such that $\sum_{1}^{n+m} c_j = 0$. *Let* S *be the symmetric group on* $\{1, \ldots, n+m\}$. *Then*

$$\frac{1}{|S|}\sum_{\sigma\in S}\Big\|\sum_{1}^{n+m} c_{\sigma(j)}e_j\Big\|_p^p \leqslant C_p^p\Big(\sum_{1}^{n+m}|c_j|^2\Big)^{p/2}.$$

(This is a consequence of Lemma 1 above and Lemma 2, p. 261 of [3].)

The proof of the proposition is now easy. Let $c_1 = \ldots = c_n = a$, $c_{n+1} = \ldots = c_{n+m} = -b$. Then it follows from Lemma 2 that there exists $\sigma \in S$ such that

$$\Big\|\sum_{1}^{n+m} c_{\sigma(j)}e_j\Big\|_p \leqslant C_p(na^2+mb^2)^{1/2}.$$

Let $E = \{j: 1 \leqslant \sigma(j) \leqslant n\}$, $F = \{j: n+1 \leqslant \sigma(j) \leqslant n+m\}$. Then (ii) of the proposition holds. Statement (i) of the proposition is an immediate consequence of Lemma 1.

Added in proof. The theorem of this paper can also be found in the work of G. Bachelis and J. Gilbert, *Banach algebras with Rider subalgebras*, Bull. Inst. Math. Acad. Sinica 7 (1979), pp. 333–347.

References

[1] G. Bachelis and B. Barnes, *Closed subalgebras of group algebras of a compact group*, Symposia Math. 22 (1977), pp. 249–265.
[2] J.-P. Kahane, *Idempotents and closed subalgebras of* $A(Z)$, in: *Function algebras*, Scott Foresman, Chicago 1966, pp. 198–207.
[3] D. Oberlin, *Random rearrangements of Fourier coefficients*, Ann. of Math. 106 (1977), pp. 259–268.
[4] D. Rider, *Closed subalgebras of* $L^1(T)$, Duke Math. J. 36 (1969), pp. 105–115.
[5] W. Rudin, *Fourier analysis on groups*, Interscience, New York 1962.
[6] A. Zygmund, *Trigonometric series*, Vol. II, Cambridge University Press, Cambridge, England 1959.

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Identity of Taylor's joint spectrum and Dash's joint spectrum

by

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Abstract. In this note we show that Dash's joint spectrum coincides with Taylor's joint spectrum in the case of normal operators, and operators on a finite-dimensional Hilbert space.

1. Since the concept of joint spectrum for a family of operators was first introduced by R. Arens and A. P. Calderón [1], some authors have asserted its definition and properties. The typical and successful definitions among them have been carried out by J. L. Taylor [7] and A. T. Dash [4].

Throughout this note, let H be a complex Hilbert space and $B(H)$ be the algebra of all continuous linear operators on H. Let $A = (A_1, \ldots, A_n) \subset B(H)$ be an n-tuple of commuting operators, and A', A'' be commutant, double commutant algebra of the set $\{A_1, \ldots, A_n\}$ in $B(H)$, respectively.

Now, let $\{e_1, \ldots, e_n\}$ be a set of indeterminates and denote by E^n the exterior algebra, over the complex field C, with the generators $e_1, \ldots, e_n$. E_p^n will stand for the space of elements of degree p in E^n $(p = 1, \ldots, n)$. Then we denote by $E_p^n(H)$ the tensor product $H\otimes E_p^n$. An element $x\otimes e_{j_1}\wedge \ldots \wedge e_{j_p} \in E_p^n(H)$ will be written simply $xe_{j_1}\wedge \ldots \wedge e_{j_p}$. The space $E_p^n(H)$ can be endowed with a natural structure of a Hilbert space.

