STUDIA MATHEMATICA, T. LXX. (1982)

The best constants in the Khintchine inequality

by

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Abstract. Let (r_n) denote the sequence of Rademacher functions. It is shown that the best possible constants A_p and B_p in the Khintchine inequality

$$A_p\,(\sum_{k=1}^{n} a_k^2)^{1/2} < (\int_0^1 |\sum_{k=1}^{n} a_k r_k|^p\, dt)^{1/p} < B_p(\sum_{k=1}^{n} a_k^2)^{1/2}$$

are given by

$$A_p = \begin{cases} 2^{1/2-1/p} & 0 < p \leqslant p_0, \\ 2^{1/2}(\Gamma((p+1)/2)/\sqrt{\pi})^{1/p} & p_0 < p < 2, \\ 1 & 2 \leqslant p < \infty, \end{cases}$$

and

$$B_p = \begin{cases} 1 & 0 < p \leqslant 2, \\ 2^{1/2}(\Gamma((p+1)/2)/\sqrt{\pi}^{1/p} & 2 < p < \infty, \end{cases}$$

where p_0 is the solution of the equation $\Gamma((p+1)/2) = \sqrt{\pi}/2$ in the interval $]1, 2[$. $p_0 \approx 1{,}84742$.

Introduction. Let $r_n(t) = \operatorname{sign}(\sin 2^n \pi t)$, $n = 1, 2, \ldots$ denote the Rademacher functions. The Khintchine inequality states that for any $p \in\,]0, \infty[$ there exist constants A_p and B_p, such that

$$(*) \qquad A_p\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2} \leqslant \Big(\int_0^1 \Big|\sum_{k=1}^{n} a_k r_k\Big|^p dt\Big)^{1/p} \leqslant B_p\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2}$$

for arbitrary $n \in \boldsymbol{N}$ and $a_1, \ldots, a_n \in \boldsymbol{R}$.

The Khintchine inequality can also be expressed in the following way: Let ε_n be a sequence of mutually independent random variables with distribution $P\{1\} = P\{-1\} = 1/2$ (Bernoulli distribution). Then

$$A_p\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2} \leqslant E\Big(\Big|\sum_{k=1}^{n} a_k\, \varepsilon_k\Big|^p\Big)^{1/p} \leqslant B_p\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2},$$

where $E(\cdot)$ denotes the mean of a random variable.

A special case of this inequality was studied by Khintchine [6] in 1923. He used it to estimate the asymptotic behaviour of certain random walks.

More systematic treatments of the inequality were given by Littlewood, Paley and Zygmund in 1930 (cf. [7], [9]).

In the last decade, the Khintchine inequality and its generalization to certain classes of Banach Spaces, have become an important tool in the study of geometric properties of Banach spaces, see [8] and references given there.

In the following A_p and B_p will denote the best constants, for which the Khintchine inequality (*) is true. Although the exact values of A_p and B_p have no importance in the applications of the Khintchine inequality, the problem of computing the best constants has from time to time gained some interest. It is elementary to prove that $B_p = 1$ for $0 < p \leqslant 2$, and $A_p = 1$ for $2 \leqslant p < \infty$. In 1961 Stečkin [10] proved that $B_{2n} = ((2n-1)!!)^{1/2n}$, $n \in N$. This result was recently extended by Young [12], who computed B_p for $p \in [3, \infty[$. Finally Szarek [11] proved that $A_1 = 1/\sqrt{2}$. This was a long outstanding conjecture of Littlewood (cf. [3]).

In the present paper, we introduce a new method which enables us to compute the best constants in the remaining cases (A_p for $0 < p < 2$, $p \neq 1$ and B_p for $2 < p < 3$), and to give a simple proof of Szarek's result $A_1 = 1/\sqrt{2}$.

We found that

$$A_p = \begin{cases} 2^{1/2-1/p} & 0 < p \leqslant p_0, \\ 2^{1/2}\left(\Gamma((p+1)/2)/\sqrt{\pi}\right)^{1/p} & p_0 < p < 2 \end{cases}$$

and

$$B_p = 2^{1/2}\left(\Gamma((p+1)/2)/\pi\right)^{1/p} \qquad 2 < p < \infty,$$

where p_0 is the solution of the equation $\Gamma((p+1)/2) = \sqrt{\pi}/2$ in the interval $]1, 2[$. It is not hard to prove that A_p cannot exceed the number indicated, and that B_p cannot be less than $2^{1/2}\left(\Gamma((p+1)/2)/\sqrt{\pi}\right)^{1/p}$, so the problem is to give sufficiently good lower estimates for A_p and sufficiently good upper estimates for B_p. Consider first the case $0 < p < 2$. We define a function $F_p(s)$, $s > 0$ by

$$F_p(s) = C_p \int_0^\infty \left(1 - |\cos(t/\sqrt{s})|^s\right) t^{-p-1}\, dt,$$

where $C_p = \frac{2}{\pi}\sin(p\pi/2)\Gamma(p+1)$, and prove that when $\sum_{k=1}^n a_k^2 = 1$,

$$(**) \qquad E\left(\left|\sum_{k=1}^n a_k r_k\right|^p\right)^{1/p} \geqslant \sum_{k=1}^n a_k^2 F_p(a_k^{-2}).$$

If n is an even integer, and $|a_k| = 1/\sqrt{n}$ for all k, one has equality in (**).

Inequality (**) reduces the problem of giving lower estimates of A_p from a multi-variable problem to a single-variable problem, namely to investigate the function $F_p(s)$. When $p = 1$, we find that $F_1(s) = \frac{2}{\sqrt{\pi s}}\Gamma\left(\frac{s+1}{2}\right)/\Gamma(s/2)$. This function is increasing, and $F_1(2) = 1/\sqrt{2}$. Hence if $\sum a_k^2 = 1$ and $|a_k| \leqslant 1/\sqrt{2}$ for all k, we get

$$E\left(\left|\sum_{k=1}^n a_k r_k\right|\right) \geqslant \sum_{k=1}^n a_k^2 F_1(2) = 1/\sqrt{2}.$$

On the other hand, if $|a_i| > 1/\sqrt{2}$ for some i, it is clear that $E(|\sum_{k=1}^n a_k r_k|) > 1/\sqrt{2}$. This proves that $A_1 \geqslant 1/\sqrt{2}$.

For $0 < p < 2$, $p \neq 1$, the idea of the proof is the same. However, one runs into severe technical problems, because $F_p(s)$ cannot be expressed in terms of simple functions, when $p \neq 1$. Also when $p > 1$, $F_p(s)$ is not a monotone function of s.

The key Lemma 2.13 tells that $F_p(s) \geqslant F_p(\infty)$, when $p_0 \leqslant p < 2$ and $s \geqslant 2$ ($F_p(\infty) = \lim_{t\to\infty} F_p(t)$). The proof is based on a series expansion of $F_{p_0}(s)$ in the interval $s \in [2, 4]$ and an asymptotic estimate of $F_{p_0}(s)$ for $s \to \infty$. As for $p = 1$, the case $\sum_{k=1}^n a_k^2 = 1$ and $|a_i| > 1/\sqrt{2}$ for some i, has to be treated separately, and when $p \neq 1$, this case is far from being trivial (see Section 3).

The computation of B_p for $2 < p < 4$ is quite parallel to the computation of A_p for $0 < p < 2$. We define functions $F_p(s)$ similar to those defined in the case $0 < p < 2$, and prove that when $\sum_{k=1}^n a_k^2 = 1$ and $2 < p < 4$, then

$$E\left(\left|\sum a_k r_k\right|^p\right)^{1/p} \leqslant \sum_{k=1}^n a_k^2 F_p(a_k^{-2}).$$

The key Lemma 4.15 tells that $F_p(s) \leqslant F_p(\infty)$, when $2 < p < 4$, and $s \geqslant \sqrt{2}$, and the proof is based on a detailed investigation of the function $\chi(s) = \frac{d}{dp}F_p(s)|_{p=2}$.

As mentioned previously, Young [12] computed B_p for $p \geqslant 3$ by a different method. For the sake of completeness, we show in Section 5 how one can compute B_p for $p \geqslant 4$, by using the value of B_p obtained for $2 < p < 4$.

The results of this paper have been announced in [2]. The remaining part of the paper is divided in the following five sections:

§ 1. Case $p = 1$.
§ 2. Case $0 < p < 2$, and $|a_k| \leqslant 1/\sqrt{2}$ (*) for all k.
§ 3. Case $0 < p < 2$, and $|a_k| > 1/\sqrt{2}$ (*) for some k.
§ 4. Case $2 < p < 4$.
§ 5. Case $p \geqslant 4$.

§ 1. Case $p = 1$. In this section we shall give a short proof of Szarek's result [11]:

THEOREM 1.1. *For* $p = 1$, *the best lower constant in the Khintchine inequality is* $A_1 = 1/\sqrt{2}$.

Note first that $A_1 \leqslant 1/\sqrt{2}$, because

$$\int_0^1 |r_1(t) + r_2(t)|\,dt = 1 = \frac{1}{\sqrt{2}}(1+1)^{1/2}.$$

In the following we consider the Rademacher functions as a series of independent random variables on the probability space $[0, 1]$, with distribution $P\{1\} = P\{-1\} = 1/2$ (Bernoulli distribution).

LEMMA 1.2. *Let X be a random variable with symmetric distribution; then*

$$E(|X|) = \frac{2}{\pi}\int_0^\infty \big(1-\varphi_X(t)\big)t^{-2}\,dt,$$

where $\varphi_X(t) = E(e^{itX})$ *is the characteristic function for* X.

Proof. Note first that for $x \in \boldsymbol{R}$ one has

$$\text{(1a)} \qquad |x| = \frac{2}{\pi}\int_0^\infty (1-\cos xt)t^{-2}\,dt.$$

In fact, by the substitution $u = |x|t$, we have that

$$\int_0^\infty (1-\cos xt)t^{-2}\,dt = |x|\int_0^\infty (1-\cos u)u^{-2}\,du.$$

By partial integration

$$\int_0^\infty (1-\cos u)u^{-2}\,du = \int_0^\infty \frac{\sin u}{u}\,du = \pi/2$$

which proves (1a). Let X be a random variable on a probability space $(\Omega, d\omega)$, with symmetric distribution (i.e $P\{X \in A\} = P\{X \in -A\}$).

(*) We use the normalization $\sum_{k=1}^n a_k^2 = 1$.

Then $E(e^{itX}) = E(e^{-itX})$, and thus

$$\varphi_X(t) = E\big(\cos(tX)\big).$$

Hence by (1a)

$$\frac{2}{\pi}\int_0^\infty \big(1-\varphi_X(t)\big)t^{-2}\,dt = \frac{2}{\pi}\int_0^\infty\Big(\int_\Omega \big(1-\cos\big(tX(\omega)\big)\big)\,d\omega\Big)t^{-2}\,dt$$
$$= \frac{2}{\pi}\int_\Omega\Big(\int_0^\infty \big(1-\cos\big(tX(\omega)\big)\big)t^{-2}\,dt\Big)\,d\omega$$
$$= \int_\Omega |X(\omega)|\,d\omega = E(|X|).$$

Note that the formula is also valid if some of the integrals are infinite. This proves the lemma.

LEMMA 1.3. *Put*

$$F(s) = \frac{2}{\pi}\int_0^\infty \big(1-|\cos(t/\sqrt{s})|^s\big)t^{-2}\,dt, \qquad s > 0.$$

If $\sum_{k=1}^n a_k^2 = 1$, *then*

$$E\Big|\sum_{k=1}^n a_k r_k\Big| \geqslant \sum_{k=1}^n a_k^2 F(a_k^{-2}).$$

(*When* $a_k = 0$, *we put* $a_k^2F(a_k^{-2}) = 0$.)

Proof. Since $E(|\sum_{k=1}^n a_k r_k|)$ is invariant under permutation of the coefficients, it is no loss of generality to assume that $a_k \neq 0$ for any k. The characteristic function of the random variable $X = \sum_{k=1}^n a_k r_k$ is

$$\varphi_X(t) = E(e^{itX}) = \prod_{k=1}^n E(e^{ia_k r_k t}) = \prod_{k=1}^n \cos a_k t$$

because the Rademacher functions are independent random variables. Hence by Lemma 1.2

$$E(|X|) = \frac{2}{\pi}\int_0^\infty \Big(1-\prod_{k=1}^n \cos a_k t\Big)t^{-2}\,dt.$$

For non-negative numbers $s_1, \ldots, s_n$ and positive numbers $\alpha_1, \ldots, \alpha_n$ with $\sum_{k=1}^n \alpha_k = 1$, one has

$$s_1^{\alpha_1}\cdot\ldots\cdot s_n^{\alpha_n} \leqslant \alpha_1 s_1 + \ldots + \alpha_n s_n.$$

Using this formula for $a_k = a_k^2$, and $s_k = |\cos a_k t|^{a_k^{-2}}$, we get

$$1-\prod_{k=1}^{n}\cos a_k t \geqslant 1-\prod_{k=1}^{n}|\cos a_k t|$$

$$\geqslant 1-\sum_{k=1}^{n} a_k^2|\cos a_k t|^{a_k^{-2}}$$

$$= \sum_{k=1}^{n} a_k^2\left(1-|\cos a_k t|^{a_k^{-2}}\right).$$

Hence

$$E(X) \geqslant \sum_{k=1}^{n} a_k^2\left(\frac{2}{\pi}\int_0^\infty \left(1-|\cos a_k t|^{a_k^{-2}}\right)t^{-2}\,dt\right)$$

$$= \sum_{k=1}^{n} a_k^2 F(a_k^{-2}).$$

This proves Lemma 1.3.

LEMMA 1.4. *Let F be the function defined in Lemma 1.3. Then*

(1) $$F(s) = \frac{2}{\sqrt{\pi s}}\Gamma\big((s+1)/2\big)/\Gamma(s/2), \qquad s>0.$$

(2) *F is an increasing function of s.*

Proof. By definition

$$F(s) = \frac{2}{\pi}\int_0^\infty \left(1-|\cos(t/\sqrt{s})|^s\right)t^{-2}\,dt$$

$$= \frac{2}{\pi\sqrt{s}}\int_0^\infty \left(1-|\cos t|^s\right)t^{-2}\,dt$$

$$= \frac{1}{\pi\sqrt{s}}\int_{-\infty}^\infty \left(1-|\cos t|^s\right)t^{-2}\,dt.$$

Using the formula

$$\frac{1}{\sin^2 t} = \sum_{n=-\infty}^{\infty}\frac{1}{(t+n\pi)^2}$$

and Fubinis theorem, we get

$$F(s) = \frac{1}{\pi\sqrt{s}}\sum_{n=-\infty}^{\infty}\int_{n\pi-\pi/2}^{n\pi+\pi/2}\left(1-|\cos t|^s\right)t^{-2}\,dt$$

$$= \frac{1}{\pi\sqrt{s}}\sum_{n=-\infty}^{\infty}\int_{-\pi/2}^{\pi/2}\left(1-(\cos t)^s\right)\frac{dt}{(t+n\pi)^2}$$

$$= \frac{1}{\pi\sqrt{s}}\int_{-\pi/2}^{\pi/2}\left(1-(\cos t)^s\right)\frac{dt}{\sin^2 t}$$

or

$$F(s) = \frac{2}{\pi\sqrt{s}}\int_0^{\pi/2}\left(1-(\cos t)^s\right)\frac{dt}{\sin^2 t}.$$

Since

$$\int\frac{dt}{\sin^2 t} = -1/\mathrm{tg}\,t,$$

one gets by partial integration that

$$\int_0^{\pi/2}\left(1-(\cos t)^s\right)\frac{dt}{\sin^2 t}$$

$$= -\lim_{\varepsilon\to 0}\frac{1-(\cos t)^s}{\mathrm{tg}\,t}\Bigg|_{\varepsilon}^{\pi/2} + s\int_0^{\pi/2}\frac{(\cos t)^{s-1}\sin t}{\mathrm{tg}\,t}\,dt$$

$$= s\int_0^{\pi/2}(\cos t)^s\,dt$$

$$= s\frac{\sqrt{\pi}}{2}\Gamma\big((s+1)/2\big)/\Gamma(s/2+1) = \sqrt{\pi}\,\Gamma\big((s+1)/2\big)/\Gamma(s/2)$$

(cf. [5], p. 593, formula 620). Hence $F(s) = \frac{2}{\sqrt{\pi s}}\Gamma\big((s+1)/2\big)/\Gamma(s/2)$.

To prove (2) we expand $F(s)$ in an infinite product: From the equality $x\Gamma(x) = \Gamma(x+1)$, and the convexity of $\mathrm{Log}\,\Gamma(x)$, we get

$$\Gamma\big((s+1)/2\big)^2 \leqslant \Gamma(s/2)\,\Gamma\big((s+2)/2\big) = \frac{s}{2}\Gamma(s/2)^2$$

and

$$\Gamma\big((s+1)/2\big)^2 = \frac{s-1}{2}\Gamma\big((s-1)/2\big)\Gamma\big((s+1)/2\big) \geqslant \frac{s-1}{2}\Gamma(s/2)^2.$$

Hence

$$\sqrt{(s-1)/2} \leqslant \Gamma\big((s+1)/2\big)/\Gamma(s/2) = \sqrt{s/2}.$$

This proves, that

$$\lim_{s\to\infty} F(s) = \sqrt{2/\pi}.$$

Again, using the equation $x\Gamma(x) = \Gamma(x+1)$ we have

$$F(s+2) = \sqrt{\frac{s}{s+2}}\,\frac{s+1}{s}\,F(s) = \big(1-1/(s+1)^2\big)^{-1/2}F(s).$$

Hence for $n \in N$

$$F(s+2n) = \prod_{k=0}^{n-1}\big(1-1/(s+2k+1)^2\big)^{-1/2}F(s)$$

which in the limit $n\to\infty$, gives

$$\sqrt{2/\pi} = \prod_{k=0}^{\infty}\big(1-1/(s+2k+1)^2\big)^{-1/2}F(s)$$

or

$$F(s) = \sqrt{2/\pi}\prod_{k=0}^{\infty}\big(1-1/(s+2k+1)^2\big)^{1/2}.$$

This formula shows, that f is an increasing function of s.

