frac{1}{i!j!} f^{(i)}(0)g^{(j)}(0)\Big]z^{n+l} + \varrho(f)\varrho(g).$$

Hence, if $k \in \{0, 1, \ldots, n-1\}$,

$$T_k(fg) = \sum_{l=0}^{k} \frac{1}{l!}[f^{(l)}(0)T_{k-l}(g) + g^{(l)}(0)T_{k-l}(f)] + \\ + \Big[\sum_{\substack{i,j=0\\ i+j=n+k}}^{n-1} \frac{1}{i!j!} f^{(i)}(0)g^{(j)}(0)\Big]e,$$

so that $\pi_{n+k}(\nu(fg)) = \pi_{n+k}(\nu(f)\nu(g))$. Finally,

$$T_n(fg) = \sum_{l=0}^{n} \frac{1}{l!}[f^{(l)}(0)T_{n-l}(g) + g^{(l)}(0)T_{n-l}(f)] + T_0(f)T_0(g),$$

so that $\pi_{2n}(\nu(fg)) = \pi_{2n}(\nu(f)\nu(g))$. We have shown that ν is multiplicative, as required. Certainly, ν is unital.

Fix $\varepsilon > 0$. If $f \in C^{(2n+\varepsilon)}$, then

$$\varrho(f) = \sum_{k=1}^{2n} \frac{1}{k!} z^k f^{(k)}(0) + z^{2n}\tilde{f},$$

say, where $|\tilde{f}(t)| = O(t^{\varepsilon})$ as $t \to 0+$. Thus, $S_n(z^{2n}\tilde{f}) = \tilde{f} \in L$ and $T_k(f) = f^{(n+k)}(0)/(n+k)!$ $(k = 0, \ldots, n)$. Hence, on $C^{(2n+\varepsilon)}$, ν agrees with the map

$$f \mapsto \Big(f(0), f'(0), \ldots, \frac{1}{(2n)!} f^{(2n)}(0)\Big),$$

which is clearly continuous. In particular, ν is $C^{(2n+1)}$-continuous.

Let $g_0 = z^{2n} f_0$, so that $g_0 \in M_{2n,2n}$. Then

$$\nu(g_0) = (0, 0, \ldots, 0, \nu(f_0)) \neq 0.$$

But g_0 is the limit in $C^{(2n)}$ of polynomials $p_\mu \in M_{2n,2n}$, and $\nu(p_\mu) = 0$.

This shows that ν is not $C^{(2n)}$-continuous, and concludes the proof of the theorem.

Remark. A fairly similar example shows that "$2\beta + \varepsilon$ for each $\varepsilon > 0$" cannot, in general, be replaced by "2β" in Theorem 2.2.

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Received February 6, 1979 (1511)