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An approximation problem in $L^p([0, 2\pi])$, $2 < p < \infty$

by

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Abstract. We treat the remaining case ($2 < p < \infty$) of an approximation problem earlier considered by Kahane and Rider.

For $n \in Z$, let $e_n(x)$ be the exponential e^{inx}, and for $f \in L^1([0, 2\pi])$ define the Fourier coefficient $\hat{f}(n)$ to be $(2\pi)^{-1} \int_0^{2\pi} f(x) e_{-n}(x)\,dx$. Then the Fourier series of f is $\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$.

We consider the following question. If $f \in L^p$ ($= L^p([0, 2\pi])$), is f the limit in the L^p norm of trigonometric polynomials q such that $\hat{q}(n) = \hat{q}(m)$ whenever $\hat{f}(n) = \hat{f}(m)$? For $p = 1$ a negative answer follows from results of Kahane [2], while a construction of Rider [4] gives the negative answer for $1 < p < 2$. For $p = 2$ the question is of course trivial. The purpose of this note is to give a negative answer for $2 < p < \infty$. Our method follows the broad outline of that of Kahane, but the details are quite different. We mention that this problem is closely related to a question about closed convolution subalgebras of L^p —see [5], [2], [4], [1].

THEOREM. *Fix p with $2 < p < \infty$. There exists a collection $\{E_j\}_{j=1}^{\infty}$ of pairwise disjoint finite subsets of Z and a function $f \in L^p$ such that $\hat{f}$ is constant on each E_j, $\hat{f} = 0$ off of $\bigcup_{j=1}^{\infty} E_j$, and such that f is not approximable in L^p by polynomials of the form $\sum_j b_j \sum_{n \in E_j} e_n$.*

Proof. In the following, C will denote a positive constant independent of k but which may increase from line to line. Let r be an even integer such that

$$r(1/2 - 1/p) > 1.$$

Let $n_0 = 0$, $m_0 = 1$, $n_k = (k!)^r$, $m_k = (k+2)^{r/2} n_k$ $(k = 1, 2, \ldots)$. Let $p_0 = 0$ and let $\{p_k\}_{k=1}^{\infty}$ be a sequence of positive integers which increases

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so rapidly that there exist trigonometric polynomials f_k^1, f_k^2 $(k = 1, 2, \ldots)$ satisfying

(1)
$$\hat{f}_k^1(n) = \begin{cases} 1 & \text{if} \quad 1 \leqslant n \leqslant p_{k-1}+m_{k-1}+n_{k-1}, \\ 0 & \text{if} \quad p_k < n, \end{cases}$$
$$\hat{f}_k^2(n) = \begin{cases} 0 & \text{if} \quad 1 \leqslant n \leqslant p_{k-1}+m_{k-1}+n_{k-1}, \\ 1 & \text{if} \quad p_k+1 \leqslant n \leqslant p_k+n_k+m_k, \end{cases}$$
$$\|f_k^1\|_1 \; (= \|f_k^1\|_{L^1}), \qquad \|f_k^2\|_1 \leqslant 1+1/k^2.$$

We can also assume $p_k \geqslant 2p_{k-1}$, $k = 1, 2, \ldots$

Let $E_1^{**} = \{1\}$. For $k \geqslant 1$ we will find disjoint subsets E_k^* and E_{k+1}^{**} of $\{p_k+1, \ldots, p_k+n_k+m_k\}$ such that (2)–(4) below will hold.

(2) If $I_k^* = \sum_{n\in E_k^*} e_n$, $I_k^{**} = \sum_{n\in E_k^{**}} e_n$, then $\sum_{k=1}^{\infty} \frac{\|I_k^{**}\|_p}{\|I_k^*\|_p} < \infty$.

(3) There exists a sequence $\{a_k\}_{k=1}^{\infty}$ of positive numbers such that the series $\sum_{k=1}^{\infty} (-1)^k a_k (I_k^{**}+I_k^*) \doteq \sum_{k=1}^{\infty} (-1)^k a_k I_k$ is the Fourier series of an L^p function f.

(4)
$$\frac{1}{\|I_k^*\|_p} = o(a_k).$$

Let us now establish

(5) If $q = \sum_{k=1}^{K} b_k I_k$, then $|b_k| \leqslant \frac{C\|q\|_p}{\|I_k^*\|_p}$.

