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Spectral extension and power independence in measure algebras

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Abstract. Suppose that $\mu_1, \ldots, \mu_n$ are measures in some subalgebra N of the convolution algebra $M(G)$, where G is a locally compact abelian group. When is the joint spectrum of $\mu_1, \ldots, \mu_n$ in N the same as their joint spectrum in the whole Banach algebra? It is shown that the independent power hypotheses which are effective in the case $n = 1$ can be adapted to give positive results, but that there are serious obstructions to multidimensional spectral extension theorems.

1. Introduction. The underlying problem is the practical one of determining spectral properties of those bounded linear operators on L^1 which commute with translations. In view of Wendel's representation of these L^1-multipliers as convolution operators, one may, of course, regard much of the difficulty as that of spectral extension in $M(G)$, the commutative Banach algebra of all regular bounded Borel measures on a locally compact group G.

Joseph Taylor's cohomological investigations (summarized in [12]) have led, in particular, to a deep but simply stated criterion for extension of spectral *values*; while a much shallower result of W. Moran and the present author, [4], gives a condition for extension of individual *homomorphisms* (generalized characters) which has proved to be a powerful tool in applications. Abstract convolution measure algebra results of these two types from part of a methodology which has been developed over the years by many authors, notably Wiener and Pitt, Williamson, Šreider, Hewitt and Kakutani, Varopoulos. A characteristic technique has been to use measure theoretic singularity to deny the existence of algebraic relations, then to use the resulting "algebraic independence" to demonstrate that there is no obstruction to extension. In this regard we feel that Williamson's analysis of the Wiener–Pitt phenomenon in terms of "independent power" elements which need not themselves be based on independent subsets of the line, [14], has been particularly influential.

Here we show that existing notions of (polynomial) power independence fail completely to give joint spectral extension results. We substitute a stronger concept of "full polynomial independence" which avoids the straightforward obstructions. However our positive results are proved only for elements based on independent sets of the underlying group and we exhibit a new obstruction phenomenon of a more subtle kind.

The theorems just discussed focus on simultaneous extension of spectral values over sets of measures rather than on extension problems for generalized characters themselves. The latter question is also considered and it is shown, in particular, that a conjecture of Moran and the author (listed as Problem 845 in [6]) is true.

Section 2 organizes the technical background necessary for a discussion of spectral extension — the results there are essentially known. In Section 3 we settle the conjecture mentioned earlier and show the obvious ways in which spectral extension can fail. We give a more satisfactory definition of polynomial independence in Section 4 and obtain various positive results. Section 5 describes some unsuspected structure in $M(G)$ which places limitations on extension theorems.

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2. Technical background.

(2.1) *Spectra, homomorphisms generalized characters.* In order to discuss spectral extension it is necessary to decide upon an appropriate notion of subobject and to fix some notation.

We follow Taylor in defining an *L-subalgebra* of $M(G)$ as a norm-closed subalgebra N with the additional property that $\mu \in M(G)$, $\nu \in N$, $\mu \ll \nu$ (μ is absolutely continuous with respect to ν) implies that $\mu \in N$. It will be a technical convenience to focus attention on unital L-subalgebras so we make the convention that, given a subset X of $M(G)$, $N(X)$ denotes the L-subalgebra of $M(G)$ generated by X and $\delta(0)$. (This convention is intended to embrace mild abuses such as $N(\mu_1, \mu_2)$ for $N(\{\mu_1, \mu_2\})$ and $N(\mu; A)$ for the L-subalgebra generated by μ and the atomic measures, $\{\delta(g): g \in G\}$, on G. Because $N(\|\mu\|^{-1}|\mu|) = N(\mu)$, provided that μ is non-zero, we lose nothing by normalizing all generators to be probability measures.)

Given a unital L-subalgebra N and μ in N, we define the *spectrum*, $S_N(\mu)$ of μ in N, in the usual way, by

$$S_N(\mu) = \{z \in C: z\delta(0) - \mu \text{ has no inverse in } N\}.$$

We write also, $\Delta(N)$, for the Gelfand space of N, so that

$$S_N(\mu) = \{\varphi(z): \varphi \in \Delta(N)\}.$$

Given $\mu_1, \ldots, \mu_k$ in N we define the *joint spectrum*, $S_N(\mu_1, \ldots, \mu_k)$, of these measures in N, by

$$S_N(\mu) = \{(\varphi(\mu_1), \ldots, \varphi(\mu_k)): \varphi \in \Delta(N)\}.$$

In the case where $N = M(G)$, we write simply $S(\mu_1, \ldots, \mu_k)$ for $S_N(\mu_1, \ldots, \mu_k)$ and, for $N = N(\mu_1, \ldots, \mu_k)$, we write $S_{\text{loc}}(\mu_1, \ldots, \mu_k)$ for $S_N(\mu_1, \ldots, \mu_k)$.

It is a further convenience to represent the homomorphisms of N as *generalized characters*. Thus we regard each φ in $\Delta(N)$ as an element $\varphi = (\varphi_\mu)_{\mu\in N}$ of the product $\prod_{\mu\in N} L^\infty(\mu)$, subject to the consistency conditions

(i) $\mu \ll \nu$ implies $\varphi_\mu(x) = \varphi_\nu(x)$ (μ a.e. x),

(ii) $\varphi_{\mu*\nu}(x+y) = \varphi_\mu(x)\varphi_\nu(y)$ ($\mu\times\nu$ a.e. (x, y)).

Moreover the subset of all φ in $\prod_{\mu\in N} L^\infty(\mu)$ which satisfy (i), (ii), and $\sup_{\mu\in N} \|\varphi_\mu\|_\infty = 1$, endowed with the product of the $\sigma(L^\infty(\mu), L^1(\mu))$-topologies, may be regarded as a concrete realization of $\Delta(N)$. (This description, due to Šreider, [10], amounts essentially to a recognition that $\Delta(N)$ is the projective limit of the maximal ideal spaces of the single generator L-subalgebras of N. Thus the content is formal, but the formalism is particularly appropriate to spectral extension.)

For $\mu_1, \ldots, \mu_k \in M(G)$, we write

$$\Delta_{\text{loc}}(\mu_1, \ldots, \mu_k) = \{(\varphi_{\mu_1}, \ldots, \varphi_{\mu_k}): \varphi \in \Delta N(\mu_1, \ldots, \mu_k)\},$$

$$\Delta_{\text{loc}}(\mu_1, \ldots, \mu_k; A) = \{(\varphi_{\mu_1}, \ldots, \varphi_{\mu_k}): \varphi \in \Delta N(\mu_1, \ldots, \mu_k; A)\},$$

$$\Delta(\mu_1, \ldots, \mu_k) = \{(\varphi_{\mu_1}, \ldots, \varphi_{\mu_k}): \varphi \in \Delta M(G)\},$$

and regard these sets as topological subspaces of $\prod_{i=1}^k L^\infty(\mu_i)$ endowed with the product of the $\sigma(L^\infty(\mu_i), L^1(\mu_i))$-topologies.

