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$C(K)$ norming subsets of $C[0,1]^*$

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Abstract. It is shown that if a bounded subset of $C[0,1]^*$ norms a subspace of $C[0,1]$ which is isomorphic to $C_0(\omega^{\omega^\alpha})$, for some $\alpha < \omega_1$, then there is a subspace of $C[0,1]$ isometric to $C_0(\omega^{\omega^\alpha})$ which is also normed by this set. The techniques employed also yield a new proof that there is a bounded linear operator form $C_0(\omega^{\omega^2})$ onto itself which is not an isomorphism when restricted to any subspace of $C_0(\omega^{\omega^2})$ isomorphic to $C_0(\omega^{\omega^2})$.

0. Introduction. Several authors have addressed the question of determining conditions on a subset of $C[0,1]^*$ which will ensure that the subset norms a subspace isomorphic to $C(K)$, the continuous functions on some compact metric space K. Necessary and sufficient conditions for the cases $K = [0,1]$, $[1,\omega]$, and $[1,\omega^\omega]$ (the ordinals less than of equal to ω, resp., ω^ω, with the order topology) have been given by Rosenthal [12], Pełczyński [10], and the author [2], respectively. Recently, J. Wolfe [15] introduced a sufficient conditon (the definitions will be given shortly) for the case of K homeomorphic to the ordinals less than or equal to ω^{ω^α}, any $\alpha < \omega_1$. The condition he gave is closely tied to the isometric structure of the $C(K)$ space and thus the necessity of the condition is far from obvious. In this paper we show that the Wolfe condition does yield a necessary and sufficient condition. As a corollary we deduce the first result stated in the abstract.

We also apply the Wolfe condition to the bounded linear operator T from $C_0(\omega^{\omega^2})$ onto $C_0(\omega^{\omega^2})$ constructed in the author's dissertation [1], to give a simpler argument that there is no subspace Y of $C_0(\omega^{\omega^2})$ such that Y is isomorphic to $C_0(\omega^{\omega^2})$ and the restriction of T to Y is an isomorphism. (For any ordinal α, $C(\omega^\alpha)$, resp., $C_0(\omega^\alpha)$, is the space of continuous functions on the ordinals less than or equal to ω^α with the order topology, resp., and vanishing at ω^α.) This also shows that the Szlenk index condition used in [2] and the Wolfe condition can be quite different.

We now give the definitions used by Wolfe [15] so that we may state our results precisely. The first is an inductive definition of a de-

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creasing family of subsets of the Cartesian product of a subset of $C(K)^*$ and the topology $\mathcal{T}$ of K.

DEFINITION. Let K be a compact Hausdorff space, $\varepsilon > 0$, and let B be a subset of $C(K)^*$ (which we identify with the finite signed regular Borel measures on K). Let

$$P_0(\varepsilon, B) = \{(\mu, G)\colon \mu \in B, G \in \mathcal{T}, \text{ and } |\mu|(G) \geqslant \varepsilon\}.$$

If $P_\alpha(\varepsilon, B)$ has been defined we let

$$P_{\alpha+1}(\varepsilon, B) = \{(\mu, B)\colon (\mu, G) \in P_0(\varepsilon, B) \text{ and there is a sequence } \{(\mu_n, G_n)\colon n \in N\} \subset P_\alpha(\varepsilon, B) \text{ such that } \mu_n \xrightarrow{w^*} \mu, G_n \cap G_{n'} = \emptyset, \text{ for } n \neq n', \text{ and } \overline{\bigcup G_n} \subset G\}.$$

For a limit ordinal β let

$$P_\beta(\varepsilon, B) = \{(\mu, G)\colon (\mu, G) \in P_0(\varepsilon, B) \text{ and there are a sequence of ordinals } \alpha_n \uparrow \alpha \text{ and an element } (\mu_n, G_n) \in P_{\alpha_n}(\varepsilon, B) \text{ for each } n, \text{ such that } \mu_n \xrightarrow{w^*} \mu, G_n \cap G_{n'} = \emptyset, \text{ for } n \neq n', \text{ and } \overline{\bigcup G_n} \subset G\}.$$

With this definition and a minor modification of Wolfe's proof of Theorem 1 of [15] we get,

THEOREM 0.1. *Let K be a compact Hausdorff space and let B be a w^* metrizable set of measures in the unit ball of $C(K)^*$. Then, for every ε, $1 \geqslant \varepsilon > 0$, there is an integer $N = N(\varepsilon)$ such that if $\beta < \omega_1$, $P_{\beta n}(\varepsilon, B) \neq \emptyset$, for some $n > N$, then there is a subspace Y of $C(K)$ which is normed by B such that Y is isometric to $C(\omega^\beta)$.*

Moreover, the norming constant, i.e.,

$$\sup\{\lambda\colon \sup\{|\langle b, f\rangle|\colon b \in B\} \geqslant \lambda\|f\| \text{ for all } f \in Y\},$$

depends only on ε.

The difference between our Theorem 0.1 and Wolfe's theorem is in the definition of the sets $P_\alpha(\varepsilon, B)$. Wolfe required that $\mu(G) \geqslant \varepsilon$ rather than $|\mu|(G) \geqslant \varepsilon$. In the remark following Lemma 1.1 we show how to deduce our theorem from Wolfe's.

Theorem 0.1 raises a question. Is it possible that there is a subset of $B_{C(K)^*}$ which norms subspaces of $C(K)$ uniformly isomorphic to $C(\omega^{\alpha_n})$, $\alpha_n \uparrow \omega^\gamma$, but which does not norm a subspace isomorphic to $C(\omega^{\omega^\gamma})$? To find such an example one might attempt to construct a set B such that for some $\varepsilon > 0$, $P_{\alpha_n}(\varepsilon, B) \neq \emptyset$ for all n but $P_{\omega^\gamma}(\delta, B) = \emptyset$ for all $\delta > 0$. One such set is $B_{C_0(\omega^{\omega^\gamma})^*}$, however it obviously norms $C_0(\omega^{\omega^\gamma})$. In Section 1 we show that this example is typical by proving

THEOREM 0.2. *Let K be a compact Hausdorff space and let B be a w^* metrizable set of measures in the unit ball of $C(K)^*$. If there is an $\varepsilon > 0$ such that $P_{\alpha_n}(\varepsilon, B) \neq \emptyset$ for a sequence of ordinals $\{\alpha_n\colon n \in N\}$ with $\alpha_n \uparrow \omega^\alpha$, then there is a subspace Y of $C(K)$ such that Y is isometric to $C_0(\omega^{\omega^\alpha})$ and Y is normed by B. Moreover, the norming constant depends only on ε.*

The significance of this result lies in the fact that $C(\omega^{\alpha_n})$ is not isomorphic to $C_0(\omega^{\omega^\alpha})$ for any n. Indeed, Bessaga and Pełczyński [5] proved that for $\beta < \gamma < \omega_1$, $C(\beta)$ is isomorphic to $C(\gamma)$ if and only if $\beta^\omega > \gamma$. This implies that $C(\omega^\varrho)$ is isomorphic to $C_0(\omega^{\omega^\alpha})$ if and only if $\omega^\alpha \leqslant \varrho < \omega^{\alpha+1}$. (It is easy to see that $C_0(\gamma)$ is isomorphic to $C(\gamma)$ for any $\gamma \geqslant \omega$.) In [15] Wolfe proves a result similar to Theorem 0.2 but under a formally stronger hypothesis.

Theorem 0.2 suggests that a necessary and sufficient condition for a set $B \subset B_{C(K)^*}$ to norm a subspace isomorphic to $C(\omega^{\omega^\alpha})$ is that there exist an $\varepsilon > 0$ such that $P_\beta(\varepsilon, B) \neq \emptyset$ for all $\beta < \omega^\alpha$. (One might call the largest such α for a given ε, the ε Wolfe index.) Under some mild (and natural) restrictions on B we are able to show that this is the case. In Section 2 we prove

THEOREM 0.3. *Let K be a compact Hausdorff space, B a bounded w^* closed w^* metrizable subset of $C(K)^*$, and $\tilde{B}$ the w^* closed convex symmetric hull of B. Then, if there is a subspace Y of $C(K)$ such that B norms Y and Y is isomorphic to $C_0(\omega^{\omega^\alpha})$, then there is an $\varepsilon > 0$ such that $P_\gamma(\varepsilon, \tilde{B}) \neq \emptyset$ for all $\gamma < \omega^\alpha$.*

A natural application of our result is to $T^*B_{X^*}$ where T is an operator from $C(K)$ into a Banach space X. Thus we get

COROLLARY 0.4. *If T is a bounded linear operator from $C(K)$ into a separable Banach space X such that there is a subspace Y of $C(K)$ such that Y is isomorphic to $C_0(\omega^{\omega^\alpha})$, for some $\alpha < \omega_1$, and the restriction of T to Y is an isomorphism, then there is a subspace Z of $C(K)$ such that Z is isometric to $C_0(\omega^{\omega^\alpha})$ and the restriction of T to Z is an isomorphism.*

COROLLARY 0.5. *Let $T\colon C(K) \to X$ be a bounded linear operator and let X be a separable Banach space. Then, there is a subspace Y of $C(K)$ such that Y is isometric to $C_0(\omega^{\omega^\alpha})$ and the restriction of T to Y is an isomorphism if and only if there is an $\varepsilon > 0$ such that $P_\gamma(\varepsilon, T^*B_{X^*}) \neq \emptyset$ for all $\gamma < \omega^\alpha$.*

COROLLARY 0.6. *Let B be a bounded w^* closed w^* metrizable subset of $C(K)^*$. Then if B norms a subspace of $C(K)$ isomorphic to $C_0(\omega^{\omega^\alpha})$, for some $\alpha < \omega_1$, there is a subspace isometric to $C_0(\omega^{\omega^\alpha})$ which is also normed by B.*

Corollary 0.6 follows from the observation that for all $f \in C(K)$,

$$\sup\{|\langle b, f\rangle|\colon b \in B\} = \sup\{|\langle b, f\rangle|\colon b \in \overline{\mathrm{co}}^{w^*}(B \cup -B)\}.$$

The third and fourth sections of this paper are devoted to the aforementioned example of an operator from $C_0(\omega^{\omega^2})$ onto itself which isomorphically preserves no copy of $C_0(\omega^{\omega^2})$. By using Corollary 0.5 we are able to give a simpler proof of this than we gave in [1]. In the last section we list some open questions and make some concluding remarks.

We will use standard Banach space notation as may be found in the book of Lindenstrauss and Tzafriri [8]. For the definitions of the ordinals and their arithmetic the reader may wish to consult [13]. Several notational conventions from our earlier papers, e.g., [2], will be used here as well. For any ordinal α and topological space A, $A^{(\alpha)}$ will denote the αth derived set of A and $A^{d(\alpha)} = A^{(\alpha)} - A^{(\alpha+1)}$. If $B \subset A$, A is a set of ordinals, and $\beta \in B$, $\beta^- = \sup\{\alpha < \beta\colon \alpha \in B\}$. (The set B will be specified whenever this notation is used.) If α and β are ordinals, $[\alpha, \beta] = \{\gamma\colon \alpha \leqslant \gamma \leqslant \beta\}$, $[\alpha, \beta) = \{\gamma\colon \alpha \leqslant \gamma < \beta\}$, etc. Finally if a subset A of $C(K)^*$ is written in the form $A = \{\mu_\alpha\colon \alpha \in \Lambda\}$ for some set of ordinals Λ, then we mean that the map $\mu_\alpha \to \alpha$ is a homeomorphism where A is considered in the w^* topology and Λ has the topology inherited from $[1, \sup \Lambda]$ with the order topology.

1. An extension of Wolfe's Theorem.

Before we begin the proof of Theorem 0.2, let us recall the basic notions used by Wolfe in the proof of Theorem 0.1.

DEFINITION. Let $\gamma < \omega_1$ and K be a compact Hausdorff space. A family $\mathscr{F}$ of nonempty open subset of K is a *γ-family* if for each α with $0 \leqslant \alpha \leqslant \gamma$ there is a subfamily $\mathscr{F}_\alpha$ of $\mathscr{F}$ such that $\mathscr{F}$ has the following properties:

(0) If $G_1, G_2 \in \mathscr{F}$, $G_1 \cap G_2 = \emptyset$, $G_1 \subset G_2$, or $G_2 \subset G_1$:

(1) $\mathscr{F} = \bigcup \{\mathscr{F}_\alpha\colon \alpha \leqslant \gamma\}$;

(2) $\mathscr{F}_\gamma$ is a singleton set, say G_γ, and if $G \in \mathscr{F}$, $G \subset G_\gamma$;

(3) If $\alpha < \gamma$, $\mathscr{F}_\alpha$ is an infinite family of disjoint open sets;

(4) If $G \in \mathscr{F}_\beta$ and $0 \leqslant \alpha < \beta \leqslant \gamma$, the set $\{H\colon H \subset G$ and $H \in \mathscr{F}_\alpha\}$ is infinite;

(5) If $G \in \mathscr{F}_{\beta+1}$, then there is a sequence $(G_n)_{n=1}^\infty$ of disjoint sets in $\mathscr{F}_\beta$ such that $\overline{\bigcup G_n} \subset G$;

(6) If $G \in \mathscr{F}_\beta$ and β is a limit ordinal, then there exist a sequence $\beta_n \uparrow \beta$ and a sequence $(G_n)_{n=1}^\infty$ of disjoint sets such that $G_n \in \mathscr{F}_{\beta_n}$ for each n and $\overline{\bigcup G_n} \subset G$.

If there is also a set of measures $\mathscr{M}$ (on K) associated with $\mathscr{F}$, i.e., for each $G \in \mathscr{F}$ there is a measure $\mu_G \in \mathscr{M}$, which satisfy

(5′) If $G \in \mathscr{F}_{\beta+1}$, then there is a sequence $(G_n)_{n=1}^\infty$ of disjoint sets with $G_n \in \mathscr{F}_\beta$ for all n, $\overline{\bigcup G_n} \subset G$, and $\mu_{G_n} \xrightarrow{w^*} \mu_G$;

(6′) If $G \in \mathscr{F}_\beta$ where β is a limit ordinal, then there exists a sequence $(G_n)_{n=1}^\infty$, of disjoint subsets of G and $\beta_n \uparrow \beta$ such that $G_n \in \mathscr{F}_{\beta_n}$, $\mu_{G_n} \xrightarrow{w^*} \mu_G$, and $\overline{\bigcup G_n} \subset G$;

(7) For each $G \in \mathscr{F}$, $|\mu_G|(G) \geqslant \varepsilon$,

then the *γ-family* is said to have *ε-measures*.

The following lemma contains some fundamental relations between the Wolfe index and γ-families with measures.

LEMMA 1.0. *Let $\gamma < \omega_1$ and B a w^* metrizable subset of $C(K)^*$. The following are equivalent:*

(a) *There is an $\varepsilon > 0$ such that $P_\gamma(\varepsilon, B) \neq \emptyset$.*

(b) *There is a γ-family $\mathscr{F}$ with ε-measures in B.*

(c) *There is a γ-family $\mathscr{F} = \{G_\alpha\colon \alpha \leqslant \omega^\gamma\}$ with ε-measures $\{\mu_\alpha\colon \alpha \leqslant \omega^\gamma\} \subset B$ (i.e., $\mu_\alpha = \mu_{G_\alpha}$, and $\mu_\alpha \to \alpha$ defines a homeomorphism).*

(d) *There is a γ-family $\mathscr{F} = \{G_\alpha\colon \alpha \leqslant \omega^\gamma\}$ with ε-measures $\{\mu_\alpha\colon \alpha \leqslant \omega^\gamma\}$, and for each $\alpha \leqslant \omega^\gamma$, $\bigcup\{G_\tau\colon G_\tau \subsetneqq G_\alpha\} \subset G_\alpha$, there is a neighborhood $\mathscr{N}_\alpha$ of α (in $[1, \omega^\gamma]$) such that $G_\alpha \supset \overline{\bigcup\{G_\tau\colon \tau \in \mathscr{N}_\alpha - \{\alpha\}\}}$, and $\{\tau\colon G_\tau \supseteq G_\alpha\}$ is finite.*

The proof is an exercise in transfinite induction and we leave it to the reader. The next lemma is quite easy.

LEMMA 1.1. *Let $\mathscr{F}$ be a γ family as in* (d) *of Lemma* 1.0. *Then if A is a closed subset of $[1, \omega^\gamma]$ and A is homeomorphic to $[1, \omega^\beta]$, then $\mathscr{G} = \{G_\alpha \in \mathscr{F}\colon \alpha \in A\}$ contains a β-family with ε-measures in $\{\mu_\alpha\colon \alpha \in A\}$.*

As a convenience we will assume that whenever we use a γ-family it satisfies the conditions of (d) of Lemma 1.0.