With (4) this will show that no sequence of such q's can approximate f in L^p.

By (1), $f_K^2 * q = b_K I_K^*$, $f_K^1 * q = q - b_K I_K^*$. Since $\|f_K^j\|_1 \leqslant 1+1/K^2$, it follows that
$$|b_k|\;\|I_K^*\|_p \leqslant (1+1/K^2)\,\|q\|_p,$$
$$\|q - b_K I_K^*\|_p \leqslant (1+1/K^2)\,\|q\|_p.$$

The first of these gives $\|b_K I_K^{**}\|_p \leqslant (1+1/K^2)\frac{\|I_K^{**}\|_p}{\|I_K^*\|_p}\|q\|_p$, which combines with the second to give
$$\Big\|\sum_{k=1}^{K-1} b_k I_k\Big\|_p = \|q - b_k I_k\|_p \leqslant (1+1/K^2)\left(1+\frac{\|I_K^{**}\|_p}{\|I_K^*\|_p}\right)\|q\|_p.$$

Iterating this process, we get $(1 \leqslant k \leqslant K-1)$
$$|b_k|\;\|I_k^*\|_p \leqslant \prod_{l=k}^{K}(1+1/l^2)\prod_{l=k+1}^{K}\left(1+\frac{\|I_l^{**}\|_p}{\|I_l^*\|_p}\right)\|q\|_p.$$

With (2), this gives (5). Thus it remains to find the sequences $\{a_k\}$, $\{E_k^*\}$, $\{E_K^{**}\}$ which satisfy (2)–(4).

Let $a_k = n_{k+1}^{-1/2} = [(k+1)!]^{-r/2}$. We will choose disjoint subsets E_k^*, E_{k+1}^{**} of $\{p_k+1, \ldots, p_k+n_k+m_k\}$ with $|E_{k+1}^{**}| = m_k$ and $|E_k^*| = n_k$ such that

(6)
$$C^{-1} n_k (m_k+n_k)^{-1/p} \leqslant \|I_k^*\|_p,$$

(7)
$$\|a_k I_k^* - a_{k+1} I_{k+1}^{**}\|_p \leqslant C(a_k^2 n_k + a_{k+1}^2 m_k)^{1/2}.$$

Since $|E_k^{**}| = m_{k-1}$, we have $\|I_k^{**}\|_p \leqslant m_{k-1}^{1-1/p}$ by the Hausdorff–Young theorem. Since $r(1/2-1/p) > 1$, (2) is a consequence of (6), while (4) follows from (6) and the definition of a_k. Statement (3) follows from (7) since $r \geqslant 4$ implies that $\sum_{k=1}^{\infty} (a_k^2 n_k + a_{k+1}^2 m_k)^{1/2} < \infty$. Thus it remains to establish (6) and (7). Since $n_k a_k = m_k a_{k+1}$, these follow from the proposition below.

Proposition. *Suppose $2 < p < \infty$. Let n, m be positive integers and let a, b be positive numbers such that $na = mb$. Then there exists a partition of the set $\{1, 2, \ldots, n+m\}$ into two sets E and F with $|E| = n$, $|F| = m$ such that*

(i) $\|\sum_{j\in E} e_j\|_p \geqslant A_p^{-1} n (n+m)^{-1/p}$,

(ii) $\|\sum_{j\in E} a e_j - \sum_{j\in F} b e_j\|_p \leqslant A_p (na^2+mb^2)^{1/2}$.

Here A_p depends only on p.

The proof of the proposition depends on two lemmas.

Lemma 1. *For $1 < p < \infty$, there is a positive constant B_p (depending only on p) such that for any positive integers n, m and any choice of numbers c_j, $1 \leqslant j \leqslant n+m$,*
$$B_p^{-1}\Big\|\sum_{1}^{n+m} c_j e_j\Big\|_p \leqslant \left[\frac{1}{n+m}\sum_{l=1}^{n+m}\Big|\sum_{j=1}^{n+m} c_j e^{2\pi i j l/(n+m)}\Big|^p\right]^{1/p} \leqslant B_p\Big\|\sum_{1}^{n+m} c_j e_j\Big\|_p.$$

(See Theorem 7.10, p. 30 of [6].)