Proof of Theorem 1.1 By the homogeneity of the Khintchine inequality, we may assume that $\sum_{k=1}^{n} a_k^2 = 1$.

Assume first that $|a_i| > 1/\sqrt{2}$ for some i; then by the Hölder inequality,

$$E\Big(\Big|\sum_{k=1}^{n} a_k r_k\Big|\Big) = \Big\|\sum_{k=1}^{n} a_k r_k\Big\|_1$$
$$\geqslant \Big|\int_0^1 r_i(t)\Big(\sum_{k=1}^{n} a_k r_k(t)\Big)\,dt\Big|$$
$$= |a_i| > 1/\sqrt{2}.$$

Assume next that $|a_k| \leqslant 1/\sqrt{2}$ for any k. By Lemma 1.3 and 1.4 we have

$$E\Big(\Big|\sum_{k=1}^{n} a_k r_k\Big|\Big) \geqslant \sum_{k=1}^{n} a_k^2 F(a_k^{-2}) \geqslant \sum_{k=1}^{n} a_k^2 F(2) = F(2).$$

Since $F(2) = \frac{2}{\sqrt{2\pi}}\Gamma(3/2)/\Gamma(1) = 1/\sqrt{2}$, the proof is complete.

§ 2. Case $0 < p < 2$ and $|a_i| \leqslant 1/\sqrt{2}$ for all i. In Sections 2 and 3 we shall prove:

THEOREM A. *Let $0 < p \leqslant 2$. The best lower constant in the Khintchine inequality is given by*

$$A_p = \begin{cases} 2^{1/2-1/p} & 0 < p \leqslant p_0, \\ 2^{1/2}\big(\Gamma((p+1)/2)\pi^{-1/2}\big)^{1/p} & p_0 \leqslant p \leqslant 2, \end{cases}$$

where p_0 is the non-trivial solution to the equation $\Gamma\big((p+1)/2\big) = \Gamma(3/2)$. One has $p_0 \approx 1.84742$.

This section takes care of the case, where $|a_k| \leqslant 1/\sqrt{2}$ for all k (we use the normalization $\sum_{k=1}^{n} a_k^2 = 1$).

In [10] Stečkin proved that $A_p \leqslant 2^{1/2}\big(\Gamma((p+1)/2)\pi^{-1/2}\big)^{1/p}$. For completeness, we give below a more direct proof of this inequality.

LEMMA 2.1.

$$A_p \leqslant \begin{cases} 2^{1/2-1/p} & 0 < p \leqslant p_0, \\ 2^{1/2}\big(\Gamma((p+1)/2)\pi^{-1/2}\big)^{1/p} & p_0 \leqslant p \leqslant 2. \end{cases}$$

Proof. Let $0 < p \leqslant 2$. Put $X_n = \frac{1}{\sqrt{n}}(r_1+r_2+\ldots+r_n)$, and let Z be a normal distributed random variable with density $\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$. From the central limit law of probability it follows that X_n converges in distribution to Z, i.e. for any continuous *bounded* function f

$$\lim_{n\to\infty} E\big(f(X_n)\big) = E\big(f(Z)\big). \tag{2a}$$

If g is a *positive*, but not necessarily bounded function, one gets by approximating g from below by continuous bounded functions that

$$\liminf_{n\to\infty} E\big(g(X_n)\big) \geqslant E\big(g(Z)\big). \tag{2b}$$

Applying (2b) to the functions $g_1(x) = |x|^p$ and $g_2(x) = x^2-|x|^p+1$, it follows that, for $0 < p \leqslant 2$,

$$E(|Z|^p) \leqslant \liminf_{n\to\infty} E(|X_n|^p) \leqslant \limsup_{n\to\infty} E(|X_n|^p) \leqslant E(|Z|^p).$$

Here we have used that $E(X_n^2) = E(Z^2) = 1$. Hence

$$\lim_{n\to\infty} E(|X_n|^p) = E(|Z|^p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|^p\exp(-x^2/2)\,dx = 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}.$$

This proves that

(2c) $$A_p \leqslant \left(2^{p/2}\Gamma\left((p+1)/2\right)\pi^{-1/2}\right)^{1/p}.$$

Putting $n = 2$ and $a_1 = a_2 = 1/\sqrt{2}$ in the Khintchine inequality, one gets easily $A_p \leqslant (2^{(p-2)/2})^{1/p} = 2^{1/2-1/p}$. Therefore

(2d) $$A_p \leqslant 2^{1/2}\min\left\{\Gamma\left((p+1)/2\right)\pi^{-1/2}, 1/2\right\}^{1/p}, \quad 0 < p \leqslant 2.$$

Since the Γ-function is convex, the equation $\Gamma\left((p+1)/2\right) = \sqrt{\pi}/2$ has only two solutions, namely the trivial solution $p = 2$, and a solution $p = p_0 < 2$. By 2nd order interpolation in the table [5], p. 726, one finds $p_0 \approx 1{,}84742$. Clearly $\Gamma\left((p+1)/2\right) > \sqrt{\pi}/2$ for $0 < p < p_0$, while $\Gamma\left((p+1)/2\right) < \sqrt{\pi}/2$ for $p_0 < p < 2$. This proves the lemma.

Remark 2.2. To avoid round off errors in later numerical computations involving p_0, we have computed p_0 to 10 correct decimals. We used the method described in [1], pp. 106–114 to compute $\Gamma(x)$ with the necessary accuracy, and found

$$p_0 = 1{,}84741.63361.$$

In the following, we put $C_p = \frac{2}{\pi}\sin(p\pi/2)\Gamma(p+1)$, $p > 0$.

Lemma 2.3. *Let X be a real random variable with symmetric distribution, such that $E(X^2) < \infty$. For $0 < p < 2$,*

(2e) $$E(|X|^p) = C_p\int_0^\infty \left(1-\varphi_X(t)\right)t^{-p-1}\,dt,$$

where $\varphi_X(t) = E(e^{itX})$ is the characteristic function for X.

Proof. We show first that for $x \in \boldsymbol{R}$ and $0 < p < 2$,

(2f) $$|x|^p = C_p\int_0^\infty (1-\cos xt)t^{-p-1}\,dt.$$

For $|x| \neq 0$, the substitution $u = |x|t$, shows that

(2g) $$\int_0^\infty (1-\cos xt)t^{-p-1}dt = |x|^p\int_0^\infty (1-\cos u)u^{-p-1}\,du.$$

By partial integration one gets

$$\int_0^\infty (1-\cos u)u^{-p-1}\,du = \frac{1}{p}\int_0^\infty \frac{\sin u}{u^p}\,du = \frac{1}{(2/\pi)\sin(\pi p/2)\Gamma(p+1)}.$$

(The last equality is trivial for $p = 1$. For $0 < p < 1$ it follows from [5], p. 594, formula 631. When $1 < p < 2$ it follows from [5], p. 594, formula 632, by partial integration.) This proves formula (2f). However, as in Section 1, (2e) can be derived from (2f) using Fubini's Theorem.

Lemma 2.4. *Let $0 < p < 2$. Put*

$$F_p(s) = C_p\int_0^\infty \left(1-|\cos(t/\sqrt{s})|^s\right)t^{-p-1}\,dt, \quad s > 0.$$

(1) *If $\sum_{k=1}^n a_k^2 = 1$, then*

$$E\left(\left|\sum_{k=1}^n a_k r_k\right|^p\right) \geqslant \sum_{k=1}^n a_k^2 F_p(a_k^{-2}).$$

(2) $$F_p(2n) = E\left(\left|\frac{1}{\sqrt{2n}}\sum_{k=1}^{2n} r_k\right|^p\right), \quad n \in \boldsymbol{N}.$$

(3) $$\lim_{s\to\infty} F_p(s) = C_p\int_0^\infty \left(1-\exp(-t^2/2)\right)t^{-p-1}dt = 2^{p/2}\Gamma\left((p+1)/2\right)\pi^{-1/2}.$$

Proof. (1) Same as the proof of Lemma 1.3.

(2) Let $X_m = \frac{1}{\sqrt{m}}\sum_{k=1}^m r_k$, $m = 1, 2, \ldots$ Then the characteristic function for X_m is $\left(\cos(t/\sqrt{m})\right)^m$. Hence

$$E(|X_{2n}|^p) = C_p\int_0^\infty \left(1-\left(\cos(t/\sqrt{2n})^{2n}\right)\right)t^{-p-1}dt = F_p(2n),$$

$n = 1, 2, \ldots$ which proves (2).

(3) For fixed $t \in \boldsymbol{R}$, we have

$$\lim_{s\to\infty}\log\left(|\cos(t/\sqrt{s})|^s\right) = \lim_{s\to\infty}\left(s\log\left(\cos(t/\sqrt{s})\right)\right) = -t^2/2.$$

Hence

$$|\cos(t/\sqrt{s})|^s \to \exp(-t^2/2) \quad \text{for} \quad s \to \infty.$$

For $s \geqslant 1$ and $t \leqslant \sqrt{2}$, we have $t/\sqrt{s} \leqslant \sqrt{2}$ and hence

$$|\cos(t/\sqrt{s})|^s = \left(\cos(t/\sqrt{s})\right)^s \geqslant (1-t^2/2s)^s \geqslant 1-t^2/2.$$

Therefore

(2h) $$\exp(-t^2/2) - |\cos(t/\sqrt{s})|^s \leqslant (1-t^2/2+t^4/8)-(1-t^2/2) = t^4/8.$$

However, $\cos u \leqslant \exp(-u^2/2)$ for $|u| \leqslant \pi/2$, and thus

$$|\cos(t/\sqrt{s})|^s \leqslant \exp(-t^2/2) \quad \text{for} \quad s \geqslant 1 \text{ and } t \leqslant \sqrt{2}.$$

Hence for $s \geqslant 1$, and $t \in \mathbf{R}$,

(2i) $$\big|(\exp(-t^2/2) - |\cos(t/\sqrt{s})|^s\big| \leqslant \begin{cases} t^4/8, & t \leqslant \sqrt{2}, \\ 1, & t \geqslant \sqrt{2}. \end{cases}$$

Lebesgue's Theorem about majorized convergence gives now easily that

(2j) $$\lim_{s\to\infty} F_p(s) = C_p \int_0^\infty (1-\exp(-t^2/2))t^{-p-1}\,dt.$$

However, $\exp(-t^2/2)$ is the characteristic function for a normal distributed random variable Z, with density $\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$. Hence by Lemma 2.3,

$$C_p \int_0^\infty (1-\exp(-t^2/2))t^{-p-1}\,dt = E(|Z|^p) = 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}.$$

This proves (3).

Remark 2.5. We put $F_p(\infty) = \lim_{s\to\infty} F_p(s) = 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}$. From Lemma 2.4 (2) it follows that $F_p(2) = 2^{(p-2)/2}$. Hence $F_p(2) < F_p(\infty)$ for $0 < p < p_0$, $F_{p_0}(2) = F_{p_0}(\infty)$ and $F_p(2) > F_p(\infty)$, for $p_0 < p < 2$. In the following we shall prove that $F_p(s) \geqslant F_p(\infty)$ when $p_0 \leqslant p < 2$ and $s \geqslant 2$. This will, using Lemma 2.4 (1) and (3), prove that

$$E\Big(\Big|\sum a_k r_k\Big|^p\Big)^{1/p} \geqslant F_p(\infty)^{1/p} = 2^{1/2}\Big(\Gamma\big((p+1)/2\big)\pi^{-1/2}\Big)^{1/p},$$

when $p_0 \leqslant p < 2$, $\sum_{k=1}^n a_k^2 = 1$ and $|a_k| \leqslant 1/\sqrt{2}$ for all k.

We will first treat the case $p = p_0$. Figure 2.1 below shows the graph of $F_{p_0}(s)$. The drawing is based only on the values $F_{p_0}(2n)$, $n = 1, 2 \ldots$ which are easily computed from Lemma 2.4 (2).

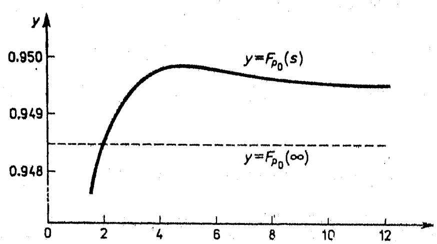


Figure 2.1

Note that the variation of $F_{p_0}(s)$ for $s \geqslant 2$ is very small (of order 10^{-3}). It is of course possible to prove $F_{p_0}(s) \geqslant F_{p_0}(\infty)$ for single values of $s > 2$ by computing the integral in Lemma 2.4 numerically. However, to prove analytically that $F_{p_0}(s) \geqslant F_{p_0}(\infty)$ for all $s \geqslant 2$, turns out to be quite complicated, because $F_p(s)$ cannot be expressed in terms of simple functions, when $p \neq 1$. The proof will be divided in two cases, namely $s \in [2, 4]$ and $s > 4$.

Lemma 2.6. *Let $1 \leqslant p \leqslant 3$. Then*

$$\sum_{n=-\infty}^{\infty} \frac{1}{|t+n\pi|^{p+1}} \leqslant 3^{(1-p)/2}|\sin t|^{1-3p}, \qquad t \neq n\pi.$$

Proof. It is well known that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(t+n\pi)^2} = \frac{1}{\sin^2 t}.$$

By differentiating this equation twice, one gets

$$\sum_{n=-\infty}^{\infty} \frac{1}{(t+n\pi)^4} = \frac{1}{\sin^4 t} - \frac{2}{3}\frac{1}{\sin^2 t}.$$

Since $\frac{1}{3}x^3 - x + \frac{2}{3} = \frac{1}{3}(x-1)^2(x+2) \geqslant 0$ when $x \geqslant 1$, it follows that

$$x - \tfrac{2}{3} \leqslant \tfrac{1}{3}x^3 \quad \text{when} \quad x \geqslant 1.$$

Hence

$$\sum_{n=-\infty}^{\infty} \frac{1}{(t+n\pi)^4} = \frac{1}{\sin^2 t}\left(\frac{1}{\sin^2 t} - \frac{2}{3}\right) \leqslant \frac{1}{3}\frac{1}{\sin^8 t}.$$

By Hölder's inequality we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{|t+n\pi|^q} \leqslant \left(\sum_{n=-\infty}^{\infty} \frac{1}{(t+n\pi)^2}\right)^{(4-q)/2} \left(\sum_{n=-\infty}^{\infty} \frac{1}{(t+n\pi)^4}\right)^{(q-2)/2}$$
$$\leqslant 3^{(2-q)/2}(\sin^2 t)^{(q-4)/2}(\sin^8 t)^{(2-q)/2} = 3^{(2-q)/2}|\sin t|^{4-3q}.$$

Substituting $q = p+1$, the assertion follows.

Lemma 2.7. *For $0 < p < 2$, put*

$$a_n^{(p)} = \int_0^\infty (\sin t)^{2n} t^{-p-1}\,dt$$

and

$$\beta_n^{(p)} = \int_0^\infty (1-\exp(-t^2))^n t^{-p-1}\,dt$$

for $n = 1, 2, \ldots$ *Then*

$$\alpha_1^{(p_0)} = \beta_1^{(p_0)}, \quad \textit{and} \quad \alpha_n^{(p_0)} < \beta_n^{(p_0)} \textit{ for } n \geqslant 2.$$

Proof. We will first prove the formulas

$$\alpha_n^{(p)} = \frac{2^{p+1}}{C_p 4^n} \sum_{k=1}^{n} (-1)^{k-1} \binom{2n}{n-k} k^p, \tag{2k}$$

$$\beta_n^{(p)} = \frac{2^p \Gamma((p+1)/2)}{C_p \sqrt{\pi}} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k^{p/2}. \tag{2l}$$

Using Euler's formula $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$ one gets

$$\sin^{2n} t = \frac{1}{4^n} \sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k} \cos(2kt).$$

Since $\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k} = 0$, we have also

$$\sin^{2n} t = \frac{1}{4^n} \sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k} (\cos(2kt) - 1).$$

Since the term at $k = 0$ vanishes, and since $\binom{2n}{n+k} = \binom{2n}{n-k}$, we get

$$\sin^{2n} t = \frac{2}{4^n} \sum_{k=1}^{n} (-1)^{k-1} \binom{2n}{n-k} (1 - \cos(2kt)). \tag{2m}$$

From (2f) with $x = 2k$ we have

$$(2k)^p = C_p \int_0^\infty (1 - \cos 2kt) t^{-p-1} dt.$$

Hence using (2m):

$$C_p \alpha_n^{(p)} = C_p \int_0^\infty (\sin t)^{2n} t^{-p-1} dt = \frac{2^{p+1}}{4^n} \sum_{k=1}^{n} (-1)^{k-1} \binom{2n}{n-k} k^p.$$

This proves (2k). To prove (2l) we use

$$(1 - \exp(-t^2))^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \exp(-kt^2).$$

Since $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$, we have also

$$(1 - \exp(-t^2))^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\exp(-kt^2) - 1) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (1 - \exp(-kt^2)). \tag{2n}$$

Using Lemma 2.4 (3), we get

$$C_p \int_0^\infty (1 - \exp(-kt^2)) t^{-p-1} dt = C_p (2k)^{p/2} \int_0^\infty (1 - \exp(-t^2/2)) t^{-p-1} dt = (2k)^{p/2} 2^{p/2} \Gamma((p+1)/2) \pi^{-1/2}.$$

Therefore by (2n)

$$C_p \beta_n^{(p)} = C_p \int_0^\infty (1 - \exp(-t^2))^n t^{-p-1} dt = \frac{2^p \Gamma((p+1)/2)}{\sqrt{\pi}} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k^{p/2}.$$

This proves (2l). From (2k) and (2l) we have

$$\alpha_1^{(p)} = \frac{2^{p+1}}{4 C_p} \quad \text{and} \quad \beta_1^{(p)} = \frac{2^p \Gamma((p+1)/2)}{C_p \sqrt{\pi}}.$$

Since $\Gamma((p_0+1)/2) = \sqrt{\pi}/2$, it follows that $\alpha_1^{(p_0)} = \beta_1^{(p_0)}$.