We define the map $\delta_A^p\colon E_p^n(H) \to E_{p-1}^n(H)$ such that

$$\delta_A^p(y) = \sum_{i=1}^{p}(-1)^{i-1}A_{j_i}xe_{j_1}\wedge \ldots \wedge \check{e}_{j_i}\wedge \ldots \wedge e_{j_p}$$

for $y = xe_{j_1}\wedge \ldots \wedge e_{j_p} \in E_p^n(H)$, where the symbol ˇ means that the corresponding letter is omitted. For convenience, we put $\delta_A^0 = \delta_A^{n+1} = 0$. Then we see that every δ_A^p is a continuous linear map and $\delta_A^p\circ\delta_A^{p+1} = 0$. Thus, we get the sequence

$$0 \xrightarrow{\delta_A^{n+1}} E_n^n(H) \xrightarrow{\delta_A^n} E_{n-1}^n(H) \xrightarrow{\delta_A^{n-1}} \ldots \xrightarrow{\delta_A^2} E_1^n(H) \xrightarrow{\delta_A^1} E_0^n(H) \xrightarrow{\delta_A^0} 0.$$

Then, J. L. Taylor [7] (cf. [2]) has defined A to be *non-singular* if sequence $(*)$ is exact; i.e. $\operatorname{im}\delta_A^p = \ker\delta_A^{p-1}$ for all $p = 1, \ldots, n+1$. And he has defined the *joint spectrum* $S_p(A, H)$ of A, to be the set of the point $z = (z_1, \ldots, z_n)$ of C^n such that $z - A = (z_1 - A_1, \ldots, z_n - A_n)$ is singular.

On the other hand, A. T. Dash [4] has defined the *joint spectrum* $\sigma(A)$ of A as follows: a point $z = (z_1, \ldots, z_n)$ of C^n is in $\sigma(A)$ if and only if for all $B_1, \ldots, B_n$ in A''

$$\sum_{i=1}^{n} B_i(z_i - A_i) \neq I,$$

where I denotes the identity operator. Furthermore, a point $z = (z_1, \ldots, z_n)$ of C^n is in the *joint approximate point spectrum* $\sigma_\pi(A)$ of A if and only if there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\|(z_i - A_i)x_k\| \to 0 \quad (k \to \infty), \quad i = 1, \ldots, n,$$

and moreover, a point $z = (z_1, \ldots, z_n)$ of C^n is in the *joint point spectrum* $\sigma_p(A)$ of A if and only if for some non-zero vector x in H

$$A_i x = z_i x, \quad i = 1, \ldots, n.$$

And a point $z = (z_1, \ldots, z_n)$ of C^n is in the *joint approximate compression spectrum* $\sigma_\varrho(A)$ of A if and only if there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\|(z_i - A_i)^* x_k\| \to 0 \quad (k \to \infty), \quad i = 1, \ldots, n.$$

It is well known that $\sigma_\pi(A)$, $\sigma(A)$ and $S_p(A, H)$ are non-empty compact sets, and that $S_p(A, H) \subset \sigma(A)$ (see Lemma 1 in [7]). Further, it is evident that $\sigma_p(A) \subset \sigma_\pi(A) \subset \sigma(A) \subset \sigma(A_1) \times \ldots \times \sigma(A_n)$ and $\sigma_\pi(A) \cup \sigma_\varrho(A) \subset \sigma(A)$ (cf. [4], [7], [9]).

2. In this section we shall show that Taylor's joint spectrum $S_p(A, H)$ coincides with Dash's joint spectrum $\sigma(A)$ in the case of commuting normal operators. Firstly we shall generalize Vasilescu's lemma (Lemma 2.1 in [8]) to n-tuple case.

LEMMA. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of commuting operators on H. If $A = (A_1, \ldots, A_n)$ is non-singular, then $A_1^*A_1 + \ldots + A_n^*A_n$ and $A_1A_1^* + \ldots + A_nA_n^*$ are invertible.*

Proof. We can identify $E_1^n(H)$ and $E_0^n(H)$ with $H \oplus \ldots \oplus H$ and H, respectively. And so, if $y = x_1 \oplus \ldots \oplus x_n$, then $\delta_A^1(y) = A_1x_1 + \ldots + A_nx_n$. And the dual map δ_A^{1*} of δ_A^1 is the map from H to $H \oplus \ldots \oplus H$ such that

$$\delta_A^{1*}(x) = A_1^*x \oplus \ldots \oplus A_n^*x$$

for $x \in H$. We can easily verify that $A_1^*A_1 + \ldots + A_n^*A_n$ and $A_1A_1^* + \ldots + A_nA_n^*$ are invertible if $A = (A_1, \ldots, A_n)$ is non-singular, in the same way as Vasilescu.