PROPOSITION 1. *$\Delta_{\text{loc}}(\mu_1, \ldots, \mu_k)$ is topologically isomorphic with $\Delta N(\mu_1, \ldots, \mu_k)$.*

Proof. The obvious restriction map from $\Delta N(\mu_1, \ldots, \mu_k)$ to $\Delta_{\text{loc}}(\mu_1, \ldots, \mu_k)$ is continuous and surjective so that the content of the proposition reduces to the assertion that this map is injective. Because $N(\mu_1, \ldots, \mu_k) = N(\omega)$, where $\omega = \exp(\sum_{i=1}^k |\mu_i|)$, every measure in $N(\mu_1, \ldots, \mu_k)$ is absolutely continuous with respect to some (infinite) linear combination of the monomials $|\mu_1|^{n_1} * |\mu_2|^{n_2} * \ldots * |\mu_k|^{n_k}$ (where we write $|\mu|^0 = \delta(0)$). It follows from the consistency conditions (i), (ii) of the definition of generalized character that each φ in $\Delta N(\mu_1, \ldots, \mu_k)$ is determined by the coordinate functions $\varphi_{\mu_1}, \ldots, \varphi_{\mu_k}$. The result follows.

(2.2) *Local spectra and extension.* Spectral extension results are effective only where we can compute local spectra and there is a serious difficulty in this regard in the case of $M(G)$. In particular there is no distinction between a single generator L-subalgebra and a countably generated L-subalgebra unless the mode of presentation is taken into account. In fact, given $\{\mu_n\}_{n=1}^{\infty}$ in $M(G)$, we may choose (positive) normalizing constants $\{a_n\}_{n=1}^{\infty}$ such that $\omega = \exp(\sum_{n=1}^{\infty} a_n|\mu_n|)$ is a probability measure in $M(G)$. Then $\omega^2 \ll \omega$ and

$$N(\{\mu_n\}_{n=1}^{\infty}) = N(\omega) = L^1(\omega).$$

Now we obtain the simple looking statement that

$$\Delta_{\mathrm{loc}}(\omega) = \{f \in L^{\infty}(\omega)\colon f(x+y) = f(x)f(y) \ (\omega\times\omega \text{ a.e. } (x, y))\}.$$

When it comes to the practical problem of checking the criterion on the right hand side of the last formula, we are at once faced with the problem of describing the $\omega\times\omega$ null sets and typically this involves finding suitable generators $\{\mu_n\}_{n=1}^{\infty}$ for $L^1(\omega)$! Moreover even when this has been done the semicharacter property of f typically involves delicate arithmetic —algebraic considerations. (See e.g. [7].) Accordingly the practical localization problem is *not* that of describing $\Delta_{\mathrm{loc}}(\omega)$ for every probability measure ω in $M(G)$ which satisfies $\omega^2 \ll \omega$, because this project is unrealistically ambitious. An apparently more reasonable programme is to seek to describe "simple" classes of ω for which it is possible to deduce information concerning $\Delta_{\mathrm{loc}}(\omega)$ and $S_{\mathrm{loc}}(\omega)$, then to find and use extension theorems to convert this to information concerning $\Delta(\omega)$, $S(\omega)$, $\Delta_{\mathrm{loc}}(\omega_1, \omega_2)$, $\Delta(\omega_1, \omega_2)$, $S(\omega_1, \omega_2)$, etc.

Progress to date has not involved *joint* spectral extension theorems but here we state the two main spectral extension theorems which are already known. Both are valid in general convolution measure algebras (the second remains valid under even weaker hypotheses) but we confine attention to $M(G)$.

In order to state Taylor's result we recall his definition that the *spine* of $M(G)$ is the L-subalgebra generated by the group algebras, i.e. the spine is $\bigoplus_{\tau} L^1(G_\tau)$, where τ ranges over the set T of all locally compact group topologies on G which are at least as fine as the original topology. The spine is closed under the involution, $\tilde{}$, defined by

$$\tilde{\mu}(E) = \overline{\mu(-E)} \quad (E \text{ Borel}, \ \mu \in M(G)).$$

THEOREM 1 (Taylor). *Let N be a unital L-subalgebra of $M(G)$ such that the intersection of N with the spine of $M(G)$ is $\tilde{}$-closed. Then, for every μ in N,*

$$S(\mu) = S_N(\mu).$$

COROLLARY. *If μ in $M(G)$ is hermitian* [*i.e.* $\mu = \tilde{\mu}$], *then*

$$S(\mu) = S_{\mathrm{loc}}(\mu).$$

Despite its simple appearance the corollary was proved by a long, ingenious, and highly technical excursion through cohomology theory and the proof has defied all attempts at simplification. The next result is as simple as it looks but gives more information in those particular situations where it applies.

THEOREM 2 (Brown–Moran). *Let N be a unital L-subalgebra of $M(G)$. Suppose that φ belongs to $\Delta(N)$ and satisfies*

$$|\varphi_\mu(x)| = 1 \quad (\mu \text{ a.e. } x), \quad \textit{for all } \mu \textit{ in } N.$$

Then there exists ψ in $\Delta(M(G))$ such that $\psi(\mu) = \varphi(\mu)$, for all μ in N.

COROLLARY. *For each μ in $M(G)$, the unimodular elements of $\Delta_{\mathrm{loc}}(\mu)$ belong to $\Delta(\mu)$.*

We noted that extension theorems should be combined with descriptions of local spectra for "simple" measures. Moreover it is not unreasonable to describe "simple" in terms of the local spectrum, provided there are results ensuring that measures of the appropriate kind exist in abundance.

A probability measure μ in $M(G)$ is said to be *monotrochic* (respectively *strongly monotrochic*) if each element of $\Delta(\mu)$ (resp. $\Delta_{\mathrm{loc}}(\mu)$) has constant modulus. μ is said to be *tame* (respectively *strongly tame*) if each element of $\Delta(\mu)$ (resp. $\Delta_{\mathrm{loc}}(\mu)$) is of the form $a\gamma$, where a is a complex number whose modulus does not exceed one and γ belongs to $G\hat{}$, the dual of G.

The existence of large classes of strongly tame measures (most Riesz products and certain infinite convolutions) was established in [1]. (It should be noted that the hypothesis that the support of δ_i contains 0 was omitted from the statement of Proposition 7 of that paper, and that the formula in the tenth line of the proof should read

$$*\big(\delta(d_j) * \tau_N\big)^{n_j} * \tau_N^{n_j'} = *\big(\delta(d_j') * \tau_N\big)^{n_j'} * \tau_N^{n_j}.)$$

Convergent convolutions of discrete probability measures are monotrochic (as noted in [5]) and the ideas go back to an example, for the case $G = \mathbf{Z}(2)^{\omega}$, given in [11].

If μ is a monotrochic probability measure in $M(G)$ then either μ belongs to $\mathrm{Rad}\, L^1(G_\tau)$ for some τ in T or $S(\mu)$ is the unit disc.

(2.3) *The rôle of independence.* Apart from the fact we need the next proposition later, we write the proof in such a way as to focus on the problems present in the general determination of local spectra. That makes it possible to abbreviate some later proofs. The proposition itself is essentially well known and may be regarded as an elaboration of results of Hewitt and Kakutani in [9]. (See also Šreider [10].) The *existence* of a perfect independent subset of G was, of course, a significant result of the original papers.

PROPOSITION 2. *Let K be a perfect subset of G which is independent in the sense that, for $n_1, \ldots, n_r \in Z$, $k_1, \ldots, k_r \in K$,*

$$n_1k_1 + n_2k_2 + \ldots + n_rk_r = 0 \quad \text{implies} \quad n_ik_i = 0 \quad (1 \leqslant i \leqslant r).$$

Let $K_1, \ldots, K_s$ be disjoint subsets of K and let $\mu_1, \ldots, \mu_s$ be continuous probability measures concentrated on $K_1 \cup (-K_1), \ldots, K_s \cup (-K_s)$, respectively. Then $\Delta(N(\mu_1, \ldots, \mu_s; A))$ is isomorphic with $\prod_{i=1}^{s} L_D^\infty(\mu_i) \times G_d\hat{}$.