The following list gives the values of $\alpha_n^{(p_0)}$ and $\beta_n^{(p_0)}$ for $1 \leqslant n \leqslant 15$, computed from (2k) and (2l) (maximal error $\leqslant 10^{-5}$). To avoid accumulated round off errors in the sums, we have used the value $p_0 = 1{,}84741.63361$ (cf. Remark 2.2).

n	$\alpha_n^{(p_0)}$	$\beta_n^{(p_0)}$	n	$\alpha_n^{(p_0)}$	$\beta_n^{(p_0)}$
1	6,82084	6,82084	9	0,19358	0,24391
2	0,68455	0,70265	10	0,18157	0,23359
3	0,43470	0,46279	11	0,17154	0,22502
4	0,33880	0,37347	12	0,16289	0,21776
5	0,28587	0,32529	13	0,15560	0,21151
6	0,25147	0,29453	14	0,14912	0,20604
7	0,22692	0,27287	15	0,14339	0,20122
8	0,20830	0,25664			

One observes that $\alpha_n^{(p_0)} < \beta_n^{(p_0)}$ for $2 \leqslant n \leqslant 15$. To treat the case $n > 15$' we will prove that for $1 \leqslant p < 2$ and $n \geqslant 3$:

$$\alpha_n^{(p)} \leqslant \sqrt{\pi/2}\ 3^{(1-p)/2} (2n-5)^{-1/2}, \tag{2o}$$

$$\beta_n^{(p)} \geqslant \frac{1}{p} (1 + \log n)^{-p/2}. \tag{2p}$$

From Lemma 2.6 we have for $1 \leqslant p < 2$ that

$$\sum_{n=-\infty}^{\infty} |t+n\pi|^{-p-1} \leqslant 3^{(1-p)/2}|\sin t|^{1-3p} \leqslant 3^{(1-p)/2}|\sin t|^{-5}.$$

Hence

$$\begin{aligned}\alpha_n^{(p)} &= \int_0^\infty (\sin t)^{2n} t^{-p-1} = \tfrac{1}{2}\int_{-\infty}^{\infty} (\sin t)^{2n}|t|^{-p-1}\,dt \\ &= \tfrac{1}{2}\int_0^\pi (\sin t)^{2n}\Big(\sum_{n=-\infty}^{\infty}|t+n\pi|^{-p-1}\Big)\,dt \leqslant \tfrac{1}{2}3^{(1-p)/2}\int_0^\pi (\sin t)^{2n-5}\,dt \\ &= \tfrac{1}{2}\,3^{(1-p)/2}\int_{-\pi/2}^{\pi/2}(\cos t)^{2n-5}\,dt \leqslant \tfrac{1}{2}3^{(1-p)/2}\int_{-\infty}^{\infty}\exp\big(-(2n-5)t^2/2\big)\,dt \\ &= \sqrt{\pi/2}\,3^{(1-p)/2}(2n-5)^{-1/2}.\end{aligned}$$

This proves (2o).

$$\begin{aligned}\beta_n^{(p)} &= \int_0^\infty (1-e^{-t^2})^n t^{-p-1}\,dt = \tfrac{1}{2}\int_0^\infty (1-e^{-u})^n u^{-(p+2)/2}\,du \\ &\geqslant \tfrac{1}{2}\int_{\log n}^{\infty}(1-e^{-u})^n u^{-(p+2)/2}\,du \geqslant \tfrac{1}{2}\int_{\log n}^{\infty}(1-ne^{-u})u^{-(p+2)/2}\,du \\ &= \tfrac{1}{2}\int_{1+\log n}^{\infty} u^{-(p+2)/2}\,du + \tfrac{1}{2}\Big(\int_{\log n}^{1+\log n}(1-ne^{-u})u^{-(p+2)/2}\,du - \\ &\qquad - \int_{1+\log n}^{\infty} ne^{-u}u^{-(p+2)/2}\,du\Big).\end{aligned}$$

Using

$$u^{-(p+2)/2} \geqslant (1+\log n)^{-(p+2)/2} \quad \text{for} \quad u \leqslant 1+\log n$$

and

$$u^{-(p+2)/2} \leqslant (1+\log n)^{-(p+2)/2} \quad \text{for} \quad u \geqslant 1+\log n,$$

it follows that

$$\begin{aligned}\beta_n^{(p)} &\geqslant \tfrac{1}{2}\int_{1+\log n}^{\infty} u^{-(p+2)/2}du + \\ &\quad + \tfrac{1}{2}(1+\log n)^{-(p+2)/2}\Big(\int_{\log n}^{1+\log n}(1-ne^{-u})\,du - \int_{1+\log n}^{\infty} ne^{-u}\,du\Big) \\ &= \frac{1}{p}(1+\log n)^{-p/2} + \tfrac{1}{2}(1+\log n)^{-(p+2)/2}(e^{-1}-e^{-1}) \\ &= \frac{1}{p}(1+\log n)^{-p/2}.\end{aligned}$$

This proves (2p).

When $n \geqslant 15$, $n - \frac{5}{2} = n(1-5/2n) \geqslant n(1-\frac{5}{30}) = \frac{5}{6}n$.

Hence by (2o) and (2p):

$$\alpha_n^{(p)} \leqslant \left(\frac{\pi\cdot 3^{1-p}}{4(n-5/2)}\right)^{1/2} \leqslant \left(\frac{\pi}{10}3^{2-p}\right)^{1/2} n^{-1/2}$$

and

$$\text{(2q)} \qquad \alpha_n^{(p)}/\beta_n^{(p)} \leqslant p\left(\frac{\pi}{10}3^{2-p}\right)^{1/2}[(1+\log n)n^{-1/p}]^{p/2}.$$

The function $t \to (1+\log t)t^{-1/p}$ is decreasing for $t > e$, because when $t > e$,

$$\frac{d}{dt}[(1+\log t)t^{-1/p}] = t^{-1-1/p}\Big(1-\frac{1}{p}(1+\log t)\Big) < t^{-1-1/p}(1-2/p) < 0.$$

For $p = p_0$ and $n = 15$, the right side of (2q) has the value $0{,}97\,548\ldots$ Hence

$$\alpha_n^{(p_0)}/\beta_n^{(p_0)} \leqslant 0{,}97\,548\ldots \quad \text{for} \quad n \geqslant 15.$$

This proves Lemma 2.7.

LEMMA 2.8. *Let $0 < p < 2$. For $0 < s \leqslant 4$:*

$$\text{(2r)} \qquad F_p(s) = F_p(\infty) + C_p s^{-p/2}\sum_{k=1}^{\infty}(\beta_k^{(p)} - \alpha_k^{(p)})(-1)^k\binom{s/2}{k}.$$

Proof. For $s > 0$ we have

$$\begin{aligned}\text{(2s)} \qquad 1-|\cos t|^s &= 1-(1-\sin^2 t)^{s/2} \\ &= 1-\sum_{k=0}^{\infty}(-1)^k\binom{s/2}{k}\sin^{2k}t \\ &= \frac{s}{2}\sin^2 t + \sum_{k=2}^{\infty}(-1)^{k-1}\binom{s/2}{k}\sin^{2k}t.\end{aligned}$$

When $s \in \,]0, 4]$, the sign of $(-1)^{k-1}\binom{s/2}{k}$ is independent of k for $k \geqslant 2$. Therefore by (2s)

$$\begin{aligned}\text{(2t)} \qquad \sum_{k=2}^{\infty}\Big|(-1)^{k-1}\binom{s/2}{k}\sin^{2k}t\Big| &= \Big|\sum_{k=2}^{\infty}(-1)^{k-1}\binom{s/2}{k}\sin^{2k}t\Big| \\ &= \Big|1-|\cos t|^s - \frac{s}{2}\sin^2 t\Big|.\end{aligned}$$

Since

$$1-|\cos t|^s-\frac{s}{2}\sin^2 t = O(t^4) \quad \text{for} \quad t\to 0,$$

it follows, using Lebesgue's Theorem of majorized convergence on (2s) that

$$\text{(2u)}\qquad \int_0^\infty (1-|\cos t|^s)t^{-p-1}dt = \sum_{k=1}^\infty (-1)^{k-1}\binom{s/2}{k}\int_0^\infty (\sin t)^{2k}t^{-p-1}dt$$

$$= \sum_{k=1}^\infty (-1)^{k-1}\binom{s/2}{k}\alpha_k^{(p)}.$$

A similar argument applied to the expansion

$$1-\exp(-st^2/2) = 1-\big(1-(1-e^{-t^2})\big)^{s/2}$$

$$= \frac{s}{2}(1-e^{-t^2})+\sum_{k=2}^\infty (-1)^{k-1}\binom{s/2}{k}(1-e^{-t^2})^k$$

gives that

$$\text{(2v)}\qquad \int_0^\infty \big(1-\exp(-st^2/2)\big)t^{-p-1}dt = \sum_{k=1}^\infty (-1)^{k-1}\binom{s/2}{k}\int_0^\infty (1-e^{-t^2})^k t^{-p-1}dt$$

$$= \sum_{k=1}^\infty (-1)^{k-1}\binom{s/2}{k}\beta_k^{(p)}.$$

From Lemma 2.4 (3) we have, for $s>0$,

$$\text{(2x)}\qquad F_p(s)-F_p(\infty) = C_p\int_0^\infty \big(\exp(-t^2/2)-|\cos(t/\sqrt{s})|^s\big)t^{-p-1}dt$$

$$= C_p s^{-p/2}\int_0^\infty \big(\exp(-st^2/2)-|\cos t|^s\big)t^{-p-1}dt.$$

Combining (2u), (2v) and (2x), we get (2r).

LEMMA 2.9.

$$F_{p_0}(s)\geqslant F_{p_0}(\infty) \quad \textit{for} \quad s\in[2,4].$$

Proof. When $s\in[2,4]$, $(-1)^k\binom{s/2}{k}\geqslant 0$ for $k\geqslant 2$. Hence the statement is an immediate consequence of Lemma 2.7 and Lemma 2.8.

Note that since $(-1)^k\binom{s/2}{k}<0$ for $0<s<2$, it follows that $F_{p_0}(s) < F_{p_0}(\infty)$ for $0<s<2$.

To treat the case $s>4$, we introduce the functions

$$\text{(2y)}\qquad a_p(s) = \int_0^\infty \big(\exp(-st^2/2)-|\cos t|^s\chi_{[0,\pi/2]}(t)\big)t^{-p-1}dt,$$

$$\text{(2z)}\qquad b_p(s) = \int_{\pi/2}^\infty |\cos t|^s t^{-p-1}dt.$$

From formula (2x) one gets, for $s>0$,

$$\text{(2aa)}\qquad F_p(s) = F_p(\infty)+C_p s^{-p/2}\big(a_p(s)-b_p(s)\big).$$

LEMMA 2.10. *Let* $0<p<2$. *Then*

$$\text{(1)}\qquad a_p(s)\geqslant (s/2)^{p/2}\left(\frac{1}{6s}\Gamma\big((4-p)/2\big)-\frac{1}{36s^2}\Gamma\big((8-p)/2\big)\right),$$

$$\text{(2)}\qquad b_p(s)\leqslant \sqrt{2\pi}\,(k_1^{(p)}s^{-1/2}+k_2^{(p)}s^{-3/2}),$$

where

$$k_1^{(p)} = \pi^{-p-1}\zeta(p+1)$$

and

$$k_2^{(p)} = (2/\pi)^{p+3}\big((1-2^{-p})\zeta(p+1)-1/2\big)$$

(ζ is the Riemann Zeta function).

Proof. *For* $0\leqslant t<\pi/2$

$$-\log(\cos t) = \int_0^t \operatorname{tg}u\,du \geqslant \int_0^t (u+u^3/3)\,du = t^2/2+t^4/12.$$

Hence

$$(\cos t)^s\leqslant \exp(-st^2/2-st^4/12), \qquad 0\leqslant t<\pi/2,$$

which implies that for $t\geqslant 0$

$$\exp(-st^2/2)-|\cos t|^s\chi_{[0,\pi/2]}(t)\geqslant \exp(-st^2/2)\big(1-\exp(-st^4/12)\big)$$

$$\geqslant \exp(-st^2/2)\big(st^4/12-\tfrac{1}{2}(st^4/12)^2\big).$$

Hence

$$a_p(s)\geqslant \int_0^\infty \exp(-st^2/2)\big(st^4/12-\tfrac{1}{2}(st^4/12)^2\big)t^{-p-1}dt$$

$$= \int_0^\infty e^{-u}(u^2/3s-u^4/18s^2)(2u/s)^{-(p+1)/2}\frac{du}{\sqrt{2su}}$$

$$= (s/2)^{p/2}\left(\frac{1}{6s}\Gamma\big((4-p)/2\big)-\frac{1}{36s^2}\Gamma\big((8-p)/2\big)\right).$$

This proves (1). Consider now

$$b_p(s) = \int_{\pi/2}^{\infty} |\cos t|^s t^{-p-1} dt = \sum_{n=1}^{\infty} \int_{n\pi-\pi/2}^{n\pi+\pi/2} |\cos t|^s t^{-p-1} dt$$
$$= \int_{-\pi/2}^{\pi/2} (\cos t)^s \Big(\sum_{n=1}^{\infty} (t+n\pi)^{-p-1} \Big) dt.$$

Since $t \to \cos t$ is an even function, we have

(2ab) $$b_p(s) = \int_{-\pi/2}^{\pi/2} (\cos t)^s \varphi_p(t) dt$$

where

(2ac) $$\varphi_p(t) = \tfrac{1}{2} \sum_{n=1}^{\infty} \big((n\pi+t)^{-p-1} + (n\pi-t)^{-p-1} \big), \qquad |t| < \pi.$$

The function φ_p is even, and it is easily seen that all the even terms in its Taylor expansion around $t = 0$ are positive. Writing

$$\varphi_p(t) = \sum_{n=0}^{\infty} b_n t^{2n}, \qquad |t| < \pi,$$

where $b_n \geqslant 0$ for any n, then clearly

(2ad) $$\varphi_p(t) \leqslant b_0 + \Big(\sum_{n=1}^{\infty} b_{2n} (\pi/2)^{2n-2} \Big) t^2 \quad \text{for} \quad |t| \leqslant \pi/2.$$

Since $\varphi_p(0) = b_0$ and $\varphi_p(\pi/2) = b_0 + \sum_{n=1}^{\infty} b_{2n}(\pi/2)^{2n}$, we have

(2ae) $$\sum_{n=1}^{\infty} b_{2n} (\pi/2)^{2n-2} = (2/\pi)^2 \big(\varphi_p(\pi/2) - \varphi_p(0) \big).$$

From (2ac) one has $\varphi_p(0) = \sum_{n=1}^{\infty} (n\pi)^{-p-1} = \pi^{-p-1}\zeta(p+1)$. Similarly

$$\varphi(\pi/2) = (2/\pi)^{p+1} \Big(\tfrac{1}{2} \sum_{n=1}^{\infty} (2n+1)^{-p-1} + \tfrac{1}{2} \sum_{n=1}^{\infty} (2n-1)^{-p-1} \Big)$$
$$= (2/\pi)^{p+1} \Big(\sum_{n=1}^{\infty} (2n-1)^{-p-1} - \tfrac{1}{2} \Big)$$
$$= (2/\pi)^{p+1} \big((1-2^{-p-1}) \zeta(p+1) - \tfrac{1}{2} \big).$$

Since $\varphi(0) = (2/\pi)^{p+1} 2^{-p-1} \zeta(p+1)$, we get

$$(2/\pi)^2 \big(\varphi(\pi/2) - \varphi(0) \big) = (2/\pi)^{p+3} \big((1-2^{-p}) \zeta(p+1) - \tfrac{1}{2} \big).$$

Hence by (2ad) and (2ae)

(2af) $$\varphi_p(t) \leqslant k_1^{(p)} + k_2^{(p)} t^2, \qquad |t| \leqslant \pi/2,$$

where

$$k_1^{(p)} = \pi^{-p-1} \zeta(p+1)$$

and

$$k_2^{(p)} = (2/\pi)^{p+3} \big((1-2^{-p}) \zeta(p+1) - \tfrac{1}{2} \big).$$

Since $(\cos t)^s \leqslant \exp(-st^2/2)$, $|t| < \pi/2$, we have

$$\int_{-\pi/2}^{\pi/2} (\cos t)^s dt \leqslant \int_{-\infty}^{\infty} \exp(-st^2/2) dt = \sqrt{2\pi} s^{-1/2}$$

and

$$\int_{-\pi/2}^{\pi/2} (\cos t)^s t^2 dt \leqslant \int_{-\infty}^{\infty} t^2 \exp(-st^2/2) dt = \sqrt{2\pi} s^{-3/2}.$$

Therefore

$$b_p(s) = \int_{-\pi/2}^{\pi/2} (\cos t)^s \varphi_p(t) dt \leqslant \int_{-\pi/2}^{\pi/2} (\cos t)^s (k_1^{(p)} + k_2^{(p)} t^2) dt$$
$$\leqslant \sqrt{2\pi} (k_1^{(p)} s^{-1/2} + k_2^{(p)} s^{-3/2}).$$

This proves (2).

Lemma 2.11. *Let $s \geqslant 2$. Then $F_{p_0}(s) \geqslant F_{p_0}(\infty)$.*

Proof. It remains to prove that $F_{p_0}(s) \geqslant F_{p_0}(\infty)$ for $s > 4$. By formula (2aa) it is equivalent to prove that $a_p(s) \geqslant b_p(s)$ for any $s > 4$.

