Damit ist (6) und nach den obigen Überlegungen $x \in W_A$ nachgewiesen. Es gilt also $L_A = W_A$ (vgl. auch (2)).

FOLGERUNG 1. *Ist A konvergenztreu und gilt $L_A \neq W_A$, so ist A μ-stetig.*

Die Beziehung $L_A \neq W_A$ ist z.B. erfüllt, wenn A fastkoregulär ($F_A \neq W_A$) oder nicht ersetzbar ($L_A \neq F_A$; vgl. Korollar 2 von Bennett [2]) ist.

Beweis. Die Aussage ist trivial, falls A nicht μ-eindeutig ist. Ist A μ-eindeutig, so ist A genau dann μ-stetig, wenn $G = G_1 + G_2$, der Kern von μ_A, $\beta(c_A^\dagger, c_A)$-abgeschlossen ist. Letzteres folgt aber unter der Voraussetzung $L_A \neq W_A$ aus Theorem 2 und dem obigen Satz.

FOLGERUNG 2. *Ist A konvergenztreu und abschnittsbeschränkt (d.h. $L_A = c_A$), so ist A μ-stetig.*

Beweis. Wäre A nicht μ-stetig, so erhielten wir $L_A = W_A$ und damit $\lim_A \in G_2$. Dies hätte aber wegen (4) die Nicht-μ-Eindeutigkeit und damit die μ-Stetigkeit von A zur Folge.

Das Ergebnis in Folgerung 2 wurde von Wilansky [8] für den Fall „c_A ist BK-Raum" bewiesen.

Zum Abschluß sei noch auf das obige Beispiel hingewiesen: A ist μ-eindeutig, und es gilt $L_A = W_A$; weiter ist A auch μ-stetig, da A reversibel ist (vgl. Theorem 3 und die Bemerkung zu (1)).

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Operators which respect norm intervals

by

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Abstract. Let X, Y be real normed linear spaces. For $x_1, x_2 \in X$ let $[x_1; x_2]$ be the intersection of all closed balls containing x_1 and x_2. $[x_1; x_2]$ is called the norm interval between x_1 and x_2. A continuous linear operator $T: X \to Y$ is said to respect norm intervals if $T([x_1; x_2]) \subset [Tx_1; Tx_2]$ for $x_1, x_2 \in X$. We investigate the collection of these operators. For example, every extreme functional on X respects norm intervals and, conversely, the set of norm interval respecting functionals can be obtained from the extreme functionals by means of a Krein–Milman type theorem.

Norm interval respecting operators on function modules are investigated in more detail. As corollaries we obtain a result of Cunningham and Roy which characterizes the extreme functionals on function modules, theorems of the Banach–Stone type and characterizations for extreme operators between spaces of continuous function.

0. Introduction. Let X be a real normed linear space. If x_1, x_2 are elements of X, $[x_1, x_2]$ (the *norm interval* between x_1 and x_2) denotes the intersection of all closed balls which contain x_1 and x_2. Some basic properties and examples are considered in Section 1. In particular, we characterize the norm intervals in function modules by means of the norm intervals of the components.

The aim of this paper is the investigation of continuous linear operators $T: X \to Y$ between real normed linear spaces X, Y which respect norm intervals, i.e. operators for which $T([x_1; x_2]) \subset [Tx_1; Tx_2]$ for $x_1, x_2 \in X$. We prove that this class contains, for example, extreme functionals, isometries with dense range, and M-bounded operators. Some general properties of norm interval respecting operators are established in Section 2. Section 3 contains a theorem by which the norm interval respecting functionals on X can be obtained from the extreme functionals. We apply this theorem to investigate the norm interval respecting functionals on some classes of Banach spaces.

Finally, in Section 4, we consider norm interval respecting operators between function modules. The main theorems (Theorem 4.3, Theorem 4.7) admit a number of corollaries; for example we obtain a characterization of extreme functionals on function modules (a result which is due to Cunningham and Roy) and Banach–Stone type theorems as well as characterizations of M-bounded and extreme operators for spaces of continuous functions.