From the above lemma, we can get the next theorem.

THEOREM 1. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of commuting operators on H. Then $\sigma_\pi(A) \cup \sigma_\varrho(A) \subset S_p(A, H)$.*

Proceeding further, we need the following theorem gained by Dash. But we shall give a simple proof.

THEOREM A (A. T. Dash [4]). *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of commuting normal operators. Then $\sigma(A) = \sigma_\pi(A)$.*

Proof. It is sufficient to prove that $\sigma(A) \subset \sigma_\pi(A)$. If $0 = (0, \ldots, 0)$ is not in $\sigma_\pi(A)$, then $\sum_{i=1}^{n} A_i^*A_i$ is invertible. For every B in $\{A_1, \ldots, A_n\}'$,

$$\Big(\sum_{i=1}^{n} A_i^*A_i\Big)^{-1} A_j^* B = B\Big(\sum_{i=1}^{n} A_i^*A_i\Big)^{-1} A_j^*.$$

So, $(\sum_{i=1}^{n} A_i^*A_i)^{-1}A_j^*$ belongs to A'' for each $j = 1, \ldots, n$, and

$$\Big(\sum_{i=1}^{n} A_i^*A_i\Big)^{-1} A_1^*A_1 + \ldots + \Big(\sum_{i=1}^{n} A_i^*A_i\Big)^{-1} A_n^*A_n = I.$$

Therefore, 0 is not in $\sigma(A)$. So, we get the proof.

Thus through Theorem 1 and Theorem A we have the following theorem.

THEOREM 2. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of commuting normal operators. Then $S_p(A, H) = \sigma(A) = \sigma_\pi(A)$.*

COROLLARY 1. *A is non-singular if and only if $A_1^*A_1 + \ldots + A_n^*A_n$ is invertible on H.*

We denote the *joint numerical range* $W(A)$ of $A = (A_1, \ldots, A_n)$ by

$$W(A) = \{((A_1x, x), \ldots, (A_nx, x)) : \|x\| = 1\}.$$

And, for a set W in the n-dimensional unitary space C^n, we denote the convex hull and the closure of W by $\operatorname{conv} W$ and $\overline{W}$, respectively. Then we get the next corollary.

COROLLARY 2. *Under the same assumption as in Theorem 2, we get that $\operatorname{conv} S_p(A, H) = \overline{W(A)}$.*

Proof. It is known that $\operatorname{conv}\sigma(A) = \overline{W(A)}$ (see [5]). Hence, $\operatorname{conv} S_p(A, H) = \overline{W(A)}$.

3. The aim of this section is to show that Taylor's joint spectrum coincides with Dash's joint spectrum in the case of commuting operators also on a finite-dimensional Hilbert space. For the sake of its proof we need the results gained in [3]. About Proposition B, we quote it without the proof, but about Theorem D we shall give a simple proof of it, based upon Engel's Theorem.

PROPOSITION B (M. Chō and M. Takaguchi [3]). *Let* $A = (A_1, \ldots, A_n)$ *be an n-tuple of commuting matrices and* $S_1, \ldots, S_n$ *be semi-simple part of* $A_1, \ldots, A_n$, *respectively. Then*

$$\sigma(A_1, \ldots, A_n) \subset \sigma(S_1, \ldots, S_n).$$

Moreover,

$$\sigma(S_1, \ldots, S_n) = \sigma_p(S_1, \ldots, S_n).$$

Proof. See Lemma 4 in [3].