Remark. From these lemmas and Lemma 2.5 we can deduce our Theorem 0.1 from Wolfe's original as follows: By the equivalence of (a) and (d) of Lemma 1.0 we can find a βn-family $\mathscr{F} = \{G_\tau\colon \tau \leqslant \omega^{\beta n}\}$ with ε-measures $\{\mu_\tau\colon \tau \leqslant \omega^{\beta n}\}$. For every $\varrho > 0$ and integer L, if N is sufficiently large, by Lemma 2.5, there is a subset $\{\nu_\tau\colon \tau \leqslant \omega^{\beta L}\}$ of $\{\mu_\tau\colon \tau \leqslant \omega^\alpha\}$ such that $\big|\|\nu_\tau\| - \|\nu_{\tau'}\|\big| < \varrho$ for all $\tau, \tau' \leqslant \omega^{\beta L}$. Choose a continuous function g and an open set G such that

$$\bar{G} \subset \{|g| = 1\}, \quad |\nu_{\omega^{\beta L}}|(G) > \|\nu_{\omega^{\beta L}}\| - \varrho,$$

and

$$\int g \, d\nu_{\omega^{\beta L}} > \|\nu_{\omega^{\beta L}}\| - \varrho.$$

Pass to a closed neighborhood $\mathscr{N}$ of $\omega^{\beta L}$ such that

$$\int g \, d\nu_\tau > \|\nu_{\omega^{\beta L}}\| - 2\varrho > \|\nu_\tau\| - 3\varrho$$

and

$$|\nu_\tau|(G) > \|\nu_{\omega\beta L}\| - \varrho, \text{ for all } \tau \in \mathcal{N}.$$

Now observe that

$$|\nu_\tau|(\bar{G}^c) \leqslant \|\nu_\tau\| - |\nu_\tau|(G) < \|\nu_{\omega\beta L}\| + \varrho - \|\nu_{\omega\beta L}\| + \varrho = 2\varrho,$$

and thus

$$g\cdot\nu_\tau(G_\tau \cap \bar{G}) > |\nu_\tau|(G_\tau \cap \bar{G}) - 3\varrho > \varepsilon - 5\varrho.$$

By considering the sets $\{G_\tau: \tau \in \mathcal{N}\}$ and the measures $\{g\cdot\nu_\tau: \tau \in \mathcal{N}\}$ we can find a βL-family with $\varepsilon - 5\varrho$ measures in the sense of Wolfe.

In his proof of Theorem 0.1, Wolfe uses the fact that $P_\alpha(\varepsilon, B) \neq \emptyset$ to construct an α-family with ε-measures from B. Then by a series of refinements, he constructs a β-family $\mathcal{F}$ with associated measure so that $X = \overline{\operatorname{span}}\{1_G: G \in \mathcal{F}\}$ is isometric to $C(\omega^\beta)$ (where X is considered as a subspace of the bounded measurable functions on K) and is normed by the associated measures. He then takes continuous approximations to the functions in X to get the required subspace Y.

For our proof of Theorem 0.2 we do not need the specific construction of the proof of Theorem 0.1; we will just use the result. It is sufficient for us to show:

(1.0) for every $\delta > 0$ there are a sequence of pairs of disjoint open sets $\{(G_n, H_n): n \in N\}$, with $\bar{G}_n \subset H_n$, and subsets B_n of B such that $P_{\alpha_n}(\varepsilon - \delta, B_n|_{\bar{G}_n}) \neq \emptyset$ and $|\mu|\big(\bigcup\{H_k: k \neq n\} \cup (H_n - \bar{G}_n)\big) < \delta$ for all $\mu \in B_n$.

(Here $B_n|_{\bar{G}_n} = \{\mu|_{\bar{G}_n}: \mu \in B_n\}$.)

Indeed, by Theorem 0.1 we can find a sequence $\beta_n \uparrow \omega^\alpha$ and subspaces Y_n of $C(\bar{G}_n)$ isometric to $C(\omega^{\omega^{\beta_n}})$, uniformly normed, with constant ϱ, by B_n. For each n let X_n be a subspace of $C(K)$ isometric to $C(\omega^{\omega^{\beta_n}})$ so that the restriction of the functions in X_n to $\bar{G}_n$ is Y_n and all of the functions in X_n are supported in H_n. (See [8] page 169.) We claim that $\overline{\operatorname{span}}\{X_n: n \in N\}$ is the required subspace Y. If $f = \sum_{i=1}^n f_i$, $f_i \in X_i$, $i = 1, 2, \ldots, n$, then $\|f\| = \|f_j\|$ for some j. Hence

$$\begin{aligned}\sup\{|\langle\mu, f\rangle|: \mu \in B\} &\geqslant \sup\{|\langle\mu, f\rangle|: \mu \in B_j\}\\ &\geqslant \sup\{|\langle\mu, f_j\rangle|: \mu \in B_j\} - \sup\{|\langle\mu, \textstyle\sum_{i\neq j} f_i\rangle|: \mu \in B_j\}\\ &\geqslant \sup\{|\langle\mu, f_j|_{\bar{G}_n}\rangle| - \|f_j\|\,|\mu|(H_n - \bar{G}_n): \mu \in B_j\} -\\ &\qquad - \|f_j\|\,|\mu|\big(\textstyle\bigcup\{H_i: i \neq j\}\big)\\ &\geqslant \varrho\|f_j\| - \delta\|f_j\| = (\varrho - \delta)\|f\|.\end{aligned}$$

Clearly Y is isometric to $C_0(\omega^{\omega^\alpha})$ and the claim is proved.

To prove (1.0) we use some of the ideas from [2]. If A is a bounded subset of $L_1(B, \Sigma, \mu)$ for some probability measure μ, let

$$\lambda(A, \mu) = \sup\{\varepsilon|\ \forall\delta > 0\ \exists S \in \Sigma \text{ and } \nu \in A \text{ such that } \mu(S) < \delta \text{ and } |\nu|(S) \geqslant \varepsilon\}.$$

For us we will take A to be a separable subset of B such that $P_{\alpha_n}(\varepsilon, A) \neq \emptyset$ for all n and μ a probability measure on K such that for all $\nu \in A$, $\nu \ll \mu$.

We will also need the following lemma whose proof we omit (Lemma 1.5 of [2]).

Lemma 1.2. *Let $(A_n)_{n=1}^\infty$ be a sequence of disjoint w^*-closed subsets of $B_{C(K)^*}$, K a compact Hausdorff space, and let μ be a probability measure on K such that $\bigcup_{n=1}^\infty A_n \subset L_1(\mu)$. Further assume that for each n $A_n = \{\mu_{n,\gamma}: \gamma \leqslant \omega^{\beta(n)}\}$ for some ordinal $\beta(n)$ and that there are constants $a > \xi \geqslant 0$ such that*

$$\lambda\Big(\bigcup_{n=1}^\infty A_n, \mu\Big) = a \quad \text{and} \quad \lambda\big((\mu_{n,\omega^{\beta(n)}})_{n=1}^\infty, \mu\big) \geqslant a - \xi.$$

Then for every $\delta > 0$ there exist an infinite subset M of N, disjoint open subsets $\{H_n: n \in M\}$ of K, and subsets $\{\nu_{n,\gamma}: \gamma \leqslant \omega^{\beta(n)}\}$ of $\{\mu_{n,\gamma}: \gamma \leqslant \omega^{\beta(n)}\}$ for each $n \in M$ such that

$$\lambda\big(\bigcup\{\nu_{n,\gamma}|_{H_n^c}: \gamma \leqslant \omega^{\beta(n)},\ n \in M\}, \mu\big) \leqslant \xi + \delta.$$

Proof of Theorem 0.2. Let

$$\tau_0 = \sup\{\tau|\ \exists\{\nu_n: n \in N\} \text{ such that } \lambda(\{\nu_n: n \in N\}, \mu) \geqslant \tau \text{ and } \nu_n \in P_{\alpha_n}(\varepsilon, A) \text{ for all } n\}.$$

The supremum is attained, so let $\{\nu_n: n \in N\}$ be an attaining sequence. Since $\nu_n \in P_{\alpha_n}(\varepsilon, A)$, for each n, we can find disjoint subsets $\{A_n: n \in N\}$ of A such that A_n is homeomorphic to $[1, \omega^{\alpha_n}]$, $A_n^{(\alpha_n)} = \{\nu_n\}$, and $P_{\alpha_n}(\varepsilon, A_n) \neq \emptyset$, (use (d) of Lemma 1.0). Moreover, we can assume that $A_n \subset P_{\alpha_n}(\varepsilon, A)$ (by passing to a subsequence $\{\alpha_n': n \in N\}$ of $\{\alpha_n: n \in N\}$ such that $\alpha_n' - \alpha_{n-1}' > \alpha_n$ and choosing $\nu_n \in P_{\alpha_n'}(\varepsilon, A)$).

We claim that there is an integer k such that

$$\lambda\big(\bigcup\{A_n: n \geqslant k\}, \mu\big) - \tau_0 < \delta/8.$$

If not, then we can find a sequence $\{\mu_l: l \in N\}$ such that $\mu_l \in \bigcup\{A_n: n \geqslant l\}$ and $\lambda(\{\mu_l: l \in N\}, \mu) \geqslant \tau_0 + \delta/8$. But, then $\mu_l \in P_{\alpha_l}(\varepsilon, A)$ for all l, so that $\tau_0 \geqslant \lambda(\{\mu_l: l \in N\}, \mu) \geqslant \tau_0 + \delta/8$ —a contradiction. Hence the hypothesis of Lemma 1.2 is satisfied, $(\tau_0 \leqslant a \leqslant \tau_0 + \delta/8)$, and we get sequences of open sets $\{H_n\colon n \in M\}$ and subsets $\{\nu_{n,\gamma}\colon \gamma \leqslant \omega^{\alpha_n}\}$ of A_n with

$$P_{\alpha_n}(\varepsilon, \{\nu_{n,\gamma}\colon \gamma \leqslant \omega^{\alpha_n}\}) \neq \emptyset \quad \text{for all } n \in M$$

and

$$\lambda\big(\bigcup\{\nu_{n,\gamma}|_{H_n^c}\colon \gamma \leqslant \omega^{\alpha_n}, n \in M\}, \mu\big) < \delta/8 + \delta/8 = \delta/4.$$

Choose an integer l sufficiently large so that

$$|\nu_{n,\gamma}||_{H_n^c}\big(\bigcup\{H_k\colon k \geqslant l, k \in M\}\big) < \delta/3$$

and

$$|\nu_{n,\gamma}|(H_n) < \tau_0 + \delta/4 \quad \text{for all } \gamma \leqslant \omega^{\alpha_n}, n \in M,\ n \geqslant l.$$

For each n in M find an open set $G_n \subset \bar{G}_n \subset H_n$ and a w^* neighborhood $\{\nu_{n,\gamma}\colon \gamma \geqslant \gamma_n\}$ of $\nu_{n,\omega^{\alpha_n}}$ such that

$$|\nu_{n,\gamma}|(G_n) \geqslant |\nu_{n,\omega^{\alpha_n}}|(G_n) - \delta/6 > |\nu_{n,\omega^{\alpha_n}}|(H_n) - \delta/4$$

for all $\gamma \geqslant \gamma_n$. Thus if we let $B_n = \{\nu_{n+1,\gamma}\colon \gamma \geqslant \gamma_{n+1}\}$,

$$|\nu|\big(\bigcup\{H_k\colon k \neq n+1\} \cup (H_{n+1} - G_{n+1})\big) < \delta/3 + \delta/2 < \delta,$$

for all $\nu \in B_n$, and we need only show that $P_{\alpha_n}(\varepsilon - \delta, B_n|_{\bar{G}_{n+1}}) \neq \emptyset$ to complete the proof of (1.0). This will follow from

LEMMA 1.3. *Let B be a bounded w^* metrizable subset of $C(K)^*$, with $B \subset L_1(\mu)$, H a closed subset of K, and $\varepsilon, \delta > 0$ such that $P_\alpha(\varepsilon, B) \neq \emptyset$ and $\lambda(B|_{H^c}, \mu) < \delta$ for some $\alpha < \omega_1$. Then $P_\gamma(\varepsilon - \delta, B|_H) \neq \emptyset$ for all $\gamma < \alpha$.*

Proof. By Lemma 1.0 (d) there is an α-family $\mathscr{F} = \{G_\beta\colon \beta \leqslant \omega^\alpha\}$ with ε-measures $\{\mu_\beta\colon \beta \leqslant \omega^\alpha\}$ in B. We claim that the cardinality of $D = \{\mu_\beta\colon |\mu_\beta|(G_\beta \cap H^c) \geqslant \delta\}$ is finite. If not, then let $\{\mu\beta_n\colon n \in N\} \subset D$ be a w^* convergent sequence with limit μ_β. By Lemma 1.1 $\{G_{\beta_n}\colon n \in N\} \cup \cup\{G_\beta\}$ contains a 1-family with ε-measures. However, this means that there is an infinite set $M \subset N$ such that $\{G_{\beta_n}\colon n \in M\}$ is a set of disjoint sets and $|\mu_{\beta_n}|(G_{\beta_n} \cap H^c) \geqslant \delta$ for all $n \in M$. This contradicts $\lambda(B|_{H^c}, \mu) < \delta$. Clearly $\mathscr{F}' = \{G_\beta \cap H\colon \mu_\beta \notin D\}$ contains γ families with $(\varepsilon - \delta)$-measures (relative to H) for all $\gamma < \alpha$, completing the proof of the lemma.

To finish our proof of Theorem 0.2, observe that $\lambda(B_n|_{H_{n+1}^c}, \mu) < \delta/4$ and for all $\nu \in B_n$, $|\nu|(H_{n+1} - \bar{G}_{n+1}) < \delta/2$. Thus $\lambda(B_n|_{\bar{G}_{n+1}^c}, \mu) < \delta$ and Lemma 1.3 yields $P_{\alpha_n}(\varepsilon - \delta, B|_{\bar{G}_{n+1}}) \neq \emptyset$.

2. Proof of Theorem 0.3. In our proof of Theorem 0.3 we will use a slight modification of the notion of γ-family. In the following definition and throughout this section K will be a compact Hausdorff space and B will be a convex bounded, w^* closed, w^* metrizable subset of $C(K)^*$.

DEFINITION. A family of open subsets of K will be called a *$\hat{\gamma}$-family* if for each $\alpha < \gamma$ there is a subset $\mathscr{F}_\alpha$ of $\mathscr{F}$ satisfying

(1′) $\mathscr{F} = \bigcup\{\mathscr{F}_\alpha\colon \alpha < \gamma\}$

and properties (0), (3), (4), (5), and (6) of the definition of γ-family. As in the case of γ-families, if there are measures satisfying (5′), (6′), and (7) associated with the $\hat{\gamma}$-family we will say that it has *ε-measures*.

Clearly if there exists a $\hat{\gamma}$-family $\mathscr{F}$ with ε-measures from B, for some $\varepsilon > 0$, then $P_\beta(\varepsilon, B) \neq \emptyset$ for all $\beta < \gamma$. Thus we may restate Theorem 0.3 in the (formally) stronger form:

THEOREM 0.3(a). *Suppose that there is a subspace Y of $C(K)$ such that Y is isomorphic to $C_0(\omega^{\omega^\alpha})$ and B norms Y, then for some $\varepsilon > 0$ there is an $\widehat{\omega^\alpha}$-family with ε-measures from B.*

That this conclusion is equivalent to the assertion that for some $\varepsilon > 0$, $P_\beta(\varepsilon, B) \neq \emptyset$ for all $\beta < \omega^\alpha$ follows from an argument similar to that given in the previous section and will be implicitly contained within our proof of Theorem 0.3(a).

We divide the proof of Theorem 0.3(a) into two parts —Propositions 2.0 and 2.1. The proofs of these propositions are inductive and depend on the establishment of *both* for smaller ordinals.

PROPOSITION 2.0. *Let $\gamma = \omega^\alpha$, for some ordinal $\alpha < \omega_1$, $b > 0$, and T be an isomorphism from $C(\omega^\gamma)$ into $C(K)$ with $\|T\| \leqslant b$. Let A be a subset of $aB_{C(K)^*}$, for some $a > 0$, such that $T^*|_A$ is a homeomorphism onto the set of point masses on $[1, \omega^\gamma]$.*

Then for every $\delta > 0$ and any ε and c satisfying $\delta < c < 1/a$ and $0 < \varepsilon < (1 - ca)/b$ there exist a $\hat{\gamma}$-family $\mathscr{F}$ with $(\varepsilon - \delta)$-measures chosen from A and a $\hat{\gamma}$-family of clopen sets $\mathscr{G}$ with 1-measures chosen from the point mass measures on $[1, \omega^\gamma]$ such that there is a Boolean algebra isomorphism $\varphi\colon \mathscr{G} \to \mathscr{F}$ with

$$\varphi(G) \subset \{x\colon |T1_G(x)| > c - \tau(G)\},$$

$$|\mu_{\varphi(G)}|\big(\{x\colon |T1_G(x)| > c - \tau(G)\} - \varphi(G)\big) < \delta,$$

for some $\tau(G) < \delta$, and $T^*\mu_{\varphi(G)} = \nu_G$ for all $G \in \mathscr{G}$. (Here $\mu_{\varphi(G)}$ and ν_G are the measures associated with $\varphi(G)$ and G, respectively.)

Remark. We are abusing the term Boolean algebra isomorphism in the statement of Proposition 2.0. We mean that φ is the restriction of a Boolean algebra isomorphism $\tilde{\varphi}$, from the Boolean algebra generated by $\mathscr{G}$ to that generated by $\mathscr{F}$ (with operations $\cup, \cap$). This amounts to saying that φ is a bijection which preserves inclusions, since we are dealing with $\hat{\gamma}$-families.

Before we state Proposition 2.1, we will deduce Theorem 0.3(a) from Proposition 2.0.

