STUDIA MATHEMATICA, T. LXX. (1981)

Basic sequences in stable infinite type power series spaces*

by

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Abstract. Under the assumption that $\Lambda_\infty(\alpha)$ is nuclear and stable, subspaces of $\Lambda_\infty(\alpha)$ with bases are characterized. The characterization is in terms of a nuclearity condition and an inequality which the basis must satisfy.

In [2] Dubinsky characterized subspaces with bases of (s), the space of rapidly decreasing sequences. In [7] Vogt studied the same problem without the requirement that the space have a basis. In this paper we use Dubinsky's techniques to solve the same problem for $\Lambda_\infty(\alpha)$. We show that $\Lambda(\alpha, N)$-nuclearity and the condition (d_2), defined by Dubinsky [2] which must be satisfied by the basis, are necessary and sufficient.

Preliminaries.

(a) A *Köthe set* A is a collection $A = \{a^k\colon k = 1, 2, \ldots\}$ of sequences of positive numbers such that $a_n^k < a_n^{k+1}$, $k, n \in N$.

The *Köthe space* $\Lambda(A)$ is the space of scalar sequences

$$\Lambda(A) = \Big\{t = (t_n)\colon \|t\|_k = \sum_{n=1}^{\infty} |t_n|\, a_n^k < \infty \text{ for all } k \in N\Big\}$$

and is topologized by the seminorms $\|\cdot\|_k$, $k = 1, 2, \ldots$

(b) *Grothendieck–Pietsch criterion.* A Köthe space $\Lambda(A)$ is nuclear if and only if

$$\forall k \exists l \ni (a_n^k/a_n^l) \in l_1.$$

(c) Let $\alpha = (\alpha_n)$ be a nondecreasing sequence of positive numbers and $0 < r \leqslant \infty$. *The power series space* $\Lambda_r(\alpha)$ generated by α is the Köthe space $\Lambda(A)$ where $a_n^k = (r_k)^{\alpha_n}$ and (r_k) is any strictly increasing sequence of positive numbers with $\lim r_k = r$.

From (b) it follows that $\Lambda_\infty(\alpha)$ (respectively $\Lambda_r(\alpha)$, $0 < r < \infty$) is nuclear if and only if for some $c \in (0, 1)$ (respectively for all $c \in (0, 1)$) $(c^{\alpha_n}) \in l_1$.

* This paper is taken from the author's dissertation at the University of Michigan written under the direction of Professor M. S. Ramanujan.

(d) A locally convex space E is said to be *stable* if $E\times E\simeq E$. For power series spaces $\Lambda_\infty(\alpha)$ stability is equivalent to: $\sup(\alpha_{2n}/\alpha_n)<\infty$.

(e) Let E be a locally convex space. For two absolutely convex zero neighborhoods V and U with $V<U$ (i.e., $V\subset rU$ for some $r>0$), the *n-th Kolmogorov diameter* of V with respect to U is defined as

$$d_n(V,U)=\inf\{\inf\{r>0:\ V\subset rU+L\}:\ L \text{ is a linear subspace of } E \text{ with } \dim L\leqslant n\}.$$

The *diametral dimension* $\Delta(E)$ of E is then defined to be the set of all scalar sequences (t_n) such that

$$\forall U\,\exists V\ni V<U \quad\text{and}\quad \lim t_n d_{n-1}(V,U)=0.$$

It is well-known (cf. [6]) that for a nuclear power series space $\Lambda_\infty(\alpha)$,

$$\Delta(\Lambda_\infty(\alpha))=\{(t_n):\ \exists\, M\in(0,\infty)\ni(t_n)=O(M^{\alpha_n})\}.$$

(f) Let α be such that $\Lambda_\infty(\alpha)$ is nuclear. A locally convex space E is said to be *$\Lambda(\alpha, N)$-nuclear* if it has a base of absolutely convex zero neighborhoods $\mathscr{U}(E)$ such that for each $U\in\mathscr{U}(E)$, $k\in N$ there is a $V\in\mathscr{U}(E)$ such that $\sup d_{n-1}(V,U)e^{k\alpha_n}<\infty$ (cf. [5]). It is easy to check that E is $\Lambda(\alpha,N)$-nuclear if and only if $\Delta(\Lambda_\infty(\alpha))\subset\Delta(E)$.

In [2], Dubinsky defined bases of type (d_3).