($L_D^\infty(\mu_i)$ denotes the unit ball of $L^\infty(\mu_i)$ and G_d is G endowed with the discrete topology.)

COROLLARY. *Under the conditions of the proposition,*

$$\Delta(\mu_1, \ldots, \mu_s) = \Delta_{\mathrm{loc}}(\mu_1, \ldots, \mu_s; A) = \Delta_{\mathrm{loc}}(\mu_1, \ldots, \mu_s).$$

Proof of Proposition 2. To preserve the notation μ^n for the nth convolution power, let us write $\mu^{[n]}$ for the n-fold product $\mu \times \mu \times \ldots \times \mu$. It is a convenience to write also $\mu^{[0]} = \delta(0)$, $G^0 = \{0\}$. The idea is to show that given $f_i \in L_D^\infty(\mu_i)$, $i = 1, \ldots, s$; $\gamma \in G_d\hat{}$, we can build a generalized character φ on $N(\mu_1, \ldots, \mu_s; A)$ which is uniquely determined by the properties that

$$\varphi_{\delta(g)}(x) = \gamma(x) \ (\delta(g) \text{ a.e.}), \quad \text{for all } g \text{ in } G,$$

$$\varphi_{\mu_i}(x) = f_i(x) \ (\mu_i \text{ a.e.}), \quad \text{for } i = 1, \ldots, s.$$

It is clear that the rest of the proof is a straightforward exercise along the lines of the proof of Proposition 1 and that there would be no hope of proving this for arbitrary f_i in the absence of a strong independence condition. Accordingly let us fix $f_1, \ldots, f_s; \gamma$ and see what consistency conditions arise. We note that every member ν of $N(\mu_1, \ldots, \mu_s; A)$ is a countable linear combination of measures, each absolutely continuous with respect to some monomial, $\delta(g) * \mu_1^{n_1} * \ldots * \mu_s^{n_s}$ so the consistency conditions must be imposed on subsets of full measure with respect to product measures of the form $\delta(g) \times \mu_1^{[n_1]} \times \ldots \times \mu_s^{[n_s]}$. We may write the typical element of the support of the latter measure in the form $(g, v_1, \ldots, v_s)$, where each $v_i = \{x(i, 1), \ldots, x(i, n(i))\} \in G^{n_i}$. Writing $\sum$ for the operation of summing all coordinates (so that $\sum$ is a map from $G \times G^{n_1} \times \ldots \times G^{n_s}$ to G) it is clear that we must define the coordinate of φ which lies in $L^\infty(\delta(g) * \mu_1^{n_1} * \ldots * \mu_s^{n_s})$ by the formula

$$\varphi_\nu\Big(\sum(g, v_1, \ldots, v_s)\Big) = \gamma(g) \prod_{i,j} f_i(x(i, j)).$$

This imposes two sets of consistency conditions. For each fixed monomial

we must be able to factor through $\sum$ (even if $g = 0$, $n_1 = 2$, $n_i = 0$, $i > 2$, this is a serious constraint in the general case) and we must take account of overlap of different monomials. All these problems vanish in the independent case under discussion after we make the preliminary observation that we may suppose that $x(i, j) \neq x(i, j')$ for $j \neq j'$. (This is a simple consequence of Fubini's theorem because the measures μ_i are supposed continuous and we work $\delta(g) \times \mu_1^{[n_1]} \times \ldots \times \mu^{[n_s]}$ a.e.) It is now clear that the independence of K implies that $\sum(g, v_1, \ldots, v_s)$ determines the vector $(g, v_1, \ldots, v_s)$ up to a permutation of the coordinates of each v_i. Accordingly we can certainly factor through $\sum$ and the measures $\delta(g) * \mu_1^{n_1} * \ldots * \mu_s^{n_s}$, $\delta(h) * \mu_1^{m_1} * \ldots * \mu_s^{m_s}$ are mutually singular unless $g = h$, $n_i = m_i$, $i = 1, \ldots, s$.

Proof of Corollary. Because of the special nature of the result this can be tackled by purely Banach algebra methods as in the original papers. It is also possible to apply Theorem 2 because the unimodular elements of $\prod_{i=1}^{s} L_D^\infty(\mu_i)$ are (weak*) dense in that space.

On this basis of his analysis of the Wiener–Pitt phenomenon, Williamson, [14], emphasized that several results depended only on the lack of algebraic relations in $M(G)$ for measures supported on independent sets. In other words the independence phenomenon occurs in the measure algebra rather than in the underlying group and may be exhibited by measures which are not supported on independent sets. Moran and the author in [4] and later papers developed this theme by relating Williamson's concepts to properties on generalized characters. Terminology is by no means standard, so we must give some definitions.

DEFINITION 1. Let F be a family of positive measures in $M(G)$. F is called *polynomially independent* if

$$(*) \qquad \mu_1^{n_1} * \ldots * \mu_k^{n_k},\ \mu_1^{m_1} * \ldots * \mu_k^{m_k} \text{ are mutually singular},$$

whenever $\mu_1, \ldots, \mu_k$ are distinct measures in F and $(n_1, \ldots, n_k)$, $(m_1, \ldots, m_k)$ are distinct k-tuples of non-negative integers; $k = 1, 2, \ldots$ If $(*)$ can be sharpened to the assertion that every translate of $\mu_1^{n_1} * \ldots * \mu_k^{n_k}$ is singular to (every translate of) $\mu_1^{m_1} * \ldots * \mu_k^{m_k}$, then F is called *strongly polynomially independent*. In the special case where F is the singleton $\{\mu\}$, the statement that F is (strongly) polynomially independent may be recast in the form, μ has (*strongly*) *independent powers*. (As before $\mu^0 = \delta(0)$.)

The next two results are simple generalizations of results in [4], [6].

PROPOSITION 3. *The probability measures $\mu_1, \ldots, \mu_s$ in $M(G)$ are strongly polynomially independent if and only if $\Delta_{\mathrm{loc}}(\mu_1, \ldots, \mu_s; A)$ contains $\prod_{i=1}^{s} C_i$, where C_i denotes the constant functions in $L^\infty(\mu_i)$.*

Proof. Consider the proof of Proposition 2. Because there is no problem in factoring through $\sum$ for constant functions, a similar argument shows that $\prod_{i=1}^{s} C_i \subset \Delta_{\mathrm{loc}}(\mu_1, \ldots, \mu_s; A)$ when $\{\mu_1, \ldots, \mu_s\}$ is strongly polynomially independent.

Suppose conversely that $\varphi \in \Delta(N(\mu_1, \ldots, \mu_s; A))$ satisfies $\varphi_{\mu_i} \equiv r_i$, $1 \leqslant i \leqslant s$, where the r_i are distinct real numbers in the interval $]0, 1[$. Suppose that ω is absolutely continuous with respect to both $\delta(g) * \mu_1^{n_1} * \ldots \ldots * \mu_s^{n_s}$ and $\delta(h) * \mu_1^{m_1} * \ldots * \mu_s^{m_s}$. Using (i) and (ii) of the definition of generalized character we see that $|\varphi_\omega|$ is constant and equal to both $|\varphi_{\delta(g)}(g)| r_1^{n_1} \ldots r_s^{n_s}$ and $|\varphi_{\sigma(h)}(h)| r_1^{m_1} \ldots r_s^{m_s}$. Because point measures are invertible $\varphi_{\delta(g)}(g)$, $\varphi_{\delta(h)}(h)$ have modulus one, and it follows readily that $\mu_1, \ldots, \mu_s$ are strongly polynomially independent.

