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On one generalization of weakly compactly generated Banach spaces*

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Abstract. The weak topology of Banach spaces is studied in context of the descriptive theory of sets. In this way the WCG property is generalized without losing the most useful properties of it.

Introduction. When considering the weak topology on a Banach space, we can express some well-known notions in terms of the descriptive theory of sets. For instance, Banach space is reflexive iff it is K_σ (i.e. is a countable union of compacts) in its weak topology. Similarly WCG Banach spaces are those which contain some dense K_σ subset. In this paper we study some generalizations (in the sense of the descriptive theory of sets) of this WCG property. The methods are similar to those used when dealing with WCG Banach spaces but after proving the main theorem (Theorem 1), the handling with them is more easy.

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Notation and basic definitions. In this paper we denote by ω the set of all natural numbers and at the same time the first infinite ordinal number. If X is a Banach space, we denote by $\operatorname{dens} X$ the smallest cardinality of a norm dense subset of X, by w^*-$\operatorname{dens} X^*$ the smallest cardinality of a w^*-dense subset of X^*. By $\overline{\operatorname{sp}} A$ we denote the closed linear span of a given set $A \subset X$. Subspace of a Banach space always means closed linear subspace. We say that a Banach space X is WCG (*weakly compactly generated*) iff there is a weak compact $C \subset X$ such that $\overline{\operatorname{sp}} C = X$. If S is a set, we denote by $c_0(S)$ the set of all mappings $x\colon S \to \boldsymbol{R}$ ($\boldsymbol{R}$ = all real numbers) which vanish at infinity. The norm $\|\cdot\|$ of a Banach space X is called *locally uniformly rotund* (LUR) iff for any sequence $\{x_n\}_{n=1}^{\infty}$ in the unit sphere of X and for any x the following implication holds:

$$\|x_n + x\| \to 2 \Rightarrow x_n \to x.$$

B'' will always denote the closed unit ball in X^{**}.

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We start with our fundamental definition:

DEFINITION 1. Let T' be a topological space, $T \subset T'$. We say that T is *countably determined* (CD) *in* T' iff there are compacts $\{A_i;\ i \in \omega\}$ in T' such that: for any $x \in T$ there is $\varrho \subset \omega$ such that $x \in \bigcap\{A_i;\ i \in \varrho\} \subset T$. In this case we say that T is *determined by* $\{A_i;\ i \in \omega\}$ *in* T'.

For our purposes the following lemma is useful:

LEMMA 1. *Let T be a uniform space countably determined in a uniform space T'. Let, moreover, T be a uniform subspace of a topologicaly complete uniform space S. Then T is countably determined in S.*

Proof. Let $\{A_i;\ i \in \omega\}$ be a sequence of compacts in T' determining T in T'. For any $i \in \omega$ we set $C_i = \mathrm{cl}(A_i \cap T)$, where cl denotes the closure in S. It is easy to see that C_i are compacts in S. Let $x \in T$, we find a set $\varrho \subset \omega$ such that $x \in \bigcap_{i\in\varrho} A_i \subset T$, hence $x \in \bigcap_{i\in\varrho} C_i$. We show that $C = \bigcap_{i\in\varrho} C_i \subset T$. Let $y \in C$. So there are nets $\{y^i_a;\ a \in I_i\}$ in $A_i \cap T$, for any $i \in \varrho$, such that $y = S-\lim_{a\in I_i} y^i_a$ (limit in S). But every such net is Cauchy in T and so there are $y^i \in A_i$ such that $y^i = T'-\lim_{a\in I_i} y^i_a$. But $y^i = y^j$ for all $i, j \in \varrho$. To show this, take any W, closed element of uniformity on T'. We find an element of uniformity V on S, such that $(V \circ V) \cap (T \times T) \subset W$ ($\circ$ denotes the composition of elements of uniformity) and $a_0 \in I_i$ $\beta_0 \in I_j$ such that for any $a \in I_i$, $a \geqslant a_0$, $\beta \in I_j$, $\beta \geqslant \beta_0$: $(y^i_a, y) \in V$, $(y^j_\beta, y) \in V$. Then $(y^i_a, y^j_\beta) \in W$. So we have $(y^i, y^j) \in W$ and W being arbitrary we get $y^i = y^j$. Hence $y^1 \in \bigcap_{i\in\varrho} A_i \subset T$, so $y = T-\lim_{a\in I_1} y^1_a$, from which $y = y^1 \in T$.