Definition. A basis (x_n) in a nuclear Fréchet space E with a continuous norm is of *type* d_3 if there is a fundamental system of norms $(\|\cdot\|_k)$ such that

$$(d_3)\qquad \frac{\|x_n\|_{k+1}}{\|x_n\|_k}\leqslant\frac{\|x_n\|_{k+2}}{\|x_n\|_{k+1}},\qquad k,n\in N.$$

Preparatory constructions and the main theorem. Throughout this paper we shall assume that $\Lambda_\infty(\alpha)$ is nuclear and stable, and the topology of $\Lambda_\infty(\alpha)$ is defined by the seminorms

$$\|t\|_k=\sup|t_n|e^{k\alpha_n}$$

(from (b) it follows that this is equivalent to the nuclearity of $\Lambda_\infty(\alpha)$).

Without loss of generality we may assume that $1\leqslant\alpha_1<\alpha_2<\ldots$ (e_n) denotes the coordinate basis of $\Lambda_\infty(\alpha)$.

In the sequel we shall need the following renorming result.

Lemma 1. *Let E be a nuclear Fréchet space with a continuous norm and with a basis (x_n) of type d_3. Assume that E is $\Lambda(\alpha,N)$-nuclear and that L is a constant greater than 1. Then there exists a rearrangement $(z_n)=(x_{\pi(n)})$ of the basis and a new system $(|\cdot|_k)$ of norms defining the topology of E such that*

(i) $e^{\alpha_{2n}}<\dfrac{|z_n|_2}{|z_n|_1}$, $n\in N$,

(ii) $\left(\dfrac{|z_n|_{k+1}}{|z_n|_k}\right)^L\leqslant\dfrac{|z_n|_{k+2}}{|z_n|_{k+1}}$, $\quad k,n\in N$.

Proof. The construction will be performed in two steps. We shall use the symbols $\|\cdot\|_k$ to denote the original norms, satisfying together with the basis (x_n) the condition (d_3).

Step 1. From the nuclearity of E it follows that

$$U_j=\Big\{t=\sum t_nx_n:\ \sup_n|t_n|\,\|x_n\|_j<1\Big\},\qquad j\in N$$

is a fundamental system of zero neighborhoods in E. The $\Lambda_\infty(\alpha)$ being stable is isomorphic to $\Lambda_\infty(\beta)$ with $\beta_n=\alpha_{2n}$. Hence E is $\Lambda(\beta,N)$-nuclear. This yields, in particular, that there exists a $j\in N$ such that

$$\sup d_{n-1}(U_{j+1},U_1)e^{\alpha_{2n}}<\infty.$$

Let $(\pi(n))$ be a permutation of N such that $(\|x_{\pi(n)}\|_1/\|x_{\pi(n)}\|_{j+1})$ is nonincreasing. Then (cf. [6]) we get

$$d_{n-1}(U_{j+1},U_1)=\|z_n\|_1/\|z_n\|_{j+1}\quad\text{for}\quad z_n=x_{\pi(n)},\ n\in N,$$

whence

$$\sup e^{\alpha_{2n}}\frac{\|z_n\|_1}{\|z_n\|_{j+1}}<C<\infty.$$

Replacing the norms $\|\cdot\|_k$ by $p_k(\cdot)=C^{k-1}\|\cdot\|_{j(k-1)+1}$ we obtain the new system satisfying the inequalities

$$e^{\alpha_{2n}}<p_2(z_n)/p_1(z_n),\qquad n\in N,$$

and the condition (d_3) for the new basis (z_n).

Step 2. We conclude the construction by letting

$$|\cdot|_k=p_{1+q+\ldots+q^k}(\cdot),\qquad k\in N,$$

where q is any integer greater than L.

Lemma 2. *Let (a_n^k) be an infinite matrix of positive numbers such that*

$$\frac{a_n^{k+1}}{a_n^k}<\frac{a_{n+1}^{k+1}}{a_{n+1}^k},\qquad n,k\in N.$$

Given numbers $t_1,\ldots,t_p$ we define, for $k\in N$,

$$q^k(t_1,\ldots,t_p)=\max\{q:\ \max_{1\leqslant i\leqslant p}|t_i|a_i^k=|t_q|a_q^k\}.$$

Then if $0<q^1<\ldots<q^m\leqslant p$ are integers, it is possible to choose numbers $t_1,\ldots,t_p$ with $t_{q^1}\neq0$ but otherwise arbitrary, $t_i=0$ for $i\neq q^1,\ldots,q^m$ and

$$|t_{q^k}|\frac{a_{q^k}^{k+1}}{a_{q^{k+1}}^{k+1}}<|t_{q^{k+1}}|<|t_{q^k}|\frac{a_{q^k}^k}{a_{q^{k+1}}^k},\qquad k=1,2,\ldots,m-1.$$

Moreover, if any such choice is made, then

$$q^k(t_1, \ldots, t_p) = q^k, \quad k = 1, \ldots, m.$$

Proof. See [1] and [2].