PROPOSITION 4. *The probability measures $\mu_1, \ldots, \mu_s$ in $M(G)$ are polynomially independent if and only if $\Delta(\mu_1, \ldots, \mu_s)$ contains $\prod_{i=1}^{s} U_i$, where U_i denotes the unimodular constant functions in $L^\infty(\mu_i)$.*

Proof. At the local level the result is proved in a manner similar to the last proposition. Extension is guaranteed by the corollary to Theorem 2.

The next result is an easy consequence of Theorem 1 and Proposition 3. It can also be proved via Proposition 4 and the corollary of Theorem 2 a route which is more elementary and entirely so when $G = R$.

PROPOSITION 5. *Let μ be probability measure in $M(G)$. If either μ is hermitian and has independent powers or if μ has strongly independent powers, then $S(\mu) = D$.*

3. Failure of extension.

(3.1) *Measures with strongly independent powers.* In [6], the one-dimensional version of Proposition 3 was applied to certain Bernoulli convolutions (such as Cantor measure) to prove strong independence in a quick indirect way. In view of the remarks at the end of (2.2) we now know that a Bernoulli convolution or a Riesz product, μ, enjoys the property of having strongly independent powers if and only if $\Delta(\mu)$ contains some constant function lying strictly between zero and one. Does this extend to all probability measures? A negative answer was conjectured in [6] and we now confirm this. Thus we show that Proposition 3 needs to be localized although Proposition 4 does not.

LEMMA 1. *Let ν_1, ν_2 be strongly polynomially independent probability measures and let $\mu = \frac{1}{2}(\nu_1 + \nu_1^2 * \nu_2)$. Then μ is strongly independent but $\Delta(\mu)$ contains no element whose modulus is a constant strictly between* 0 *and* 1. *Moreover there is an idempotent in $\Delta_{\mathrm{loc}}(\mu, A) \setminus \Delta(\mu)$.*

Proof. Note first that $\{\nu_1, \nu_1^2 * \nu_2\}$ is a strongly polynomially independent set. In fact $\nu_1^p (\nu_1^2 * \nu_2)^q$ can overlap $\nu_1^r (\nu_1^2 * \nu_2)^s$ only if $p + 2q = r + 2s$ and $q = s$, in which case $p = r$ and $q = s$. It is also easy to see that $\tau_1 + \tau_2$ is strongly independent if $\{\tau_1, \tau_2\}$ is strongly polynomially independent (because in the latter event $\tau_1^p \tau_2^q$ certainly cannot overlap $\tau_1^r \tau_2^s$ unless $p + q = r + s$) and it follows that μ is strongly independent.

Now suppose that $\varphi \in \Delta M(G)$ and that $|\varphi_\mu|$ equals (μ a.e.) some constant r. Because $\nu_1 \ll \mu$, it follows that $|\varphi_{\nu_1}| = r$ (ν_1 a.e.). Hence for (g_1, g_2, g_3) in a set of full measure for $\nu_1 \times \nu_1 \times \nu_2$,

$$|\varphi_{\nu_1 * \nu_1 * \nu_2}(g_1 + g_2 + g_3)| = r^2 |\varphi_{\nu_2}(g_3)| \leqslant r^2.$$

The fact that $\nu_1^2 * \nu_1 \ll \mu$ now forces $r^2 = r$ and we have shown that $\Delta(\mu)$ admits no constant whose modulus lies strictly between 0 and 1.

A similar argument shows that the idempotent which is zero on the support of ν_1 and one on the support of ν_2 cannot belong to $\Delta(\mu)$. On the other hand this idempotent *does* belong to $\Delta_{\mathrm{loc}}(\mu, A)$, as can be seen by an application of Proposition 3 with $\mu_1 = \nu_1$, $\mu_2 = \nu_1^2 * \nu_2$. This completes the proof of the lemma.

Remarks. (i) The lemma gives a strong affirmation of the conjecture in the sense that we may choose μ to be a hermitian measure in $M_0(G)$. Perhaps the easiest choice is to take ν_1, ν_2 concentrated on $K_1 \cup (-K_1)$, $K_2 \cup (-K_2)$ as in Proposition 2, where K is chosed by Körner's techniques to be a set of multiplicity. It is also possible to use Riesz products.

(ii) The last part of the statement indicates a limitation on generalizations of Theorem 2, but similar counter-examples were already known.

Faced with these examples we may feel that it was quite unrealistic to demand a constant function in the first place. However the next result, which is much more delicate, comes close to showing that Proposition 5 cannot be improved in any generality.

PROPOSITION 6. *Let G be a non-discrete locally compact abelian group. Then there is a strongly independent hermitian measure, μ, in $M_0(G)$ with the property that for each $\varphi \in \Delta M(G)$ such that $\varphi_\mu \not\equiv 0$,*

$$\operatorname{ess\,sup}\{|\varphi_\mu(g)| \colon g \in \operatorname{support} \mu\} = 1.$$

Proof. We take from [3] the existence of a family $(\nu_t)_{t>0}$ of tame singular hermitian probability measures indexed by the positive reals and lying in $M_0(G)$ such that

(i) $\delta(g) * \nu_t \perp \nu_s$, whenever $s \neq t$,

(ii) ν_{s+t}, $\nu_s * \nu_t$ are equivalent measures.

Let us choose a countable set, X, of positive real numbers which has zero as a limit point but which is independent over the rationals. We enumerate X as $(x_n)_{n=1}^\infty$ and define the measure μ by

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \nu_{x_n}.$$

To check that μ has strongly independent powers, we consider an overlap between $\delta(g)*\mu^p$, μ^q. In view of (i) this would force (at least) one relationship of the form

$$\sum_{n=i}^{\infty} a_n x_n = \sum_{n=1}^{\infty} b_n x_n, \qquad \sum_{n=1}^{\infty} a_n = p, \qquad \sum_{n=1}^{\infty} b_n = q,$$

where the a_n, b_n are non-negative integers. Each relation is of course finite and the independence of X ensures that $p = q$.

Now consider any $\varphi \in \Delta M(G)$ which is not identically zero on μ. Then there is some n such that $\varphi_{\nu_n} \neq 0$ and, by tameness, $|\varphi_{\nu_n}|$ is a non-zero constant. In fact, for each $t > 0$, $|\varphi_{\nu_t}|$ is a non-zero constant and (in consequence of (ii))

$$|\varphi_{\nu_t}|\,|\varphi_{\nu_s}| = |\varphi_{\nu_{t+s}}| \qquad (s, t > 0).$$

It follows that there exists some k with $0 < k \leqslant 1$ such that

$$|\varphi_{\nu_t}| = k^t \qquad (t > 0).$$

Now it is clear that

$$\operatorname{ess\,sup}\{|\varphi_\mu(g)|:\ g \in \operatorname{support}\mu\} = \sup_n k^{x_n} = 1.$$

This completes the proof.