Let T be an isomorphism from $C(\omega^{\omega^\alpha})$ onto Y. Since Y is normed by B, there is a constant ϱ such that

$$\sup\{|\langle b, y\rangle|: b \in B\} \geqslant \varrho\|y\|$$

for all $y \in Y$. It follows from the Hahn–Banach Theorem that

$$T^*B \supset \frac{\varrho}{\|T^{-1}\|} B_{C(\omega^{\omega^\alpha})^*}.$$

By Proposition 2 of [10] there is a subset A of B such that $\|T^{-1}\|\varrho^{-1}T^*$ is a homeomorphism from A onto some closed subset S of $\{\delta_\beta: \beta \leqslant \omega^{\omega^\alpha}\}$ with $S^{(\omega^\alpha)} \neq \varphi$. By Lemma 2.3 there is an isometry L from $C(S)$ into $C(\omega^{\omega^\alpha})$ such that $L^*s = \delta_s$ for all $s \in S$ and thus $\|T^{-1}\|\varrho^{-1}TL$ satisfies the hypothesis of Proposition 2.0. Hence there is an $\widehat{\omega^\alpha}$-family with $(\varepsilon-\delta)$-measures from B, as claimed.

Proposition 2.1. *If Proposition 2.0 holds for β, i.e., $\gamma = \omega^\beta$, then for every $\delta > 0$ and $n \in N$ there is an integer $N = N(\delta, n)$ such that if T is an isomorphism from $C(\omega^{\gamma N})$ into $C(K)$ with $\|T\| \leqslant b$, and A is a subset of $aB_{C(K)^*}$ such that $T^*|_A$ is a homeomorphism onto the pointmass measures on $[1, \omega^{\gamma N}]$, then for every $\delta > 0$, and c and ε satisfying $\delta < c < 1/a$ and $0 < \varepsilon < (1-ca)/b$, there exist a $\widehat{\gamma n}$-family $\mathscr{F}$ with $(\varepsilon-\delta)$-measures from A and a $\widehat{\gamma n}$-family of clopen sets $\mathscr{G}$ with 1-measures from the point masses on $[1, \omega^{\gamma N}]$ such that there is a Boolean algebra isomorphism $\varphi: \mathscr{G} \to \mathscr{F}$ with*

$$\varphi(G) \subset \{x: |T1_G(x)| > c-\tau(G)\},$$

$$|\mu_{\varphi(G)}|(\{x: |T1_G(x)| > c-\tau(G)\} - \varphi(G)) < \delta,$$

for some $\tau(G) < \delta$, and $T^*\mu_{\varphi(G)} = \delta_G$.

The basic induction argument follows these steps. We prove Proposition 2.0 for $\alpha = 0$, $\omega^0 = 1 = \gamma$, then we assume its validity and the validity of Proposition 2.1 for all ordinals $\alpha' < \alpha$, and prove Proposition 2.0 for α. (Actually we will not use Proposition 2.1 when α is a limit ordinal.) Then, we will prove Proposition 2.1 for α using Proposition 2.0 (for α) to complete the inductive cycle.

Remark. For the cases $\alpha = 0,1$, Proposition 2.0 could be deduced from earlier results of Pełczyński [9] and this author [2]. The techniques used here are really derived from those.

Proof of Proposition 2.0. Fix a probability measure μ such that $A \subset L_1(\mu)$. Let $\alpha = 0$, $0 < \tau < \delta$, and T be an isomorphism of $C(\omega)$ into

$C(K)$ as in the hypothesis. If $\mu_n = T^{*-1}\delta_n$ ($\mu_n \in A$), then

$$|\mu_n|(\{x: |T1_{\{n\}}|(x) > c-\tau\}) > \varepsilon.$$

Indeed,

$$1 = |\langle \mu_n, T1_{\{n\}}\rangle| \leqslant \|T\|\,|\mu_n|(\{x: |T1_{\{n\}}|(x) > c-\tau\}) + (c-\tau)\|\mu_n\|$$
$$\leqslant b\,|\mu_n|(\{x: |T1_{\{n\}}|(x) > c-\tau\}) + (c-\tau)a.$$

Hence

$$|\mu_n|(\{x: |T1_{\{n\}}|(x) > c-\tau\}) \geqslant \frac{1}{b}(1-ca+\tau a) > \varepsilon.$$

We claim that $\lambda(\{\mu_n: n \in N\}, \mu) = \varrho \geqslant \varepsilon$. Observe that since $\|\sum \pm 1_{\{n\}}\| = 1$ at most $\|T\|/c-\tau$ of the sets $\{x: |T1_{\{n\}}|(x) > c-\tau\}$ can contain a given point. Thus $\mu(\{x: |T1_{\{n\}}|(x) > c-\tau\})$ goes to zero as n goes to infinity, establishing the claim.

By Lemma 1.2 (with $\beta(n) = 0$) there is an infinite set $M \subset N$ and disjoint open sets $\{H_m: m \in M\}$ such that

$$\lambda(\{\mu_{m|H_m^c}: m \in M\}, \mu) < \delta/4.$$

Let $G_m = H_m \cap \{x: |T1_{\{m\}}|(x) > c-\tau\}$ for all $m \in M$. To complete our argument for the $\alpha = 0$ we need to show that there is an infinite subset $L \subset M$ such that

$$|\mu_m|(G_m) > \varepsilon-\delta \quad \text{and} \quad |\mu_m|(\{x: |T1_{\{m\}}|(x) > c-\tau\} - G_m) < \delta,$$

for all $m \in L$. Then, $\mathscr{F} = \{G_m: m \in L\}$ and $\mathscr{G} = \{\{m\}: m \in L\}$ will be our $\hat{1}$-families with measures $\{\mu_m: m \in L\}$ and $\{\delta_m: m \in L\}$, respectively.

To see that L exists, consider

$$L^c = \{m: m \in M \text{ and } |\mu_m|(\{x: |T1_{\{m\}}|(x) > c-\tau\} - G_m) \geqslant \delta\}.$$

If L^c is infinite,

$$\lambda(\{\mu_m|_{H_m^c}: m \in M\}, \mu) \geqslant \lambda(\{\mu_m|_{H_m^c \cap \{x: |T1_{\{m\}}|(x) > c-\tau\}}\}, \mu) \geqslant \delta$$

—contradicting our choice of $\{H_m: m \in M\}$. Since

$$|\mu_m|(\{x: |T1_{\{m\}}|(x) > c-\tau\}) \geqslant \varepsilon, \quad |\mu_m|(G_m) \geqslant \varepsilon-\delta$$

for all $m \in L$, and the first step of our induction is complete.

The proof of Proposition 2.0 for the limit ordinal case is similar to the proof for successor ordinals. Thus we will present only the argument for successor ordinals and leave the adaption of the proof to the limit ordinal case to the reader. In both cases the argument is like that for Theorem 0.2, except that we must now build $\widehat{\omega^\alpha}$-families. To do this we will need the following lemmas:

LEMMA 2.2. *Let K be a compact Hausdorff space and let H be a closed subset of K. Suppose that for some $\gamma < \omega_1$, $\mathscr{F}$ is a $\hat{\gamma}$-family relative to H, i.e., if $F \in \mathscr{F}$, $F \subset H$ and is open relative to H, and that G is an open set in K with $G \supset H$.*

Then, there is a $\hat{\gamma}$-family $\mathscr{G}$ (relative to K) such that for all F in $\mathscr{F}$ there is a set G_F in $\mathscr{G}$ such that if $F_1 \subsetneqq F \subsetneqq F_2$, $F_1, F_2 \in \mathscr{F}$, then $F_1 \subset G_F \cap H \subset F_2$.

Moreover, if $\mathscr{F}$ has ε-measures, then given $\delta > 0$, $\mathscr{G}$ may be chosen to have $(\varepsilon - \delta)$-measures and such that $|\mu_F|(F - G_F) < \delta$, for all $F \in \mathscr{F}$.

Proof of Lemma 2.2. For each set $F \in \mathscr{F}$ let f_F be a continuous function on $H \cup G^c$ such that

$$f_F(h) = \begin{cases} 1 & \text{if} \quad h \in \overline{\{E \subsetneqq F,\ E \in \mathscr{F}\}}, \\ 0 & \text{if} \quad h \in F^c. \end{cases}$$

Let $X = [f_F\colon F \in \mathscr{F}]$ and note that X is separable. By a standard application of the Michael selection theorem (see [7] page 170) we can find a norm 1 positive linear extension operator $L\colon X \to C(K)$.

Let $\{\varepsilon_\alpha\colon \alpha < \gamma\}$ be a set of positive real numbers such that $\sum \varepsilon_\alpha < 1/2$ and define

$$\mathscr{G} = \Big\{G_F\colon G_F = \{Lf_F > 1 - \sum_{\alpha \leqslant \beta} \varepsilon_\alpha\} \quad \text{where } \beta \text{ is the unique ordinal such that } F \in \mathscr{F}_\beta\Big\}.$$

We claim that $\mathscr{G}$ is the required family, with $\mathscr{G}_\alpha = \{G_F\colon F \in \mathscr{F}_\alpha\}$, for all $\alpha < \gamma$. All of the inclusion properties follow from the corresponding properties for $\mathscr{F}$ and the positivity of L. The disjointness properties follow from $\sum \varepsilon_\alpha < 1/2$.

The "moreover" assertion can be obtained by choosing for each $F \in \mathscr{F}$, a closed subset F' such that

$$F \supset F' \supset \bigcup\{G \subsetneqq F,\ G \in \mathscr{F}\} \quad \text{and} \quad |\mu_F|(F') > |\mu_F|(F) - \delta.$$

We then require that

$$f_F(h) = \begin{cases} 1 & h \in F', \\ 0 & h \in F^c \end{cases}$$

and complete the argument as above.

Remark. In the above lemma we have referred to a $\hat{\gamma}$-family realtive to H with ε-measures. There is some ambiguity here with regard to the w^* convergence requirements and the domain of the measures. In our case, we will use measures defined on the Borel subsets of K and the w^* convergence will be relative to $C(K)$.

LEMMA 2.3. *Let $\mathcal{O}$ be a clopen subset of $[1, \alpha]$, for some ordinal α and A a closed subset of $\mathcal{O}$. Then there is a positive linear extension operator $L\colon C(A) \to C[1, \alpha]$ such that $X = L(C(A))$ is supported in $\mathcal{O}$, and consequently, X is a complemented subspace of $C[1, \alpha]$, the restriction map $R_A\colon C[1, \alpha] \to C(A)$ is an isometry from X onto $C(A)$ and LR_A is a projection onto X.*

Proof. Define a linear extension operator $L'\colon C(A) \to C[1, \alpha]$ by

$$L'f(\beta) = \begin{cases} 0 & \text{if} \quad \beta > \sup A, \\ f(a) \text{ where } a = \inf\{\gamma \in A\colon \gamma \geqslant \beta\} & \text{if} \quad \beta \leqslant \sup A \end{cases}$$

and let $R_{\mathcal{O}}$ be the restriction map onto $C(\mathcal{O})$, i.e., $R_{\mathcal{O}} f = f|_{\mathcal{O}}$. $L = R_0 L'$ satisfies the requirements of the lemma.

We are now ready to continue our proof of Proposition 2.0.

Suppose $\alpha = \beta + 1$ and let T be the given isomorphism from $C(\omega^{\omega^\alpha})$ into $C(K)$. Define

$$\varrho_0 = \sup\big\{\varrho \mid \exists\{\nu_n\colon n \in N\} \text{ such that } \lambda(\{\nu_n\colon n \in N\}, \mu) \geqslant \varrho \text{ and } \nu_n \in P_{\omega^{\beta_n}}(\varepsilon - \delta/2, A) \text{ for all } n\big\}.$$

It follows from Proposition 2.1 and the fact that $C(\omega^{\omega^\alpha}) \supset C(\omega^{\omega^{\beta_n}})$ for all n that such sequences exist. The supremum is attained, so let $\{\nu_n\colon n \in N\}$ be a sequence as above with $\lambda(\{\nu_n\colon n \in N\}, \mu) = \varrho_0$.

By passing to a subsequence if necessary we may assume that $\nu_n \in P_{\omega^{\beta_{2n}}}(\varepsilon - \delta/2, A)$, for all n. For each n choose a subset A_n of A such that A_n is homeomorphic to $[1, \omega^{\omega^{\beta_n}}]$, $A_n^{(\omega^{\beta_n})} = \{\nu_n\}$, $P_{\omega^{\beta_n}}(\varepsilon - \delta/2, A_n) \neq \emptyset$, and $A_n \subset P_{\omega^{\beta_n}}(\varepsilon - \delta/2, A)$. The same argument as we used in the proof of Theorem 0.2 shows that there is an n_0 such that $\lambda\big(\bigcup\{A_n\colon n \geqslant n_0\}, \mu\big) < \varrho_0 + \delta/16$. By Lemma 1.2 there are an infinite subset M of N, disjoint open sets $\{H_m\colon m \in M\}$, and subsets $\{\mu_{m,\gamma}\colon \gamma \leqslant \omega^{\omega^{\beta_m}}\} \subset A_m$ for all $m \in M$ such that

$$\lambda\big(\bigcup\{\mu_{m,\gamma}|_{H_m^c}\colon \gamma \leqslant \omega^{\omega^{\beta_m}},\ m \in M\}, \mu\big) < \delta/8. \tag{2.0}$$

Clearly we can assume that

$$|\mu_{m,\gamma}|(H_m) > \varrho_0 - \delta/8 \quad \text{for all } \gamma \leqslant \omega^{\omega^{\beta_m}},\ m \in M. \tag{2.1}$$

For each $k \in N$ let $n(k, \delta/16)$ be the integer given by Proposition 2.1. Choose for each k, an integer $m(k) \geqslant n(k, \delta/16)$ with $m(k) \in M$, and let

$$D_k = \{\varrho\colon \delta_\varrho = T^* \mu_{m(k),\gamma} \quad \text{for some } \gamma \in [1, \omega^{\omega^{\beta_{m(k)}}}]^{(\omega^{\beta}(m(k) - n(k, \delta/16)))}\}.$$

(Here we could use elements of any closed subset of $[1, \omega^{\omega^{\beta_{m(k)}}}]$, homeomorphic to $[1, \omega^{\omega^{\beta_{n(k,\delta/16)}}}]$.) By passing to smaller sets of the same homeomorphic type we may assume that there are disjoint clopen sets $\{\mathcal{O}_k\colon k \in N\}$ with $\mathcal{O}_k \supset D_k$ for all k. Let $X_k = L\big(C(D_k)\big)$ where L is the linear extension operator given by Lemma 2.3 (with $\mathcal{O} = \mathcal{O}_k$).

Since X_k is isometric to $C(\omega^{\omega^{\beta_{n(k,\delta/16)}}})$, we can consider $T|_{X_k}$ as an isomorphism from $C(\omega^{\omega^{\beta_{n(k,\delta/16)}}})$ into $C(K)$. By Proposition 2.1

there are $\widehat{\omega^\beta k}$ families $\mathscr{F}_k$ and $\mathscr{G}_k$ with $(\varepsilon-\delta/16)$-measures $\{\nu_{k,\gamma}\colon \gamma<\omega^{\omega^\beta k}\}$ and 1-measures $\{\delta_{k,\gamma}\colon \gamma<\omega^{\omega^\beta k}\}$, respectively, such that there is a Boolean algebra isomorphism $\varphi_k\colon \mathscr{G}_k\to\mathscr{F}_k$ with $\varphi_k(G)\subset\{x\colon |T1_G|(x)>c-\tau(G)\}$,

$$|\nu_{\varphi_k(G)}|(\{x\colon |T1_G|(x)>c-\tau(G)\}-\varphi_k(G))<\delta/16, \tag{2.2}$$

for some $\tau(G)<\delta/16$, and $T^*\mu_{\varphi(G)}=\delta_G$, for all $G\in\mathscr{G}_k$, $k\in\boldsymbol{N}$. Here the sets G in $\mathscr{G}_k$ satisfy $1_G\in X_k$.

We would like to use $\bigcup\{\mathscr{G}_k\colon k\in\boldsymbol{N}\}$ and $\bigcup\{\mathscr{F}_k\colon k\in\boldsymbol{N}\}$ as our $\widehat{\omega^{\beta+1}}$-families however there is no guarantee that if $F_k\in\mathscr{F}_k$ and $F_{k'}\in\mathscr{F}_{k'}$, $k\neq k'$, $F_k\cap F_{k'}=\emptyset$. (Since X_k is supported in $\mathcal{O}_k$, this is automatic for $\bigcup\{\mathscr{G}_k\colon k\in\boldsymbol{N}\}$.) Thus we must modify the families $\{\mathscr{F}_k\colon k\in\boldsymbol{N}\}$ slightly. We will first find closed sets $H_k''\subset H_k$ such that $\{F\cap H_k''\colon F\in\mathscr{F}_k\}$ contains an $\omega^\beta(k-1)$-family $\mathscr{F}_k'$ relative to H_k'' with $\varepsilon-3\delta/4$ measures. Then we will use Lemma 2.2 to replace $\mathscr{F}_k'$ by a family $\mathscr{F}_k''$ relative to K so that all of its sets are within H_k.