THEOREM. *Let E be a Fréchet space with a basis (x_n). Then E is isomorphic to a subspace of $\Lambda_\infty(a)$ if and only if it is $\Lambda(a, N)$-nuclear and the basis is of type* d_3.

Proof. *Necessity*. Suppose E is isomorphic to a subspace of $\Lambda_\infty(a)$. Since $\Lambda_\infty(a)$ is nuclear, E is nuclear, and from the classical Basis Theorem it follows that E is isomorphic to a Köthe space. Hence $\Delta(\Lambda_\infty(a)) \subset \Delta(E)$ which shows that E is $\Lambda(a, N)$-nuclear. Also for $x = \Sigma t_n e_n \in \Lambda_\infty(a)$,

$$\|x\|_{k+1}^2 = (\sup_n |t_n| e^{(k+1)a_n})^2 = |t_{n_0}|^2 e^{2(k+1)a_{n_0}}$$
$$= (|t_{n_0}| e^{(k+2)a_{n_0}})(|t_{n_0}| e^{ka_{n_0}}) \leqslant \|x\|_{k+2} \|x\|_k.$$

It follows that (x_n) is a basis of type d_3.

Sufficiency. Let $M = \sup(a_{2n}/a_n)$, and suppose (x_n) is a basis for E of type d_3. Applying Lemma 1 with $L = M+1$ and letting $c_j^k = |z_j|_k$, we get

$$e^{a_{2j}} < \frac{c_j^{k+1}}{c_j^k} < \left(\frac{c_j^{k+1}}{c_j^k}\right)^M < \frac{c_j^{k+2}}{c_j^{k+1}}, \quad k, j \in N. \tag{1}$$

Now we use the bijection $\gamma\colon N \times N \to N$ defined by $\gamma(j, m) = (2j-1)2^{m-1}$ to partition the coordinate basis (e_n) of $\Lambda_\infty(a)$ into countably many pairwise disjoint infinite subsequences $(e_{(j,m)})$, $(j, m) \in N \times N$. For

$$y = \sum t_n e_n = \sum \xi_{(j,m)} e_{(j,m)} \in \Lambda_\infty(a)$$

with

$$n = \gamma(j, m) \quad \text{and} \quad \xi_{(j,m)} = t_{\gamma(j,m)} = t_n,$$

we define

$$\|y\|_k^* = \sup_{(j,m)} |\xi_{(j,m)}| e^{ka_{j2m}}, \quad k \in N.$$

It is easy to check that the system $(\|\cdot\|_k^*)$ defines the topology of $\Lambda_\infty(a)$.

Next we fix $j \in N$, and by induction construct a strictly increasing sequence of positive integers $(q(k, j))$ such that

$$e^{a_{j2^{q(k,j)}}} < \frac{c_j^{k+1}}{c_j^k} < e^{a_{j2^{q(k+1,j)}}}, \quad k \in N. \tag{2}$$

To do this we first observe that it follows from (1) that $e^{a_{2j}} < c_j^2/c_j^1$, so that we may choose $q(1, j)$ to be the largest positive integer satisfying the left hand inequality (2) with $k = 1$. Suppose then that we have chosen $q(1, j) < \ldots < q(k+1, j)$ so that (2) holds and moreover, $q(k+1, j)$ is

the smallest integer such that the right hand inequality (2) holds. Then, it follows from (1) that

$$e^{a_{j2^{q(k+1,j)}}} = e^{a_{2j2^{q(k+1,j)-1}}} \leqslant e^{Ma_{j2^{q(k+1,j)-1}}}$$
$$= (e^{a_{j2^{q(k+1,j)-1}}})^M < \left(\frac{c_j^{k+1}}{c_j^k}\right)^M < \frac{c_j^{k+2}}{c_j^{k+1}}.$$

This establishes the left hand inequality (2) for k replaced by $k+1$ and also shows that if $q(k+2, j)$ is chosen to be the smallest integer such that the right hand inequality (2) holds with k replaced by $k+1$, then $q(k+1, j) < q(k+2, j)$. This completes the definition of $q(k, j)$.