NOTATION. $D(x, r)$ (resp. $B(x, r)$) denotes the closed (resp. open) ball with centre x and radius r. $D(X)$ means the closed unit ball of a normed linear space X, $\operatorname{ex} D(X)$ the subset of its extreme points.

All spaces are assumed to be nonzero.

1. Norm intervals.

1.1. PROPOSITION. *Let X be a real normed linear space, $a \in \mathbf{R}$, $z, x_1, x_2, x_3 \in X$. Then*

(i) $[x_1; x_2] = \bigcap \{B(x, r) \mid x_1, x_2 \in B(x, r)\}$
$= \bigcap \{D(x, r) \mid x_1, x_2 \in B(x, r)\}$;

(ii) $z \in [x_1; x_2]$ *implies that* $x_1 + x_2 - z \in [x_1; x_2]$;

(iii) $[x_1 + x_3; x_2 + x_3] = [x_1; x_2] + x_3$, $a[x_1; x_2] = [ax_1; ax_2]$

(in particular, all norm intervals can be constructed from norm intervals of the form $[0; x]$*).*

Proof. (ii): Let $z \in [x_1; x_2]$ and $D(x, r)$ be a ball such that $x_1, x_2 \in D(x, r)$. It follows that $x_1, x_2 \in D(x_1 + x_2 - x, r)$ so that $\|(x_1 + x_2 - x) - z\| \leqslant r$, i.e. $x_1 + x_2 - z \in D(x, r)$.

The proof of the other assertions of the proposition is routine.

1.2. LEMMA. *Let $D := D(x, r)$ be a closed ball, z, y points in X such that $z \notin D$, $y \in B(x, r)$. Define $\hat{y}$ to be the unique point in $\operatorname{co}\{y, z\}$ (= the convex hull of $\{y, z\}$) such that $\|x - \hat{y}\| = r$. Then, for $D_t := D\big(t(x - \hat{y}) + \hat{y}, tr\big)$ (all $t \geqslant 1$) we have $z \notin D_t$ and $D \subset D_t$.*

Proof. For $\lambda \in [0, 1]$, let $x_\lambda := \lambda y + (1 - \lambda) z$. $\hat{y} = x_\eta$ for a uniquely determined $\eta \in]0, 1[$, and $x_\lambda \notin D$ for $\lambda < \eta$. Since $x_\eta - (1/t)(x_\eta - z) = x_{\eta(1-1/t)}$, it follows that $z \notin D_t$ for all $t \geqslant 1$. For $\hat{x} \in D$, $t \geqslant 1$, we have $(1 - 1/t) x_\eta + (1/t)\hat{x} \in D$. This implies that $\hat{x} \in D_t$.

1.3. PROPOSITION. (i) *For $z \notin [x_1; x_2]$ there are balls D with arbitrarily large radius such that $z \notin D$, $x_1, x_2 \in D$.*

(ii) *For $z \in [x_1; x_2]$, $\mu \in [0, 1]$, we have $z \in [x_1; \mu x_2 + (1 - \mu) z]$.*

Proof. (i): Let $D := D(x, r)$ be a ball such that $z \notin D$, $x_1, x_2 \in B(x, r)$. We only have to apply the preceding lemma with $y = x_1$; $x_1, x_2 \in D \subset D_t$, but $z \notin D_t$ for every $t \geqslant 1$.

(ii): We may assume that $\mu \in]0, 1[$. Suppose that $z \notin [x_1; \mu x_2 + (1 - \mu) z]$. It follows that there is a ball $D := D(x, r)$ such that $z \notin D$, $x_1, \mu x_2 + (1 - \mu) z \in B(x, r)$. Let $x_\lambda := \lambda x_2 + (1 - \lambda) z$ for $\lambda \in [0, 1]$. If η is the unique value in $]0, 1[$ such that $\|x - x_\eta\| = r$, then $\eta < \mu$ and $x_\lambda \notin D$ for $\lambda < \eta$. It follows that $z \notin D_t$, $D \subset D_t$ for all $t \geqslant 1$ (D_t as in 1.2). In particular, for $t_0 := (1 - \eta)/(\mu - \eta)$, we have $(1 - 1/t_0) x_\eta + (1/t_0) x_2 = x_\mu \in D$ which implies that $x_1, x_2 \in D_{t_0}$ in contradiction to $z \in [x_1; x_2]$.