(3.2) *Failure of extension.* To ask for a full generalization of Taylor's extension theorem (Theorem 1) would be to ask that, given a unital L-subalgebra N of $M(G)$ whose intersection with the spine of $M(G)$ is $\tilde{\ }$-closed, one has perfect extension of generalized character, i.e. every element of $\Delta(N)$ extends to $\Delta(M(G))$. This is clearly too ambitious although we are not aware of any counter-example existing in the literature. It is a little more plausible to seek to extend Proposition 5 to the case of joint spectra. We now give a result which settles all these questions in the negative — even for $M_0(G)$. The technique is not very far removed from that used in proving Lemma 1.

PROPOSITION 7. *Let G be a non-discrete locally compact abelian group and let n be any integer strictly greater than one. There exists a strongly polynomially independent family $\{\mu_1, \ldots, \mu_n\}$ of hermitian probability measures in $M_0(G)$ such that*

$$S_{\mathrm{loc}}(\mu_1, \ldots, \mu_n; A) = S_{\mathrm{loc}}(\mu_1, \ldots, \mu_n) = D^n$$

but

$$S(\mu_{\sigma(1)}, \ldots, \mu_{\sigma(s)}) \neq D^s,$$

whenever $s \geqslant 2$ and σ permutes s of the integers $1, \ldots, n$. In fact, if $\{\nu_1, \ldots, \nu_n\}$ is a strongly polynomially independent family of hermitian probability measures in $M_0(G)$, then we may take

$$\mu_i = \mathop{*}_{j=1}^{i} \nu_j, \quad i = 1, \ldots, n.$$

The proof is left to the reader. The existence of suitable families may be established by means of independent sets of multiplicity or using Riesz products. It is clear at this point that polynomial independence is simply the wrong concept for use in joint spectral extension theorems.

4. Full polynomial independence. In view of the examples in the last section our search for joint spectral extension results should not be too ambitious. In particular, we may as well concentrate on the higher dimensional analogues of Proposition 5. Thus we will consider strongly polynomially independent families of probability measures subject to some additional hypothesis. To be satisfactory, the extra condition should be concrete in nature and, at least in principle, capable of direct verification in applications.

DEFINITION 2. We say that the family $\{\mu_i:\ i = 1, \ldots, s\}$ of probability measures in $M(G)$ is *fully polynomially independent* if it is strongly polynomially independent and has the additional property that, for each $j = 1, \ldots, s$ and every positive integer n,

$$\Big(\sum_{i=1, i\neq j}^{s} \mu_i\Big)^n \perp \mu_j * M(G).$$

The extra hypothesis is so strong that it is by no means obvious that $M(G)$ contains fully polynomially independent subsets. However the natural elementary examples arise from independence in the underlying group. The next proof is another example of the way extension results for generalized characters allow us to avoid computational arguments.

LEMMA 2. *Let K be a perfect independent subset of G and $K_1, K_2, \ldots K_s$ disjoint subsets of K. Suppose that $\mu_1, \ldots, \mu_s$ are continuous probability measures concentrated on $K_1 \cup (-K_1), \ldots, K_s \cup (-K_s)$, respectively. Then $\{\mu_1, \ldots, \mu_s\}$ is a fully polynomially independent subset of $M(G)$.*

Proof. We know from Propositions 2 and 3 that the family is strongly polynomially independent. Now fix some j in $\{1, \ldots, s\}$. In view of the corollary to Proposition 2, there is a generalized character χ in $\Delta M(G)$ such that $\chi_{\mu_i} \equiv 0$ and $\chi_{\mu_j} \equiv 1$, for $i = 1, \ldots, s$; $i \neq j$. We see that $\chi_\nu \equiv 0$, for every measure ν in the ideal generated by μ_j, and the result follows.

Now we obtain an extension theorem for fully polynomially independent subsets of certain subalgebras of $M(G)$. Given a perfect independent subset K of G, we let $N(K)$ denote the L-subalgebra of $M(G)$ generated by the discrete measures on G together with the continuous measures on $K \cup (-K)$. Although $N(K)$ is strictly smaller than the algebra of measures concentrated on the Raikov system generated by $K\cup(-K)$, it is nevertheless the case (cf. [13]) that the orthogonal complement of $N(K)$ is an ideal.

THEOREM 3. *Suppose that K is a perfect independent subset of G and that $\{\mu_1, \ldots, \mu_s\}$ is a fully polynomially independent subset of $N(K)$. Then $S(\mu_1, \ldots, \mu_s) = D^s$.*

It will be convenient to delay the proof of Theorem 3. Before proceeding, we note that full polynomial independence is not a necessary condition in such results.

PROPOSITION 8. *Let K be a perfect independent subset of G and K_1, K_2, K_3, K_4 disjoint subsets of K. Suppose that $\nu_1, \nu_2, \nu_3, \nu_4$ are continuous probability measures concentrated on $K_1\cup -K_1, K_2\cup -K_2, K_3\cup -K_3, K_4\cup -K_4$, respectively. Let $\mu_1 = \frac{1}{2}(\nu_1+\nu_2*\nu_3)$, $\mu_2 = \frac{1}{2}(\nu_2+\nu_1*\nu_4)$. Then μ_1, μ_2 are strongly polynomially independent and satisfy $S(\mu_1, \mu_2) = S_{\mathrm{loc}}(\mu_1, \mu_2) = D^2$, but are not fully polynomially independent.*

Proof. In order to verify strong polynomial independence, we consider whether every translate of a non-trivial measure of the form $\nu_1^a*(\nu_2*\nu_3)^b$ is orthogonal to all measures of the form $\nu_2^c*(\nu_1*\nu_4)^d$. In view of the terms in ν_3, ν_4, overlap would certainly be impossible unless $b = d = 0$. This reduces the problem to a comparison of ν_1^a and ν_2^c and it is clear that we have strong polynomial independence. On the other hand μ_1 fails to be singular to $\nu_3*\mu_2$, so that full polynomial independence is violated in a basic way.

To complete the proof we must check that $S(\mu_1, \mu_2)$ equals D^2. We do this using the definition of $\nu_1, \nu_2, \nu_3, \nu_4$ and the corollary of Proposition 2 to justify the assignment of arbitrary constant values z_1, z_2, z_3, z_4, of modulus not exceeding one, to the functions $\chi_{\nu_1}, \chi_{\nu_2}, \chi_{\nu_3}, \chi_{\nu_4}$ corresponding to some χ in $\Delta M(G)$. In fact let us fix $0\leqslant r_1, r_2\leqslant 1$, θ_1, θ_2 and prove that $(r_1e^{i\theta_1}, r_2e^{i\theta_2})$ belongs to $S(\mu_1, \mu_2)$. The required prescription is $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$, $z_3 = (2r_1-1)e^{i(\theta_1-\theta_2)}$, $z_4 = (2r_2-1)e^{i(\theta_2-\theta_1)}$, where we note that $|2r_i-1|\leqslant 1$, for $i = 1, 2$. The rest is straightforward.

We now give an analogue of Proposition 4. The proof is quite simple but did not present itself immediately—perhaps because it is an odd mixture of measure algebra and Banach algebra methods.