Next with j still fixed we apply Lemma 2 with $a_n^k = e^{ka_{j2^n}}$, $p = q(j, j)$, $m = j$ and $q^l = q(l, j)$ for $l = 1, \ldots, m$. We set

$$t_{q(k,j)} = \frac{c_j^k}{e^{ka_{j2^{q(k,j)}}}}, \quad k = 1, \ldots, j;\ j \in N$$

and $t_i = 0$ if $i \neq q(k, j)$, $k = 1, \ldots, j$. It is easy to check that inequality (2) s equivalent to the inequality in the lemma. Hence if we define

$$y_j = t_1 e_{(j,1)} + \ldots + t_{q(j,j)} e_{(j,q(j,j))} \in \Lambda_\infty(a),$$

then

$$\|y_j\|_l^* = \sup_{1 \leqslant i \leqslant q(j,j)} |t_i| e^{la_{j2^i}} = |t_{q(l,j)}| e^{la_{j2^{q(l,j)}}} = c_j^l = |z_j|_l, \quad l = 1, \ldots, j.$$

Finally we note that (y_j) is a block basic sequence of some permutation of the coordinate basis (e_n) in $\Lambda_\infty(a)$, so it is a basis for the space it generates which is a subspace of $\Lambda_\infty(a)$ isomorphic to E.

The author wishes to thank the referee and Professor Bessaga for their suggestions to simplify the proofs.

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Received June 14, 1978 (1438)
Revised version October 23, 1978

STUDIA MATHEMATICA, T. LXX. (1981)

$C(K)$ norming subsets of $C[0,1]^*$

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Abstract. It is shown that if a bounded subset of $C[0,1]^*$ norms a subspace of $C[0,1]$ which is isomorphic to $C_0(\omega^{\omega^\alpha})$, for some $\alpha < \omega_1$, then there is a subspace of $C[0,1]$ isometric to $C_0(\omega^{\omega^\alpha})$ which is also normed by this set. The techniques employed also yield a new proof that there is a bounded linear operator form $C_0(\omega^{\omega^2})$ onto itself which is not an isomorphism when restricted to any subspace of $C_0(\omega^{\omega^2})$ isomorphic to $C_0(\omega^{\omega^2})$.

0. Introduction. Several authors have addressed the question of determining conditions on a subset of $C[0,1]^*$ which will ensure that the subset norms a subspace isomorphic to $C(K)$, the continuous functions on some compact metric space K. Necessary and sufficient conditions for the cases $K = [0,1]$, $[1,\omega]$, and $[1,\omega^\omega]$ (the ordinals less than of equal to ω, resp., ω^ω, with the order topology) have been given by Rosenthal [12], Pełczyński [10], and the author [2], respectively. Recently, J. Wolfe [15] introduced a sufficient conditon (the definitions will be given shortly) for the case of K homeomorphic to the ordinals less than or equal to ω^{ω^α}, any $\alpha < \omega_1$. The condition he gave is closely tied to the isometric structure of the $C(K)$ space and thus the necessity of the condition is far from obvious. In this paper we show that the Wolfe condition does yield a necessary and sufficient condition. As a corollary we deduce the first result stated in the abstract.

We also apply the Wolfe condition to the bounded linear operator T from $C_0(\omega^{\omega^2})$ onto $C_0(\omega^{\omega^2})$ constructed in the author's dissertation [1], to give a simpler argument that there is no subspace Y of $C_0(\omega^{\omega^2})$ such that Y is isomorphic to $C_0(\omega^{\omega^2})$ and the restriction of T to Y is an isomorphism. (For any ordinal α, $C(\omega^\alpha)$, resp., $C_0(\omega^\alpha)$, is the space of continuous functions on the ordinals less than or equal to ω^α with the order topology, resp., and vanishing at ω^α.) This also shows that the Szlenk index condition used in [2] and the Wolfe condition can be quite different.

We now give the definitions used by Wolfe [15] so that we may state our results precisely. The first is an inductive definition of a de-

* Supported in part by NSF-MCS 7610613.