Assume first that $s \geqslant 5$. By Lemma 2.5 it suffices to prove that for $s \geqslant 5$,

(2ag) $$(s/2)^{p_0/2} \Big(\frac{1}{6s} \Gamma\big((4-p_0)/2\big) - \frac{1}{36s^2} \Gamma\big((8-p_0)/2\big) \Big)$$
$$> \sqrt{2\pi} (k_1^{(p_0)} s^{-1/2} + k_2^{(p_0)} s^{-3/2}).$$

If one multiplies (2ag) with $s^{1/2}$, one gets an increasing function on the left side and a decreasing function on the right side. Hence it is enough to check (2ag) for $s = 5$. From the table [5], p. 726:

(2ah) $$\Gamma\big((4-p_0)/2\big) = \Gamma(1{,}07629) = 0{,}96133,$$

(2ai) $$\Gamma\big((8-p_0)/2\big) = \big((6-p_0)/2\big)\big((4-p_0)/2\big)\Gamma\big((4-p_0)/2\big) = 2{,}1483$$

(correct to the number of figures indicated). Using [5], p. 609, we have $\zeta(2) = 1{,}6449$ and $\zeta(3) = 1{,}2021$. Hence by the convexity of the ζ-function:

$$\zeta(p_0+1) \leqslant \zeta(2)(2-p_0) + \zeta(3)(p_0-1) = 1{,}2696.$$

Inserting this estimate in the expressions for $k_1^{(p)}$ and $k_2^{(p)}$ we get:

$$k_1^{(p_0)} \leqslant 0{,}04876, \tag{2aj}$$

$$k_2^{(p_0)} \leqslant 0{,}04669. \tag{2ak}$$

Using the above estimates, the left side of (2ag) has for $s = 5$ the value 0,06914, while the right side is at most 0,06513 when $s = 5$. This proves $a_p(s) > b_p(s)$ for $s \geqslant 5$. It turns out that (2ag) fails for $s = 4$. Therefore we shall need a more careful estimate of $b_{p_0}(s)$ in the case $s \in [4, 5[$.

For convenience, put

$$a_p^*(s) = (s/2)^{p/2}\left(\frac{1}{6s}\Gamma\big((4-p)/2\big) - \frac{1}{36s^2}\Gamma\big((8-p)/2\big)\right).$$

Clearly

$$\frac{d}{ds}a_p^*(s) = (s/2)^{p/2}\left(-\frac{2-p}{12s^2}\Gamma\big((4-p)/2\big) + \frac{4-p}{72s^3}\Gamma\big((8-p)/2\big)\right).$$

Hence $\frac{d}{ds}a_p^*(s) > 0$ for

$$s < \frac{4-p}{6(2-p)}\,\frac{\Gamma\big((8-p)/2\big)}{\Gamma\big((4-p)/2\big)} = \frac{(4-p)^2(6-p)}{24(2-p)}. \tag{2al}$$

When $p = p_0$, the right side of (2al) is 5,2544. Hence $a_{p_0}^*(s)$ is increasing for $s \leqslant 5$. However, it is trivial that $s \to b_p(s)$ is decreasing. Thus, if we can prove that $a_{p_0}^*(4) > b_{p_0}(4)$, it will follow that $a_{p_0}(s) \geqslant a_{p_0}^*(s) > b_{p_0}(s)$ for $s \in [4, 5[$. By (2ab) and (2af):

$$b_{p_0}(4) \leqslant \int_{-\pi/2}^{\pi/2} \cos^4 t\,(k_1^{(p_0)} + k_2^{(p_0)} t^2)\,dt.$$

Using $\cos^4 t = \frac{3}{8} + \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t$, one gets

$$\int_{-\pi/2}^{\pi/2} \cos^4 t\,dt = \tfrac{3}{8}\pi \quad\text{and}\quad \int_{-\pi/2}^{\pi/2} \cos^4 t\; t^2\,dt = \tfrac{1}{64}(2\pi^3 - 15\pi).$$

Hence using the estimates (2ah)–(2ak), we have

$$b_{p_0}(4) \leqslant \tfrac{3}{8}\pi k_1^{(p_0)} + \tfrac{1}{64}(2\pi^3 - 15\pi)\,k_2^{(p_0)} = 0{,}06831$$

while $a_{p_0}^*(4) = 0{,}06891$. This proves $a_{p_0}^*(4) > b_{p_0}(4)$, and hence $a_{p_0}(s) > b_{p_0}(s)$ also for $s \in [4, 5[$.

Lemma 2.12. *Let $a_p(s)$ and $b_p(s)$ be the functions defined before Lemma 2.10. Then for $s \geqslant 2$,*

(1) *$p \to (\pi/2)^p a_p(s)$ is an increasing function of $p \in\,]0, 2[$;*

(2) *$p \to (\pi/2)^p b_p(s)$ is a decreasing function of $p \in\,]0, 2[$.*

Proof. (2): We have

$$\left(\frac{\pi}{2}\right)^p b_p(s) = \frac{2}{\pi}\int_{\pi/2}^{\infty} |\cos t|^s (\pi/2t)^{p+1}\,dt.$$

For $t > \pi/2$, $(\pi/2t)^{p+1}$ is a decreasing function of p. This proves (2). Clearly

$$(\pi/2)^p a_p(s) = \frac{2}{\pi}\int_0^{\infty} \big(\exp(-st^2/2) - |\cos t|^s \chi_{[0,\pi/2]}(t)\big)(\pi/2t)^{p+1}\,dt.$$

Hence

$$\frac{d}{dp}\big((\pi/2)^p a_p(s)\big) = \frac{2}{\pi}\int_0^{\infty} \big(\exp(-st^2/2) - |\cos t|^s \chi_{[0,\pi/2]}(t)\big)(\pi/2t)^{p+1}\log(\pi/2t)\,dt.$$

Since for $p > 0$ $x^{p+1} > x$, when $x > 1$ and $x^{p+1} < x$, when $0 < x < 1$, we have $x^{p+1}\log x \geqslant x\log x \geqslant x - 1$ for any $x > 0$. This implies that

$$\frac{d}{dp}\big((\pi/2)^p a_p(s)\big) \geqslant \frac{2}{\pi}\int_0^{\infty} \big(\exp(-st^2/2) - |\cos t|^s \chi_{[0,\pi/2]}(t)\big)(\pi/2t - 1)\,dt. \tag{2am}$$

For $t \in [0, \pi/2]$, we have by the proof of Lemma 2.10 that

$$(\cos t)^s \leqslant \exp(-st^2/2 - st^4/12).$$

Hence

$$\exp(-st^2/2) \geqslant (\cos t)^s \exp(st^4/12) \geqslant (\cos t)^s (1 + st^4/12).$$

Therefore

$$\int_0^{\pi/2} \big(\exp(-st^2/2) - (\cos t)^s\big)(\pi/2t - 1)\,dt \geqslant \int_0^{\pi/2} (\cos t)^s \frac{st^4}{12}(\pi/2t - 1)\,dt \tag{2an}$$

$$= \frac{s}{12}\int_0^{\pi/2} (\cos t)^s (\pi/2 - t)\, t^3\,dt \geqslant \frac{s}{12}\int_0^{\pi/2} (\cos t)^{s+1} \sin^3 t\,dt$$

$$= \frac{s}{12}\int_0^{\pi/2} \big((\cos t)^{s+1} - (\cos t)^{s+3}\big)\,d(\cos t)$$

$$= \frac{s}{12}\big(1/(s+2) - 1/(s+4)\big) = s/6(s+2)(s+4).$$

Moreover, using $t^2 = ((t-\pi/2)+\pi/2)^2 \geqslant \pi^2/4+\pi(t-\pi/2)$, we get

$$\text{(2ao)}\qquad \int_{\pi/2}^{\infty} \exp(-st^2/2)(1-\pi/2t)\,dt \leqslant \int_{\pi/2}^{\infty} \exp(-st^2/2)(1-\pi/2t)\frac{2t}{\pi}\,dt$$

$$= \frac{2}{\pi}\int_{\pi/2}^{\infty} \exp(-st^2/2)(t-\pi/2)\,dt$$

$$\leqslant \frac{2}{\pi}\exp(-s\pi^2/8)\int_{\pi/2}^{\infty}\exp\left(-\frac{s\pi}{2}(t-\pi/2)\right)(t-\pi/2)\,dt$$

$$= \frac{2}{\pi}\exp(-s\pi^2/8)\int_0^{\infty}\exp\left(-\frac{s\pi}{2}u\right)u\,du$$

$$= \frac{2}{\pi}(2/s\pi)^2\exp(-s\pi^2/8) = \frac{8}{s^2\pi^3}\exp(-s\pi^2/8).$$

Combining (2am), (2an) and (2ao) we get

$$\frac{d}{dp}\big((\pi/2)^p a_p(s)\big) \geqslant \frac{s}{6(s+2)(s+4)} - \frac{8}{s^2\pi^3}\exp(-s\pi^2/8)$$

$$= \frac{1}{s^2}\left(\frac{s}{6(1+2/s)(1+4/s)} - \frac{8}{\pi^3}\exp(-s\pi^2/8)\right).$$

The function in the bracket is an increasing function of s. For $s = 2$ it has the value $\frac{1}{18} - \frac{8}{\pi^3}\exp(-\pi^2/4) = 0{,}03367 > 0$. Hence $\frac{d}{dp}\big((\pi/2)^p \times$ $\times a_p(s)\big) > 0$ for $s \geqslant 0$ and $0 < p < 2$. This proves (1).

Lemma 2.13. *Let $p_0 \leqslant p < 2$. Then $F_p(s) \geqslant F_p(\infty)$ for $s \in [2, \infty[$.*

Proof. By (2aa) $F_p(s) - F_p(\infty) = C_p s^{-p/2}\big(a_p(s) - b_p(s)\big)$. Hence by Lemma 2.11, $a_{p_0}(s) - b_{p_0}(s) \geqslant 0$ for $s \geqslant 2$. By Lemma 2.12 the function

$$p \to (\pi/2)^p\big(a_p(s) - b\ \ (s)\big)$$

is increasing, when $s \geqslant 2$. Therefore $a_p(s) \geqslant b_p(s)$ for $p_0 \leqslant p < 2$ and $s \geqslant 2$. This proves the lemma.

Lemma 2.14. *Let $X = \sum_{i=1}^{n} a_i r_i$. Then $\log E(|X|^p)$ is a convex function of p.*

Proof. Let $p < q < r$. Since $|X|^q = (|X|^p)^{(r-q)/(r-p)}(|X|^r)^{(q-p)/(r-p)}$ and $(r-q)/(r-p)+(q-p)/(r-p) = 1$ one gets by the Hölder inequality that

$$\text{(2ap)}\qquad E(|X|^q) \leqslant E(|X|^p)^{(r-q)/(r-p)}E(|X|^r)^{(q-p)/(r-p)}.$$

The assertion follows now by taking the logarithm on both sides of (2ap).

Proposition 2.15. *Let $X = \sum_{i=1}^{n} a_i r_i$, where $\sum a_i^2 = 1$, and $|a_i| \leqslant 1/\sqrt{2}$ for all i. Then*

$$E(|X|^p) \geqslant \begin{cases} 2^{(p-2)/2}, & 0 < p \leqslant p_0, \\ 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}, & p_0 \leqslant p < 2. \end{cases}$$

Proof. For $p_0 \leqslant p < 2$, we have by Lemmas 2.4 and 2.13 that

$$E(|X|^p) \geqslant \sum_{i=1}^{n} a_i^2 F_p(a_i^{-2}) \geqslant \inf_{s\geqslant 2} F_p(s) = F_p(\infty) = 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}.$$

In particular, for $p = p_0$, we have

$$E(|X|^{p_0}) \geqslant 2^{(p_0-2)/2}, \quad \text{because} \quad \Gamma\big((p_0+1)/2\big) = \sqrt{\pi}/2.$$

Let now $0 < p < p_0$. Then by Lemma 2.14

$$E(|X|^{p_0}) \leqslant E(|X|^p)^{(2-p_0)/(2-p)}E(|X|^2)^{(p_0-p)/(2-p)}$$
$$= E(|X|^p)^{(2-p_0)/(2-p)}$$

or equivalently

$$E(|X|^p) \geqslant E(|X|^{p_0})^{(2-p)/(2-p_0)} \geqslant (2^{(p_0-2)/2})^{(2-p)/(2-p_0)} = 2^{(p-2)/2}.$$

This completes the proof.

§ 3. Case $0 < p < 2$ and $|a_i| > 1/\sqrt{2}$ for some i. In the case $p = 1$ a one line argument gave (see § 1) that when $X = \sum_{i=1}^{n} a_i r_i$, where $\sum a_i^2 = 1$ and $|a_i| > 1/\sqrt{2}$ for some i, then $E(|X|) > 1/\sqrt{2}$. The same argument applied to the case $1 < p < 2$ gives $E(|X|^p) > 2^{-p/2}$. However, to prove that $A_p = 2^{1/2-1/p}$ for $1 < p \leqslant p_0$, we need to prove that

$$E(|X|^p) \geqslant 2^{(p-2)/2} > 2^{-p/2}.$$

It turns out that $E(|X|^p) > 2^{(p-2)/2}$ for $0 < p < 2$ when $|a_i| > 1/\sqrt{2}$ for some i. The proof which we present in this section is not very elegant, but we have not been able to find an easier solution. We reduce the problem to prove that when $Z = \sum_{i=1}^{n} c_i r_i$ and $\sum_{i=1}^{n} c_i^2 = 1$ (no further conditions on c_i), and $b \in [0, 1]$, then

$$\text{(3a)}\qquad E\big(\psi(bZ)\big) \leqslant 0,$$

where

$$\psi(x) = \tfrac{1}{2}[(x+1)^2\log|x+1| + (x-1)^2\log|x-1|] - (\log 2 - \tfrac{1}{2}) - (\log 2 + \tfrac{1}{2})x^2.$$

Inequality (3a) is proved by constructing a polynomial Q, such that $\psi(x) \leqslant Q(x)$, $x \in \mathbf{R}$, and such that $E\big(Q(bZ)\big) \leqslant 0$ for all $b \in [0, 1]$, and all Z

of the above form. The latter inequality is verified by expressing $E(Z^{2p})$, $p \in \boldsymbol{N}$ in terms of the numbers $\varrho_r = \sum_{i=1} c_i^{2r}$, $r = 1, 2, \ldots, p$. After some investigation of the problem we found that the degree of Q has to be at least ten, and after some more work, we found that

$$Q(x) = \tfrac{1}{5676} \log \tfrac{27}{16} (x^2-1)^2 (x^6 - 28x^4 + 335x^2 - 958)$$

can be used.

LEMMA 3.1. *Put*

$$\text{(3b)} \qquad \varphi(x) = \begin{cases} \tfrac{1}{2}[(x+1)^2 \log|x+1| + (x-1)^2 \log|x-1|], & |x| \neq 1, \\ 2\log 2 & |x| = 1. \end{cases}$$

Then there exists a constant $K > 0$ such that $\forall x \in \boldsymbol{R}$

$$\text{(3c)} \quad \varphi(x) \leqslant (\log 2 - \tfrac{1}{2}) + (\log 2 + \tfrac{1}{2}) x^2 + K(x^2-1)^2 (x^6 - 28x^4 + 335x^2 - 958).$$

Proof. Put

$$\text{(3d)} \qquad \psi(x) = \varphi(x) - (\log 2 - \tfrac{1}{2}) - (\log 2 + \tfrac{1}{2}) x^2$$

and

$$P(x) = (x^2-1)^2 (x^6 - 28x^4 + 335x^2 - 958) = x^{10} - 30x^8 + 392x^6 - 1656x^4 + 2251x^2 - 958.$$

It is easily seen that ψ is differentiable, and

$$\text{(3e)} \qquad \psi'(x) = (x+1)\log|x+1| + (x-1)\log|x-1| - (2\log 2)x$$

for $|x| \neq 1$ and $\psi'(1) = \psi'(-1) = 0$. Moreover, ψ is twice differentiable except at $x = 1$ and $x = -1$,

$$\text{(3f)} \qquad \psi''(x) = \log|x^2-1| + 2 - 2\log 2, \qquad |x| \neq 1$$

and

$$\text{(3g)} \qquad \lim_{x\to 1} \psi''(x) = \lim_{x \to -1} \psi''(x) = -\infty.$$

However, ψ'' is integrable and

$$\int_a^b \psi''(x)\,dx = \psi'(b) - \psi'(a), \qquad a, b \in \boldsymbol{R},\ a < b.$$

also if 1 or -1 is contained in $[a, b]$. We now choose

$$K = \psi'(2)/P'(2) = \frac{3\log 3 - 4\log 2}{5676} = \tfrac{1}{5676} \log(27/16) > 0.$$

Put $\lambda(x) = KP(x) - \psi(x)$. We shall prove that $\lambda(x) \geqslant 0$ for $x \in \boldsymbol{R}$. Since $\lambda(x) = \lambda(-x)$, it is enough to consider $x \geqslant 0$. We need the following facts which follow by direct computation:

$$\text{(3h)} \qquad \lambda(0) > 0, \quad \lambda(1) = 0, \quad \lambda(2) > 0,$$

$$\text{(3i)} \qquad \lambda'(0) = \lambda'(1) = \lambda'(2) = 0,$$

$$\text{(3j)} \qquad \lambda''(0) < 0, \quad \lim_{t\to 1} \lambda''(t) = +\infty, \quad \lambda''(2) > 0.$$

Indeed,

$$\lambda(0) = -958 \cdot K + \log 2 - \tfrac{1}{2} = 0{,}1048 > 0,$$

$$\lambda(1) = 0 \ \text{(trivial)},$$

$$\lambda(2) = -18K - \tfrac{9}{2}\log 3 + 5\log 2 + \tfrac{3}{2} = 0{,}0203 > 0.$$

Clearly $\lambda'(0) = \lambda'(1) = 0$, and $\lambda'(2) = 0$ by the choice of K. Using

$$\text{(3k)} \qquad P''(x) = 90x^8 - 1680x^6 + 11760x^4 - 19872x^2 - 4502,$$

we get

$$\lambda''(0) = 4502 \cdot K - 2 + 2\log 2 = -0{,}1985 < 0$$

and

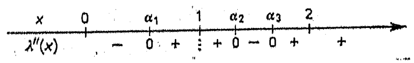
$$\lambda''(2) = 28694 \cdot K - \log 3 - 2 + \log 2 = 0{,}2407 > 0.$$

The last statement in (3j) follows from (3g). Using (3f) and (3k) we have

$$\frac{d^2}{dt^2} \lambda''(\sqrt{t}) = K \cdot 120(3t-14)^2 + 1/(t-1)^2 > 0 \qquad \text{for} \quad t > 0,\ t \neq 1.$$

Hence the function $t \to \lambda''(\sqrt{t})$ is strictly convex for $t \in]0, 1[$ and $t \in]1, \infty[$. Therefore the equation $\lambda''(t) = 0$ can have at most two solutions in each of the intervals $]0, 1[$ and $]1, \infty[$. However, by (3j) there is an odd number of roots for λ'' for $0 < t < 1$, and an even number of roots (counted with multiplicity) for $1 < t < 2$.