PROPOSITION 9. *Let μ, ν be probability measures in $M(G)$. Equivalent are*

(i) *$\mu^n \perp \nu*M(G)$ for each positive integer n,*

and

(ii) *there exists a generalized character ε in $\Delta M(G)$ such that $\varepsilon_\mu \equiv 1$ and $\varepsilon_\nu \equiv 0$.*

COROLLARY. *Let $\{\mu_1, \ldots, \mu_s\}$ be a strongly polynomially independent subset of probability measures in $M(G)$. The set is fully polynomially independent if and only if there exists, for each $i = 1, \ldots, s$, a generalized character $\varepsilon^{(i)}$ in $\Delta M(G)$ such that $\varepsilon^{(i)}_{\mu_j} \equiv 1$, $(j\neq i)$, and $\varepsilon^{(i)}_{\mu_j} \equiv 0$.*

Proof of Proposition 9. Suppose first that there is some ε in $\Delta M(G)$ with $\varepsilon_\mu \equiv 1$, $\varepsilon_\nu \equiv 0$. It follows that $\varepsilon_{\omega*\nu} \equiv 0$, for every ω in $M(G)$ and this denies the possibility of overlap between $\omega*\nu$ and any power of μ.

Accordingly we concentrate on the converse problem where we are given μ, ν satisfying (i). We can, of course, extend (i) a little to obtain the statement that every power of μ is orthogonal to the L-ideal, I, say, generated by ν. Let us consider the canonical quotient map $q\colon M(G)\to M(G)/I$ (where we regard the objects as Banach algebras rather than as measure algebras). We see that, for each positive integer n,

$$\|q(\mu^n)\| = \inf\{\|\mu^n+\omega\|:\ \omega\in I\} = 1,$$

and, therefore, that $q(\mu^n)$ has unit spectral radius in the quotient algebra $M(G)/I$. It follows, of course, that there is some complex homomorphism φ of $M(G)/I$ such that $|\varphi(q(\mu))| = 1$. Then $\chi = \varphi\circ q$ is a member of $\Delta M(G)$ with the property that $|\chi(\mu)| = 1$ and $\chi(M(G)*\nu) = 0$. Because χ can be represented as a generalized character with $\chi_\mu\in L^\infty(\mu)$ and because μ is a probability measure we see, in fact, that $\chi_\mu(g)$ is constant μ almost everywhere and has unit modulus. The formula $\varepsilon = |\chi|$ now defines a generalized character in $\Delta M(G)$ with $\varepsilon_\mu \equiv 1$ and $\varepsilon_\nu \equiv 0$.

It is easy to obtain the corollary by repeated application of the proposition. It is worth noting explicitly that the proposition is an extension theorem. The corresponding result in which (ii) becomes the similar statement with $\Delta M(G)$ replaced by $\Delta(A\oplus I)$, where A is the L-algebra generated by μ, I the L-ideal generated by ν, is the simple "local" version.

The next proposition gives the main step in the proof of Theorem 3.

PROPOSITION 10. *Let K be a perfect independent subset of G and suppose that the continuous probability measures μ, ν form a fully polynomially independent subset of $N(K)$. Then, for each r satisfying $0\leqslant r\leqslant 1$, there exists $\chi\in\Delta M(G)$ such that $\chi_\mu \equiv 1$ and $\nu^\wedge(\chi) = r$.*

Proof. We know from Proposition 9 that there exists $\varepsilon\in\Delta M(G)$ such that $\varepsilon_\mu \equiv 1$ and $\varepsilon_\nu \equiv 0$, and we can and do suppose, without loss of generality, that ε is idempotent (consider e.g. $\lim|\varepsilon|^n$). Let us regard ε as a generalized function and consider its restriction to $E = M_c(K\cup -K)$, the Banach space of continuous bounded Borel measures on $K\cup(-K)$. This gives an orthogonal splitting

$$E = E_1\oplus E_2,$$

where

$$E_1 = \{\omega\in E:\ \varepsilon_\omega\equiv 1\},\qquad E_2 = \{\omega\in E:\ \varepsilon_\omega\equiv 0\},$$

and this enables us to define a convenient family of generalized functions in the unit ball of the dual of E. In fact, for each t in $[0, 1]$, we define $\varepsilon^{(t)}$ by

$$\varepsilon^{(t)}_{\omega} = 1 \ (\omega \in E_1), \qquad \varepsilon^{(t)}_{\omega} = t \ (\omega \in E_2).$$

In fact (cf. Proposition 2 and its Corollary) we may obtain a family $\varepsilon^{(t)}$ of generalized characters of $\Delta M(G)$, whose restrictions to E give the family of generalized functions just defined. (It is even possible to choose canonical extensions, first to $N(K)$ by stipulating, say, that the generalized character is one on all discrete measures; then trivially to $M(G)$ using the fact that $N(K)^{\perp}$ is an ideal.) The important feature for us is that the map F: $[0,1] \to [0,1]$, defined by

$$F(t) = \int \varepsilon^{(t)}_{\nu}(g)\, d\nu(g) = \nu\hat{\ }(\varepsilon^{(t)}),$$

is continuous. Moreover

$$F(0) = \nu\hat{\ }(\varepsilon^{(0)}) = \nu\hat{\ }(\varepsilon) = 0,$$

$$F(1) = \int 1\, d\nu(g) = 1.$$

It follows, from the intermediate value theorem, that given r in $[0, 1]$, we may choose t so that

$$\nu\hat{\ }(\varepsilon^{(t)}) = F(t) = r.$$

Since $\varepsilon^{(t)}_{\mu} \equiv 1$, for all t, the proof is complete.

Remarks. It was not strictly necessary to use Proposition 9 in the previous proof because of our concrete knowledge of $\Delta N(K)$ — but, as μ, ν need not belong to $M_c(K \cup (-K))$, the abstract argument seems quicker.

It is now a simple matter of generalized character techniques to finish the proof of Theorem 3.

Proof of Theorem 3. Note first that the measures $\mu_1, \ldots, \mu_s$ are probability measures, by hypothesis, and continuous because of strong polynomial independence. Now fix a vector $(r_1 e^{i\theta_1}, \ldots, r_s e^{i\theta_s})$ in D^s. We know from Proposition 4 that polynomial independence guarantees the existence of a generalized character θ in $\Delta M(G)$ such that $\theta_{\mu_j} = e^{i\theta_j}$, $j = 1, \ldots, s$. Now we fix some i, and apply Proposition 10 to obtain a generalized character $\chi^{(i)}$ in $\Delta M(G)$ such that

$$\chi^{(i)}_{\mu_j} \equiv 1,\ j \neq i; \qquad \mu_i\hat{\ }(\chi^{(i)}) = r_i.$$

(The appropriate choices are $\nu = \mu_i$, $(s-1)\mu = \sum \mu_j$, where the sum is over the indices from 1 to s, with the exception of i.) Having done this for each $i = 1, \ldots, s$; we define χ in $\Delta M(G)$ by

$$\chi = \theta . \chi^{(1)} . \chi^{(2)} \ldots \chi^{(s)},$$

and it is easy to check that

$$\mu_j\hat{\ }(\chi) = r_j e^{i\theta_j}, \qquad j = i, \ldots, s.$$

This completes the proof.

5. The new obstruction. It is by now reasonably clear that the two-dimensional case is typical of multi-dimensional spectral extension problems, so we will save some effort by taking $s = 2$ from now on.