COROLLARY 1. *Let a Banach space X be in its weak topology countably determined in some uniform space T'. Then X is countably determined in X^{**} in its weak-star topology.*

Now the following definition is natural:

DEFINITION 2. A Banach space X will be called *weakly countably determined* (shortly WCD) iff X in its weak topology is countably determined in X^{**} in its weak-star topology.

Remark 1. In [6] J. Lindenstrauss gave the following question: Is it true that a Banach space is WCG iff it is Lindelöff in its weak topology? The "only if" part was proved negative by H. P. Rosenthal in [8] by finding a non WCG subspace of a WCG Banach space. The "if" part was proved independently by D. Preiss and M. Talagrand, using the notions of analytic (for definition see [3]) and $K_{\sigma\delta}$ topological space (countable intersection of countable unions of compacts). It is easy to show (using [3]) that

$$X \text{ is } K_{\sigma\delta} \text{ in } (X^{**}, w^*) \Rightarrow X \text{ is analytic in its weak topology} \Rightarrow X \text{ is WCD}.$$

Moreover, Talagrand gave an example of a Banach space which is not a subspace of any WCG Banach space but is analytic in its weak topology ([10]) or even an example of such a space which is, in its weak topology, $K_{\sigma\delta}$ ([11]). See also the example of Pol in [7].

LEMMA 2. *Let X be a WCG Banach space. Then X is $K_{\sigma\delta}$ in X^{**} in its weak-star topology.*

Proof. We simply set $A_{in} = nC + \frac{1}{i}B''$, where C is an absolutely convex weak compact fundamental in X and B'' is the closed unit ball in X^{**}. Than A_{in} are weak-star compacts in X^{**} and

$$X = \bigcap_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A_{in}.$$

LEMMA 3. *Let T be a topological space which is countably determined in some topological space T'. Then T is Lindelöff.*

Proof. Let $\mathfrak{A}$ be an open covering of T, $\mathfrak{A}'$ the system of all finite unions of elements of $\mathfrak{A}$. Let us denote

$$\mathscr{S} = \{\pi \subset \omega^\omega;\ \bigcap_{i\in\omega} A_{\pi(i)} \subset T\},$$

where A_i determine T in T'. We define $f\colon \mathscr{S} \to \exp T$ by

$$f(\pi) = \bigcap_{i\in\omega} A_{\pi(i)}, \quad \text{for all } \pi \in \mathscr{S}.$$

For any $G \in \mathfrak{A}'$ we denote $f^{-1}(G) = \{\pi \in \mathscr{S};\ f(\pi) \subset G\}$. We show that $\{f^{-1}(G);\ G \in \mathfrak{A}'\}$ is an open covering of $\mathscr{S}$ ($\mathscr{S}$ with the topology of ω^ω). First every $f^{-1}(G)$ is open in $\mathscr{S}$ because if $\pi \in f^{-1}(G)$, then there is a $i_0 \in \omega$ such that $\bigcap_{i=1}^{i_0} A_{\pi(i)} \subset G$ and so for every $\pi' \in \mathscr{S}$ so near to π that $\pi(i) = \pi'(i)$ for all $i \leqslant i_0$ we have

$$f(\pi') \subset \bigcap_{i=1}^{i_0} A_{\pi'(i)} = \bigcap_{i=1}^{i_0} A_{\pi(i)} \subset G, \quad \text{i.e.} \quad \pi' \in f^{-1}(G).$$

Secondly, for any $\pi \in \mathscr{S}$ there is a $G \in \mathfrak{A}$ such that $f(\pi) \subset G$ (from compactness of $f(\pi)$ and the fact that $\mathfrak{A}'$ is closed under finite unions). Now $\mathscr{S}$ is metric separable (because ω^ω is), so Lindelöff, and so there are $G_n \in \mathfrak{A}'$, $n \in \omega$, such that $\bigcup_{n\in\omega} f^{-1}(G_n) = \mathscr{S}$. It is easy to see that then $\bigcup_{n\in\omega} G_n = T$ (because $\bigcup_{\pi\in\mathscr{S}} f(\pi) = T$). Hence the system $\mathfrak{B} = \{G_{ni};\ n \in \omega,\ i = 1, 2, \ldots, i_n\}$, where $G_{ni} \in \mathfrak{A}$, $\bigcup_{i=1}^{i_n} G_{ni} = G_n$, is countable subcovering of $\mathfrak{A}$.