Lemma 2. *For $1 < p < \infty$, there is a positive constant C_p such that the following holds. Fix positive integers n, m and numbers c_j, $1 \leqslant j \leqslant n+m$,* *such that* $\sum_{1}^{n+m} c_j = 0$. *Let* S *be the symmetric group on* $\{1, \ldots, n+m\}$. *Then*

$$\frac{1}{|S|}\sum_{\sigma\in S}\Big\|\sum_{1}^{n+m} c_{\sigma(j)}e_j\Big\|_p^p \leqslant C_p^p\Big(\sum_{1}^{n+m}|c_j|^2\Big)^{p/2}.$$

(This is a consequence of Lemma 1 above and Lemma 2, p. 261 of [3].)

The proof of the proposition is now easy. Let $c_1 = \ldots = c_n = a$, $c_{n+1} = \ldots = c_{n+m} = -b$. Then it follows from Lemma 2 that there exists $\sigma \in S$ such that

$$\Big\|\sum_{1}^{n+m} c_{\sigma(j)}e_j\Big\|_p \leqslant C_p(na^2+mb^2)^{1/2}.$$

Let $E = \{j: 1 \leqslant \sigma(j) \leqslant n\}$, $F = \{j: n+1 \leqslant \sigma(j) \leqslant n+m\}$. Then (ii) of the proposition holds. Statement (i) of the proposition is an immediate consequence of Lemma 1.

Added in proof. The theorem of this paper can also be found in the work of G. Bachelis and J. Gilbert, *Banach algebras with Rider subalgebras*, Bull. Inst. Math. Acad. Sinica 7 (1979), pp. 333–347.

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Identity of Taylor's joint spectrum and Dash's joint spectrum

by

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Abstract. In this note we show that Dash's joint spectrum coincides with Taylor's joint spectrum in the case of normal operators, and operators on a finite-dimensional Hilbert space.

1. Since the concept of joint spectrum for a family of operators was first introduced by R. Arens and A. P. Calderón [1], some authors have asserted its definition and properties. The typical and successful definitions among them have been carried out by J. L. Taylor [7] and A. T. Dash [4].

Throughout this note, let H be a complex Hilbert space and $B(H)$ be the algebra of all continuous linear operators on H. Let $A = (A_1, \ldots, A_n) \subset B(H)$ be an n-tuple of commuting operators, and A', A'' be commutant, double commutant algebra of the set $\{A_1, \ldots, A_n\}$ in $B(H)$, respectively.

Now, let $\{e_1, \ldots, e_n\}$ be a set of indeterminates and denote by E^n the exterior algebra, over the complex field C, with the generators $e_1, \ldots, e_n$. E_p^n will stand for the space of elements of degree p in E^n $(p = 1, \ldots, n)$. Then we denote by $E_p^n(H)$ the tensor product $H \otimes E_p^n$. An element $x \otimes e_{j_1} \wedge \ldots \wedge e_{j_p} \in E_p^n(H)$ will be written simply $xe_{j_1} \wedge \ldots \wedge e_{j_p}$. The space $E_p^n(H)$ can be endowed with a natural structure of a Hilbert space.

We define the map $\delta_A^p: E_p^n(H) \to E_{p-1}^n(H)$ such that

$$\delta_A^p(y) = \sum_{i=1}^{p}(-1)^{i-1}A_{j_i}xe_{j_1}\wedge\ldots\wedge\check{e}_{j_i}\wedge\ldots\wedge e_{j_p}$$

for $y = xe_{j_1}\wedge\ldots\wedge e_{j_p} \in E_p^n(H)$, where the symbol $\check{}$ means that the corresponding letter is omitted. For convenience, we put $\delta_A^0 = \delta_A^{n+1} = 0$. Then we see that every δ_A^p is a continuous linear map and $\delta_A^p \circ \delta_A^{p+1} = 0$. Thus, we get the sequence

$$0 \xrightarrow{\delta_A^{n+1}} E_n^n(H) \xrightarrow{\delta_A^n} E_{n-1}^n(H) \xrightarrow{\delta_A^{n-1}} \ldots \xrightarrow{\delta_A^2} E_1^n(H) \xrightarrow{\delta_A^1} E_0^n(H) \xrightarrow{\delta_A^0} 0.$$