Therefore λ'' has precisely one root α_1 between 0 and 1. Assume that λ'' has no roots or a double root between 1 and 2. Then again by (3j), $\lambda''(t) \geqslant 0$ for $t \in]1, 2[$ except possibly at one point. Hence $\lambda'(2) - \lambda'(1) = \int_1^2 \lambda''(t)\,dt > 0$, which contradicts (3i). Therefore λ'' has precisely two distinct roots $\alpha_2, \alpha_3, \alpha_2 < \alpha_3$ between 1 and 2, and as λ'' has at most two roots in $]1, \infty[$, there are no roots in $[2, \infty[$. Hence we have the following signs of λ'':



Since $\lambda'(0) = \lambda'(1) = \lambda'(2) = 0$, it follows from (31) that $\lambda'(x) < 0$ for $x \in]0, 1[$. Moreover, λ' can have only one root $\beta \in]1, 2[$.

Hence $\lambda'(x)$ must have the following sign variation

Therefore the minimum of $\lambda(x)$, $x \geqslant 0$ is attained either in $x = 1$ or $x = 2$. As $\lambda(1) = 0$, and $\lambda(2) > 0$ by (3h) we have proved that $\lambda(x) \geqslant 0$ for $x \in [0, \infty[$. This completes the proof of Lemma 3.1.

Remark 3.2. Since

$$\frac{d}{dx}(x^6 - 28x^4 + 335x^2 - 958) = 2x(3x^4 - 56x^2 + 335) > 0$$

for $x > 0$, the polynomial $R(x) = x^6 - 28x^4 + 335x^2 - 958$ is strictly increasing for $x \geqslant 0$. Since $R(2) = -2$, we have $R(x) \leqslant 0$ for $x \in [0, 2]$. Hence also $P(x) \leqslant 0$ for $x \in [0, 2]$. It is not hard to check that the positive root of $R(x)$ is contained in the interval $2 < q < 2{,}01$. Figure 3.1 below shows how the polynomial $KP(x)$ "fits" with the function $\psi(x)$.

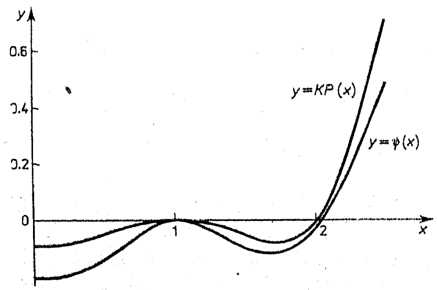


Figure 3.1

LEMMA 3.3 *Let* $Z = \sum_{i=1}^{n} a_i r_i$, *where* $\sum_{i=1}^{n} a_i^2 = 1$, *and put* $\varrho_q = \sum_{i=1}^{n} a_i^{2q}$, $q \in N$. *Then*

(1) $E(Z^2) = 1$,

(2) $E(Z^4) = 3 - 2\varrho_2$,

(3) $E(Z^6) = 15 - 30\varrho_2 + 16\varrho_3$,

(4) $E(Z^8) = 105 - 420\varrho_2 + (448\varrho_3 + 140\varrho_2^2) - 272\varrho_4$,

(5) $E(Z^{10}) = 945 - 6300\varrho_2 + (10080\varrho_3 + 6300\varrho_2^2) - (12240\varrho_4 + 6720\varrho_2\varrho_3) + 7936\varrho_5$.

Proof. (1) is trivial. (2): Using that the Rademacher functions are independent random variables, one gets easily that

$$E(Z^4) = \sum_{k=1}^{n} a_k^4 + 6 \sum_{k<l} a_k^2 a_l^2$$
$$= 3\left(\sum a_k^2\right)^2 - 2\sum a_k^4 = 3 - 2\varrho_2.$$

(3)–(5) can in principle be proved in the same way. However, for (4) and (5), the computations become so involved that an alternative method is necessary. Let $\varphi_Z(t) = \prod_{i=1}^{n} \cos a_i t$ be the characteristic function for Z. We will prove (3)–(5) by computing the Taylor series for $\varphi_Z(t)$ in two ways. Put $\beta_p = E(Z^{2p})$, $p \in N$. By differentiating the expression

$$\varphi_Z(t) = E(e^{itZ})$$

$2p$-times, one obtains

$$\beta_p = E(Z^{2p}) = (-1)^p \left[\frac{d^{2p}}{dt^{2p}} \varphi_Z(t)\right]_{t=0}.$$

Hence

$$\varphi_Z(t) = \sum_{p=0}^{\infty} (-1)^p \beta_p \frac{t^{2p}}{(2p)!}, \qquad t \in \mathbf{R}. \tag{3m}$$

Let α_p be the coefficients in the Taylor expression

$$\operatorname{tg} t = \sum_{p=1}^{\infty} \alpha_p \frac{t^{2p-1}}{(2p-1)!}, \qquad |t| < \pi/2.$$

From [5], p. 603, we get

(3n) $\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 16, \quad \alpha_4 = 272, \quad \alpha_5 = 7936, \ldots$

We have

$$\log(\cos t) = -\int_0^t \operatorname{tg} u\, du = -\sum_{p=1}^{\infty} \alpha_p \frac{t^{2p}}{(2p)!}, \qquad |t| < \pi/2.$$

Since $|a_i| \leqslant 1$, $i = 1, \ldots, n$, we get for $|t| < \pi/2$

$$\log \varphi_Z(t) = \sum_{i=1}^{n} \log(\cos a_i t)$$
$$= -\sum_{i=1}^{n} \left(\sum_{p=1}^{\infty} \alpha_p \frac{(a_i t)^{2p}}{(2p)!}\right) = -\sum_{p=1}^{\infty} \alpha_p \varrho_p \frac{t^{2p}}{(2p)!}.$$

Hence

$$\varphi_Z(t) = \exp\big(\log\varphi_Z(t)\big)$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Big(\sum_{p=1}^{\infty}a_p\varrho_p\frac{t^{2p}}{(2p)!}\Big)^k$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{p_1,\ldots,p_k}\frac{1}{(2p_1)!\ldots(2p_k)!}\Big(\prod_{j=1}^{k}a_{p_j}\varrho_{p_j}\Big)t^{2(\sum_{j=1}^{k}p_j)}.$$

Comparing this formula with (3m) we see that

$$\text{(3o)}\qquad \mathrm{E}(Z^{2p}) = \beta_p$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^{p-k}}{k!}\Big(\sum_{\substack{p_1+\ldots+p_k=p\\ p_i\geqslant 1}}\frac{(2p)!}{(2p_1)!\ldots(2p_k)!}\prod_{j=1}^{k}a_{p_j}\varrho_{p_j}\Big).$$

In the first sum one can take upper limit $k = p$, because $p_1+\ldots+p_k = p$ implies $k\leqslant p$. Inserting the values for a_p from (3n) in (3o) a tedious, but straightforward computation, gives the formulas (1)–(5). This finishes the proof.

LEMMA 3.4. *With the notation of Lemma 3.3,*

$$\text{(1)}\qquad \varrho_5\leqslant\varrho_4\leqslant\varrho_3\leqslant\varrho_2\leqslant\varrho_1 = 1,$$

$$\text{(2)}\qquad \varrho_2^2\geqslant\varrho_4,$$

$$\text{(3)}\qquad \varrho_2-5\varrho_3+8\varrho_4-4\varrho_5\geqslant 0,$$

$$\text{(4)}\qquad \varrho_2\varrho_3\geqslant\varrho_2^2-\varrho_3+\varrho_4.$$

Proof. (1) and (2) are trivial. (3) follows from

$$\varrho_2-5\varrho_3+8\varrho_4-4\varrho_5 = \sum_{i=1}^{n}a_i^4(1-a_i^2)(1-2a_i^2)^2\geqslant 0.$$

(4): Using $\sum_{i=1}^{n}a_i^2 = 1$, we get

$$\varrho_3-\varrho_4-\varrho_2(\varrho_2-\varrho_3) = \sum_{i,j}\big(a_i^2(a_j^6-a_j^8)-a_i^4(a_j^4-a_j^6)\big)$$

$$= \sum_{i,j}a_i^2(a_j^4-a_j^6)(a_j^2-a_i^2).$$

Since the diagonal terms in the double sum vanish, we can split the sum over $i<j$ and $i>j$. By intertwining the indices in the last sum, and add it to the first, we get

$$\varrho_3-\varrho_4-\varrho_2(\varrho_2-\varrho_3) = \sum_{i<j}[a_i^2(a_j^4-a_j^6)-a_j^2(a_i^4-a_i^6)](a_j^2-a_i^2)$$

$$= \sum_{i<j}a_i^2a_j^2(a_j^2-a_i^2)^2(1-a_i^2-a_j^2)\geqslant 0$$

because $1-a_i^2-a_j^2 = \sum_{k\neq i,j}a_k^2\geqslant 0$, when $i\neq j$. This proves (4).

LEMMA 3.5. *Let* $Z = \sum_{i=1}^{n}a_ir_i$, *where* $\sum_{i=1}^{n}a_i^2 = 1$, *let* $b\in[0,1]$, *and let* φ *be the function defined in Lemma 3.1. Then*

$$E\big(\varphi(bZ)\big)\leqslant(\log 2-\tfrac{1}{2})+(\log 2+\tfrac{1}{2})b^2.$$

Proof. From Lemma 3.1 we have

$$\text{(3p)}\qquad \varphi(x)\leqslant(\log 2-\tfrac{1}{2})+(\log 2+\tfrac{1}{2})x^2+KP(x),$$

where $K>0$, and

$$\text{(3q)}\qquad P(x) = (x^2-1)^2(x^6-28x^4+335x^2-958)$$

$$= x^{10}-30x^8+392x^6-1656x^4+2251x^2-958.$$

Hence, using $E\big((bZ)^2\big) = b^2$, we get by (3p)

$$\text{(3r)}\qquad E\big(\varphi(bZ)\big)\leqslant(\log 2-\tfrac{1}{2})+(\log 2+\tfrac{1}{2})b^2+KE\big(P(bZ)\big).$$

Thus it remains to prove that $E\big(P(bZ)\big)\leqslant 0$. Putting $\beta = b^2$, we get by (3q)

$$\text{(3s)}\qquad E\big(P(bZ)\big) = \beta^5E(Z^{10})-30\beta^4E(Z^8)+392\beta^3E(Z^6)-$$

$$-1656\beta^2E(Z^4)+2251\beta E(Z^2)-958.$$

Inserting Lemma 3.3 (1)–(5) in (3s), we obtain

$$\text{(3t)}\qquad E\big(P(bZ)\big) = Q_1(\beta)+Q_2(\beta)\varrho_2+Q_3(\beta)\varrho_3+Q_4(\beta)\varrho_4+$$

$$+Q_5(\beta)\varrho_5+Q_6(\beta)\varrho_2^2+Q_7(\beta)\varrho_2\varrho_3,$$

where

$$Q_1(\beta) = 945\beta^5-3150\beta^4+5880\beta^3-4968\beta^2+2251\beta-958,$$

$$Q_2(\beta) = -6300\beta^5+12600\beta^4-11760\beta^3+3312\beta^2,$$

$$Q_3(\beta) = 10080\beta^5-13440\beta^4+6272\beta^3,$$

$$Q_4(\beta) = -12240\beta^5+8160\beta^4,$$

$$Q_5(\beta) = 7936\beta^5,$$

$$Q_6(\beta) = 6300\beta^5-4200\beta^4,$$

$$Q_7(\beta) = -6720\beta^5.$$

Since $Q_7(\beta) \leqslant 0$ for $\beta \in [0,1]$, we get by Lemma 3.4(4) that

$$Q_7(\beta)\varrho_2\varrho_3 \leqslant Q_7(\beta)(\varrho_2^2 - \varrho_3 + \varrho_4).$$

Hence by (3t):

$$\text{(3u)} \qquad E(P(bZ)) \leqslant Q_1(\beta) + Q_2(\beta)\varrho_2 + [Q_3(\beta) - Q_7(\beta)]\varrho_3 + \\ + [Q_4(\beta) + Q_7(\beta)]\varrho_4 + Q_5(\beta)\varrho_5 + [Q_6(\beta) + Q_7(\beta)]\varrho_2^2.$$

However, $Q_6(\beta) + Q_7(\beta) = -420\beta^5 - 4200\beta^4 \leqslant 0$ for $\beta \in [0,1]$. Using $\varrho_2^2 \geqslant \varrho_4$, we get

$$[Q_6(\beta) + Q_7(\beta)]\varrho_2^2 \leqslant [Q_6(\beta) + Q_7(\beta)]\varrho_4.$$

Therefore by (3u)

$$\text{(3v)} \qquad E(P(bZ)) \leqslant Q_1(\beta) + Q_2(\beta)\varrho_2 + Q_3^*(\beta)\varrho_3 + Q_4^*(\beta)\varrho_4 + Q_5(\beta)\varrho_5,$$

where

$$Q_3^*(\beta) = Q_3(\beta) - Q_7(\beta) = 16800\beta^5 - 13440\beta^4 + 6272\beta^3,$$
$$Q_4^*(\beta) = Q_4(\beta) + Q_6(\beta) + 2Q_7(\beta) = -19380\beta^5 + 3960\beta^4.$$

By Lemma 3.4(3), $1984\beta^5(\varrho_2 - 5\varrho_3 + 8\varrho_4 - 4\varrho_5)$ is non-negative. If we add this term to the right side of (3v), we get

$$\text{(3w)} \qquad E(P(bZ)) \leqslant Q_1(\beta) + Q_2^{**}(\beta)\varrho_2 + Q_3^{**}(\beta)\varrho_3 + Q_4^{**}(\beta)\varrho_4,$$

where

$$Q_2^{**}(\beta) = -4316\beta^5 + 12600\beta^4 - 11760\beta^3 + 3312\beta^2,$$
$$Q_3^{**}(\beta) = 6880\beta^5 - 13440\beta^4 + 6272\beta^3,$$
$$Q_4^{**}(\beta) = -3508\beta^5 + 3960\beta^4.$$

Clearly $Q_4^{**}(\beta) \geqslant 0$ for $\beta \in [0,1]$. Since $\varrho_4 \leqslant \varrho_3$, we get from (3w):

$$\text{(3x)} \qquad E(P(bZ)) \leqslant Q_1(\beta) + Q_2^{**}(\beta)\varrho_2 + (Q_3^{**}(\beta) + Q_4^{**}(\beta))\varrho_3.$$

However, $Q_3^{**}(\beta) + Q_4^{**}(\beta) = 3372\beta^5 - 9480\beta^4 + 6272\beta^3$ is non-negative for $\beta \in [0,1]$, because the two roots of $3372\beta^2 - 9480\beta + 6272$ are both greater than 1. Since $\varrho_3 \leqslant \varrho_2$, we get from (3x):

$$\text{(3y)} \qquad E(P(bZ)) \leqslant Q_1(\beta) + (Q_2^{**}(\beta) + Q_3^{**}(\beta) + Q_4^{**}(\beta))\varrho_2.$$

Moreover,

$$Q_2^{**}(\beta) + Q_3^{**}(\beta) + Q_4^{**}(\beta) = -944\beta^5 + 3120\beta^4 - 5488\beta^3 + 3312\beta^2$$
$$= \beta^2(1-\beta)(944\beta^2 - 2176\beta + 3312) \geqslant 0$$

for $\beta \in [0,1]$, because $944\beta^2 - 2176\beta + 3312$ has negative discriminant.

Hence using $\varrho_2 \leqslant 1$ we get from (3y):

$$\text{(3z)} \qquad E(P(bZ)) \leqslant Q_1(\beta) + [Q_2^{**}(\beta) + Q_3^{**}(\beta) + Q_4^{**}(\beta)]$$
$$= \beta^5 - 30\beta^4 + 392\beta^3 - 1656\beta^2 + 2251\beta - 958.$$

By (3q), the right side of (3z) is just $P(b)$, because $\beta = b^2$. Hence for $b \in [0,1]$,

$$E(P(bZ)) \leqslant P(b) \leqslant 0 \qquad \text{(cf. Remark 3.2).}$$

This proves the lemma.

LEMMA 3.6. *Let* $X = \sum_{i=1}^{n} a_i r_i$, *such that* $\sum_{i=1}^{n} a_i^2 = 1$, *and* $|a_i| > 1/\sqrt{2}$ *for some* i. *Then*

$$E(X^2 \log|X|) \leqslant \tfrac{1}{2}\log 2$$

(*we put* $t^2\log|t| = 0$ *for* $t = 0$).