Choose an integer l such that

$$|\nu_{k,\gamma}|\,|_{H_k^c}\big(\bigcup\{H_n\colon n\geqslant l,\ n\in M\}\big)<\delta/8 \tag{2.3}$$

and

$$|\nu_{k,\gamma}|(H_k)<\varrho_0+\delta/8 \quad\text{for}\quad \gamma<\omega^{\omega^\beta k},\quad k\in M,\ k\geqslant l. \tag{2.4}$$

For each k in M let H_k' be an open set such that $H_k'\subset\overline{H_k'}\subset H_k$ and

$$|\nu_{k,\omega^{\omega^\beta(k-1)}}|(H_k')>|\nu_{k,\omega^{\omega^\beta(k-1)}}|(H_k)-\delta/8. \tag{2.5}$$

Pass to a w^* neighborhood, $\{\nu_{k,\gamma}\colon \gamma_k\leqslant\gamma\leqslant\omega^{\omega^\beta(k-1)}\}$, of $\nu_{k,\omega^{\omega^\beta(k-1)}}$ such that

$$|\nu_{k,\gamma}|(H_k')>|\nu_{k,\omega^{\omega^\beta(k-1)}}|(H_k)-\delta/8. \tag{2.6}$$

Let

$$\mathscr{G}_k'=\{G_{k,\gamma}\colon \gamma_k\leqslant\gamma\leqslant\omega^{\omega^\beta(k-1)}\}$$

and

$$\mathscr{F}_k'=\{\varphi_k(G_{k,\gamma})\cap\overline{H_k'}\colon \gamma_k\leqslant\gamma\leqslant\omega^{\omega^\beta(k-1)}\}$$

for all $k\in M$, $k\geqslant l$.

We claim that $\mathscr{F}_k'$ is an $\omega^\beta(k-1)$-family with $\varepsilon-3\delta/4$ measures. Since $\mathscr{F}_k$ is an $\widehat{\omega^\beta k}$-family, it is enough to show that $|\nu_{k,\gamma}|\big(\varphi_k(G_{k,\gamma})\cap\overline{H_k'}\big)>\varepsilon-3\delta/4$. We will in fact show more, namely, that if k is sufficiently large,

$$|\nu_{k,\gamma}|\big(\varphi_k(G_{k,\gamma})-\varphi_k(G_{k,\gamma})\cap\overline{H_k'}\big)<5\delta/8. \tag{2.7}$$

Suppose that it were not the case that there exists a $k_0\geqslant l$ such that for all $k\geqslant k_0$,

$$|\nu_{k,\gamma}|\big(\varphi_k(G_{k,\gamma})-\varphi_k(G_{k,\gamma})\cap\overline{H_k'}\big)<5\delta/8$$

for $\gamma_k\leqslant\gamma\leqslant\omega^{\omega^\beta(k-1)}$. Then, we could find a sequence $\{\nu_{k,\gamma(k)}\colon k\in\boldsymbol{N}\}$ such that

$$\overline{\lim}\,|\nu_{k,\gamma(k)}|\big(\varphi_k(G_{k,\gamma(k)})-\varphi_k(G_{k,\gamma(k)})\cap\overline{H_k'}\big)\geqslant 5\delta/8.$$

Since the subspaces X_k are disjointly supported,

$$\lim_k\mu\big(\{x\colon |T1_{G_{k,\gamma(k)}}|(x)>c-\tau(G_{k,\gamma(k)})\}\big)=0$$

and thus $\lim_k\mu\big(\varphi_k(G_{k,\gamma(k)})\big)=0$. This implies that $\lambda(\{\nu_{k,\gamma(k)}|_{\overline{H_k'^c}}\colon k\in\boldsymbol{N}\},\mu)\geqslant 5\delta/8$, but we also have that

$$\begin{aligned}|\nu_{k,\gamma(k)}|(H_k-H_k')&<|\nu_{k,\gamma(k)}|(H_k)-|\nu_{k,\omega^{\omega^\beta(k-1)}}|(H_k)+\delta/4\\&<\varrho_0+\delta/8-(\varrho_0-\delta/8)+\delta/4=\delta/2,\end{aligned}$$

by (2.6), (2.4), and (2.1), and, by (2.0),

$$\lambda(\{\nu_{k,\gamma(k)}|_{H_k^c}\colon k\in\boldsymbol{N}\},\mu)<\delta/8.$$

This contradiction gives us our k_0, so that $\mathscr{F}_k'$ is an $\omega^\beta(k-1)$-family with $(\varepsilon-3\delta/4)$-measures, for all $k\geqslant k_0$.

By Lemma 2.2 there is an $\omega^\beta(k-1)$-family $\mathscr{E}_k$ of subsets of H_k with $(\varepsilon-7\delta/8)$-measures. For each k and $\gamma_k\leqslant\gamma\leqslant\omega^{\omega^\beta(k-1)}$ let $F_{k,\gamma}$ be the set in $\mathscr{E}_k$ corresponding to $\varphi_k(G_{k,\gamma})\cap\overline{H_k'}$ and define

$$\mathscr{F}_k''=\{F_{k,\gamma}\cap\varphi_k(G_{k,\gamma})\colon \gamma_k\leqslant\gamma\leqslant\omega^{\omega^\beta(k-1)}\}.$$

It is easy to see that $\mathscr{F}_k''$ is an $\omega^\beta(k-1)$-family, and since by Lemma 2.2 $\nu_{k,\gamma}|\big(\varphi_k(G_{k,\gamma})\cap\overline{H_k'}-F_{k,\gamma}\big)<\delta/8$, $\mathscr{F}_k''$ has $(\varepsilon-7\delta/8)$-measures.

Let $\mathscr{F}=\bigcup\{\mathscr{F}_k''\colon k\in M,\ k\geqslant k_0\}$ and $\mathscr{G}=\bigcup\{\mathscr{G}_k'\colon k\in M,\ k\geqslant k_0\}$. Clearly $\mathscr{F}$ and $\mathscr{G}$ are $\widehat{\omega^{\beta+1}}$-families with $(\varepsilon-\delta)$-measures

$$\{\nu_{k,\gamma}\colon \gamma_k\leqslant\gamma\leqslant\omega^{\omega^\beta(k-1)},\ k\in M,\ k\geqslant k_0\}$$

and 1-measures

$$\{T^*\nu_{k,\gamma}\colon \gamma_k\leqslant\gamma\leqslant\omega^{\omega^\beta(k-1)},\ k\in M,\ k\geqslant k_0\},$$

respectively. Define a Boolean algebra isomorphism $\varphi\colon\mathscr{G}\to\mathscr{F}$ by

$$\varphi(G_{k,\gamma})=G_{k,\gamma}\cap\varphi_k(G_{k,\gamma}),\qquad \gamma_k\leqslant\gamma\leqslant\omega^{\omega^\beta(k-1)},\qquad k\in M,\ k\geqslant k_0.$$

Then $\varphi(G_{k,\gamma})\subset\varphi_k(G_{k,\gamma})\subset\{x\colon |T1_{G_{k,\gamma}}|(x)>c-\tau(G_{k,\gamma})\}$,

$$\begin{aligned}&|\nu_{k,\gamma}|\big(\{x\colon |T1_{G_{k,\gamma}}|(x)>c-\tau(G_{k,\gamma})\}-\varphi(G_{k,\gamma})\big)\\&\quad\leqslant|\nu_{k,\gamma}|\big(\{x\colon |T1_{G_{k,\gamma}}|(x)>c-\tau(G_{k,\gamma})\}-\varphi_k(G_{k,\gamma})\big)+\\&\qquad+|\nu_{k,\gamma}|\big(\varphi_k(G_{k,\gamma})-\varphi_k(G_{k,\gamma})\cap F_{k,\gamma}\big)<\delta/16+5\delta/8<\delta,\end{aligned}$$

(by (2.2 and (2.7)), and $\tau(G_{k,\gamma}) < \delta/16 < \delta$. Thus the proof of Proposition 2.0 is complete.

For our proof of Proposition 2.1 we will need a few technical lemmas. The first is a purely combinatorial result that was proved in [3]. We would like to thank Y. Benyamini for pointing out that this lemma could be used to simplify several of our original arguments.

LEMMA 2.4. *For every $k, n \in N$ and ordinal $\alpha < \omega_1$ there is an integer $M = M(k, n)$ such that if $[1, \omega^{\alpha M}] = \bigcup\{A_i: i = 1, 2, \ldots, k\}$, then there is an index i_0 such that A_{i_0} contains a subset homeomorphic to $[1, \omega^{\alpha n}]$.*

Our next two lemmas concern w^* convergence and its relationship to the norm.

LEMMA 2.5. *For every $\varepsilon > 0$, $n \in N$, and ordinal $\alpha < \omega_1$ there is an integer $m = m(n, \varepsilon)$ such that if $\mathscr{A} = \{\mu_\gamma: \gamma \leqslant \omega^{\alpha m}\}$ is a subset of $B_{C(K)^*}$ (in the w^* topology), then there is a subset $\mathscr{A}'$ of $\mathscr{A}$ such that $\mathscr{A}'$ is homeomorphic to $[1, \omega^{\alpha n}]$ and $\big|\|\mu\| - \|\nu\|\big| < \varepsilon$ for all $\mu, \nu \in \mathscr{A}'$.*

Proof. Let $m = M([\varepsilon^{-1}]+1, n)$, where M is given by Lemma 2.4, and consider $\{\gamma: (i+1)\varepsilon > \|\mu_\gamma\| \geqslant i\varepsilon,\ \gamma \leqslant \omega^{\alpha m}\}$, $i = 0, 1, 2, \ldots, [\varepsilon^{-1}]$. Clearly this is a partition of $[1, \omega^{\alpha m}]$ and Lemma 2.5 implies that $\{\gamma: (i_0+1)\varepsilon > \|\mu_\gamma\| \geqslant i\varepsilon,\ \gamma \leqslant \omega^{\alpha m}\}$ contains a subset homeomorphic to $[1, \omega^{\alpha n}]$, as required.

LEMMA 2.6. *Let K be compact Hausdorff space. Let $\delta, \varepsilon > 0$, $A = \{\mu_\gamma: \gamma \leqslant \omega^\beta\}$ for some $\beta < \omega_1$, and $A \subset B_{C(K)^*}$ such that $\big|\|\mu\| - \|\nu\|\big| < \varepsilon$ for all $\mu, \nu \in A$. Then*

(i) *If $f \in B_{C(K)}$, there is a neighborhood $\mathscr{N}_1$ of μ_{ω^β} in A such that*

$$\Big|\int f d|\mu| - \int f d|\mu_{\omega^\beta}|\Big| < \varepsilon + \delta, \quad \text{for all} \quad \mu \in \mathscr{N}_1;$$

(ii) *If F is a closed subset of K and $|\mu_{\omega^\beta}|(F) < \delta$, then there is a neighborhood $\mathscr{N}_2$ of μ_{ω^β} in A such that*

$$|\mu|(F) < \varepsilon + \delta, \quad \text{for all } \mu \in N_2.$$

Proof. Observe that if ν is a w^* limit point of the net $\{|\mu_\gamma|: \gamma \uparrow \omega^\beta\}$, $\nu \geqslant |\mu_{\omega^\beta}|$. Indeed, given $h \geqslant 0$ and $\delta' > 0$ choose $g \in B_{C(K)}$ such that $|\int hd|\mu_{\omega^\beta}| - \int hg\,d\mu_{\omega^\beta}| < \delta'$. Then

$$\int hd|\mu_{\omega^\beta}| - \delta' < \int hg\,d\mu_{\omega^\beta} = \lim_\gamma \int hg\,d\mu_\gamma \leqslant \varliminf_\gamma \int h|g|\,d|\mu_\gamma|$$
$$\leqslant \varliminf_\gamma \int hd|\mu_\gamma| \leqslant \int h\,d\nu.$$

On the other hand, $\|\nu\| \leqslant \varlimsup_\gamma \|\mu_\gamma\| \leqslant \|\mu_{\omega^\beta}\| + \varepsilon$, and thus $\big\||\mu_{\omega^\beta}| - \nu\big\| \leqslant \varepsilon$.

For (i) we have

$$\Big|\varlimsup_\gamma \int fd|\mu_\gamma| - \int fd|\mu|\Big| \leqslant \varepsilon\|f\| \leqslant \varepsilon$$

so that for γ sufficiently large

$$\Big|\int fd|\mu_\gamma| - \int fd|\mu|\Big| < \varepsilon + \delta.$$

If (ii) were false, $\varlimsup_\gamma |\mu_\gamma|(F) \geqslant \varepsilon + \delta$ and thus for some limit point ν of $\{|\mu_\gamma|: \gamma \uparrow \omega^\beta\}$, $|\nu|(F) \geqslant \varepsilon + \delta$. But,

$$\nu(F) \leqslant |\mu_{\omega^\beta}|(F) + \big||\mu_{\omega^\beta}| - \nu\big|(F) < \delta + \varepsilon.$$

LEMMA 2.7 (Gillis [7]). *For every $c, \varrho > 0$ and $n \in N$ there is an integer $k = k(n, c, \varrho)$ such that if $\{G_l: l = 1, 2, \ldots, k\}$ are measurable subsets of a measure space $(\Omega, \mathscr{B}, \mu)$ where μ is a positive measure of total variation less than c and $\mu(G_l) \geqslant \varrho$ for $l = 1, 2, \ldots, k$, then there are indices $l(1) < l(2) < \ldots < l(n)$ such that $\mu(\{\bigcap G_{l(i)}: i = 1, 2, \ldots, n\}) > 0$.*

We now have all the tools we need to prove Proposition 2.2. Two observations that we will employ in our argument are:

(1) For a nested sequence of clopen sets $J_{2n} \subsetneqq J_{2n-1} \ldots \subsetneqq J_2 \subsetneqq J_1$ $\|\sum_{k=1}^n (1_{J_{2k}} - 1_{J_{2k-1}})\| = 1$ and thus $\|\sum_{k=1}^n (T1_{J_{2k}} - T1_{J_{2k-1}})\|$ is bounded, independent of n.

(2) If $\mathscr{F}$ is a $\widehat{\gamma n}$-family and for each $F \in \mathscr{F}_0$ there is an $\hat{\gamma}$-family $\mathscr{F}(F)$ such that $\overline{\bigcup\{G: G \in \mathscr{F}(F)\}} \subset F$, then $\mathscr{F} \cup \bigcup\{\mathscr{F}(F): F \in \mathscr{F}_0\}$ is an $\widehat{\gamma(n+1)}$-family.

Proof of Proposition 2.1. We assume inductively that the proposition holds for $n-1$, $n \geqslant 1$, and any $\delta > 0$ and prove it for n. We have by Proposition 2.0 that we can take $N(\delta, 1) = 1$, and thus the first step of the induction has been completed.

Let $N = N(\delta, n) = m\big(k\cdot(N(\delta/3, n-1)+1),\ \delta/24\big)$ where $k = k(8\|T\|\delta^{-1}+1,\ a,\ \delta/24)$ from Lemma 2.7, $N(\delta/3, n-1)$ is defined by the inductive hypothesis, and m is determined by Lemma 2.5. To simplify our notation let $l = N(\delta/3, n-1)$ and $p = k(l+1)$. By our choice of N we can find a closed subset A' of A such that A' is homeomorphic to $[1, \omega^{\gamma p}]$ and $\big|\|\nu\| - \|\nu'\|\big| < \delta/24$ for all ν, ν' in A'. By Lemma 2.3 we can find a positive linear extension operator $L: C(A'') \to C[1, \omega^{\gamma N}]$, where $A'' = \{\tau: T^*\nu = \delta_\tau$ for some $\nu \in A'\}$, such that TL is an isomorphism of $C(A'')$ into $C(K)$ with $\|TL\| \leqslant \|T\|$, and such that $(TL)^*|_{A'}$ is a homeomorphism onto $\{\delta_\tau: \tau \in A''\}$.

In this way we can assume without loss of generality that T is defined on $C[1, \omega^{\gamma p}]$, A is homeomorphic to $[1, \omega^{\gamma p}]$, and

$$\text{(2.8)} \qquad \text{for all } \nu, \nu' \in A, \quad \big|\|\nu\| - \|\nu'\|\big| < \delta/24.$$

Let X be the subspace of $C[1,\omega^{\gamma p}]$ which is the range of the linear extension operator L: $C([1,\omega^{\gamma p}]^{(\gamma(p-l))})\to C[1,\omega^{\gamma p}]$ given by Lemma 2.3. Note that TL and

$$B=\{\mu:\ T^*\mu=\delta_\tau,\ \text{for some } \tau\in[1,\omega^{\gamma p}]^{(\gamma(p-l))}\ \text{ and } \mu\in A\}$$

satisfy the inductive hypothesis. Thus there are $\gamma\widehat{(n-1)}$-families $\mathcal{F}$ and $\mathcal{G}$ with $(\varepsilon-\delta/3)$-measures from B and 1-measures from $[1,\omega^{\gamma p}]^{(\gamma(p-l))}$, respectively, and a Boolean algebra isomorphism φ: $\mathcal{G}\to\mathcal{F}$ such that

$$\{x:\ |T1_G|(x)>c-2\delta/3-\tau(G)\}\supset\varphi(G),$$

for some $\tau(G)<\delta/3$,

$$|\mu_{\varphi(G)}|\big(\{x:\ |T1_G|(x)>c-2\delta/3-\tau(G)\}-\varphi(G)\big)<\delta/3, \tag{2.9}$$

and $K^*T^*\mu_{\varphi(G)}=\nu_G$ for all $G\in\mathcal{G}$. Formally the sets in $\mathcal{G}$ are subsets of $[1,\omega^{\gamma p}]^{(\gamma(p-l))}$, however we will consider them to be subsets of $[1,\omega^{\gamma p}]$ by the identification $G\leftrightarrow\operatorname{supp}L1_G$.