THEOREM 4. *Let G be a non-discrete compact abelian group. There exist tame hermitian fully polynomially independent measures μ_1, μ_2 in $M_0(G)$ with the property that $S(\mu_1, \mu_2)$ is of measure zero in $S_{\text{loc}}(\mu_1, \mu_2) = D^2$.*

Proof. We fix some infinite dissociate subset Λ of the dual group of G and suppose that Λ has been expressed as the disjoint union of three infinite subsets $\Lambda_1, \Lambda_2, \Lambda_3$. We suppose further that an enumeration of each of these sets has been fixed, so that we have $\Lambda_1 = \{\theta_n\}_{n=1}^{\infty}$, $\Lambda_2 = \{\varphi_n\}_{n=1}^{\infty}$, and $\Lambda_3 = (\psi_n)_{n=1}^{\infty}$. Let us fix also some sequence $(r_n)_{n=1}^{\infty}$ of real numbers satisfying $0 \leqslant r_n \leqslant \frac{1}{2}$, $n = 1, 2, \ldots$; $r_n \to 0$; and $\sum_{n=1}^{\infty} r_n^k = \infty$, for every positive integer k.

The required measures will be defined as Riesz products on Λ. Thus we exploit the fact (cf. [1]) that a tame hermitian probability measure μ arises as the weak* limit as $N \to \infty$ of

$$\prod_{n=1}^{N} \big(1+\mu\hat{\ }(\theta_n)(\theta_n+\bar{\theta}_n)\big)\big(1+\mu\hat{\ }(\varphi_n)(\varphi_n+\bar{\varphi}_n)\big)\big(1+\mu\hat{\ }(\psi_n)(\psi_n+\bar{\psi}_n)\big)\cdot\lambda,$$

where λ denotes Haar measure on G and the numbers $\mu\hat{\ }(\theta_n)$, $\mu\hat{\ }(\varphi_n)$, $\mu\hat{\ }(\psi_n)$ are chosen arbitrarily in $[0, \frac{1}{2}]$. In fact we make the following choices

$$\mu_1\hat{\ }(\theta_n) = r_n, \qquad \mu_1\hat{\ }(\varphi_n) = r_n, \qquad \mu_1\hat{\ }(\psi_n) = 0;$$

$$\mu_2\hat{\ }(\theta_n) = r_n^{\pi}, \qquad \mu_2\hat{\ }(\varphi_n) = 0, \qquad \mu_2\hat{\ }(\psi_n) = r_n.$$

(Here π does indeed denote the usual constant, but could be replaced by any real number t greater than one, such that $\{1, t\}$ is independent over the rationals.)

In checking full polynomial independence we need a criterion for singularity of measures and it is convenient to employ the following one which we gave in [2];

$$\text{(i)} \quad \sum_{n=1}^{\infty} \Big|\int X_n\, d\omega_1 - \int X_n\, d\omega_2\Big|^2 = \infty,$$

$$\text{(ii)} \quad \sum_{k=0}^{\infty} \sup_{i,n} \Big|\int X_n \bar{X}_{n+k}\, d\omega_i - \int X_n\, d\omega_i \int \bar{X}_{n+k}\, d\omega_i\Big| < \infty,$$

where the X_n are square integrable, ensures that the probability measures ω_1, ω_2 are mutually singular. (In the special case when ω_1, ω_2 are both

Riesz products, (ii) is trivially true and the criterion reduces to one given in [8].) Consider first the case where $\omega_1 = \mu_1^a$, $\omega_2 = \mu_2 * \omega$, where is a a positive integer and ω is any probability measure in $M(G)$. We choose $X_n = \varphi_n$. Although ω_2 need not be a Riesz product, condition (ii) is certainly satisfied and condition (i) becomes $\sum r_n^{2a}$ diverges. It follows that μ_1^a is singular to $M(G)*\mu_2$. A similar argument, using ψ_n, shows that each power of μ_2 is orthogonal to the ideal generated by μ_1. To complete the verification of full polynomial independence we may consider $\omega_1 = \delta(g_1)*\mu_1^a*\mu_1^b$, $\omega_2 = \delta(g_2)*\mu_1^c*\mu_2^d$, $X_n = \theta_n$. As noted earlier (ii) holds; while (i) becomes the statement that $\sum|\theta_n(g_1)r_n^{a+b\pi} - \theta_n(g_2)r_n^{c+d\pi}|^2$ diverges. The two measures will therefore be singular if $\sum|r_n^{a+b\pi} r_n^{c+d\pi}|^2$ diverges. Moreover we can write

$$|r_n^{a+b\pi} - r_n^{c+d\pi}| = r_n^x|1-r_n^y|,$$

where $x \geqslant 0$, and $y = |a-c+(b-d)\pi|$. Since $r_n \to 0$, the only way that (i) could fail would be if $y = 0$. This latter event occurs only if $a = c$ and $b = d$. We have now verified that $\{\mu_1, \mu_2\}$ is fully polynomially independent.

We now know that $S_{\mathrm{loc}}(\mu_1, \mu_2)$ equals D^2 and it remains only to check that $S(\mu_1, \mu_2)$ is of zero measure in D^2. In fact it will be shown that $S(\mu_1, \mu_2)$ is the union of the following sets:

$$\{(re^{i\theta_1}, r^\pi e^{i\theta_2}) \colon 0 \leqslant r \leqslant 1;\ \theta_0, \theta_2 \in \boldsymbol{R}\}, \quad \{(re^{i\theta_1}, 0) \colon 0 \leqslant r \leqslant 1,\ \theta_1 \in \boldsymbol{R}\},$$
$$\{(0, re^{i\theta_2}) \colon 0 \leqslant r \leqslant 1,\ \theta_2 \in \boldsymbol{R}\}.$$

The first step will be to check that, for every $\chi \in \Delta M(G)$ with $(\mu_1*\mu_2)\hat{}(\chi) \neq 0$, we have $|\chi_{\mu_2}| = |\chi_{\mu_1}|^\pi$. To see this we introduce the Riesz product ν defined like μ_1, μ_2 by the values of its transform on Λ. In fact we stipulate that $\nu\hat{}(\theta_n) = r_n$, $\nu\hat{}(\varphi_n) = \nu\hat{}(\psi_n) = 0$. Also we interpolate a one-parameter family $\{\nu_t \colon t > 0\}$ so that $\nu_1 = \nu$ and ν_{t+s} is equivalent to $\nu_t * \nu_s$, for all $s, t > 0$. (This is done as in [3]. The only reason we cannot use a one-parameter semigroup is the condition that all Fourier coefficients be not greater than one half.) Now we observe, by checking transforms, that

$$(*) \qquad \mu_1*\nu = \nu^2, \quad \mu_2*\nu = \nu_{1+\pi}, \quad \mu_1*\mu_2 = \nu_{1+\pi}.$$