Proof. We may assume that $|a_1| > 1/\sqrt{2}$. Put $Y = (1/a_1)X$. Then

$$\text{(3aa)} \qquad Y = r_1 + \sum_{k=2}^{n} b_k r_k,$$

where $b_k = a_k/a_1$. Put $b = (\sum_{k=2}^{n} b_k^2)^{1/2}$. Then

$$b^2 = a_1^{-2}\Big(\sum_{k=2}^{n} a_k^2\Big) = a_1^{-2}(1 - a_1^2) = a_1^{-2} - 1 < 1.$$

It is no loss of generality to assume $b > 0$ because $b = 0$ would imply that $X = r_1$ and hence $E(X^2\log|X|) = 0 < \frac{1}{2}\log 2$. Put now

$$\text{(3ab)} \qquad Z = \frac{1}{b}\sum_{k=2}^{n} b_k r_k = \sum_{k=2}^{n} c_k r_k,$$

where $c_k = b_k/b$, and $\sum_{k=2}^{n} c_k^2 = 1$. Then $Y = r_1 + bZ$. Since r_1 and Z are independent random variables, and since r_1 takes the values $+1$ and -1 each with probability $\frac{1}{2}$, one gets

$$\text{(3ac)} \qquad E(Y^2\log|Y|) = \tfrac{1}{2}E\big((bZ+1)^2\log|bZ+1| + (bZ-1)^2\log|bZ-1|\big)$$
$$= E(\varphi(bZ))$$

where φ is the function defined in Lemma 3.1. Since $a_1^{-2} = 1 + b^2$, we have

$$\text{(3ad)} \qquad Y^2\log|Y| = a_1^{-2}X^2\log|a_1^{-1}X|$$
$$= a_1^{-2}X^2\log|X| + a_1^{-2}\log|a_1^{-1}|X^2$$
$$= (1+b^2)X^2\log|X| + \tfrac{1}{2}(1+b^2)\log(1+b^2)X^2.$$

Since $E(X^2) = 1$ we get from (3ac) and (3ad) that

$$(1+b^2)E(X^2\log|X|) = E(Y^2\log|Y|) - \tfrac{1}{2}(1+b^2)\log(1+b^2)$$
$$= E(\varphi(bZ)) - \tfrac{1}{2}(1+b^2)\log(1+b^2).$$

Hence, using Lemma 3.5,

$$(1+b^2)E(X^2\log|X|)$$
$$\leqslant (1+b^2)\log 2 - \tfrac{1}{2}(1-b^2) - \tfrac{1}{2}(1+b^2)\log(1+b^2)$$
$$\leqslant (1+b^2)/2\log 2 - \big(1-(1+b^2)/2\big) - (1+b^2)/2\log\big((1+b^2)/2\big).$$

Using that $t\log t \geqslant t-1$ for any $t>0$, one gets

$$(1+b^2)E(X^2\log|X|) \leqslant \tfrac{1}{2}(1+b^2)\log 2,$$

or equivalently

$$E(X^2\log|X|) \leqslant \tfrac{1}{2}\log 2.$$

Proof of Theorem A. By the homogeneity of the Khintchine inequality, it suffices to consider the case $\sum_{i=1}^{n} a_i^2 = 1$. Put $X = \sum_{i=1}^{n} a_i r_i$. If $|a_i| \leqslant 1/\sqrt{2}$ for all i, we have from Proposition 2.15 that

$$E(|X|^p) \geqslant \begin{cases} 2^{(p-2)/2} & 0<p\leqslant p_0, \\ 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2} & p_0<p<2. \end{cases} \tag{3ae}$$

If $|a_i| > 1/\sqrt{2}$ for some i, we have from Lemma 3.6 that

$$\left[\frac{d}{dp}\log E(|X|^p)\right]_{p=2} = \left[\frac{d}{dp}E(|X|^p)\right]_{p=2}$$
$$= E(X^2\log|X|) \leqslant \tfrac{1}{2}\log 2.$$

Since the function $p \to \log E(|X|^p)$ is convex (Lemma 2.14), we have for $0<p\leqslant 2$ that

$$\log E(|X|^p) \geqslant \log(E|X|^2) + (p-2)\left[\frac{d}{dp}\log E(|X|^p)\right]_{p=2}$$
$$= -(2-p)E(X^2\log|X|).$$
$$\geqslant \tfrac{1}{2}(p-2)\log 2.$$

Hence $E(|X|^p) \geqslant 2^{(p-2)/2}$, $0<p\leqslant 2$.

Since $\Gamma\big((p+1)/2\big) \leqslant \sqrt{\pi}/2$ for $p_0 \leqslant p \leqslant 2$ (cf. the proof of Lemma 2.1), it follows that

$$E(|X|^p) \geqslant 2^{(p-2)/2} \geqslant 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}, \qquad p_0 \leqslant p \leqslant 2.$$

Hence (3ae) is satisfied also if $|a_i| > 1/\sqrt{2}$ for some i. This together with Lemma 2.1 shows that

$$A_p^p = \begin{cases} 2^{(p-2)/2}, & 0<p\leqslant p_0, \\ 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}, & p_0<p\leqslant 2. \end{cases}$$

Taking the pth root, we get Theorem A.

§ 4. Case $2 \leqslant p < 4$. In Sections 4 and 5 we shall prove:

THEOREM B. *Let $p \geqslant 2$. The best constant B_p in the Khintchine inequality is given by*

$$B_p = 2^{1/2}\big(\Gamma\big((p+1)/2\big)/\sqrt{\pi}\big)^{1/p}.$$

This section takes care of the case $2 \leqslant p < 4$.

LEMMA 4.1 (Stečkin [10]). *Let $p \geqslant 2$. Then $B_p \geqslant 2^{1/2}\big(\Gamma\big((p+1)/2\big)/\sqrt{\pi}\big)^{1/p}$.*

Proof. Let Z be a normal distributed random variable with density $\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$. By the proof of Lemma 2.1 (cf. formula (2b)), we have

$$\liminf_{n\to\infty} E\left(\left|\frac{r_1+\ldots+r_n}{\sqrt{n}}\right|^p\right) \geqslant E(|Z|^p) = 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}$$

from which it follows that $B_p^p \geqslant 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}$. As in § 2 we put

$$C_p = \frac{2}{\pi}\sin\frac{p\pi}{2}\,\Gamma(p+1), \qquad p>0.$$

Note that $C_p < 0$ for $2<p<4$.

LEMMA 4.2. *Let X be a real, symmetric random variable that satisfies $E(X^4) < \infty$. For $2<p<4$,*

$$E(|X|^p) = (-C_p)\int_0^\infty \big(\varphi_X(t) - 1 + \tfrac{1}{2}E(X^2)t^2\big)t^{-p-1}\,dt, \tag{4a}$$

where $\varphi_X(t) = E(e^{iXt})$ is the characteristic function for X.

Proof. We show first that for $x \in \mathbf{R}$ and $2<p<4$,

$$|x|^p = (-C_p)\int_0^\infty (\cos xt - 1 + \tfrac{1}{2}x^2t^2)\,t^{-p-1}\,dt. \tag{4b}$$

Note that $\cos xt - 1 + \frac{1}{2}x^2t^2 \geqslant 0$ for any t. Moreover, $\cos xt - 1 + \frac{1}{2}x^2t^2 = O(t^4)$ for $t \to 0$ and $O(t^2)$ for $t\to\infty$, so the integral in (4b) is bounded for $2<p<4$. For $x \neq 0$, the substitution $u = |x|t$ shows that

$$\int_0^\infty (\cos xt - 1 + \tfrac{1}{2}x^2t^2)\,t^{-p-1}\,dt = |x|^p\int_0^\infty (\cos u - 1 + \tfrac{1}{2}u^2)\,u^{-p-1}\,du.$$

By repeated partial integration, we get

$$\int_0^\infty (\cos u - 1 + \tfrac{1}{2}u^2)u^{-p-1}\,du = \frac{1}{p}\int_0^\infty (u - \sin u)u^{-p}\,du$$

$$= \frac{1}{p(p-1)}\int_0^\infty (1-\cos u)u^{-p+1}\,du.$$

Since $0 < p-2 < 2$, we get from the proof of Lemma 2.3 (cf. formula (2f)) that

$$\frac{1}{p(p-1)}\int_0^\infty (1-\cos u)u^{-p+1}\,du = \frac{1}{p(p-1)C_{p-2}}$$

$$= \left(p(p-1)\frac{2}{\pi}\sin\frac{(p-2)\pi}{2}\,\Gamma(p-1)\right)^{-1}$$

$$= \left(-\frac{2}{\pi}\sin\frac{p\pi}{2}\,\Gamma(p+1)\right)^{-1} = (-C_p)^{-1}.$$

Hence

$$\int^\infty (\cos xt - 1 + \tfrac{1}{2}x^2t^2)t^{-p-1}\,dt = (-C_p)^{-1}|x|^p,$$

which proves (4b). (4a) follows now from (4b) precisely as in the proof of Lemma 2.3.

LEMMA 4.3. *Let $2 < p < 4$. For $s > 0$ put*

$$F_p(s) = (-C_p)\int_0^\infty \left(\tfrac{1}{2}t^2 - 1 + |\cos(t/\sqrt{s})|^s\right)t^{-p-1}\,dt.$$

(1) *If $\sum_{k=1}^n a_k^2 = 1$, then*

$$E\left(\Big|\sum_{k=1}^n a_k r_k\Big|^p\right) \leqslant \sum_{k=1}^n a_k^2 F_p(a_k^{-2}).$$

(2) $$F_p(2n) = E\left(\Big|\frac{1}{\sqrt{2n}}(r_1 + \ldots + r_{2n})\Big|^p\right), \qquad n \in \mathbf{N}.$$

(3) $$\lim_{s\to\infty} F_p(s) = (-C_p)\int_0^\infty \left(\tfrac{1}{2}t^2 - 1 + \exp(-t^2/2)\right)t^{-p-1}\,dt$$

$$= 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}.$$

The proof of Lemma 4.3 is almost identic with the proof of Lemma 2.4, and will be left to the reader. For $2 < p < 4$, we put $F_p(\infty) = \lim_{s\to\infty} F_p(s) = 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}$.

Note that the same formula for $F_p(\infty)$ holds for $0 < p < 2$ (cf. Lemma 2.4). The use of the notation F_p for the function in Lemma 4.3 is justified by the following lemma.

LEMMA 4.4. *Let $0 < p < 4$, $p \neq 2$. Then*

(4c) $$F_p(s) = F_p(\infty) + C_p\int_0^\infty \left(\exp(-t^2/2) - |\cos(t/\sqrt{s})|^s\right)t^{-p-1}\,dt.$$

PROOF. When $0 < p < 2$, it follows from Lemma 2.4 (3). When $2 < p < 4$, we have from Lemma 4.3(3) that

(4d) $$-C_p\int_0^\infty \left(\tfrac{1}{2}t^2 - 1 + \exp(-t^2/2)\right)t^{-p-1}\,dt = 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi} = F_p(\infty).$$

Subtracting (4d) from the expression

$$F_p(s) = -C_p\int_0^\infty \left(\tfrac{1}{2}t^2 - 1 + |\cos(t/\sqrt{s})|^s\right)t^{-p-1}\,dt$$

one obtains (4c).

Remark 4.5. Since $C_p = 0$ for $p = 2$, and since $\Gamma(3/2) = \sqrt{\pi}/2$, (4c) is true also for $p = 2$ if one defines $F_2(s) = 1$, $s > 0$. From Lemma 4.3(1) we have

$$E\left(\Big|\sum_{k=1}^n a_k r_k\Big|^p\right) \leqslant \sup_{s\geqslant 1} F_p(s),$$

where $\sum a_k^2 = 1$. Hence if we could prove, that

$$F_p(s) \leqslant F_p(\infty), \qquad 2 < p < 4,\ s \in [1, \infty],$$

we could conclude that

$$B_p^p \leqslant F_p(\infty) = 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}, \qquad 2 < p < 4.$$

Unfortunately it turns out that $F_p(1) > F_p(\infty)$, when p comes close to 2 (cf. Remark 4.10). However, we are able to prove that $F_p(s) \leqslant F_p(\infty)$, when $s \in [\sqrt{2}, \infty[$. This will take care of the case where $|a_k| \leqslant 2^{-1/4}$ for all k. The remaining cases are treated by the following lemma.

LEMMA 4.6. *Let $2 \leqslant p \leqslant 4$, and put $X = \sum_{k=1}^n a_k r_k$, where $\sum a_k^2 = 1$. If $|a_k| > 2^{-1/4}$ for some k, then*

$$E(|X|^p) \leqslant 2^{(p-2)/2} \leqslant 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}.$$

Proof. We have $E(|X|^2) = \sum a_k^2 = 1$, and from Lemma 3.3

$$E(|X|^4) = 3 - 2\sum_{k=1}^n a_k^4.$$

Hence, when $|a_k| \geq 2^{-1/4}$, for some k, $E(|X|^4) \leqslant 2$. By Hölder's inequality we get now

$$E(|X|^p) \leqslant E(|X|^2)^{(4-p)/2} E(|X|^4)^{(p-2)/2} \leqslant 2^{(p-2)/2}$$

for $p \in [2, 4]$. Since the Γ-function is convex, and since $\Gamma((p+1)/2) = \sqrt{\pi}/2$ for $p = 2$, and $p = p_0 < 2$, we have $\Gamma((p+1)/2) \geqslant \sqrt{\pi}/2$ for $p \leqslant p_0$ and $p \geqslant 2$. Hence $2^{p/2}\Gamma((p+1)/2)/\sqrt{\pi} \geqslant 2^{(p-2)/2}$ for $p \geqslant 2$. This proves the second inequality.

The proof of $F_p(s) \leqslant F_p(\infty)$, $s \in [\sqrt{2}, \infty[$, $p \in]2, 4[$ will be based on a detailed investigation of the function

$$\chi(s) = \frac{d}{dp} F_p(s)\Big|_{p=2}.$$

LEMMA 4.7. *The function* $\chi(s) = \frac{d}{dp} F_p(s)|_{p=2}$ *is well defined for* $s \in]0, \infty]$. *Moreover,*

(1) $\chi(s) = \chi(\infty) - 2\int_0^\infty \left(\exp(-t^2/2) - |\cos(t/\sqrt{s})|^s\right) t^{-3}\, dt, \qquad s > 0;$

(2) $\qquad \chi(\infty) = \lim_{s\to\infty} \chi(s) = \frac{1}{2}(2 - \log 2 - \gamma)$

where γ *is Eulers constant;*

(3) $\qquad \chi(2) = \frac{1}{2}\log 2.$

Proof. Since $F_p(\infty) = 2^{p/2}\Gamma((p+1)/2)/\sqrt{\pi}$, we get

$$\text{(4e)} \qquad \frac{d}{dp} F_p(\infty)/F_p(\infty) = \tfrac{1}{2}\log 2 + \tfrac{1}{2}\Gamma'((p+1)/2)/\Gamma((p+1)/2).$$

By [4], 6.3.4, $\Gamma'(3/2)/\Gamma(3/2) = 2 - 2\log 2 - \gamma$. Hence, putting $p = 2$ in (4e), we get $\chi(\infty) = \frac{1}{2}(2 - \log 2 - \gamma)$. Since $C_p = 0$ for $p = 2$, and

$$\frac{d}{dp} C_p\Big|_{p=2} = \frac{2}{\pi}\frac{d}{dp}\left(\sin(p\pi/2)\Gamma(p+1)\right)\Big|_{p=2} = -\Gamma(3) = -2,$$

we get by differentiating (4c) at $p = 2$ that

$$\chi(s) = \chi(\infty) - 2\int_0^\infty \left(\exp(-t^2/2) - |\cos(t/\sqrt{s})|^s\right) t^{-3}\, dt.$$

This proves (1). However

$$\exp(-t^2/2) - |\cos(t/\sqrt{s})|^s \to 0 \quad \text{for} \quad s \to \infty$$

and for $s \geqslant 1$, $\left|\exp(-t^2/2) - |\cos(t/\sqrt{s})|^s\right| \leqslant h(t)$,

where

$$h(t) = \begin{cases} \frac{1}{8}t^4, & t \leqslant \sqrt{2}, \\ 1, & t \geqslant \sqrt{2} \end{cases} \qquad \text{(cf. proof of Lemma 2.4).}$$

Since $\int_0^\infty h(t) t^{-3}\, dt < \infty$, it follows from Lebesgue's dominated convergence theorem that $\lim_{s\to\infty} \chi(s) = \chi(\infty)$. This proves (2). From Lemma 2.4(2) and Lemma 4.3 we have

$$F_p(2) = E\left(\left|\frac{1}{\sqrt{2}}(r_1 + r_2)\right|^p\right) = 2^{(p-2)/2} \quad \text{for} \quad 0 < p < 4.$$

Hence

$$\chi(2) = \frac{d}{dp}\left(2^{(p-2)/2}\right)\Big|_{p=2} = \tfrac{1}{2}\log 2.$$

Remark 4.8. To five correct decimals, one has

$$\chi(2) = \tfrac{1}{2}\log 2 = 0{,}34657$$

and

$$\chi(\infty) = \tfrac{1}{2}(2 - \log 2 - \gamma) = 0{,}36482.$$

In the following we shall prove that $\chi(s) \leqslant \chi(\infty)$ when $s \geqslant \sqrt{2}$. Fig. 4.1 below shows the graph of $\chi(s)$. The drawing is based on the values $\chi(2n)$, $n = 1, 2, \ldots$, which are easily computed, using

$$F_p(2n) = E\left(\left|\frac{1}{\sqrt{2n}}(r_1 + r_2 + \ldots + r_{2n})\right|^p\right),$$

and a numerical computation of $\chi(1)$ based on Lemma 4.9 below.

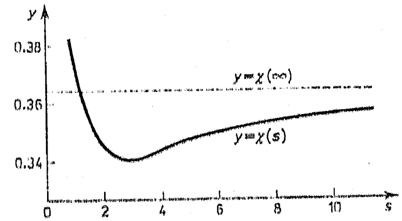


Figure 4.1

LEMMA 4.9. (1) $\chi(1) = \log 2 - \frac{2}{\pi}\log(\pi/2) - \frac{4}{\pi}\sum_{k=2}^{\infty}(-1)^k \frac{\log k}{4k^2 - 1}$.

(2) $0{,}36584 < \chi(1) < 0{,}37623$.