Let $D=\{\tau:\ \delta_\tau=\nu_G$ for some $G\in\mathcal{G}\}$. By (an easy modification of) Lemma 1.0 (b) $\Rightarrow$ (d) we can assume that D is homeomorphic to $[1,\omega^{\gamma(n-1)})$. For each

$$d\in D^{d(0)}=\{d:\ d\in D\ \text{and}\ d\ \text{is an isolated point}\},$$

$G_d\cap(d^-,d]$ is homeomorphic to $[1,\omega^\xi n]$ for some $\xi\geqslant\gamma(p-l)$ and $n\in N$. (Here G_d is the set in $\mathcal{G}$ such that $\nu_{G_d}=\delta_d$.) Thus we can choose a subset $E(d)$ of $[d^-,d]^{(\gamma(p-l-1))}\cap G_d$ such that $E(d)$ is homeomorphic to $[1,\omega^\gamma]$ and $E(d)^{(\gamma)}=\{d\}$. By passing to the subset of $E(d)$ given by Lemma 2.6(ii) we may assume that for all μ, such that $T\mu^*=\delta_\tau$ for some $\tau\in E(d)$,

$$|\mu|\big(\{x:\ |T1_{G_d}|(x)\geqslant c-\tau(G_d)-2\delta/3+\delta/24\}-\varphi(G_d)\big)<\delta/3+\delta/6=\delta/2, \tag{2.10}$$

as well. Let $L(d)$: $C(E(d))\to C[1,\omega^{\gamma p}]$ be the linear extension operator given by Lemma 2.3 with $\mathcal{O}=(d^-,d]\cap G_d$, and let $X(d)=L(d)\big(C(E(d))\big)$. As before the operator $TL(d)$ and the set

$$B(d)=\{\mu:\ T^*\mu=\delta_\tau\ \text{for some}\ \tau\in E(d)\ \text{and}\ \mu\in A\},$$

satisfy the inductive hypothesis $(n=1)$. Thus there are $\hat\gamma$-families $\mathcal{F}(d)$ and $\mathcal{G}(d)$ with $(\varepsilon-\delta/3)$-measures from $B(d)$ and 1-measures from $\{\delta_\tau$: $\tau\in E(d)\}$, respectively, and a Boolean algebra isomorphism $\varphi(d)$: $\mathcal{G}(d)\to\mathcal{F}(d)$ such that

$$\{x:\ |T1_G|(x)>c-\tau(G)\}\supset\varphi(d)(G)$$

for some $\tau(G)<\delta/3$,

$$|\mu_{\varphi(d)(G)}|\big(\{x:\ |T1_G|(x)>c-\tau(G)\}-\varphi(d)(G)\big)<\delta/3, \tag{2.11}$$

and $L(d)^*T^*\mu_{\varphi(d)(G)}=\nu_G$ for all $G\in\mathcal{G}(d)$. As above we will consider the sets in $\mathcal{G}(d)$ to be subsets of $[1,\omega^{\gamma p}]$.

We would like to use $\mathcal{F}\cup\bigcup\{\mathcal{F}(d):\ d\in D^{d(0)}\}$ as our $\widehat{\gamma n}$-family, but we do not know that if $F\in\mathcal{F}(d)$, $F\subset\varphi(G_d)$. Thus we will try to replace $\mathcal{F}(d)$ by some $\hat\gamma$-family whose sets are subsets of $\varphi(G_d)$. Since we must also have $(\varepsilon-\delta)$-measures, we may not be able to do this for these $\mathcal{F}(d)$'s. If this happens, we will consider other $\gamma\widehat{(n-1)}$-families and $\hat\gamma$-families and show that there must be a $\gamma\widehat{(n-1)}$-family $\mathcal{F}$ and $\hat\gamma$-families $\mathcal{F}(d)$ which can be modified so that $\mathcal{F}\cup\bigcup\{\mathcal{F}'(d):\ d\in D^{(0)}\}$ is a $\widehat{\gamma n}$-family with the required properties.

Consider the set

$$E=\big\{d:\ |\mu_{\varphi(d)(G)}|\big(\{x:\ |T1_H|(x)<c-2\delta/3-\tau(H)+\delta/24\}\cap\{x:\ |T1_G|(x)>c-\tau(G)\}\big)<\delta/6\ \text{for all}\ G\in\mathcal{G}(d)\ \text{and}\ H\in\mathcal{G}\ \text{with}\ H\supset G\big\}. \tag{2.12}$$

If $E\supset D^{(0)}$, we complete the argument as follows: For each d let $F(d)$ be an open subset of $\varphi(G_d)$ such that

$$|\mu_{\varphi(G_d)}|\big(F(d)\big)>|\mu_{\varphi(G_d)}|\big(\varphi(G_d)\big)-\delta/24\quad\text{and}\quad\overline{F(d)}\subset\varphi(G_d). \tag{2.13}$$

Observe that $S=\{x:\ |T1_{G_d}|(x)\geqslant c-2\delta/3-\tau(G_d)+\delta/24\}\cap F(d)^c$ is a closed set and thus by Lemma 2.6(ii) there is a closed w^* neighborhood $\mathcal{N}(d)$ of $\mu_{\varphi(G_d)}$ such that $|\mu|(S)<|\mu_{\varphi(G_d)}|(S)+\delta/12$ for all $\mu\in\mathcal{N}(d)$. Since

$$S\subset\{x:\ |T1_{G_d}|(x)>c-2\delta/3-\tau(G_d)\}-F(d),$$

it follows from (2.9) and (2.13) that $|\mu_{\varphi(G_d)}|(S)<\delta/3+\delta/24$ and hence that

$$|\mu|(S)<11\delta/24,\quad\text{for all}\ \mu\in\mathcal{N}(d). \tag{2.14}$$

By Lemma 1.1 there is a $\hat\gamma$-family contained in $\mathcal{F}(d)$ with $(\varepsilon-\delta/3)$-measures in $\mathcal{N}(d)$, so without loss of generality, we will assume that this $\hat\gamma$-family is $\mathcal{F}(d)$ itself. (This poses no problem for our assumptions regarding $\mathcal{G}(d)$ since we can replace it by the image under $\varphi_{(d)}^{-1}$ of the family in $\mathcal{F}(d)$.)

Let $\mathcal{F}''(d)=\{F\cap\overline{F(d)}:\ F\in\mathcal{F}(d)\}$. We claim that $\mathcal{F}''(d)$ is a $\hat\gamma$-family with $(\varepsilon-\delta)$-measures. The "family" properties of the $\hat\gamma$-family $\mathcal{F}(d)$ are obviously inherited by $\mathcal{F}''(d)$ if we show that $F\cap\overline{F(d)}\neq\emptyset$ for all

$F \in \mathscr{F}(d)$. This in turn will follow from

(2.15) $$|\mu_F|\big(F\cap\overline{F(d)}\big) > \varepsilon - 23\delta/24 \quad \text{for all } F \in \mathscr{F}(d).$$

We will actually show more, namely, that

(2.16) $$|\mu_F|\big(F - \overline{F(d)}\big) < 15\delta/24.$$

We have that

$$F - \overline{F(d)} \subset S \cup \big(\{x\colon |T1_{G_d}|(x) < c - 2\delta/3 - \tau(G_d) + \delta/24\} \cap \\ \cap \{x\colon |T1_{\varphi(d)^{-1}(F)}|(x) > c - \tau(\varphi(d)^{-1}(F))\}\big).$$

Thus by (2.14) and (2.12),

$$|\mu_F|\big(F - \overline{F(d)}\big) \leqslant |\mu_F|(S) + |\mu_F|\big(\{x\colon |T1_{G_d}|(x) < c - 2\delta/3 - \tau(G_d) + \delta/24\} \cap \\ \cap \{x\colon |T1_{\varphi(d)^{-1}(F)}|(x) > c - \tau(\varphi(d)^{-1}(F))\}\big) < 11\delta/24 + \delta/6 = 15\delta/24.$$

Since $\mathscr{F}(d)$ has $(\varepsilon - \delta/3)$-measures, it follows that $|\mu_F|\big(F \cap \overline{F(d)}\big) > \varepsilon - \delta/3 - 15\delta/24 = \varepsilon - 23\delta/24$, establishing our claim. Note that we have as well that

(2.17) $$|\mu_F|\big(\{x\colon |T1_{\varphi(d)^{-1}(F)}|(x) > c - \tau(\varphi(d)^{-1}(F))\} - (F \cap \overline{F(d)})\big) \\ < \delta/3 + 15\delta/24 < \delta$$

by (2.11) and (2.16).

It only remains to replace $\mathscr{F}''(d)$ by an $\hat{\gamma}$-family of subsets of $\varphi(G_d)$, i.e., a $\hat{\gamma}$-family relative to K. This we can accomplish by Lemma 2.2. Indeed, let $F'(d)$ be an open subset of K such that $\overline{F(d)} \subset F'(d) \subset \overline{F'(d)} \subset \varphi(G_d)$. By Lemma 2.2 there is a $\hat{\gamma}$-family $\mathscr{F}'''(d)$, of subsets of $F'(d)$ with $(\varepsilon - 23\delta/24 - \delta/24)$-measures. Let

$$\mathscr{F}'(d) = \{F' \cap F\colon F \in \mathscr{F}(d),\ F' \in \mathscr{F}'''(d), \\ \text{and } F' \text{ is the set corresponding to } F \cap \overline{F(d)}\}$$

and note that $|\mu_F|(F' \cap F) > \varepsilon - \delta$ and

$$|\mu_F|\big(\{x\colon |T1_{\varphi(d)^{-1}(F)}|(x) > c - \tau(\varphi(d)^{-1}(F))\} - F' \cap F\big) < \delta,$$

since $|\mu_F|\big(F \cap \overline{F(d)} - F'\big) < \delta/24$ and $\mathscr{F}''(d)$ satisfies (2.15) and (2.17).

Let

$$\mathscr{G}' = \mathscr{G} \cup \big(\bigcup \{\mathscr{G}(d)\colon d \in D^{(0)}\}\big)$$

and

$$\mathscr{F}' = \mathscr{F} \cup \big(\bigcup \{\mathscr{F}'(d)\colon d \in D^{(0)}\}\big),$$

and observe that they are both $\widehat{\gamma n}$-families. (Everything is obvious except perhaps (6) which follows from $F \subset F'(d) \subset \overline{F'(d)} \subset \varphi(G_d)$.)

$$\{\nu_G\colon G \in \mathscr{G}\} \cup \{\nu_G\colon G \in \mathscr{G}(d),\ d \in D^{(0)}\}$$

and

$$\{\mu_{\varphi(G)}\colon G \in \mathscr{G}\} \cup \{\mu_F\colon F \in \mathscr{F}(d),\ d \in D^{(0)}\}$$

are 1-measures from the point masses and $(\varepsilon - \delta)$-measures from A, respectively, which makes $\mathscr{G}'$ and $\mathscr{F}'$, $\widehat{\gamma n}$-families with the appropriate measures. Define $\varphi'\colon \mathscr{G}' \to \mathscr{F}'$ by

$$\varphi'(G) = \begin{cases} \varphi(G) & \text{if } G \in \mathscr{G}, \\ \varphi(d)(G) \cap \varphi(d)(G)' & \text{if } G \in \mathscr{G}(d). \end{cases}$$

(Here $\varphi(d)(G)' \in \mathscr{F}'''(d)$ and corresponds to $\varphi(d)(G) \cap \overline{F(d)}$ under Lemma 2.2.) Clearly φ' is a Boolean algebra isomorphism and we have already verifed the measure theoretic requirements. Thus if $E \supset D^0$, we are done.

If $E \not\supset D^{(0)}$, then there is a $d \in D^{(0)}$, and a set $G \in \mathscr{G}(d)$ and $H \in \mathscr{G}_0$ such that $H \supset G$ and

$$|\mu_{\varphi(d)(G)}|\big(\{x\colon |T1_H|(x) \leqslant c - 2\delta/3 - \tau(H) + \delta/24\} \cap \\ \cap \{x\colon |T1_G|(x) > c - \tau(G)\}\big) \geqslant \delta/6.$$

By Lemma 2.5 we can find a clopen neighborhood $\mathscr{N}$ of $\mu_{\varphi(d)(G)}$ in A such that for all $\mu \in \mathscr{N}$

(2.18) $$|\mu|\big(\{x\colon |T1_H|(x) < c - 15\delta/24\} \cap \{x\colon |T1_G|(x) > c - \tau(G)\}\big) \\ \geqslant \delta/6 - 3\delta/24 = \delta/24$$

and thus

$$|\mu|(\{x\colon |T1_H - T1_G| > \delta/4\}) > \delta/24.$$

Let $J(1) = H$ and $J(2) = G$.

We now repeat our earlier argument using subspaces of $C[1, \omega^{\gamma p}]$ which are supported in $I_1 = J(2) \cap \{\gamma\colon \delta_\gamma \in T^*\mathscr{N}\}$. First we observe that $I_1^{(\gamma(p-(l+1)))} \neq \emptyset$, and thus we can choose a subset B of $I_1^{(\gamma(p-2l-1))}$ such that B is homeomorphic to $[1, \omega^{\gamma l}]$. As before we use Lemma 2.3 to find a subspace of $C[1, \omega^{\gamma p}]$ isometric to $C(B)$ and an extension operator L. We apply the inductive hypothesis to TL to get $\widehat{\gamma(n-1)}$-families $\mathscr{F}$ and $\mathscr{G}$, just as before ((2.9)). Let $D = \{\gamma\colon \delta_\gamma = \nu_G \text{ for some } G \in \mathscr{G}\}$ and for each $d \in D^{(0)}$, let $G(d)$ be the set in $\mathscr{G}_0$ satisfying $\delta_d = \nu_{G(d)}$. For each such d we then find a subset $E(d)$ of $\big((d^-, d] \cap G(d)\big)^{(\gamma(p-2(l+1)))}$ such that $E(d)$ is homeomorphic to $[1, \omega^\gamma]$ and $E(d)^{(\gamma)} = \{d\}$. Again we apply Lemma 2.3 to get an extension operator $L(d)\colon C(E(d)) \to C[1, \omega^{\gamma p}]$ and use the induction hypothesis (on $TL(d)$) to get $\hat{\gamma}$-families $\mathscr{F}(d)$ and $\mathscr{G}(d)$ as before ((2.11)). Let E be defined as in (2.12). If $E \supset D^{(0)}$, we complete the argument as we indicated earlier; if not, we get clopen sets G and H such that $G \subsetneqq H \subsetneqq J(2)$ and we get a measure $\mu_{\varphi(d)(G)}$ (for some $d \in D^{(0)}$) such that

$$|\mu_{\varphi(d)(G)}|\big(\{x\colon |T1_H - T1_G|(x) > \delta/4\}\big) \geqslant \delta/6.$$

Since $T^*\mu_{\varphi(d)(G)} \in \mathcal{N} \cap \{\delta_\gamma:\ \gamma \in J(2)\}$, by (2.18),

$$|\mu_{\varphi(d)(G)}|\big(\{x:\ |T1_{J(1)} - T1_{J(2)}|(x) > \delta/4\}\big) \geqslant \delta/24$$

as well. By the same argument as before we can find a neighborhood $\mathcal{N}(1)$ of $\mu_{\varphi(d)(G)}$ in $\mathcal{N}$ such that

$$|\mu|\big(\{x:\ |T1_H - T1_G|(x) > \delta/4\}\big) > \delta/24 \quad \text{for all } \mu \in \mathcal{N}(1).$$

Let $J(3) = H$ and $J(4) = G$.

By our choice of p we can carry out this process $k = k(8\|T\|\delta^{-1} + 1, a, \delta/24)$ times, if we do not succeed at some stage. Thus we would have clopen sets $J(1) \supsetneqq J(2) \supsetneqq \ldots \supsetneqq J(2k)$ and a measure μ in A such that for each i, $1 \leqslant i \leqslant k$,

$$|\mu|\big(\{x:\ |T1_{J(2i-1)} - T1_{J(2i)}|(x) > \delta/4\}\big) > \delta/24.$$

By Lemma 2.7 and the definition of k there are $[1+8\|T\|\delta^{-1}]$ of the sets, $\{x:\ |T1_{J(2i-1)} - T1_{J(2i)}|(x) > \delta/4\}$, $i = 1, 2, \ldots, k$, which have a common point, say x_0. Also for at least half of these $(T1_{J(2i-1)} - T1_{J(2i)})(x_0) > \delta/4$ or for at least half $(T1_{J(2i-1)} - T1_{J(2i)})(x_0) < -\delta/4$. For simplicity let us assume that the value at x_0 is larger than $\delta/4$ for $i = 1, 2, \ldots, i_0$, $i_0 \geqslant \frac{1}{2}[1+8\|T\|\delta^{-1}]$. Then

$$\Big\|T\sum_{i=1}^{i_0}(1_{J(2i-1)} - 1_{J(2i)})\Big\| \geqslant \Big|\sum_{i=1}^{i_0}(T1_{J(2i-1)} - T1_{J(2i)})(x_0)\Big|$$

$$> \frac{\delta}{4}\frac{1}{2}[1+8\|T\|\delta] > \|T\|,$$

but

$$\Big\|\sum_{i=1}^{i_0} 1_{J(2i-1)} - 1_{J(2i)}\Big\| = 1.$$

This contradiction shows that we must have $E \supseteq D^{(0)}$ at some stage.