1.4. EXAMPLES. (a) For $X = \mathbf{R}$ the norm intervals are the usual closed intervals.

(b) It can easily be shown that $[x_1; x_2] = \operatorname{co}\{x_1, x_2\}$ if X is a Hilbert space. More generally we will prove in Proposition 3.2 (iii) below that this is the case whenever the extreme functionals on X are weak*-dense in the dual unit sphere (in particular, every smooth space has this property). We will say that X *has small norm intervals* if $[x_1; x_2] = \operatorname{co}\{x_1, x_2\}$ for $x_1, x_2 \in X$.

(c) If K is a compact Hausdorff space, it is easy to see that (in the Banach space CK) $[f; g] = \{h \mid f \wedge g \leqslant h \leqslant f \vee g\}$ for $f, g \in CK$.

(d) If K is a compact convex subset of a locally convex Hausdorff space, then (in the Banach space AK)

$$[f; g] = \{h \mid (f \wedge g)^\vee \leqslant h \leqslant (f \vee g)^\wedge\} \quad \text{for} \quad f, g \in AK.$$

(e) It is clear that the definition of $[x_1; x_2]$ depends on the whole space X. If necessary, we have to write $[x_1; x_2]_X$ instead of $[x_1; x_2]$. In general, $[x_1; x_2]_Y \supsetneqq Y \cap [x_1; x_2]_X$ if Y is a subspace of X containing x_1 and x_2. However, it can easily be shown that $[x_1; x_2]_Y = Y \cap [x_1; x_2]_X$ if Y is a dense subspace of X containing x_1, x_2.

The structure of the norm intervals in example 1.4(c) is a special case of a result for function modules. Function modules have been introduced by Cunningham ([7]). They play an important role in M-structure theory (cf. for example [4], [5], [7]). Note that every Banach space can be regarded as a function module over a "maximal" base space.

1.5. DEFINITION. Let K be a compact Hausdorff space (the *base space*), $(X_k)_{k \in K}$ a family of real Banach spaces indexed by the points of K (the X_k are called the *component spaces*). A *function module* X is a closed subspace of the Banach space product of the X_k (i.e. the space of all bounded families $\big(x(k)\big)_{k \in K}$, provided with the supremum norm) for which the following properties are satisfied:

(i) $k \mapsto \|x(k)\|$ is upper semicontinuous for every $x \in X$;

(ii) $hx \in X$ for $x \in X$ and $h \in CK$ $\big((hx)(k) := h(k) x(k)\big)$;

(iii) $\{x(k) \mid x \in X\} = X_k$ for every $k \in K$;

(iv) $\{k \mid X_k \neq 0\}$ is dense in K.

Note. The most important properties are (i) and (ii). (iii) is always automatically satisfied ([7], p. 621) and (iv) has been included to avoid trivial situations.

1.6. PROPOSITION. *Let X be a function module over K with component spaces X_k. Then, for $x_1, x_2 \in X$,*

$$[x_1; x_2] = \{z \mid z \in X,\ z(k) \in [x_1(k); x_2(k)] \text{ for every } k \in K\} =: [x_1; x_2]_{\text{f.m.}}\,.$$

Proof. The inclusion $[x_1; x_2]_{\text{f.m.}} \subset [x_1; x_2]$ is obviously valid. Conversely, let $z \in X$ be given such that there is a $k \in K$ for which $z(k)$

$\notin [x_1(k); x_2(k)]$. We choose $x_k \in X_k$, $r > 0$ such that

$$x_1(k),\ x_2(k) \in B(x_k, r), \qquad \|z(k) - x_k\| > r.$$

By 1.3(i) we may assume that $r \geqslant \|x_1\| + \|x_2\|$. Let x be an element of X such that $x(k) = x_k$ (1.5(iii)). $\|(x_i - x)(k)\| < r$ $(i = 1, 2)$, so that there is a neighbourhood U of k with $\|(x_i - x)(l)\| < r$ for $i = 1, 2$, $l \in U$. We choose a function $h \in CK$ such that $0 \leqslant h \leqslant 1$, $h(k) = 1$, $h(l) = 0$ for $l \notin U$. Then the ball $D\big(x_1 + h(x - x_1), r\big)$ contains x_1 and x_2 but not z, i.e. $z \notin [x_1; x_2]$.