Proof. For $p \in]0,1[$ we have by Lemma 2.4 that

$$F_p(1) = C_p \int_0^\infty (1-|\cos t|)\, t^{-p-1}\, dt. \tag{4f}$$

We can expand $|\cos t|$ in a Fourier series

$$|\cos t| \sim a_0 + \sum_{n=1}^\infty a_n \cos 2nt,$$

where

$$a_0 = \frac{2}{\pi} \quad \text{and} \quad a_n = \frac{4}{\pi}\,\frac{(-1)^{n-1}}{4n^2-1},\ n \geq 1. \tag{4g}$$

Since $\sum_{n=1}^\infty |a_n| < \infty$, we have in fact

$$|\cos t| = a_0 + \sum_{n=1}^\infty a_n \cos 2nt. \tag{4h}$$

In particular, $1 = a_0 + \sum_{n=1}^\infty a_n$. Subtracting (4h), we get

$$1 - |\cos t| = \sum_{n=1}^\infty a_n (1-\cos 2nt). \tag{4i}$$

Since

$$\int_0^\infty (1-\cos 2nt)\, t^{-p-1}\, dt = (2n)^p \int_0^\infty (1-\cos t)\, t^{-p-1}\, dt = C_p^{-1}(2n)^p \tag{4j}$$

we have for $p \in]0,1[$ that

$$\int_0^\infty \Big(\sum_{n=1}^\infty |a_n|(1-\cos 2nt)\, t^{-p-1}\Big) dt = \sum_{n=1}^\infty |a_n| \int_0^\infty (1-\cos 2nt)\, t^{-p-1}\, dt = C_p^{-1} \sum_{n=1}^\infty |a_n| (2n)^p < \infty.$$

Hence, using Lebesgue's dominated convergence theorem, we obtain by (4f) and (4i) that

$$F_p(1) = C_p \int_0^\infty \Big(\sum_{n=1}^\infty a_n (1-\cos 2nt)\, t^{-p-1}\Big) dt = C_p \sum_{n=1}^\infty a_n \int_0^\infty (1-\cos 2nt)\, t^{-p-1}\, dt = \sum_{n=1}^\infty a_n (2n)^p = \frac{4}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \frac{(2n)^p}{4n^2-1}.$$

Using the identity $(2n)^p/(4n^2-1) = (2n)^{p-2} + (2n)^{p-2}/(4n^2-1)$, we get for $p \in]0,1[$ that

$$F_p(1) = \frac{2^p}{\pi}\Big(\sum_{n=1}^\infty (-1)^{n-1} n^{p-2} + \sum_{n=1}^\infty (-1)^{n-1} n^{p-2}/(4n^2-1)\Big). \tag{4k}$$

Since $\sum_{n=1}^\infty (-1)^{n-1} n^{-s} = (1-2^{1-s}) \sum_{n=1}^\infty n^{-s} = (1-2^{1-s})\zeta(s)$ for $s > 1$, where $\zeta(s)$ is the Riemann zeta function, it follows that, for $0 < p < 1$,

$$F_p(1) = \frac{2^p}{\pi}\Big[(1-2^{p-1})\zeta(2-p) + \sum_{n=1}^\infty (-1)^n n^{p-2}/(4n^2-1)\Big]. \tag{4l}$$

The function $a \to \sum_{n=1}^\infty (-1)^n n^{a-2}/(4n^2-1)$ is well defined and analytic in the complex halfplan $\mathrm{Re}(a) < 3$. It is well known that ζ is an analytic function in $C\setminus\{1\}$ with a simple pole in 1. Therefore the right side of (4l) can be extended to an analytic function in the half plane $\mathrm{Re}(p) < 3$. From Lemma 4.4 it follows easily that $p \to F_p(1)$ can be extended to an analytic function in the strip $0 < \mathrm{Re}(p) < 4$. By uniqueness of analytic continuation we conclude that (4l) is true also for $p \in]1,3[$. Differentiating (4l) termwise at $p = 2$, one gets, remembering that $F_2(1) = 1$,

$$\chi(1) = \log 2 + \frac{4}{\pi}[\zeta'(0) - 2\log 2\,\zeta(0)] + \frac{4}{\pi}\sum_{n=2}^\infty (-1)^{n-1}\log n/(4n^2-1). \tag{4m}$$

(The termwise differentiation is legal, because both the sum in (4l) and the termwise differentiated sum is absolutely convergent.) From [1], p. 12 and p. 60, we have $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log 2\pi$. Since the term for $n = 1$ in (4m) vanishes, we obtain

$$\chi(1) = \log 2 - \frac{2}{\pi}\log(\pi/2) - \frac{4}{\pi}\sum_{n=2}^\infty (-1)^n \log n/(4n^2-1). \tag{4n}$$

This proves (1). The terms in the infinite sum in (4n) have alternating signs. Moreover, the absolute values of the terms are strictly decreasing, because for $t \geq 2$

$$\frac{d}{dt}\Big(\frac{\log t}{4t^2-1}\Big) = \frac{4t^2-1-8t^2\log t}{t(4t^2-1)^2} < \frac{4t(1-2\log t)}{(4t^2-1)^2} < 0.$$

Hence, for $n \in \boldsymbol{N}$, $n \geqslant 2$, we have

$$\sum_{k=2}^{2n-1} \frac{(-1)^k}{4k^2-1}\log k \leqslant \sum_{k=2}^{\infty} \frac{(-1)^k}{4k^2-1}\log k \leqslant \sum_{k=2}^{2n} \frac{(-1)^k}{4k^2-1}\log k.$$

Putting $n = 4$ we get

$$0{,}023119 < \sum_{k=2}^{\infty} \frac{(-1)^k}{4k^2-1}\log k < 0{,}031275.$$

Inserting these bounds in (4n) we obtain

$$0{,}36584 < \chi(1) < 0{,}37623.$$

Remark 4.10. By Lemma 4.9 and Remark 4.8, $\chi(1) > 0{,}36584 > \chi(\infty)$. Hence, when p is sufficiently close to 2 ($p > 2$), one has $F_p(1) > F_p(\infty)$.

Lemma 4.11. *For* $n = 2, 3, 4, \ldots$, *put*

$$\alpha_n^{(2)} = \int_0^\infty \sin^{2n} t\, t^{-3}\,dt, \quad \beta_n^{(2)} = \int_0^\infty \bigl(1-\exp(-t^2)\bigr)^n t^{-3}\,dt.$$

Then $\alpha_2^{(2)} = \beta_2^{(2)}$, *and* $\alpha_n^{(2)} < \beta_n^{(2)}$ *for* $n \geqslant 3$.

Proof. Since $\sin^{2n} t \leqslant O(t^4)$ and $\bigl(1-\exp(-t^2)\bigr)^n \leqslant O(t^4)$ for $t \to 0$ when $n \geqslant 2$, it is clear that the above integrals are bounded. Moreover,

$$\alpha_n^{(2)} = \lim_{p\to 2-} \alpha_n^{(p)} \quad \text{and} \quad \beta_n^{(2)} = \lim_{p \to 2-} \beta_n^{(p)},$$

where $\alpha_n^{(p)}, \beta_n^{(p)}$, $0 < p < 2$ are defined as in Lemma 2.7. We have, for $0 < p < 2$,

$$C_p \alpha_n^{(p)} = \frac{2^{p+1}}{4^n} \sum_{k=1}^{n} (-1)^{k-1} \binom{2n}{n-k} k^p \tag{4o}$$

and

$$C_p \beta_n^{(p)} = \frac{2^p \Gamma\bigl((p+1)/2\bigr)}{\sqrt{\pi}} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k^{p/2}. \tag{4p}$$

Since $\lim\limits_{p\to 2} C_p = 0$, we get from (4o) and (4p) that

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{2n}{n-k} k^2 = \lim_{p \to 2-} \sum_{k=1}^{n} (-1)^{k-1} \binom{2n}{n-k} k^p = 0$$

and

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k = \lim_{p\to 2-} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k^{p/2} = 0.$$

Since $\left.\dfrac{dC_p}{dp}\right|_{p=2} = -2$, it follows by differentiating (4o) and (4p) on both sides at $p = 2$ that

$$-2\alpha_n^{(2)} = \lim_{p\to 2} \left(\frac{2^{p+1}}{4^n}\right) \sum_{k=1}^{n} (-1)^{k-1} \binom{2n}{n-k} k^2 \log k$$

and

$$-2\beta_n^{(2)} = \lim_{p\to 2} \bigl(2^p \Gamma\bigl((p+1)/2\bigr)\pi^{-1/2}\bigr) \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{k}{2} \log k.$$

Hence

$$\alpha_n^{(2)} = 4^{1-n} \sum_{k=2}^{n} (-1)^k \binom{2n}{n-k} k^2 \log k \tag{4q}$$

and

$$\beta_n^{(2)} = \tfrac{1}{2} \sum_{k=2}^{n} (-1)^k \binom{n}{k} k \log k. \tag{4r}$$

Putting $n = 2$ one finds $\alpha_n^{(2)} = \beta_n^{(2)} = \log 2$. The following list gives the values of $\alpha_n^{(2)}$ and $\beta_n^{(2)}$, for $2 \leqslant n \leqslant 20$ computed from (4q) and (4r), (maximal error $\leqslant 10^{-5}$).

n	$\alpha_n^{(2)}$	$\beta_n^{(2)}$	n	$\alpha_n^{(2)}$	$\beta_n^{(2)}$
1	—	—	11	0,16006	0,19454
2	0,69315	0,69315	12	0,15194	0,18768
3	0,42175	0,43152	13	0,14493	0,18179
4	0,32364	0,33980	14	0,13880	0,17667
5	0,27087	0,29164	15	0,13339	0,17215
6	0,23709	0,26138	16	0,12856	0,16814
7	0,21322	0,24032	17	0,12422	0,16454
8	0,19525	0,22467	18	0,12029	0,16129
9	0,18111	0,21248	19	0,11670	0,15833
10	0,16963	0,20266	20	0,11342	0,15563

One observes that $\alpha_n^{(2)} < \beta_n^{(2)}$ for $3 \leqslant n \leqslant 20$. The estimates (2o) and (2p) in the proof of Lemma 2.7 give in the limit $p \to 2-$ for $n \geqslant 3$ that

$$\alpha_n^{(2)} \leqslant \sqrt{\pi/2}\, 3^{-1/2}(2n-5)^{-1/2} = \bigl(\pi/12\,(n-5/2)\bigr)^{1/2}, \tag{4s}$$

$$\beta_n^{(2)} \geqslant \tfrac{1}{2}(1+\log n)^{-1}. \tag{4t}$$

When $n \geqslant 20$, $n-5/2 = n(1-5/2n) \geqslant n(1-5/40) = \frac{7}{8}n$. Hence, for $n \geqslant 20$,

$$\alpha_n^{(2)}/\beta_n^{(2)} \leqslant (\pi/12\cdot\tfrac{7}{8}n)^{1/2}2(1+\log n) = \sqrt{8\pi/21}(1+\log n)n^{-1/2}.$$

The function $t \to (1+\log t)t^{-1/2}$ is decreasing for $t > e$, because

$$\frac{d}{dt}\big((1+\log t)t^{-1/2}\big) = t^{-3/2}\big(1-\tfrac{1}{2}(1+\log t)\big) < 0$$

when $t > e$. Therefore

$$\alpha_n^{(2)}/\beta_n^{(2)} \leqslant \sqrt{8\pi/21}\,\frac{1+\log 20}{\sqrt{20}} = 0{,}9774\ldots < 1$$

for $n \geqslant 20$. This proves the lemma.

Lemma 4.12 *Let* $0 < s \leqslant 4$. *Then*

$$\chi(s) = \tfrac{1}{2}\log 2 + \frac{2}{s}\sum_{k=3}^{\infty}(\beta_k^{(2)}-\alpha_k^{(2)})(-1)^{k-1}\binom{s/2}{k}. \tag{4u}$$

Proof. Let $0 < p < 2$. From Lemma 2.8 we have the expansion

$$F_p(s) = F_p(\infty)+C_p s^{-p/2}\sum_{k=1}^{\infty}(\beta_k^{(p)}-\alpha_k^{(p)})(-1)^k\binom{s/2}{k}, \qquad 0 < s \leqslant 4. \tag{4v}$$

From (4o) and (4p) it follows that

$$\begin{aligned}\beta_1^{(p)}-\alpha_1^{(p)} &= C_p^{-1}2^{p/2}\big(2^{p/2}\Gamma((p+1)/2)\pi^{-1/2}-2^{(p-2)/2}\big)\\ &= C_p^{-1}2^{p/2}\big(F_p(\infty)-F_p(2)\big).\end{aligned}$$

Since $C_2 = 0$, we get

$$\lim_{p\to 2}\frac{C_p}{p-2} = \left[\frac{d}{dp}C_p\right]_{p=2} = -2.$$

Hence

$$\begin{aligned}\lim_{p\to 2-}(\beta_1^{(p)}-\alpha_1^{(p)}) &= \lim_{p\to 2}\left(\frac{p-2}{C_p}\cdot 2^{p/2}\left(\frac{F_p(\infty)-1}{p-2}-\frac{F_p(2)-1}{p-2}\right)\right)\\ &= -\big(\chi(\infty)-\chi(2)\big).\end{aligned}$$

Differentiating (4v) after p at $p = 2$, one obtains

$$\chi(s) = \chi(\infty)+\frac{2}{s}\lim_{p\to 2-}\sum_{k=1}^{\infty}(\beta_k^{(p)}-\alpha_k^{(p)})(-1)^{k-1}\binom{s/2}{k}.$$

The 1st term in the sum is $\frac{s}{2}(\beta_1^{(p)}-\alpha_1^{(p)})$. Hence using $\lim_{p\to 2-}(\beta_1^{(p)}-\alpha_1^{(p)}) = \chi(2)-\chi(\infty)$, we get

$$\chi(s) = \chi(2)+\frac{2}{s}\lim_{p\to 2-}\sum_{k=2}^{\infty}(\beta_k^{(p)}-\alpha_k^{(p)})(-1)^{k-1}\binom{s/2}{k}. \tag{4w}$$

Put

$$M_1 = \sup\{\alpha_2^{(p)}|\ p\in[1,2]\} < \infty$$

and

$$M_2 = \sup\{\beta_2^{(p)}|\ p\in[1,2]\} < \infty.$$

Then clearly $\alpha_k^{(p)} \leqslant M_1$, and $\beta_k^{(p)} \leqslant M_2$ for $p\in[1,2]$ and $k\geqslant 2$. Consider now the Taylor expansion

$$(1-t)^{s/2} = 1-\frac{s}{2}t+\sum_{k=2}^{\infty}(-1)^k\binom{s/2}{k}t^k, \qquad |t|<1.$$

When $s\in\,]0,4[$, the sign of $(-1)^k\binom{s/2}{k}$, $k\geqslant 2$ is independent of k. Therefore for $t\in[0,1[$,

$$\sum_{k=2}^{\infty}\left|\binom{s/2}{k}\right|t^k = \left|\sum_{k=2}^{\infty}(-1)^k\binom{s/2}{k}t^k\right| = |(1-t)^{s/2}-1-(s/2)t|.$$

In the limit $t\to 1$ one gets $\sum_{k=2}^{\infty}\left|\binom{s/2}{k}\right| = |1-s/2| \leqslant 1$, when $s\in\,]0,4]$. Hence the sequence $(\beta_k^{(p)}-\alpha_k^{(p)})(-1)^{k-1}\binom{s/2}{k}$ is dominated by the summable sequence $(M_1+M_2)\left|\binom{s/2}{k}\right|$ for $p\in[1,2]$. Therefore the limit in (4w) can be taken termwise, i.e.

$$\chi(s) = \chi(2)+\frac{2}{s}\sum_{k=2}^{\infty}(\beta_k^{(2)}-\alpha_k^{(2)})(-1)^{k-1}\binom{s/2}{k}.$$

Since $\beta_2^{(2)} = \alpha_2^{(2)}$ by Lemma 4.11, we have proved (4u).

Lemma 4.13. $\chi(s) \leqslant \chi(\infty)$ *for* $s \geqslant \sqrt{2}$.

Proof. We divide the proof in 3 cases (1) $s\in[\sqrt{2},2]$, (2) $s\in[2,4]$ and (3) $s\in[4,\infty[$.

(1) One has

$$(-1)^{k-1}\frac{2}{s}\binom{s/2}{k} = \frac{1}{2^{k-1}k!}\cdot(2-s)(4-s)\cdot\ldots\cdot(2k-2-s), \qquad k\geqslant 2. \tag{4x}$$

Hence $(-1)^{k-1}\frac{2}{s}\binom{s/2}{k} \geqslant 0$ for $s \in]0, 2]$. Moreover,

$$\frac{d^2}{ds^2}\left((-1)^{k-1}\frac{2}{s}\binom{s/2}{k}\right) = \frac{1}{2^{k-1}k!}\sum_{i\neq j}\prod_{\substack{l\neq i\\ l\neq j}}(2l-s) \geqslant 0 \quad \text{for} \quad s\in[0,2]$$

$(i, j, l = 1, 2, \ldots, k-1)$. From the formula

(4y) $$\chi(s) = \tfrac{1}{2}\log 2 + \frac{2}{s}\sum_{k=3}^{\infty}(\beta_k^{(2)}-\alpha_k^{(2)})(-1)^{k-1}\binom{s/2}{k}$$

and the fact that $\beta_k^{(2)} > \alpha_k^{(2)}$, $k = 3, 4, \ldots$ (Lemma 4.11), it follows now [hat $\chi''(s) \geqslant 0$ for $s \in]0, 2]$, i.e. χ is a convex function in the interval t0, 2]. Hence, for $s \in [1, 2]$,

$$\chi(s) \leqslant (2-s)\chi(1)+(s-1)\chi(2) = \chi(1)+(s-1)(\chi(2)-\chi(1)).$$

By Remark 4.8 and Lemma 4.9, $\chi(1) > \chi(2)$. Hence, for $s \in [\sqrt{2}, 2]$,

$$\chi(s) \leqslant \chi(1)+(\sqrt{2}-1)(\chi(2)-\chi(1))$$
$$= (2-\sqrt{2})\chi(1)+(\sqrt{2}-1)\chi(2).$$

Inserting $\chi(2) = \frac{1}{2}\log 2$ and $\chi(1) < 0{,}37623$, one obtains

$$\chi(s) < 0{,}36395 < \chi(\infty) \quad \text{for} \quad s\in[\sqrt{2}, 2[.$$

(cf. Remark 4.8).