Let us fix some $\chi \in \Delta M(G)$ such that neither $\mu_1\hat{}(\chi)$ nor $\mu_2\hat{}(\chi)$ is zero. It follows from the preceding relations $(*)$ that $\nu_{1+\pi}(\chi) \neq 0$ and hence that $\nu\hat{}(\chi) \neq 0$. Because ν is a tame probability measure, it follows that $\nu\hat{}(|\chi|) \neq 0$. Arguing as in the proof of Proposition 6, we see that there is some positive r with

$$|\chi_{\nu_t}| = r^t, \quad \text{for all } t > 0.$$

We apply $(*)$ again to deduce that

$$|\chi_{\mu_1}|r = r^2, \quad |\chi_{\mu_2}|r = r^{1+\pi};$$

and hence that

$$(**) \qquad |\chi_{\mu_1}| = r, \quad |\chi_{\mu_2}| = r^\pi.$$

Now we are in a position to determine $S(\mu_1, \mu_2)$. By Proposition 5 $S(\mu_1*\mu_2)$ equals D. On the other hand, apart from the number one, the Fourier–Stieltjes transform of $\mu_1*\mu_2$ lies wholly inside the disc of radius one quarter. Because μ_1, μ_2 are tame we see that there must be values of r arbitrarily close to one, for which $|\chi_{\mu_1}*\mu_2|$ equals $r^{1+\pi}$. By considering powers of the corresponding generalized characters and applying Proposition 4, we see that $S(\mu_1, \mu_2)$ contains the set $\{(re^{i\theta_1}, r^\pi e^{i\theta_2}) \colon 0 \leqslant r \leqslant 1;\ \theta_1, \theta_2 \text{ real}\}$. Also, by Proposition 9, there exists $\varepsilon^{(1)}, \varepsilon^{(2)} \in \Delta M(G)$ such that $\varepsilon_{\mu_1}^{(1)} = \varepsilon_{\mu_2}^{(2)} = 0$, $\varepsilon_{\mu_2}^{(1)} = \varepsilon_{\mu_2}^{(2)} = 1$, so that $S(\mu_1, \mu_2)$ contains also the sets $D \times \{0\}$, $\{0\} \times D$. Let us now verify that we have described all possible elements of $S(\mu_1, \mu_2)$. The only conceivable difficulty is associated with homomorphisms, χ, which do not annihilate $\mu_1*\mu_2$. Any such χ must satisfy formula $(**)$ for some r in $]0,1[$. However, in view of tameness and the original definitions of the measures, this shows

$$\chi_{\mu_1} = e^{i\alpha_1} r\gamma_1, \quad \chi_{\mu_2} = e^{i\alpha_2} r^\pi \gamma_2, \quad \chi_\nu = e^{i\alpha_3} r\gamma_3,$$

for $\gamma_1, \gamma_2, \gamma_3$ in $G\hat{}$, and real $\alpha_1, \alpha_2, \alpha_3, r$, with $0 < r \leqslant 1$. However we evaluate $(*)$ on the generalized character $\bar{\gamma}_3\chi$ to obtain

$$|\mu_2\hat{}(\gamma_1\bar{\gamma}_3)| = 1 = |\mu_1\hat{}(\gamma_2\bar{\gamma}_3)|,$$

and hence may take $\gamma_1 = \gamma_2 = \gamma_3$. It follows that

$$|\mu_2\hat{}(\chi)| = |\mu_2\hat{}(\chi)|^\pi$$

and the proof is complete.

Consideration of the proofs of Theorems 3,4 indicates that the obstruction to joint spectral extension results is a certain lack of connectedness in the maximal ideal space. We can make this more explicit by considering a pair μ_1, μ_2 of probability measures in any abstract convolution measure algebra N. Strong polynomial independence corresponds to the existence of generalized characters $\chi^{(\theta_1,\theta_2)}$ taking the values $e^{i\theta_1}, e^{i\theta_2}$ on μ_1, μ_2, respectively. Full polynomial independence corresponds to the added existence of generalized characters $\varepsilon^{(1)}, \varepsilon^{(2)}$ with $\varepsilon_{\mu_1}^{(1)} = 1$, $\varepsilon_{\mu_2}^{(1)} = 0$, $\varepsilon_{\mu_1}^{(2)} = 1$, $\varepsilon_{\mu_2}^{(2)} = 0$. We can guarantee spectral extension by demanding that $\varepsilon^{(1)}, \varepsilon^{(2)}$ be arc-connected to $\chi^{(0,0)}$ in

$$\{\psi \in \Delta N \colon \psi \geqslant 0,\ \mu_1\hat{}(\psi) = 1\}, \quad \{\psi \in \Delta N \colon \psi \geqslant 0,\ \mu_2\hat{}(\psi) = 1\},$$

respectively. We have not stated this extension theorem as a formal result because we have been unable to translate it into a concrete condition like full or strong polynomial independence.

The referee has made the interesting remark that yet another condition for polynomial independence can be considered:

$$(***)\qquad \mu_1^{n_1}*\mu_2^{n_2}\perp\left(\mu_1^{n_1+1}*M(G)\right)\cup\left(\mu_2^{n_2+1}*M(G)\right),$$

for all positive integers n_1, n_2. In this connexion he notes a complementary example to Proposition 4 which can be obtained by similar arguments. In fact there exist Riesz products ϱ_1, ϱ_2 in $M_0(G)$ such that $S(\varrho_1, \varrho_2) = D^2$. Retaining the notation that $\Lambda_1 = \{\theta_n\}$, $\Lambda_2 = \{\varphi_n\}$ are disjoint subsets of some infinite dissociate set Λ, we may characterize ϱ_1, ϱ_2 by $\varrho_1\hat{\ }(\theta_n) = s_n = \varrho_2\hat{\ }(\varphi_n)$, $\varrho_1\hat{\ }(\varphi_n) = t_n = \varrho_2\hat{\ }(\theta_n)$ where $(s_n), (t_n)$ are null sequences whose entries are real numbers lying between 0 and $\frac{1}{2}$, so chosen that $\sum s_n^k, \sum t_n^k$ diverge and $\lim\limits_{n\to\infty} t_n s_n^{-k} = 0$ for all positive integers k. Now let ϱ be the Riesz product on Λ with $\varrho\hat{\ }(\theta) = t_n$, $\varrho\hat{\ }(\varphi_n) = 0$ and note that $\varrho*\varrho_2 = \varrho^2$, $\varrho \in \varrho_1^k*M(G)$, for all positive integers k. Given a real number r with $0<r<1$, there exists χ such that $\chi_\varrho = r$. It follows that $\chi_{\varrho_2} = r$ and that $\chi_{\varrho_1} = 1$. After a similar argument, with the rôles of θ_n, φ_n interchanged, we see that $S(\varrho_1, \varrho_2) = D^2$.

The simplest way to find examples of monotrochic measures ϱ_1, ϱ_2 satisfying $S(\varrho_1, \varrho_2) = D^2$ seems to be to consider a pair of Bernoulli convolutions

$$\varrho_1 = \mathop{*}_{n=1}^{\infty} \tfrac{1}{2}\left(\delta(x_n)+\delta(-x_n)\right),\qquad \varrho_2 = \mathop{*}_{n=1}^{\infty} \tfrac{1}{2}\left(\delta(y_n)+\delta(-y_n)\right)$$

for suitably chosen x_n, y_n. (For example we may fix some sequence (a_n) of positive integers such that a_n tends to infinity, write $p_n = a_1a_2\ldots a_n$, and choose x_n to be the reciprocal of p_{2n-1}, y_n to be the reciprocal of p_{2n}. Assuming, for convenience, that 8 divides a_1, we can see that $\exp(2\pi i p_{2n-1}/8)$ converges to $1/\sqrt{2}$ in the $\sigma\left(L^\infty(\varrho_1), L^1(\varrho_1)\right)$-topology and to 1 in the $\sigma\left(L^\infty(\varrho_2), L^1(\varrho_2)\right)$-topology. Also we reverse the rôles of ϱ_1, ϱ_2 when considering $\exp\left(2\pi i p_{2n}/8\right)$).

The preceding paragraphs indicate several directions for further work. Very little is known concerning concrete examples (in particular for other values of x_n, y_n in the Bernoulli convolution case) of measures satisfying $S(\varrho_1, \varrho_2) = D^2$. Variants of "full polynomial independence" such as condition (***) present themselves and the relationship with connectedness properties of the maximal ideal space may reward further investigation.

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