3. Technical lemmas. In the previous sections we have proved severa results relating the Wolfe index to subspaces of $C[0,1]$ isometric to $C_0(\omega^{\omega^a})$. This raises the question of the relationship between the Wolfe index and the Szlenk index as applied in [2]. Those results were obtained only for $C(\omega^\omega)$ and an example was produced [1] to show that the direct generalization of those results was impossible. In Section 4 we will present that example with a new (and shorter) argument which makes use of the Wolfe index. It will follow then that there is a subset L of $B_{C_0(\omega^{\omega^2})^*}$ for which (the Szlenk index) $\eta(\frac{1}{2}, B_{C_0(\omega^{\omega^2})}, L) \geqslant \omega^2$, but for every $\varepsilon > 0$, $P_{\omega n}(\varepsilon, \overline{\mathrm{co}}^{w^*}(L \cup -L)) = \emptyset$ for n sufficiently large.

In this section we will prepare the way for the presentation of the example by proving several lemmas of a general nature. As motivation for the lemmas let us note that the example is obtained by choosing a subset L of the probability measures on $[1, \omega^{\omega^2})$ such that $L \cup \{0\}$ (in the w^* topology) is homeomorphic to $[1, \omega^{\omega^2}]$ and such that the map T: $C_0(\omega^{\omega^2}) \to C_0(L)$ defined by $(Tf)(l) = l(f)$, for all $l \in L$, is surjective. By Theorem 0.3 it is sufficient to show that for every $\varepsilon > 0$ $P_{\omega n}(\varepsilon, \mathrm{co}(\pm L)) = \emptyset$, for n sufficiently large, in order to conclude that L does not norm a subspace of $C_0(\omega^{\omega^2})$ isomorphic to $C_0(\omega^{\omega^2})$. Our primary goal in this section is to show that it is sufficient to prove that $P_{\omega n}(\varepsilon, L) = \emptyset$, for n sufficiently large.

LEMMA 3.0. *Let K be a countable compact metric space and let $\varepsilon > 0$. For every $\delta > 0$ and ordinal $a < \omega_1$, if $\{\mu_\gamma:\ \gamma \leqslant \omega^a\}$ is a subset of $B_{C(K)^*}$ satisfying $\big|\|\mu_\gamma\| - \|\mu_{\gamma'}\|\big| < \varepsilon$ for all $\gamma, \gamma' \leqslant \omega^a$, then there is a constant c, a closed subset A of $[1, \omega^a]$ homeomorphic to $[1, \omega^a]$, and elements $\{\nu_\gamma:\ \gamma \in A\}$ of $B_{C(K)^*}$ such that for all $\gamma \in A$*

(1) $\|\mu_\gamma - \nu_\gamma\| < \varepsilon + \delta$;

(2) $\|\nu_\gamma\| = c \geqslant \|\mu_{\omega^a}\| - \delta$.

Proof. By replacing $\{\mu_\gamma:\ \gamma \leqslant \omega^a\}$ with a set $\{\mu'_\gamma:\ \gamma \leqslant \omega^a\}$ such that μ'_γ has finite support and $\|\mu_\gamma - \mu'_\gamma\| < \delta/2$, for all γ, we may assume that each μ_γ has finite support. With this assumption we will show that we can choose $c = \|\mu_{\omega^a}\|$.

For the case $a = 0$, the lemma is trivial, so assume that it is true for β and we will prove it for $a = \beta + 1$.

Let $\mu_{\omega^a} = \sum_{n=1}^{N} a_n \delta_{x(n)}$, $\{x(1), x(2), \ldots\} \subset K$. Because K is countable and therefore totally disconnected, we can choose disjoint clopen sets G_n, $n = 1, 2, \ldots, N$ such that $x(n) \in G_n$, for each n. For each n let $R_n\mu_\gamma = \mu_\gamma|_{G_n}$.

Choose an integer k_1 such that $\|R_n\mu_\gamma\| > \|R_n\mu_{\omega^a}\|(1-\delta/4)$ for all $\gamma > \omega^\beta k_1$ and $n = 1, 2, \ldots, N$. For each $k > k_1$, choose an ordinal γ_k such that $\|R_n\mu_\gamma\| > \|R_n\mu_{\omega^\beta k}\|(1-\delta/4)$ for all γ, $\gamma_k < \gamma < \omega^\beta k$.

Each of the sets $\{\mu_\gamma:\ \gamma_k < \gamma \leqslant \omega^\beta k\}$ satisfies the inductive hypothesis, and thus, there are sets $A_k \subset (\gamma_k, \omega^\beta k]$, and measures $\{\nu'_\gamma:\ \gamma \in A_k\}$ such that A_k is homeomorphic to $[1, \omega^\beta]$ and for all $\gamma \in A_k$,

$$\|\nu'_\gamma\| = \|\mu_{\omega^\beta k}\|,$$

and

$$\|\nu'_\gamma - \mu_\gamma\| < \sup\big\{\big|\|\mu_\gamma\| - \|\mu_{\gamma'}\|\big|:\ \gamma_k < \gamma, \gamma' \leqslant \omega^\beta k\big\} + \delta/4$$

for all $k > k_1$.

Choose $k_2 \geqslant k_1$ such that for all $\gamma \in \bigcup\{A_k:\ k > k_2\}$

(3.1) $$|\nu'_\gamma(G_n) - a_n| < |a_n|\delta/4, \quad n = 1, 2, \ldots, N.$$

For each $\gamma \in \bigcup \{A_k:\ k > k_2\}$ let

$$\nu_\gamma = \sum_{n=1}^{N} a_n \|R_n(\nu'_\gamma)^{\varepsilon(n)}\|^{-1} R_n(\nu'_\gamma)^{\varepsilon(n)},$$

where $\varepsilon(n) = \operatorname{sgn} a_n$ and $(\nu'_\gamma)^+ - (\nu'_\gamma)^-$ is the Jordan decomposition of ν_γ. We have that

$$\|\nu_\gamma\| = \sum_{n=1}^{N} |a_n| = \|\mu_{\omega^\alpha}\|$$

and

$$\|\nu_\gamma - \mu_\gamma\| \leqslant \|\nu_\gamma - \nu'_\gamma\| + \|\nu'_\gamma - \mu_\gamma\|.$$

For each n

$$\begin{aligned}
&\|R_n(\nu_\gamma - \nu'_\gamma)\|\\
&= |a_n[(\nu'_\gamma)^{\varepsilon(n)}(G_n)]^{-1}(\nu'_\gamma)^{\varepsilon(n)}(G_n) - \varepsilon(n)(\nu'_\gamma)^{\varepsilon(n)}(G_n)| + |\nu'_\gamma|(G_n) - (\nu'_\gamma)^{\varepsilon(n)}(G_n)\\
&= \big||a_n| - (\nu'_\gamma)^{\varepsilon(n)}(G_n)\big| + |a_n| - (\nu'_\gamma)^{\varepsilon(n)}(G_n) + |\nu'_\gamma|(G_n) - |a_n|\\
&< 2|a_n|\delta/4 + |\nu'_\gamma|(G_n) - |a_n|,
\end{aligned}$$

because

$$(\nu'_\gamma)^{\varepsilon(n)}(G_n) \geqslant \varepsilon(n)\nu'_\gamma(G_n) > \varepsilon(n)a_n - |a_n|\delta/4 = |a_n|(1-\delta/4),$$

by (3.1). Hence, if $\gamma \in A_k$, for $k > k_2$,

$$\begin{aligned}
\|\nu_\gamma - \nu'_\gamma\| &= \sum_{n=1}^{N} \|R_n(\nu_\gamma - \nu'_\gamma)\| + |\nu'_\gamma|\big(\bigcup\{G_n:\ n = 1, 2, \ldots, N\}\big)^c\\
&< (\delta/2)\sum_{n=1}^{N}|a_n| + \|\nu'_\gamma\| - \sum_{n=1}^{N}|a_n| \leqslant (\delta/2)\|\mu_{\omega^\alpha}\| + \|\mu_{\omega^\beta}\| - \|\mu_{\omega^\alpha}\|.
\end{aligned}$$

Also

$$\begin{aligned}
\|\nu'_\gamma - \mu_\gamma\| &< \sup\{\big|\|\mu_\gamma\| - \|\mu_{\tau'}\|\big|:\ \gamma_k < \tau,\ \tau' \leqslant \omega^\beta k\} + \delta/4\\
&\leqslant \sup\{\|\mu_\tau\|:\ \gamma_k < \tau \leqslant \omega^\beta k\} - \|\mu_{\omega^\beta k}\| + \delta/4 + \delta/4.
\end{aligned}$$

Combining these estimates we have that

$$\|\nu_\gamma - \mu_\gamma\| < \frac{\delta}{2}\|\mu_{\omega^\alpha}\| + \frac{\delta}{2} + \sup\{\|\mu_\tau\|:\ \gamma_k < \tau \leqslant \omega^\beta k\} - \|\mu_{\omega^\alpha}\| \leqslant \delta + \varepsilon,$$

proving our claim for $\beta+1$. The proof for the case when α is a limit ordinal is similar and we leave it to the reader.

LEMMA 3.1. *Let K be a compact metric space and let L be a countable w^* closed subset of the positive elements in $B_{C(K)^*}$. Suppose that for some $\alpha < \omega_1$ and $\varepsilon > 0$, $P_\alpha(\varepsilon, \overline{\operatorname{co}} L) \neq \emptyset$, then $P_\alpha(\varepsilon, L) \neq \emptyset$.*

Proof. By Lemma 1.0(d) there is an α-family $\{G_\gamma:\ \gamma \leqslant \omega^\alpha\}$ with ε-measures $\{\mu_\gamma:\ \gamma \leqslant \omega^\alpha\} \subset \overline{\operatorname{co}} L$. Because L is countable, for each γ, μ_γ is a (possibly infinite) convex combination of elements of L, i.e., $\mu_\gamma = \sum_{i=1}^{\infty} \lambda_{\gamma,i} l_i$, where $l_i \in L$, $\lambda_{\gamma,i} \geqslant 0$, for all i, and $\sum_{i=1}^{\infty} \lambda_{\gamma i} = 1$. Choose for each $\gamma \in [1, \omega^\alpha]^{d(0)}$ an element $l(\gamma)$ in the expansion of μ_γ, i.e., $l_\gamma(\gamma) = l_i$ and $\lambda_{\gamma i} \neq 0$, such that $l(\gamma)(G_\gamma) \geqslant \varepsilon$. We claim that there is a w^* closed subset A of $[1, \omega^\alpha]$ such that $\{G_\gamma:\ \gamma \in A\}$ is an α-family with ε-measures $\overline{\{l(\gamma):\ \gamma \in A^{d(0)}\}}$.

Observe that if $\{\gamma_i\}_{i=1}^{\infty} \subset [1, \omega^\alpha]^{d(0)}$, $\lim_{i\to\infty}\gamma_i = \gamma$, and $w^*\lim l(\gamma_i) = l$, then $\bigcup_{i=i_0} G_{\gamma_i} \subset G_\gamma$, for some i_0 and $l(G_\gamma) \geqslant \underline{\lim}_{i\to\infty} l(\gamma_i)[G_{\gamma_i}] \geqslant \varepsilon$. It follows from this that

$$\overline{\{l(\gamma):\ \gamma \in [1, \omega^\alpha]^{d(0)}\}}^{(\alpha)} \neq \emptyset$$

and thus the set A can be easily constructed by induction.

Our next lemmas almost achieves our goal. The difficulty is that 0 is in the w^* closure of the set we construct in Section 4. However, this problem is not difficult to overcome and we leave it until Section 4.

LEMMA 3.2. *Let L be a w^* closed countable subset of $\{\mu:\ \mu \in B_{C(K)^*},\ \mu > 0\}$ for some countable compact metric space K. Suppose that the evaluation map $T:\ C(K) \to C(L)$ defined by $(Tf)(l) = l(f)$, for all $l \in L$, is surjective. Then, there is an $\varepsilon > 0$ such that $P_\gamma\big(\varepsilon, \overline{\operatorname{co}}(\pm L)\big) \neq \emptyset$, for all $\gamma < \omega^\alpha$ if and only if there is an $\varepsilon' > 0$ such that $P_\gamma(\varepsilon', L) \neq \emptyset$ for all $\gamma < \omega^\alpha$.*

Proof. Clearly if $P_\gamma(\varepsilon', L) \neq \emptyset$, $P_\gamma\big(\varepsilon', \overline{\operatorname{co}}(\pm L)\big) \neq \emptyset$, thus one direction is trivial. Assume $P_\gamma\big(\varepsilon, \overline{\operatorname{co}}(\pm L)\big) \neq \emptyset$, for all $\gamma < \omega^\alpha$ and let $\alpha_n \uparrow \omega^\alpha$. From Lemma 1.0 it follows that for each n, we can find an α_n-family with ε-measures $\{\nu'_{n,\gamma}:\ \gamma \leqslant \omega^{\alpha_n}\}$ in $\overline{\operatorname{co}}(\pm L)$. By Lemma 2.5 for each $k \in \boldsymbol{N}$ we can find an integer $n(k)$ such that $\{\nu'_{n(k),\gamma}:\ \gamma \leqslant \omega^{\alpha_{n(k)}}\}$ contains a subset $\{\nu_{k,\gamma}:\ \gamma \leqslant \omega^{\alpha_k}\}$ such that

$$\big|\|\nu_{k,\gamma}\| - \|\nu_{k,\gamma'}\|\big| < \varepsilon/8\|T^{*-1}\|.$$

($\|T^{*-1}\| < \infty$ because T is surjective, and thus T^* is an isomorphism.) Moreover, by Lemma 1.1 we may assume that we have an α_k-family $\{G_{k,\gamma}:\ \gamma \leqslant \omega^{\alpha_k}\}$ so that $\{\nu_{k,\gamma}:\ \gamma \leqslant \omega^{\alpha_k}\}$ are associated ε-measures.

For each $\gamma \leqslant \omega^{\alpha_k}$, let $\tau_{k,\gamma} = T^{*-1}\nu_{k,\gamma}$ and note that

$$\|\tau_{k,\gamma} - \tau_{k,\gamma'}\| < \|T^{*-1}\|\,\|\nu_{k,\gamma} - \nu_{k,\gamma'}\| < \varepsilon/8,$$

for all $\gamma, \gamma' \leqslant \omega^{\alpha_n}$. Thus by Lemma 3.0 there are a closed subset A of $[1, \omega^{\alpha_k}]$, homeomorphic to $[1, \omega^{\alpha_k}]$, a constant c_k, and measures $\{\varphi_{k,\gamma}:\ \gamma \in A\} \subset B_{C(L)^*}$ such that $\|\varphi_{k,\gamma} - \tau_{k,\gamma}\| < \varepsilon/4$ and $\|\varphi_{k,\gamma}\| = c_k > \|\tau_{k,\omega^{\alpha_k}}\| - \varepsilon/8$, for all $\gamma \in A$. Observe that

$$|T^*\varphi_{k,\gamma}|(G_{k,\gamma}) \geqslant |\nu_{k,\gamma}|(G_{k,\gamma}) - |T^*\varphi_{k,\gamma} - T^*\tau_{k,\gamma}|(G_{k,\gamma}) \geqslant \varepsilon - \|T^*\|\varepsilon/4 = 3\varepsilon/4.$$

Consequently, $\{G_{k,\gamma}\colon \gamma\in A\}$ is an α_k family with $3\varepsilon/4$ measures $\{T^*\varphi_{k,\gamma}\colon \gamma\in A\}$ such that $\|\varphi_{k,\gamma}\| = c_k$, for all $\gamma\in A$.

Next note that it follows from Lemma 2.6(i) that if $\mathscr{A}$ is a w^* closed subset of $\{\mu\colon \mu\in C(L)^*$ and $\|\mu\| = c_k\}$, (k fixed), then the map $\mu\to|\mu|$ is w^* continuous from $\mathscr{A}$ to $C(L)^*$. Hence $\{T^*|\varphi_{k,\gamma}|\colon \gamma\in A\}$ is homeomorphic to $[1, \omega^{\alpha_k}]$ and

$$T^*|\varphi_{k,\gamma}|(G_{k,\gamma}) \geqslant |T^*\varphi_{k,\gamma}|(G_{k,\gamma}) \geqslant 3\varepsilon/4$$

for all $\gamma\in A$. Thus $\{G_{k,\gamma}\colon \gamma\in A\}$ is an α_k-family with $3\varepsilon/4$-measures $\{T^*|\varphi_{k,\gamma}|\colon \gamma\in A\}$ in $\overline{\mathrm{co}}\,L$.

By Lemma 1.0, $P_{\alpha_k}(3\varepsilon/4, \overline{\mathrm{co}}\,L)\neq\emptyset$ for each $k\in N$, and so by Lemma 3.1, $P_{\alpha_k}(3\varepsilon/4, L)\neq\emptyset$, for each $k\in N$, as well. Because $\alpha_k\uparrow\omega^\alpha$, $P_\gamma(3\varepsilon/4, L)\neq\emptyset$ for all $\gamma<\omega^\alpha$, and the proof is complete.