(2) Let now $s \in [2, 4]$. Using formula (4y) and the fact that $(-1)^{k-1}\frac{2}{s}\binom{s/2}{k} \leqslant 0$ for $s \in [2, 4]$ one gets $\chi(s) \leqslant \frac{1}{2}\log 2 < \chi(\infty)$ for $s \in [2, 4]$.

(3) From Lemma 4.7 we have

(4z) $$\chi(s) = \chi(\infty) - 2\int_0^\infty (\exp(-t^2/2) - |\cos(t/\sqrt{s})|^s)t^{-3}dt$$

$$= \chi(\infty) - \frac{2}{s}\int_0^\infty (\exp(-st^2/2) - |\cos t|^s)t^{-3}dt.$$

The functions

(4aa) $$a_p(s) = \int_0^\infty (\exp(-st^2/2) - |\cos t|^s \chi_{[0,\pi/2]}(t))\, t^{-p-1}dt,$$

(4ab) $$b_p(s) = \int_{\pi/2}^\infty |\cos t|^s t^{-p-1}dt$$

defined in § 2 (formulas (2y) and (2z)), are well defined also for $2 \leqslant p < 4$. By (4z) we have

(4ac) $$\chi(s) = \chi(\infty) - \frac{2}{s}(a_2(s) - b_2(s)).$$

From Lemma 2.10, we get in the limit $p \to 2$ — that

$$a_2(s) \geqslant \frac{s}{2}\left(\frac{1}{6s}\Gamma(1) - \frac{1}{36s^2}\Gamma(3)\right) = \frac{1}{12} - \frac{1}{36s}$$

and

$$b_2(s) < \sqrt{2\pi}(k_1^{(2)}s^{-1/2} + k_2^{(2)}s^{-3/2}),$$

where $k_1^{(2)} = \pi^{-3}\zeta(3)$ and

$$k_2^{(2)} = 4\pi^{-5}((2^3-2)\zeta(3)-2^2) = 4\pi^{-5}(6\zeta(3)-4).$$

Using $\zeta(3) = 1{,}20205\ldots$ (cf. [5], p. 811), one gets

$$k_1^{(2)} < 0{,}0388 \quad \text{and} \quad k_2^{(2)} < 0{,}0420.$$

Hence

$$a_2(s) - b_2(s) > \frac{1}{12} - \frac{1}{36s} - \sqrt{2\pi}(0{,}0388s^{-1/2} + 0{,}0420s^{-3/2}).$$

The right side of the inequality is an increasing function of s. Hence for $s \geqslant 4$

$$a_2(s) - b_2(s) > (\tfrac{1}{12} - \tfrac{1}{36\cdot 4}) - \sqrt{2\pi}(0{,}0388\cdot 4^{-1/2} + 0{,}0420\cdot 4^{-3/2})$$
$$= 0{,}07639 - 0{,}06178 > 0.$$

Therefore by (4ac), $\chi(s) \leqslant \chi(\infty)$ for $s \in [4, \infty[$. This proves the lemma.

LEMMA 4.14. *Let $a_p(s)$ and $b_p(s)$ be the function defined by the formulas* (4aa) *and* (4ab). *Then for $s \geqslant 2$,*

(1) $p \to (\pi/2)^p a_p(s)$ *is an increasing function of* $p \in [2, 4[$;

(2) $p \to (\pi/2)^p b_p(s)$ *is a decreasing function of* $p \in [2, 4[$.

Proof. The proof is essentially the same as the proof of Lemma 2.12. However, a slight change is necessary because the estimates given there do not work for $s < 2$.

(2): We have easily that

$$(\pi/2)^p b_p(s) = \frac{2}{\pi}\int_{\pi/2}^\infty |\cos t|^s(\pi/2t)^{p+1}dt$$

is a decreasing function of p.

(1):

$$(\pi/2)^p a_p(s) = \frac{2}{\pi}\int_0^\infty \big(\exp(-st^2/2) - |\cos t|^s \chi_{[0,\pi/2]}(t)\big)(\pi/2t)^{p+1}\,dt.$$

Hence

$$\frac{d}{dp}\big((\pi/2)^p a_p(s)\big)$$

$$= \frac{2}{\pi}\int_0^\infty \big(\exp(-st^2/2) - |\cos t|^s \chi_{[0,\pi/2]}(t)\big)(\pi/2t)^{p+1}\log(\pi/2t)\,dt.$$

Since $x\log x \geqslant x-1$, and since $x^p \geqslant x^2$ when $x \geqslant 1$ and $x^p \leqslant x^2$ when $x \leqslant 1$ for $p \in [2, 4[$, it follows that

$$x^{p+1}\log x = x^p(x\log x) \geqslant x^2(x-1)$$

for any $x > 0$. This implies that

$$\text{(4ad)}\quad \frac{d}{dp}\big((\pi/2)^p a_p(s)\big)$$

$$\geqslant \frac{2}{\pi}\int_0^\infty \big(\exp(-st^2/2) - |\cos t|^s \chi_{[0,\pi/2]}(t)\big)(\pi/2t)^2(\pi/2t-1)\,dt.$$

Now from the proof of Lemma 2.12 we have

$$\exp(-st^2/2) \geqslant (\cos t)^s(1+st^4/12) \quad \text{for} \quad t \in [0, \pi/2], \quad s > 0.$$

Therefore

$$\text{(4ae)}\quad \int_0^{\pi/2} \big(\exp(-st^2/2) - (\cos t)^s(\pi/2t)^2(\pi/2t-1)\big)\,dt$$

$$\geqslant \int_0^{\pi/2} (\cos t)^s \frac{st^4}{12}(\pi/2t)^2(\pi/2t-1)\,dt$$

$$= \frac{s}{12}(\pi/2)^2\int_0^{\pi/2} (\cos t)^s t(\pi/2-t)\,dt$$

$$\geqslant \frac{s}{12}(\pi/2)^2\int_0^{\pi/2} (\cos t)^{s+1}\sin t\,dt = \frac{\pi^2}{4}\cdot\frac{s}{12(s+2)}.$$

As in the proof of Lemma 2.12 we have

$$\text{(4af)}\quad \int_{\pi/2}^\infty \exp(-st^2/2)(\pi/2t)^2(1-\pi/2t)\,dt \leqslant \int_{\pi/2}^\infty \exp(-st^2/2)(1-\pi/2t)\,dt$$

$$\leqslant \frac{8}{\pi^3 s^2}\exp(-s\pi^2/8).$$

Combining (4ad), (4ae), and (4af), one gets

$$\frac{d}{dp}\big((\pi/2)^p a_p(s)\big) \geqslant \frac{\pi^2}{4}\cdot\frac{s}{12(s+2)} - \frac{8}{\pi^3 s^2}\exp(-s\pi^2/8).$$

The function on the right side is an increasing function of s. For $s = \sqrt{2}$ it has the value

$$\frac{\pi^2}{4}\,\frac{\sqrt{2}}{12(\sqrt{2}+2)} - \frac{4}{\pi^3}\exp(-\sqrt{2}\pi^2/8) = 0{,}06263\ldots > 0.$$

This proves that $\frac{d}{dp}\big((\pi/2)^p a_p(s)\big) > 0$ for $s \geqslant \sqrt{2}$ and $p \in [2, 4[$.

LEMMA 4.15. *Let $p \in]2, 4[$. Then $F_p(s) \leqslant F_p(\infty)$ for $s \geqslant \sqrt{2}$.*

Proof. By Lemma 4.4 and formulas (4aa) and (4ab) we have

$$\text{(4ag)}\quad F_p(s) = F_p(\infty) + C_p s^{-p/2}\int_0^\infty \big(\exp(-st^2/2) - |\cos t|^s\big)t^{-p-1}\,dt$$

$$= F_p(\infty) + C_p s^{-p/2}\big(a_p(s) - b_p(s)\big).$$

Similry, by Lemma 4.7

$$\chi(s) = \chi(\infty) - \frac{2}{s}\int_0^\infty \big(\exp(-st^2/2) - |\cos t|^s\big)t^{-3}\,dt$$

$$= \chi(\infty) - \frac{2}{s}\big(a_2(s) - b_2(s)\big).$$

By Lemma 4.13 $\chi(s) \leqslant \chi(\infty)$ for $s \geqslant \sqrt{2}$. Hence $a_2(s) \geqslant b_2(s)$ for $s \geqslant \sqrt{2}$. Lemma 4.14 gives now $a_p(s) \geqslant b_p(s)$ for $s \geqslant \sqrt{2}$, $p \in]2, 4[$. Since $C_p < 0$ for $p \in]2, 4[$, we get by (4ad) that

$$F_p(s) \leqslant F_p(\infty), \quad s \geqslant \sqrt{2}.$$

PROPOSITION 4.16. *For $p \in [2, 4[$, $B_p^p = 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}$.*

Proof. The case $p = 2$ is trivial. Let $2 < p < 4$. By Lemma 4.1

$$B_p^p \geqslant 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}.$$

If $X = \sum_{k=1}^{n} a_k r_k$, $\sum_{k=1}^{n} a_k^2 = 1$, and $|a_k| \leqslant 2^{-1/4}$ for all k, one has by Lemma 4.3(1) and Lemma 4.15 that

$$E(|X|^p) = \sum_{k=1}^{n} a_k^2 F_p(a_k^{-2}) \leqslant \sum_{k=1}^{n} a_k^2 F_p(\infty)$$

$$= F_p(\infty) = 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}.$$

This together with lemma 4.6 proves that $B_p^n \leqslant 2^{p/2}\Gamma\big((p+1)/2\big)\pi^{-1/2}$.

§ 5. Case $4 \leqslant p < \infty$. The case $p \geqslant 4$ will be reduced to the case $2 \leqslant p < 4$, by use of the following lemma:

LEMMA 5.1. *Let $p > 0$, and $a_1, \ldots, a_n \in \mathbf{R}$. Then*

$$\text{(5a)} \qquad E\Big(\Big|\sum_{k=1}^{n} a_k r_k\Big|^{p+2}\Big) = (p+1)\sum_{j=1}^{n} a_j^2 \int_0^1 E\Big(\Big|s a_j r_j + \sum_{k\neq j} a_k r_k\Big|^p\Big)\,ds.$$

Proof. Assume first $0 < p < 2$. Put $X = \sum_{k=1}^{n} a_k r_k$, and $a = (\sum_{k=1}^{n} a_k^2)^{1/2}$. Then $E(X^2) = \sum_{k=1}^{n} a_k^2 = a^2$. Hence by Lemma 4.2,

$$\text{(5b)} \qquad E(|X|^{p+2}) = -C_{p+2}\int_0^\infty \big(\varphi_X(t) - 1 + \tfrac{1}{2}a^2t^2\big)t^{-p-3}\,dt,$$

where $\varphi_X(t) = \prod_{k=1}^{n} \cos a_k t$ is the characteristic function for X. We have

$$-C_{p+2} = -\frac{2}{\pi}\sin\frac{(p+2)\pi}{2}\,\Gamma(p+3)$$

$$= \frac{2}{\pi}\sin\frac{p\pi}{2}(p+1)(p+2)\Gamma(p+1) = (p+1)(p+2)C_p.$$

Since $\varphi_X(t) - 1 + \frac{1}{2}a^2t^2 = O(t^4)$ for $t \to 0$, we get by partial integration of (5b), that

$$\text{(5c)} \qquad E(|X|^{p+2}) = (p+1)(p+2)C_p\int_0^\infty \big(\varphi_X'(t) + a^2 t\big)\frac{t^{-p-2}}{p+2}\,dt$$

$$= (p+1)C_p\int_0^\infty \big(\varphi_X'(t)/t + a^2\big)t^{-p-1}\,dt.$$

Now

$$\frac{1}{t}\varphi_X'(t) = -\frac{1}{t}\sum_{j=1}^{n} a_j \sin a_j t \prod_{j\neq k}\cos a_k t$$

$$= -\sum_{j=1}^{n} a_j^2\frac{\sin a_j t}{a_j t}\prod_{j\neq k}\cos a_k t.$$

Using $\sin t/t = \int_0^1 \cos st\,ds$, we have

$$\frac{1}{t}\varphi_X'(t) = -\sum_{j=1}^{n} a_j^2\int_0^1\Big(\cos s a_j t\prod_{k\neq j}\cos a_k t\Big)\,ds$$

and hence

$$\text{(5d)} \qquad a^2 + \frac{1}{t}\varphi_X'(t) = \sum_{j=1}^{n} a_j^2\int_0^1\Big(1 - \cos s a_j t\prod_{k\neq j}\cos a_k t\Big)\,ds.$$

Since $1 - \cos s a_j t\prod_{k\neq j}\cos a_k t \geqslant 0$, we can apply Fubini's Theorem in the following computation. By (5c) and (5d),

$$E(|X|^{p+2}) = (p+1)C_p\int_0^\infty\Big(\sum_{j=1}^{n} a_j^2\int_0^1\Big(1 - \cos s a_j t\prod_{k\neq j}\cos a_k t\Big)\,ds\Big)t^{-p-1}\,dt$$

$$= (p+1)\sum_{j=1}^{n} a_j^2\int_0^1\Big(C_p\int_0^\infty\Big(1 - \cos s a_j t\prod_{k\neq j}\cos a_k t\Big)t^{-p-1}\,dt\Big)\,ds.$$

However: $\cos s a_j t\prod_{k\neq j}\cos a_k t$ is the characteristic function of $s a_j r_j + \sum_{k\neq j} a_k r_k$. Hence, by Lemma 2.3,

$$E\Big(\Big|s a_j r_j + \sum_{k\neq j} a_k r_k\Big|^p\Big) = C_p\int_0^\infty\Big(1 - \cos s a_j t\prod_{k\neq j}\cos a_k t\Big)t^{-p-1}\,dt.$$

Therefore

$$E(|X|^{p+2}) = (p+1)\sum_{j=1}^{n} a_j^2\int_0^1 E\Big(\Big|s a_j r_j + \sum_{k\neq j} a_k r_k\Big|^p\Big)\,ds.$$

This proves formula (5a) for $0 < p < 2$. However, it is easily seen that both

$$E\Big(\Big|\sum_{k=1}^{n} a_k r_k\Big|^{p+2}\Big) \quad\text{and}\quad (p+1)\sum_{j=1}^{n} a_j^2\int_0^1 E\Big(\Big|s a_j r_j + \sum_{k\neq j} a_k r_k\Big|^p\Big)\,ds$$

can be extended to analytic functions of p in the complex halfplane $\mathrm{Re}(p) > 0$. Hence by uniqueness of analytic continuation (5a) is true for any $p > 0$.

LEMMA 5.2. *Let $p \geqslant 1$, and let $a_1, \ldots, a_n \in \mathbf{R}$ such that $\sum_{k=1}^{n} a_k^2 = 1$. Then*

$$E\Big(\Big|\sum_{k=1}^{n} a_k r_k\Big|^{p+2}\Big) \leqslant (p+1)E\Big(\Big|\sum_{k=1}^{n} a_k r_k\Big|^p\Big).$$

Proof. Since $sa_jr_j+\sum_{k\neq j} a_kr_k$, $0\leqslant s\leqslant 1$, is a convex combination of $a_jr_j+\sum_{k\neq j}a_kr_k$ and $-a_jr_j+\sum_{k\neq j}a_kr_k$, we have

$$\Big\|sa_jr_j+\sum_{k\neq j}a_kr_k\Big\|_p\leqslant \max\Big\{\Big\|a_jr_j+\sum_{k\neq j}a_kr_k\Big\|_p, \Big\|-a_jr_j+\sum_{k\neq j}a_kr_k\Big\|_p\Big\} = \Big\|\sum_{k=1}^{n}a_kr_k\Big\|_p$$

(p-norm in $L^p(0,1)$). Equivalently

$$E\Big(\Big|sa_jr_j+\sum_{k\neq j}a_kr_k\Big|^p\Big)\leqslant E\Big(\Big|\sum_{k=1}^{n}a_kr_k\Big|^p\Big).$$

Hence, by Lemma 5.1,

$$E\Big(\Big|\sum_{k=1}^{n}a_kr_k\Big|^{p+2}\Big)\leqslant (p+1)\sum_{j=1}^{n}a_j^2E\Big(\Big|\sum_{k=1}^{n}a_kr_k\Big|^p\Big)=(p+1)E\Big(\Big|\sum_{k=1}^{n}a_kr_k\Big|^p\Big),$$

where we have used $\sum a_j^2=1$.

Proof of Theorem B. Let $p\geqslant 2$. From Lemma 4.1 we have $B_p^p\geqslant 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}$. We prove by induction in $k\in \mathbf{N}$ that $2k\leqslant p<2(k+1)$ implies $B_p^p\leqslant 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}$. When $k=1$, this follows from Proposition 4.16. Moreover, if $B_p^p\leqslant 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}$ for $2k\leqslant p<2(k+1)$ for some fixed $k\in \mathbf{N}$, we have by Lemma 5.2 that $B_{p+2}^{p+2}\leqslant (p+1)B_p^p$ and hence

$$B_{p+2}^{p+2}\leqslant (p+1)2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}$$
$$=2\cdot 2^{p/2}\Gamma\big((p+1)/2\big)\frac{(p+1)}{2}/\sqrt{\pi}=2^{(p+2)/2}\Gamma\big((p+3)/2\big)/\sqrt{\pi}$$

or equivalently

$$B_p^p\leqslant 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}\quad\text{for}\quad 2(k+1)\leqslant p<2(k+2).$$

Therefore $B_p^p\leqslant 2^{p/2}\Gamma\big((p+1)/2\big)/\sqrt{\pi}$ for $p\in\bigcup_{k=1}^{\infty}[2k, 2(k+1)[\,=[2,\infty[$. This completes the proof.

Acknowledgement. We wish to thank Prof. Niels J. Nielsen for encouragement during the work on this paper.

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Received April 23, 1979 (1538)