1.7. Note. It follows that a function module has small intervals only if $K = \{k_0\}$ and X_{k_0} has small norm intervals. This implies that a Banach space X cannot have small norm intervals if the centralizer of X is non-trivial (we recall that X has a function module representation where CK is isometrically isomorphic to the centralizer; see [2] or [4] for definitions). The converse is not true (consider f. ex. l_3^1).

2. Operators which respect norm intervals.

2.1. Definition. Let X and Y be real normed linear spaces, $T: X \to Y$ a linear continuous operator. We say that T respects norm intervals or that T is a *norm interval respecting operator* (abbr. nir-*operator*) if $T([x_1; x_2]) \subset [Tx_1; Tx_2]$ for $x_1, x_2 \in X$. If $Y = \boldsymbol{R}$, T will be called an nir-*functional*.

2.2. Proposition. *Let $T: X \to Y$ be a linear continuous operator.*

(i) *T is an* nir-*operator iff $T([0; x]) \subset [0; Tx]$ for every $x \in X$.*

(ii) *If $Y = \boldsymbol{R}$, the following assertions are equivalent:*

(a) *T is an* nir-*functional;*

(b) *there is a $c \in \boldsymbol{R}$ such that $T([0; x]) \subset [0; Tx]$ for every $x \in X$ such that $Tx = c$;*

(c) $[0; x] \subset \ker T$ *whenever* $x \in \ker T$.

Proof. (i): This follows at once from 1.1(iii).

(ii): (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) are trivially satisfied. (b) $\Rightarrow$ (a): We will consider the two cases $c \neq 0$ and $c = 0$.

$c \neq 0$: 1.1(iii) easily implies that $T([0; x]) \subset [0; Tx]$ for every x such that $Tx \neq 0$. It remains to show that $[0; x] \subset \ker T$ for $x \in \ker T$. Suppose that there are $x, z \in X$, $Tx = 0$, $z \in [0; x]$, $Tz \neq 0$. By 1.3(ii) we have $z \in [0; (x+z)/2]$ so that $Tz \in \big[0; T\big((x+z)/2\big)\big] = [0; Tz/2]$, a contradiction.

$c = 0$: We have to show that $Tx \neq 0$ implies that $T([0; x]) \subset [0; Tx]$. Let $x, z \in X$ be given such that $z \in [0; x]$, $Tx \neq 0$. Suppose that $Tz \notin [0; Tx]$. W.l.o.g. we may assume that $Tz < 0 < Tx$ (if necessary we consider $-x$ instead of x and $x - z$ instead of z; cf. 1.1(ii)). We choose a μ in $]0, 1[$ such that $T\big(\mu x + (1-\mu)z\big) = 0$. Since $z \in [0; \mu x + (1-\mu)z]$ (1.3(ii)) we get $Tz = 0$ in contrast to our assumption.

2.3. Examples. Each of the following mappings T is an nir-operator (resp. an nir-functional):

(a) T is a linear continuous operator from X to Y, and X has small norm intervals (1.4(b)).

(b) X is a function module, and $T = \pi_k: X \to X_k$ is the projection $x \mapsto x(k)$ onto the kth component.

(c) T is an isometric isomorphism from X onto Y (more generally: T is an isometric isomorphism from X onto a dense subspace of Y).

(d) $X = Y$, and T is an M-bounded operator (i.e. there is a $\lambda \in \boldsymbol{R}$ such that $Tx \in D$ for every closed ball D which contains $\pm\lambda x$; cf. [2] for more details).

(e) $Y = \boldsymbol{R}$, and T is an extreme point in the unit ball $D(X')$ of the dual X' of X.