Now we turn our attention to the properties of WCD Banach spaces. We introduce the following definitions:

DEFINITION 3. (a) A Banach space X *has projectional resolution of identity* (PRI) iff there is a family $\{P_a;\ \omega \leqslant a \leqslant \mu\}$ of continuous linear projections from X into X, where μ is the first ordinal number of cardinality $\operatorname{dens} X$, satisfying:

(i) $\|P_\alpha\| = 1$ for any α s.t. $\omega \leqslant \alpha \leqslant \mu$,

(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ for any α, β s.t. $\omega \leqslant \beta \leqslant \alpha \leqslant \mu$,

(iii) $\operatorname{dens} P_\alpha(X) \leqslant \operatorname{card} \alpha$ for any α s.t. $\omega \leqslant \alpha \leqslant \mu$,

(iv) for all $x \in X$ the mapping $P_\alpha(x)$ is continuous on $\{\alpha;\ \omega \leqslant \alpha \leqslant \mu\}$ in the usual order topology on ordinal numbers,

(v) P_μ is identity on X.

(b) Let $\{A_i;\ i \in \omega\}$ be a family of subsets of X^{**}. We say that X *has a* PRI *subordinated to* $\{A_i;\ i \in \omega\}$ iff X has PRI $\{P_\alpha;\ \omega \leqslant \alpha \leqslant \mu\}$ and there are linear continuous operators $T_\alpha\colon X^{**} \to X^{**}$ such that $T_\alpha/X = P_\alpha$ and $T_\alpha(A_i) \subset A_i$ for all $i \in \omega$ and α s.t. $\omega \leqslant \alpha \leqslant \mu$.

To prove the main theorem we must somewhat refine the sets A_i from Definition 1.

Lemma 4. *Let X be a* WCD *Banach space. Then there are absolutely convex w^*-compacts A_i, $i \in \omega$, in X^{**} with nonvoid norm interiors which determine X in X^{**}.*

Proof. Let $\{C_i;\ i \in \omega\}$ be a sequence of w^*-compacts in X^{**} determining X and closed under finite intersections. Then it suffices to take for A_i the algebraic sum of the w^*-closure of the absolutely convex hull of C_i and $\frac{1}{i}B''$.

Theorem 1. *Let X be a* WCD *Banach space and $\{A_i;\ i \in \omega\}$ be the w^*-compacts from Lemma 4. Then X has a* PRI *subordinated to $\{A_i;\ i \in \omega\}$.*

Proof. We prove the theorem in several steps similarly as in [1].

Step I. Let E be a finite-dimensional subspace of X, $f_1, \ldots, f_m \in X^*$, $m \in \omega$, let $n, s \in \omega$. Then there exist C—a separable subspace of X^{**}—such that for any $\varepsilon > 0$ and any finite-dimensional subspace Z in X containing E, $\dim(Z/E) = n$, there is a linear continuous operator $T\colon Z \to C$ satisfying:

(i) $T(e) = e$ for all $e \in E$,

(ii) $T(Z \cap A_i) \subset (1+\varepsilon)A_i$ for all $i \leqslant s$,

(iii) $|f_k(z) - f_k(Tz)| \leqslant \varepsilon\|z\|$ for all $k \leqslant m$ and $z \in Z$,

(iv) $T(Z \cap X) \subset X$.

Proof of this step is only a slight generalization of Lemma 2 from [4] for finite number of norms (we take Minkowski functionals of A_i, $i \leqslant s$).

Step II. Let $E, f_1, \ldots, f_m, m, n$ be as in Step I. Then there exists a separable subspace C in X^{**} such that for any Z as in Step I there is $T\colon Z \to X^{**}$ satisfying:

(i) $T(e) = e$ for all $e \in E$,

(ii) $T(Z \cap A_i) \subset A_i$ for all $i \in \omega$,

(iii) $f_k(z) = f_k(Tz)$ for all $k \leqslant m$, $z \in Z$,

(iv) $T(Z \cap X) \subset C$.

Proof of Step II. For any $s \in \omega$ we find C_s separable and the operators $T_s\colon Z \to C_s$ from Step I, $\varepsilon = 1/s$. Let $C = \overline{\operatorname{sp}}(\bigcup_{s\in\omega} C_s)$; then C is separable. For any $s \in \omega$, $T_s(Z \cap B'') \subset 2B''$, so we can consider $\{T_s/(Z \cap B'');\ s \in \omega\}$ as a net in $(2B'')^{Z\cap B''}$ (B'' with w^*-topology). Hence this net has a certain point of condensation T. We extend T homogenously on Z and denote again by T. Evidently T satisfies (i)–(iii). But from (ii) it follows that $T(Z \cap X) \subset X$ and so, for any $x \in X$, $T(x)$ is a weak limit of some net in C, which gives (iv).