Out next lemma will be used to show that $P_{\omega n}(\varepsilon, L)=\emptyset$ for n sufficiently large, where L is the set constructed in Section 4.

Before stating the lemma we need an additional definition. If $u = \sum_{i=1}^{N} a_i\delta_{t_i}$, $t_i\in F$, for some compact Hausdorff space F, then a family of disjoint open sets $\{A_i\colon i = 1, 2, \ldots, N\}$ will be called a *distinguishing family for* μ, if for each i there is an ordinal α_i such that $A_i^{(\alpha_i)} = \{t_i\}$.

LEMMA 3.3. *Let $\{G_\gamma\colon \gamma\leqslant\omega^\omega\}$ be an ω family with ε-measures $\{k_\gamma\colon \gamma\leqslant\omega^\omega\}\subset B_{C(F)^*}$ where F is homeomorphic to $[1, \omega^{\omega m}]$. If*

$$k_{\omega^\omega} = \sum_{i=0}^{m}\sum_{j=1}^{N_i} A_{ij}\delta_{t_{ij}}$$

with

$$t_{ij}\in F(i) = F^{(\omega i)} - F^{(\omega(i+1))}\quad \text{for}\quad 1\leqslant j\leqslant N_i,\ 0\leqslant i\leqslant m,$$

and

$$\{A_{ij}\colon 1\leqslant j\leqslant N_i,\ 0\leqslant i\leqslant m\}$$

is a distinguishing family for k_{ω^ω}, then for every $\delta>0$ and integer N there exists a subset $\mathscr{A}$ of $[1, \omega^\omega)$ homeomorphic to $[1, \omega^N]$ such that

$$|k_\beta|\Big(\bigcup\{A_{ij}\cap F(i)\cap G_\beta\colon 1\leqslant j\leqslant N_i,\ 0\leqslant i\leqslant m\}\Big)<\delta,\quad \text{for all } \beta\in\mathscr{A}.$$

Proof. We will use induction on N. Suppose $N=0$. For each i and j let r_{ij} be the unique integer such that

$$t_{ij}\in F(i)^{(r_{ij})} - F(i)^{(r_{ij}+1)} = F^{(\omega i+r_{ij})} - F^{(\omega i+r_{ij}+1)}.$$

Let

$$r = \max\{r_{ij}\colon 1\leqslant j\leqslant N_i,\ 0\leqslant i\leqslant m\}$$

and

$$A = \bigcup\{A_{ij}\cap F(i)\colon 1\leqslant j\leqslant N_i,\ 0\leqslant i\leqslant m\}.$$

Consider the map $T\colon C(F)\to C(A)$ defined by $Tf = f|_A$.

Note that $A^{(r+1)} = \emptyset$ and thus, by the result of Bessaga and Pełczyński [5], there is no isomorphism from $C(\omega^\omega)$ into $C(A)$. It follows from Theorem 0.2 that $\{G_\gamma\colon \gamma\leqslant\omega^\omega\}$ with the measures $\{k_\gamma|_A\colon \gamma\leqslant\omega^\omega\}$ is not an ω-family with δ-measures. Hence there is an ordinal γ_0 such that $|k_{\gamma_0}|(G_{\gamma_0}\cap A)<\delta$, as required.

Now assume that the result is true for $N-1$. By Theorem 0.1 there is an integer M such that $P_M(\delta, T^*B_{C(A)^*}) = \emptyset$. $\{G_\gamma\colon \gamma\in[1, \omega^\omega]^{(M+1)}\}$ is an ω family with ε-measures $\{k_\gamma\colon \gamma\in[1, \omega^\omega]^{(M+1)}\}$ and so by the inductive hypothesis we can find a subset $\mathscr{A}_1$ of $[1, \omega^\omega]^{(M+1)}$ such that $\mathscr{A}_1$ is homeomorphic to $[1, \omega^{N-1}]$ and $|k_\gamma|(G_\gamma\cap A)<\delta$ for all $\gamma\in\mathscr{A}_1$. For each $\gamma\in\mathscr{A}_1^{(0)}$ we can find a sequence $(\gamma_n)\subset[1, \omega^\omega]^{(M)}$ such that $\bigcup\{G_{\gamma_n}\colon n\in N\}\subset G_\gamma$ and $\gamma_n\to\gamma$. Because $P_M(\delta, T^*B_{C(A)^*}) = \emptyset$, for each n, $|k_{\tau_n}|(G_{\tau_n}\cap A)<\delta$ for some ordinal $\tau_n = \tau_n(\gamma)$ such that $G_{\tau_n}\subset G_{\gamma_n}$. Clearly $\lim\tau_n = \gamma$ and thus the union of $\mathscr{A}_1$ and $\{\tau_n(\gamma)\colon \gamma\in\mathscr{A}_1^{(0)},\ n\in N\}$ is the required set $\mathscr{A}$.

4. The example. We now present the example of a subset L of $B_{C_0(\omega^{\omega^2})^*}$ such that $P_{\omega n}(\varepsilon, \overline{\mathrm{co}}(L\cup -L)) = \emptyset$ but the Szlenk index $\eta(\frac{1}{2}, B_{C_0(\omega^{\omega^2})}, L) \geqslant\omega^2$.

To construct the set L, it is convenient to use a space K homeomorphic to $[1, \omega^{\omega^2}]$. To this end we need some additional notation.

For each $\alpha\in[1, \omega^\omega]^{d(0)}$ let F_α be a compact Hausdorff space. We define $\sum_{\alpha<\omega^\omega} F_\alpha$ to be the set

$$\bigcup\{F_\alpha\colon \alpha\in[1, \omega^\omega]^{d(0)}\}\cup[1, \omega^\omega]^{(1)}$$

with the topology generated by sets of the form

$$\bigcup\{F_\alpha\colon \beta_1<\alpha<\beta_2\}\cup\big([1, \omega^\omega]^{(1)}\cap(\beta_1, \beta_2)\big)\cup\Big(\bigcup\{G_\alpha\colon \alpha\in[1, \omega^\omega]^{d(0)}\}\Big)$$

where $\beta_1, \beta_2\in[0, \omega^\omega+1]$ and for each α, G_α is an open subset of F_α. It is easy to verify that $\sum_{\alpha<\omega^\omega} F_\alpha$ is a compact Hausdorff space.

Intuitively, this topology is the natural topology obtained when one replaces the isolated points of $[1, \omega^\omega]$ with the compact Hausdorff spaces F_α, $\alpha\in[1, \omega^\omega]^{d(0)}$. For example, if each F_α is homeomorphic to $[1, \omega^\omega]$, then $\sum_{\alpha<\omega^\omega} F_\alpha$ is homeomorphic to $[1, \omega^{\omega^2}]$. (One may view this topology as the procupine topology on

$$Y = \bigcup\{F_\alpha\colon \alpha\in[1, \omega^\omega]^{d(0)}\}\cup[1, \omega^\omega]^{(1)}$$

with index space $[1, \omega^\omega]$, and selection $S\colon [1, \omega^\omega] \to Y$ defined by

$$S(\alpha) = \begin{cases} \alpha & \text{if} \quad \alpha \in [1, \omega^\omega]^{(1)}, \\ s_\alpha \ \text{ where } s_\alpha \text{ is any point of } F_\alpha & \text{if} \quad \alpha \in [1, \omega^\omega]^{d(0)}. \end{cases}$$

See page 327 of [6] for details.) Also we will need the one point compactification of a topological space F which we will denote by $\mathrm{pt}(F)$.

Our space K will be $\mathrm{pt}(\bigcup_{n=1}^{\infty} K_n)$ where the spaces $(K_n)_{n=1}^{\infty}$ are disjoint and defined inductively as follows:

Let $K_1 = [1, \omega^\omega]$. For each $\alpha \in [1, \omega^\omega]^{d(0)}$ let $F(2, \alpha, 1)$ and $F(2, \alpha, -1)$ be homeomorphic to K_1. Define

$$K(2, 1) = \sum_{\alpha < \omega^\omega} F(2, \alpha, 1)$$

and

$$K(2, -1) = \mathrm{pt}\big(\bigcup \{F(2, \alpha, -1) - F(2, \alpha, -1)^{(\omega)}\colon \alpha \in [1, \omega^\omega]^{d(0)}\}\big).$$

The space K_2 is the space obtained by identifying the point in $K(2, 1)^{(\omega 2)}$ with the point in $K(2, -1)^{(\omega)}$ in the disjoint union of $K(2, 1)$ and $K(2, -1)$. Figure 1 below gives an intuitive picture of K_2.

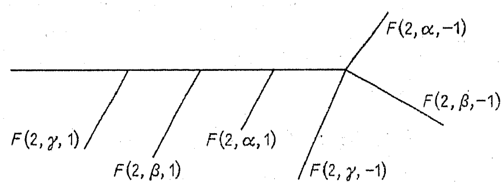

Fig. 1

The horizontal line represents $K_2^{(\omega)}$ which is homeomorphic to $[1, \omega^\omega]$. For each isolated point of $K_2^{(\omega)}$ there is a subset of K_2 homeomorphic to $[1, \omega^\omega]$ with the isolated point corresponding to ω^ω. Three such subsets are represented by the three parallel slanted segments terminating in $K_2^{(\omega)}$. Also for each isolated point of $K_2^{(\omega)}$ there is a segment terminating at the right end point of the segment representing $K_2^{(\omega)}$. These segments represent

$$\mathrm{pt}\big(\bigcup \{F(2, \alpha, -1) - F(2, \alpha, -1)^{(\omega)}\colon \alpha \in [1, \omega^\omega]^{d(0)}\}\big).$$

Suppose we have defined K_n. For each $\alpha \in [1, \omega^\omega]^{d(0)}$ let $F(n+1, \alpha, 1)$ and $F(n+1, \alpha, -1)$ be homeomorphic to K_n. Define

$$K(n+1, 1) = \sum_{\alpha < \omega^\omega} F(n+1, \alpha, 1)$$

and

$$K(n+1, -1) = \mathrm{pt}\big(\bigcup \{F(n+1, \alpha, -1) - F(n+1, \alpha, -1)^{(\omega n)}\colon \alpha \in [1, \omega^\omega]^{d(0)}\}\big).$$

Let K_{n+1} be the space obtained from the disjoint union of $K(n+1, 1)$ and $K(n+1, -1)$ by identifying the point in $K(n+1, 1)^{(\omega(n+1))}$ with the point in $K(n+1, -1)^{(\omega n)}$. We illustrate K_4 in Figure 2.

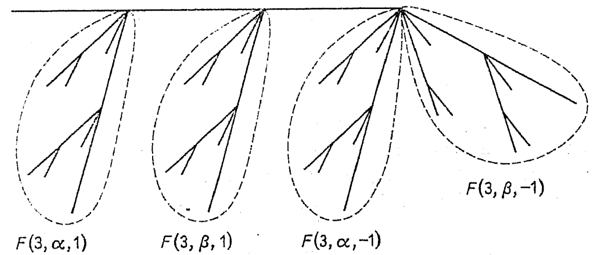

Fig. 2

As in Figure 1 the horizontal line segment represents $K_4^{(\omega 3)}$, a subset of K_4 homeomorphic to $[1, \omega^\omega]$. The areas encircled by dotted lines indicate subsets homeomorphic to K_3 which correspond to the isolated points of $K_4^{(\omega 3)}$. The right hand endpoint of the horizontal line segment represents the attaching point of $K(4, 1)$ and $K(4, -1)$.

Note that K_n and $K(n, 1)$ are homeomorphic to $[1, \omega^{\omega n}]$ and that $K(n, -1)$ is homeomorphic to $[1, \omega^{\omega(n-1)}]$. Perhaps the easiest way to see this is to observe that each space is a countable compact metric space and thus by the Mazurkiewicz–Sierpiński Theorem [9] is determined by its derived sets.

We will need some notation for the points of each K_n. Since the space K_n is composed of copies of K_{n-1}, we will label the points of K_n using the labeling of K_{n-1}. K_1 is homeomorphic to $[1, \omega^\omega]$ so let $K_1 = \{(1, \alpha)\colon \alpha \leqslant \omega^\omega\}$. (Precisely speaking, we are defining a homeomorphism and labeling the points of K_1 via the homeomorphism.) Next consider K_2. Denote the single point in $K_2^{(\omega 2)}$ by $(2, \omega^\omega)$ and recall that the rest of K_2 is

$$[K(2, 1) - K(2, 1)^{(\omega 2)}] \cup [K(2, -1) - K(2, -1)^{(\omega)}].$$

Let

$$[K(2,1)-K(2,1)^{(\omega 2)}]^{(\omega)} = \{(2, \alpha): \alpha < \omega^\omega\}.$$

($[K(2,1)-K(2,1)^{(\omega 2)}]^{(\omega)}$ is homeomorphic to $[1, \omega^\omega)$.) The remaining points of K_2 are points of $F(2, \alpha, 1)-F(2, \alpha, 1)^{(\omega)}$ or $F(2, \alpha, -1)-F(2, \alpha, -1)^{(\omega)}$ for some $\alpha \in [1, \omega^\omega]^{d(0)}$. Since each is homeomorphic to $K_1 - K_1^{(\omega)}$ let

$$F(2, \alpha, \varepsilon)-F(2, \alpha, \varepsilon)^{(\omega)} = \{(2, \alpha, \varepsilon, \beta): (1, \beta) \in K_1 - K_1^{(\omega)}\}$$

for $\varepsilon = 1$ or -1.

Suppose the points of K_{n-1} have been labeled and consider K_n. $K_n^{(\omega(n-1))}$ is homeomorphic to $[1, \omega^\omega]$, so let $K_n^{\omega((n-1))} = \{(n, \alpha): \alpha \leqslant \omega^\omega\}$. The remaining points of K_n belong to $F(n, \alpha, \varepsilon)-F(n, \alpha, \varepsilon)^{(\omega(n-1))}$ for $\varepsilon = 1$ or -1 and $\alpha \in [1, \omega^\omega]^{d(0)}$. Each of these homeomorphic to $K_{n-1}-K_{n-1}^{(\omega(n-1))}$. Let

$$F(n, \alpha, \varepsilon)-F(n, \alpha, \varepsilon)^{(\omega(n-1))} = \{(n, \alpha, \varepsilon, \bar\beta): (n-1, \bar\beta) \in K_{n-1} - K_{n-1}^{(\omega(n-1))}\}$$

for $\varepsilon = 1$ or -1 and $\alpha \in [1, \omega^\omega]^{d(0)}$. (Here $\bar\beta$ is a tuple.)

We will define the subset L of $B_{C_0(K)^*}$ as follows: For each integer $n > 0$, integer s, with $n-1 \geqslant s > 0$, and $s+1$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ such that $\alpha_i \in [1, \omega^\omega]^{d(0)}$ for $0 \leqslant i < s$ and $\alpha_s \in [1, \omega^\omega)$ let

$$l(n, \alpha_0, \alpha_1, \ldots, \alpha_s) = \frac{1}{2^{s+1}}\Big(\delta(n, \omega^\omega) + \sum_{k=0}^{s} \sum_{\varepsilon_r = \pm 1} \delta(n, \alpha_0, \varepsilon_1, \alpha_1, \ldots, \varepsilon_k, \alpha_k)\Big)$$

where $\sum_{\varepsilon_r=\pm1}$ means the sum over all choices of signs and $\delta(t)$ is the point mass measure at t. Also let $l(n, \omega^\omega) = \delta(n, \omega^\omega)$. Our set L will be the w^*-closure of $\bigcup_{n=1}^{\infty} L_n$ where

$$L_n = \{l(n, \alpha_0, \alpha_1, \ldots, \alpha_s): \alpha_i \in [1, \omega^\omega]^{d(0)} \text{ for } 0 \leqslant i < s \text{ and } \alpha_s \in [1, \omega^\omega),\ s = 0, 1, 2, \ldots, n-1\} \cup \{l(n, \omega^\omega)\}.$$

Note that $\overline{\bigcup\{L_n: n \in N\}}^{w^*} = \bigcup\{L_n: n \in N\} \cup \{0\}$ since $l \in L_n$ is supported in K_n. To see that L is homeomorphic to $[1, \omega^{\omega^2}]$ we only need to show that L_n is homeomorphic to $[1, \omega^{\omega n}]$ for each n.

Define a map $\varphi_n: L_n \to [1, \omega^{\omega n}]$ by

$$\varphi_n\big(l(n, \alpha_0, \alpha_1, \ldots, \alpha_s)\big) = \omega^{\omega(n-1)}(\alpha_0 - 1) + \omega^{\omega(n-2)}(\alpha_1 - 1) + \ldots + \omega^{\omega(n-s-1)}\alpha_s$$

for $\alpha_i \in [1, \omega^\omega]^{d(0)}$, $0 \leqslant i < s$ and $\alpha_s \in [1, \omega^\omega)$, $s = 0, 1, 2, \ldots, n-1$ and $\varphi_n\big(l(n, \omega^\omega)\big) = \omega^{\omega n}$. We leave it to the reader to verify that this is a homeomorphism.