The next theorem is the simple case of the Engel's Theorem (cf. Theorem 2.4, p. 135 in [6]). In this case we can simply verify.

THEOREM C. *Let* $N_1, \ldots, N_n$ *be commuting nilpotent matrices. If non-zero subspace* V *is invariant for* $N_1, \ldots, N_n$, *then there exists a non-zero vector* x *in* V *such that*

$$N_i x = 0, \quad i = 1, \ldots, n.$$

Proof. We shall prove inductively. For $n = 1$, since $N_1(V) \subsetneq V$, we get $\ker N_1 \cap V \supsetneq \{0\}$. So it is true for $n = 1$. We assume that it is true for $n-1$. We put

$$W = \{x \in V: N_i x = 0,\ i = 1, \ldots, n-1\},$$

then W is non-zero invariant subspace for N_n. In the same way as above there exists a non-zero vector x in W such that $N_n x = 0$. Therefore, it is true for n. So we get the proof.

THEOREM D (M. Chō and M. Takaguchi [3]). *Let* $A = (A_1, \ldots, A_n)$ *be an n-tuple of commuting matrices. Then* $\sigma(A) = \sigma_p(A)$.

Proof. We decompose these matrices $A_1, \ldots, A_n$ such that

$$A_i = S_i + N_i, \quad i = 1, \ldots, n,$$

where, for each i, S_i, N_i are semi-simple part, nilpotent part of A_i, respectively. Then from Proposition B we get that

$$\sigma(A_1, \ldots, A_n) \subset \sigma(S_1, \ldots, S_n) = \sigma_p(S_1, \ldots, S_n).$$

If the point $z = (z_1, \ldots, z_n)$ is in $\sigma(S_1, \ldots, S_n)$, then

$$V = \{x: S_i x = z_i x,\ i = 1, \ldots, n\}$$

is non-zero subspace and invariant for $N_1, \ldots, N_n$. Hence, from Theorem C there exists a non-zero vector x in V such that

$$N_i x = 0, \quad i = 1, \ldots, n.$$

Therefore, for this vector x,

$$A_i x = z_i x, \quad i = 1, \ldots, n.$$

So we get $\sigma(S_1, \ldots, S_n) \subset \sigma_p(A_1, \ldots, A_n)$, and hence the proof is complete.

THEOREM 3. *Let* $A = (A_1, \ldots, A_n)$ *be an n-tuple of commuting matrices. Then*

$$\sigma(A) = S_p(A, H).$$

Proof. We may only prove that $\sigma_p(A) \subset S_p(A, H)$. If the point $z = (z_1, \ldots, z_n)$ is in $\sigma_p(A)$, then $\ker \delta^n_{z-A}$ is non-zero subspace. Thus

$$\operatorname{im} \delta^{n+1}_{z-A} \subsetneq \ker \delta^n_{z-A}.$$

So z belongs to $S_p(A, H)$. Thus the proof is complete.

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References

[1] R. Arens and A. P. Calderón, *Analytic function of several Banach algebra elements*, Ann. of Math. 62 (1955), pp. 204–216.
[2] Z. Ceaușescu, and F.-H. Vasilescu, *Tensor products and Taylor's joint spectrum*, Studia Math. 62 (1978), pp. 305–311.
[3] M. Chō and M. Takaguchi, *Joint spectra of matrices*, (to appear).
[4] A. T. Dash, *Joint spectra*, Studia Math. 45 (1973), pp. 225–237.
[5] — *Joint numerical range*, Glasnik Mat. 7, 1 (1972), pp. 75–81.
[6] S. Helgason, *Differential geometry and symmetric space*, Academic Press, 1962.
[7] J. L. Taylor, *A joint spectrum for several commuting operators*, J. Functional Analysis 6 (1970), pp. 172–191.
[8] F.-H. Vasilescu, *On pairs of commuting operators*, Studia Math. 62 (1978), pp. 203–207.
[9] W. Żelazko, *On a problem concerning joint approximate point spectra*, ibid. 45 (1973), pp. 239–240.

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