Step III. Let E, F be finite-dimensional subspaces of X, X^*, repectively. Then there is a separable subspace C in X^{**} and a continuous linear operator $T\colon X^{**} \to X^{**}$ satisfying:

(i) $T(e) = e$ for all $e \in E$,

(ii) $T(A_i) \subset A_i$ for all $i \in \omega$,

(iii) $T^*(f) = f$ for all $f \in F$,

(iv) $T(X) \subset C$.

Proof. From Step II, if we take for $f_1, \ldots, f_m$ any basis of F, $n \in \omega$, we get C_n, separable subspaces in X^{**}. Let $C = \overline{\operatorname{sp}}(\bigcup_{n\in\omega} C_n)$. For any finite-dimensional Z containing E we have $T_Z\colon Z \to X^{**}$ from Step II, $n = \dim(Z/E)$. Extending every T_Z by zero outside of Z and then restricting to B'', we can again as in Step II find a point of condensation in $(B'')^{B''}$, homogenous extensions of which satisfy (i)–(iv).

Step IV. Let E be a separable subspace of X, F a w^*-separable subspace of X^*. Then we can find a linear continuous operator $T\colon X^{**} \to X^{**}$ such that:

(i) $P = T/X$ is projection into X,

(ii) $E \subset P(X)$,

(iii) $F \subset P^*(X)$,

(iv) $T(A_i) \subset A_i$ for any $i \in \omega$,

(v) $P(X)$ is separable.

Proof of Step IV. We use Step III and procede as in [1] but we limit in $(B'')^{B''}$ and use the fact that all A_i are preserved in this limiting (so we get that X is preserved).

Step V. Let E, F are subspaces of X, X^*, respectively, let $\operatorname{dens} E \leqslant \aleph$, $w^*\text{-}\operatorname{dens} F \leqslant \aleph$, where $\aleph$ is a given cardinal number. Then there is a linear continuous operator $T\colon X^{**} \to X^{**}$ such that:

(i) $P = T/X$ is projection into X,

(ii) $E \subset P(X^*)$,

(iii) $F \subset P^*(X)$,

(iv) $T(A_i) \subset A_i$ for any $i \in \omega$,

(v) $\operatorname{dens} P(X) \leqslant \aleph$.

Proof of Step V. By transfinite induction over $\aleph$ using Step IV.

Step VI. (Proof of the theorem). We construct PRI suborinated to $\{A_i;\ i \in \omega\}$ by transfinite induction over $\operatorname{dens} X$. The first separable projection we have found in Step IV. Then we procede again as in [1] but limit again in $(B'')^{B''}$. Norm one of all members of our PRI we assure, if necessary, simply by adding B'' between the members of $\{A_i;\ i \in \omega\}$.

Corollary 2. *Let X be a* WCD *Banach space. Then X has the weak-star cardinality property, i.e. for any subspace Y in X* w^*-dens Y^* = dens Y.

Proof. Since WCD property is hereditary on closed subspaces, it suffices to show that $\operatorname{dens} X = w^*\text{-}\operatorname{dens} X^*$. In Step V of Theorem 1 we set $E = \{0\}$, $F = X^*$, $\aleph = w^*\text{-}\operatorname{dens} X^*$. So we can find a projection P: $X \to X$ such that $X^* \subset P^*(X^*)$, $\operatorname{dens} P(X) \leqslant \aleph$. But then P^* is identity on X^*, so P is the identity on X. This immediately gives: $\operatorname{dens} X = \operatorname{dens} P(X) \leqslant w^*\text{-}\operatorname{dens} X^*$. The other inequality holds in every Banach space.

Corollary 3. *Let X be a* WCD *Banach space. Then*

(a) *There is a set Γ and a linear continuous one-to-one operator L from X into $c_0(\Gamma)$;*

(b) *X has an equivalent locally uniformly rotund norm.*

Proof. (a) is proved in [6], (b) in [12], both for WCG Banach spaces. But those proofs clearly work in every Banach space subspaces of which have PRI (all subspaces or even only all complemented subspaces). Theorem 1 shows that WCD Banach spaces are of that type.