Let us examine the set L more closely. First observe that the Szlenk index $\eta(\frac{1}{2}, B_{C_0(K)}, L) \geqslant \omega^2$, (see [14] for definitions). Indeed, if we let $Q_\alpha(\varepsilon, B_{C_0(K)}, L)$ denote the αth, ε Szlenk set, then $L^{(\alpha)} \subset Q_\alpha(\frac{1}{2}, B_{C_0(\omega^{\omega^2})}, L)$ for all $\alpha \leqslant \omega^2$. Since L is homeomorphic to $[1, \omega^{\omega^2}]$, $L^{(\omega^2)} \neq \emptyset$ and thus $\eta(\frac{1}{2}, B_{C_0(\omega^{\omega^2})}, L) \geqslant \omega^2$.

Actually we can prove as lightly stronger result. Define an operator $T: C_0(K) \to C_0(L)$ by $Tf(l) = l(Tf)$, i.e., by evaluation. We claim that T is *onto*. (From the properties of the Szlenk index [14], it would follow from this that the Szlenk index is not less than ω^2.) To show that this is the case, we will establish that T^* is an isomorphism.

Observe that $T^*\delta_l = l$ for each $l \in L$ and hence range $T^* \subset [L]$. Moreover $[L] = [M]$ where

$$M = \bigcup_{n=1}^{\infty}\Big\{\frac{1}{2^s}\sum_{\varepsilon_r=\pm1}\delta(n, \alpha_0, \varepsilon_1, \alpha_1, \ldots, \varepsilon_s, \alpha_s):$$
$$\alpha_r \in [1, \omega^\omega]^{d(0)} \text{ for } 0 \leqslant r < s,\ \alpha_s \in [1, \omega^\omega),\ s = 1, 2, \ldots, n-1 \text{ or } \alpha_s \in [1, \omega^\omega] \text{ for } s = 0\Big\}.$$

The measures in M are disjointly supported so that $[M]$ is isometric to l_1. We define an inverse S for T^* as follows: Let

$$S(m) = \begin{cases} \delta_m & \text{if } m = \delta(\omega^\omega, n) \text{ for some } n \in N, \\ 2\delta_{l(n,\alpha_0,\alpha_1,\ldots,\alpha_s)} - \delta_{l(n,\alpha_0,\alpha_1,\ldots,\alpha_{s-1})} & \text{if } m = \dfrac{1}{2^s}\displaystyle\sum_{\varepsilon_r=\pm1}\delta(n, \alpha_0, \varepsilon_1, \alpha_1, \ldots, \varepsilon_s, \alpha_s) \\ & \text{for some } (\alpha_r)_{n=1}^{s-1} \subset [1, \omega^\omega]^{d(0)},\ \alpha_s \in [1, \omega^\omega), \\ & s = 0, 1, 2, \ldots, n-1, \\ & n = 1, 2, \ldots \end{cases}$$

and extend linearly. Clearly $\|S\| \leqslant 3$ and an easy computation shows that S is an inverse for T^*.

We are now ready to show that for every $\varepsilon > 0$, there is an integer n such that $P_{\omega n}(\varepsilon, \overline{\text{co}}(\pm L)) = \emptyset$. First observe that it is sufficient to show that for every $\varepsilon > 0$ there is an integer n such that

$$P_{\omega n}\big(\varepsilon, \overline{\text{co}}(\pm \bigcup\{L_r: 1 \leqslant r \leqslant s\})\big) = \emptyset \quad \text{for all } s.$$

Indeed, if $P_{\omega n}(\varepsilon, \overline{\text{co}}(\pm L)) \neq \emptyset$, for all n, then as we argued in the proof of Lemma 3.2 for each n there is an ωn-family $\{G_\gamma: \gamma \leqslant \omega^{\omega n}\}$ with ε-measures $\{\mu_\gamma: \gamma \leqslant \omega^{\omega n}\}$ in $\overline{\text{co}}(\pm L)$ such that $\big|\|\mu_\gamma\| - \|\mu_{\gamma'}\|\big| < \varepsilon/8$, for all $\gamma, \gamma' \leqslant \omega^{\omega n}$. Fix n and choose s sufficiently large so that

$$|\mu_{\omega^{\omega n}}|\big(\bigcup\{L_r: 1 \leqslant r \leqslant s\}\big) > \|\mu_{\omega^{\omega n}}\| - \varepsilon/8.$$

Because $\bigcup\{L_r\colon 1\leqslant r\leqslant s\}$ is clopen, there is a neighborhood $\mathcal{N}$ of $\mu_{\omega^\omega n}$ such that

$$|\mu_\gamma|\big(\bigcup\{L_r\colon 1\leqslant r\leqslant s\}\big) > \|\mu_{\omega^\omega n}\|-\varepsilon/8$$

for all $\mu_\gamma\in\mathcal{N}$. Consequently

$$|\mu_\gamma|\big(\bigcup\{L_r\colon s<r\}\big) < \|\mu_\gamma\|-\|\mu_{\omega^\omega n}\|+\varepsilon/8\leqslant\varepsilon/4$$

and thus

$$\{G_\gamma\cap\big(\bigcup\{L_r\colon 1\leqslant r\leqslant s\}\big)\colon \mu_\gamma\in\mathcal{N}\}$$

is an ωn family with $3\varepsilon/4$ measures $\{\mu_\gamma|_{\cup\{L_r\colon 1\leqslant r\leqslant s\}}\colon \mu_\gamma\in\mathcal{N}\}$ in $\overline{\mathrm{co}}\big(\pm\bigcup\{L_r\colon 1\leqslant r\leqslant s\}\big)$. By Lemma 1.0

$$P_{\omega n}\big(3\varepsilon/4,\ \overline{\mathrm{co}}\big(\pm\bigcup\{L_r\colon 1\leqslant r\leqslant s\}\big)\big)\neq\emptyset$$

proving our claim.

Next observe that the evaluation map

$$\tilde{T}\colon C_0(K)\to C\big(\bigcup\{L_r\colon 1\leqslant r\leqslant s\}\big)$$

is onto and thus by Lemma 3.2 it is sufficient to show that for every $\varepsilon>0$, there is an integer n such that $P_{\omega n}\big(\varepsilon,\bigcup\{L_r\colon 1\leqslant r\leqslant s\}\big)=\emptyset$, for all s. Because each set L_r is open, it follows as well that we need only show that $P_{\omega n}(\varepsilon, L_r)=\emptyset$, for all r. This will be a consequence of the following result:

Proposition 4.0. *Let $\mathcal{G}=\{G_\gamma\colon \gamma\leqslant\omega^\omega\}$ be an ω-family of open subsets of K_r with ε-measures $\{\mu_\gamma\colon \gamma\leqslant\omega^\omega\}$ in L_r. Then, there is an ordinal $\gamma_0<\omega^\omega$ such that $\mu_{\gamma_0}(G_{\gamma_0})\leqslant\frac{1}{2}\mu_{\omega^\omega}(G_{\omega^\omega})$.*

Proof. Let $\{A_{ij}\colon 1\leqslant j\leqslant N_i,\ 0\leqslant i\leqslant m\}$ be a distinguishing family of open sets for μ_{ω^ω} (as in Lemma 3.3) such that for each i and j either $A_{ij}\subset G_{\omega^\omega}$ or $A_{ij}\cap G_{\omega^\omega}=\emptyset$. By passing to an appropriate subset of $\{\mu_\gamma\colon\gamma\leqslant\omega^\omega\}$ we may assume that

$$\mu_\gamma(A_{ij})=\mu_{\omega^\omega}(A_{ij})\quad\text{for all } i \text{ and } j. \tag{4.0}$$

Let $N>r+1$. Then by Lemma 3.3 there is a subset $\mathcal{A}$ of $[1,\omega^\omega]$ such that $\mathcal{A}$ is homeomorphic to $[1,\omega^N]$ and

$$\mu_\beta\big(\bigcup\{A_{ij}\cap K_r(i)\cap G_\beta\colon 1\leqslant j\leqslant N_i,\ 0\leqslant i\leqslant m\}\big)<\varepsilon \tag{4.1}$$

for all $\beta\in\mathcal{A}$. Because $N>r+1$ and $L_r^{(\omega(r+1))}=\emptyset$, there is a convergent sequence $\{\mu_{\alpha(n)}\colon n\in\mathbf{N}\}$ with limit μ_α and $\alpha(n)\in\mathcal{A}$ for all n, such that

$$\{\mu_\alpha\}\cup\{\mu_{\alpha(n)}\colon n\in\mathbf{N}\}\subset L_r^{(\omega k)}-L_r^{(\omega(k+1))}$$

for some integer k.

Let $\mu_{\omega^\omega}=l\big(r,\beta(0),\beta(1),\ldots,\beta(s)\big)$ where $\beta(i)\in[1,\omega^\omega)^{d(0)}$, $i=0,1,2,\ldots,s-1$, $\beta(s)\in[1,\omega^\omega)$, and note that $\mu_{\omega^\omega}\notin L_r^{(\omega k)}-L_r^{(\omega(k+1))}$. Indeed, if $\mu_{\omega^\omega}\in L_r^{(\omega k)}-L_r^{(\omega(k+1))}$, by (4.0) and (4.1),

$$\mu_\beta(G_\beta)=\mu_\beta\big(\bigcup\{A_{ij}\cap K_r(i)\cap G_\beta\colon 1\leqslant j\leqslant N_i,\ 0\leqslant i\leqslant m\}\big)<\varepsilon$$

for all $\beta\in\mathcal{A}$, contradicting the fact that $\{\mu_\gamma\colon\gamma\leqslant\omega^\omega\}$ are ε-measures. It follows then that

$$\mu_\alpha=l\big(r,\tau(0),\tau(1),\ldots,\tau(r-k-1)\big)$$

and

$$\mu_{\alpha(n)}=l\big(r,\tau(0),\tau(1),\ldots,\tau(r-k-2),\tau(n,r-k-1)\big)$$

where $\tau(i)=\beta(i)$, $i=0,1,2,\ldots,s-1$, $\tau(s)\leqslant\beta(s)$, $s<r-k-1$, $\tau(i)\in[1,\omega^\omega)^{d(0)}$, $i=0,1,2,\ldots,r-k-2$, and $\tau(r-k-1)$, $\tau(n,r-k-1)\in[1,\omega^\omega)$.

Consider a point

$$\big(r,\beta(0),\varepsilon(1),\ldots,\beta(t)\big)\in\operatorname{supp}\mu_{\omega^\omega}\cap A_{r-t-1,j}.$$

Note that

$$\mu_{\omega^\omega}|_{A_{r-t-1,j}}=\frac{1}{2^{s+1}}\,\delta\big(r,\beta(0),\varepsilon(1),\ldots,\beta(t)\big),$$

$$\begin{aligned}\mu_\alpha|_{A_{r-t-1,j}}=\frac{1}{2^{r-k}}\sum_{p=s+1}^{r-k-1}\ \sum_{\substack{\varepsilon(m)=\pm1\\ m<s+1}}&\delta\big(r,\tau(0),\varepsilon(1),\ldots,\tau(t),1,\tau(t+1),\\ &-1,\ldots,-1,\tau(s+1),\varepsilon(s+2),\ldots,\tau(p)\big)+\\ &+\frac{1}{2^{r-k}}\,\delta\big(r,\beta(0),\varepsilon(1),\ldots,\beta(t)\big),\end{aligned}$$

and

$$\begin{aligned}\mu_{\alpha(n)|A_{r-t-1,j}}=\frac{1}{2^{r-k}}\sum_{p=s+1}^{r-k-2}\ \sum_{\substack{\varepsilon(m)=\pm1\\ m>s+1}}&\delta\big(r,\tau(0),\varepsilon(1),\ldots,\tau(t),1,\tau(t+1),\\ &-1,\ldots,-1,\tau(s+1),\varepsilon(s+2),\ldots,\tau(p)\big)+\\ &+\frac{1}{2^{r-k}}\sum_{\substack{\varepsilon(m)=\pm1\\ m>s+1}}\delta\big(r,\tau(0),\varepsilon(1),\ldots,\tau(t),1,\tau(t+1),-1,\ldots\\ &\ldots,-1,\tau(s+1),\varepsilon(s+2),\ldots,\varepsilon(r-k-1)\tau(n,r-k-1)\big)+\\ &+\frac{1}{2^{r-k}}\,\delta\big(r,\beta(0),\varepsilon(1),\ldots,\beta(t)\big).\end{aligned}$$

Because the sets $G_{\alpha(n)}$ are disjoint, $\mu_\alpha(G_{\alpha(n)})=0$ for n sufficiently large. Observe that

$$\|(\mu_\alpha\wedge\mu_{\alpha(n)})|_{A_{r-t-1,j}}\|=\tfrac{1}{2}\|\mu_{\alpha(n)}|_{A_{r-t-1,j}}\|$$

and thus

$$\mu_{a(n)}|_{A_{ij}}(G_{a(n)}) \leqslant \tfrac{1}{2}\mu_{a(n)}(A_{ij} \cap G_{\omega^\omega}) = \tfrac{1}{2}\mu_{\omega^\omega}(A_{ij} \cap G_{\omega^\omega})$$

for all i and j. Summing over i and j we have that

$$\mu_{a(n)}(G_{a(n)}) = \sum_{i,j} \mu_{a(n)}(A_{ij} \cap G_{a(n)}) \leqslant \tfrac{1}{2} \sum_{i,j} \mu_{a(n)}(A_{i,j} \cap G_{\omega^\omega})$$
$$= \tfrac{1}{2} \sum_{i,j} \mu_{\omega^\omega}(A_{ij} \cap G_{\omega^\omega}) = \tfrac{1}{2}\mu_{\omega^\omega}(G_{\omega^\omega}),$$

as claimed.

Now we will show that $P_{\omega n}(1/2^{n-1}, L_r) = \emptyset$ for every n and r. Indeed, if $P_{\omega n}(1/2^{n-1}, L_r) \neq \emptyset$, by Lemma 1.0 there would be an ωn-family $\{G_\gamma: \gamma \leqslant \omega^{\omega n}\}$ with $1/2^{n-1}$ measures $\{\mu_\gamma: \gamma \leqslant \omega^{\omega n}\}$ in L_r. By repeated application of Proposition 4.0 we could find a sequence of ordinals

$$\omega^{\omega n} = \gamma_n \geqslant \gamma_{n-1} \geqslant \ldots \geqslant \gamma_0$$

such that

$$\mu_{\gamma_k}(G_{\gamma_k}) \geqslant 2\mu_{\gamma_{k-1}}(G_{\gamma_{k-1}}),$$

and

$$G_{\gamma_{k-1}} \in \{G_\gamma: G_\gamma \subset G_{\gamma_k} \text{ and } \gamma \in (\gamma_k^-, \gamma_k]^{(\omega(k-1))}\},$$

for $k = 1, 2, \ldots, n$.

However, this would imply that

$$1 \geqslant \mu_{\gamma_n}(G_{\gamma_n}) \geqslant 2^k \mu_{\gamma_{n-k}}(G_{\gamma_{n-k}}) \geqslant 2^n \mu_{\gamma_0}(G_{\gamma_0}) \geqslant 2.$$

5. Remarks and open problems. In the previous sections we have shown that the Wolfe index provides a necessary and sufficient condition for an operator on $C[0,1]$ to be an isomorphism when restricted to a subspace isomorphic to $C(\omega^{\omega^\alpha})$, $\alpha < \omega_1$. One application of this result is to the problem of determining the complemented subspaces of $C[0,1]$. The natural guess is that each of these spaces is isomorphic to a $C(K)$ space. Our work here and Rosenthal's theorem [10] tell us which $C(K)$ space it must be, if it is a $C(K)$ space. Thus the conjecture could be verified by a positive solution of

PROBLEM 1. If Q is a projection on $C[0,1]$ and there is an ordinal $\alpha < \omega_1$ such that for each $\varepsilon > 0$ there is a $\beta < \omega^{\alpha+1}$ for which $P_\beta(\varepsilon, Q^* B_{C[0,1]^*}) = \emptyset$, is $Q(C[0,1])$ isomorphic to a complemented subspace of $C(\omega^{\omega^\alpha})$?

We have shown in Section 3 that the Szlenk index and the Wolfe index of an operator can be quite different. From the results of [2] and Corollary 0.5 it follows that if T is an operator on $C[0,1]$, then there is an $\varepsilon > 0$ such that $\eta(\varepsilon, B_{C[0,1]}, T^* B_{C[0,1]}) \geqslant \omega$ if and only if there is a $\delta > 0$ such that $P_n(\delta, T^* B_{C[0,1]^*}) \neq \emptyset$ for all n. Thus sometimes the two indices give the same information. In particular if the conjecture about complemented subspaces of $C[0,1]$ is correct, then the indices must give the same information for projections on $C[0,1]$.

PROBLEM 2. Is there a class of operators $\mathscr{A}$ on $C[0,1]$ which contains the projections such that for any $T \in \mathscr{A}$ if $\eta(\varepsilon, B_{C[0,1]}, T^* B_{C[0,1]}) \geqslant \omega^{\omega^\alpha}$, then there is an $\varepsilon' > 0$ such that $P_\beta(\varepsilon', T^* B_{C[0,1]}) \neq \emptyset$ for all $\beta < \omega^\alpha$?

An affirmative solution to Problem 2 would also give an affirmative solution to our next problem as well (see [3]).

PROBLEM 3. If Q is a projection on $C[0,1]$ and for each $\varepsilon > 0$, $P_\beta(\varepsilon, Q^*(B_{C[0,1]^*})) = \emptyset$ for some $\beta < \omega^{\alpha+1}$, is $Q(C[0,1])$ isomorphic to a quotient of $C(\omega^{\omega^\alpha})$?

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