Theorem 2. *Let X be a* WCD *Banach space. Then the following conditions are equivalent:*

(I) *every subspace $Y \subset X$ has a projectional resolution of identity $\{P_\alpha;\ \omega \leqslant \alpha \leqslant \mu\}$ such that $\{P_\alpha^*;\ \omega \leqslant \alpha \leqslant \mu\}$ is* PRI *on Y^*,*

(II) *X has an equivalent Fréchet differentiable norm (at any nonzero point),*

(III) *X has an equivalent norm, the dual norm of which is* LUR,

(IV) *every subspace $Y \subset X$ has a shrinking Markuševič basis, i.e. a biorthogonal pair (H_1, H_2), $H_1 \subset Y$, $H_2 \subset Y^*$, such that $\overline{\operatorname{sp}} H_1 = Y$ and $\overline{\operatorname{sp}} H_2 = Y^*$.*

Proof. (IV) $\Rightarrow$ (III): If Y has a shrinking Markuševič basis, then X is WCG — see for instance [5] or [12]. So we can use [5] where (IV) $\Rightarrow$ (III) is proved for WCG Banach spaces.

(III) $\Rightarrow$ (II): It holds in every Banach space.

(II) $\Rightarrow$ (I): Let Y be a subspace of X, we take PRI $\{P_\alpha;\ \omega \leqslant \alpha \leqslant \mu\}$ on Y from Theorem 1. If we take on X and so on Y this equivalent Fréchet differentiable norm $|||\cdot|||$, $\{P_\alpha;\ \omega \leqslant \alpha \leqslant \mu\}$ satisfies the conditions (ii)–(v) from Definition 3(a) and (i) is weakened to (i)′: there is $K > 0$ such that $|||P_\alpha||| \leqslant K$ for all α, $\omega \leqslant \alpha \leqslant \mu$. Now it is shown in [5] (with $|||P||| = 1$, but it is not esential) that then $\{P_\alpha^*;\ \omega \leqslant \alpha \leqslant \mu\}$ also satisfies (i)′, (ii)–(v) in the norm dual to $|||\cdot|||$. So it is PRI on X^* in the norm dual to the previous norm.

(I) $\Rightarrow$ (IV): By transfinite induction over dens X:

1. If X is separable, then X^* is separable. It follows simply from the same number of projections in those PRI on X and X^*. In that case it is well-known fact that X has a shrinking Markuševič basis.

2. Let $\operatorname{dens} X > \aleph_0$ and let X satisfy (I). Let every WCD Banach space Z which satisfies (I) and has density smaller than X has a shrinking Markuševič basis. Take a PRI $\{P_\alpha;\ \omega \leqslant \alpha \leqslant \mu\}$ from (I) for X. For any α, $\omega \leqslant \alpha < \mu$, Banach space $(P_{\alpha+1} - P_\alpha)(X)$ has a shrinking Markuševič basis (H_α^1, H_α^2). Set $H^1 = \bigcup_{\omega \leqslant \alpha < \mu} H_\alpha^1$, $H^2 = \bigcup_{\omega \leqslant \alpha < \mu} (P_{\alpha+1} - P_\alpha)^*(H_\alpha^2)$ (we consider $(P_{\alpha+1} - P_\alpha)$ as mapping from X onto $(P_{\alpha+1} - P_\alpha)(X)$ and in this sense we take $(P_{\alpha+1} - P_\alpha)^*$). Then we show that the pair (H^1, H^2) is a shrinking Markuševič basis of X. It is easy to show that (H^1, H^2) is biorthogonal. $\overline{\operatorname{sp}} H^1$ is X because it contains all subspaces $(P_{\alpha+1} - P_\alpha)(X)$ the union of which is dense in X, as follows from (iv) in Definition 3. Similarly $\overline{\operatorname{sp}} H^2$ contains all subspaces $\overline{\operatorname{sp}}(P_{\alpha+1} - P_\alpha)^*(H_\alpha^2) = (P_{\alpha+1}^* - P_\alpha^*)(X^*)$ for all α, $\omega \leqslant \alpha < \mu$, and so $\overline{\operatorname{sp}} H^2 = X^*$. Thus we proved that every Banach space which is WCD and (I) has a shrinking Markuševič basis. But both these properties are hereditary and so X satisfies (IV).

Close connection of WCD and WCG Banach spaces show also these theorems:

Theorem 3. (a) *Let X be a* WCD *Banach space, Y a Banach space, and f a linear continuous mapping of X onto Y. Then Y is* WCD.

(b) *Let X be a Banach space such that X^* is* WCD. *Then X has the cardinality property,* i.e. dens Y = dens Y^* *for any subspace $Y \subset X$.*

Proof: (a) Take a family $\{A_i;\ i \in \omega\}$ of w^*-compacts in X^{**} determining X in X^{**} and closed under finite intersections. Let $C_i = f^{**}(A_i)$ for all $i \in \omega$. C_i are w^*-compacts in Y^{**}. Let $y \in Y$. We find $x \in X$ s.t. $f(x) = y$ and $\varrho \subset \omega$ s.t. $x \in \bigcap_{i \in \varrho} A_i \subset X$, $A_i \subset A_j$ for any $j \leqslant i$, $i, j \in \varrho$. Then $y \in \bigcap_{i \in \varrho} C_i$. Moreover, for any $z \in \bigcap_{i \in \varrho} C_i$ we find $x_i \in A_i$, $i \in \varrho$, such that $f^{**}(x_i) = z$. We can take $\{x_i;\ i \in \varrho\}$ bounded and so it has a w^*-condensation point — say x. But then $x \in \bigcap_{i \in \varrho} A_i$ so $x \in X$ and so $z = w^*\text{-}\lim_{\alpha \in I} f^{**}(x_\alpha)$

$= f^{**}(x)$, where $\{x_\alpha;\ \alpha \in I\}$ is a subnet of $\{x_i;\ i \in \varrho\}$ w^*-converging to x, hence $z \in Y$.

(b) In every Banach space Y $\operatorname{dens} Y \leqslant \operatorname{dens} Y^*$. Now when Y is subspace of X, Y^* is isometrically isomorphic to $X^*/Y^\perp$, where $Y^\perp = \{g \in X^*;\ g(y) = 0 \text{ for all } y \in Y\}$, so Y^* is a linear continuous image of X^*. Hence using (a) and Corollary 2 $\operatorname{dens} Y^* \leqslant w^*\text{-}\operatorname{dens} Y^{**} \leqslant \operatorname{dens} Y$.

Remark 2. In [9] it was proved that the cardinality property of X is equivalent with the Radon–Nikodým property of X. So every WCD dual has the Radon–Nikodým property.

THEOREM 4. *Let X and X^* be both* WCD *Banach spaces. Then X is* WCG.

Proof. Using Corollary 3(b), we can suppose that X has LUR norm. Let $\{A_i;\ i \in \omega\}$, $\{C_i;\ i \in \omega\}$ be a weak-star compacts determining X, X^* in X^{**}, X^{***}, respectively. From Theorem 1 we find a PRI $\{P_\alpha;\ \omega \leqslant \alpha \leqslant \mu\}$ on X^* subordinated to $\{A_i^0;\ i \in \omega\} \cup \{C_i;\ i \in \omega\}$. Here A_i^0 means the polar set of A_i in X^{***}. We show that then $\{P_\alpha^*/X;\ \omega \leqslant \alpha \leqslant \mu\}$ is PRI on X. From the preservation of all A_i by all P_α it follows that $P_\alpha^*(X) \subset X$ for all α, $\omega \leqslant \alpha \leqslant \mu$. P_α^*/X clearly satisfy (i), (ii), (v) from Definition 3. To prove (iii) take any α: $\omega \leqslant \alpha \leqslant \mu$. Because $\big(P_\alpha^*(X)\big)^*$ is isometrically isomorphic with $P_\alpha(X^*)$, from Theorem 3 we get:

$$\operatorname{dens} P_\alpha^*(X) = \operatorname{dens}\big(P_\alpha^*(X)\big)^* = \operatorname{dens} P_\alpha(X^*) \leqslant \operatorname{card} \alpha.$$

To prove (iv) take any $x \in X$ and α — a limit ordinal number, $\omega \leqslant \alpha \leqslant \mu$. It is easy to show that then $P_\beta^*(x)$ converge weakly to $P_\alpha^*(x)$ and $\|P_\beta^*(x)\|$ converge to $\|P_\alpha^*(x)\|$ over $\beta < \alpha$. Hence using the LUR property of the norm of X, $P_\beta^*(x)$ converge to $P_\alpha^*(x)$ in norm.

We have proved that X has a PRI, the dual projections of which form a PRI on X^*. But if Y is a subspace of X, then Y and Y^* are both WCD and so we have showed that X satisfies condition (I) from Theorem 2 and so (using again [5]) X is WCG.

Open problems.

(1) X is WCD $\Rightarrow$ X is analytic in its weak topology?

(2) X is analytic in its weak topology $\Rightarrow$ X is $K_{\sigma\delta}$ in its weak topology?

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