(f) $X = AK$ (K as in 1.4(d)), $Y = \boldsymbol{R}$, and T is the evaluation map associated with an extreme point of K.

Proof. (a): Obvious.

(b): This follows at once from 1.6.

(c): This is a consequence of the remark in 1.4(e).

(d): Every M-bounded operator is (up to isometric isomorphism) a multiplication operator associated with a real-valued continuous function on a suitable function module (this is just the function module representation theorem; cf. [7], [4]). Thus, the assertion is an easy consequence of 1.6.

Another proof which does not depend on the function module representation theorem can be given using Proposition 2.6 below: for $p \in \operatorname{ex} D(X')$, there is a $\lambda_p \in \boldsymbol{R}$ such that $p \circ T = \lambda_p p$ ([2], Th. 4.8, [6], Th. 2.3). 2.3(e) and 2.6(ii) imply that $p \circ T$ is an nir-functional so that T must be an nir-operator by 2.6(i).

(e): Let p be an extreme point in the unit ball of X', $x \in X$. By 2.2(ii) we only have to show that $p(x) = 0$ yields $p|_{[0;x]} = 0$. For $\varepsilon > 0$, consider

$$K_1 := \operatorname{co}\big(\{(q, 0) \mid q \in D(X')\} \cup \{(q, q(x)) \mid q \in D(X')\}\big)$$

and

$$K_2 := \{(p, \varepsilon)\}$$

(in $X' \times \boldsymbol{R}$, X' provided with the weak*-topology). Let $(w, a) \in X \times \boldsymbol{R} \cong (X' \times \boldsymbol{R})'$ be a functional which strictly separates the disjoint compact convex sets K_1 and K_2:

$$p(w) + a\varepsilon > c > q(w),\ q(w) + aq(x)$$

for a suitable $c \in \boldsymbol{R}$ and all $q \in D(X')$. $0 \in D(X')$ implies that $c > 0$, and

$p \in D(X')$ gives $a > 0$. With $r := c/a$ it follows that

$$\|w/a\| = \max\{q(w/a) \mid q \in D(X')\} < r,$$
$$\|x + w/a\| = \max\{q(x + w/a) \mid q \in D(X')\} < r$$

so that $0, x \in D(-w/a, r)$. Hence $[0; x] \subset D(-w/a, r)$ so that, for $y \in [0; x]$, $p(y + w/a) \leqslant r$ and consequently $p(y) \leqslant c/a - p(w)/a < \varepsilon$. Considering $-p$ instead of p we get $p|_{[0;x]} \geqslant -\varepsilon$ as well so that, since ε was arbitrary, $p|_{[0;x]} = 0$.

(This result is a special case of [6], Prop. 2.2; this proposition has been the starting point for our investigations.)

(f): This follows from (e), since $\mathrm{ex}\, D((AK)') = \{\pm\delta_k \mid k \in \mathrm{ex}\, K\}$ (δ_k denotes the evaluation map $f \mapsto f(k)$). An independent proof can be given using a well-known fact from Choquet theory: for $f \in CK$ and $k \in \mathrm{ex}\, K$ we have $f^\wedge(k) = f(k) = f^\vee(k)$ ([1], I.4.1). Thus our assertion is an immediate consequence of 1.4(d).

2.4. Proposition. *Let X be a real normed linear space, $x \in X$. Then*

$$[0; x] = \{z \mid z \in X,\ p(z) \in [0; p(x)] \text{ for every } p \in \mathrm{ex}\, D(X')\}.$$

More generally: If N is a norm-defining family of nir-*functionals* (i.e. *every $p \in N$ is an* nir-*functional, and* $\|x\| = \sup\{|p(x)|/\|p\| \mid p \in N \setminus \{0\}\}$ *for* $x \in X$), *then N determines norm intervals*:

$$[0; x] = \{z \mid z \in X, p(z) \in [0; p(x)] \text{ for every } p \in N\} \quad (\text{all } x \in X).$$

Proof. "$\subset$" is clear by definition (resp. a consequence of 2.3(e)) Conversely, let $z \notin [0; x]$. There is a ball $D(y, r)$ such that $0, x \in D(y, r)$, $\|y - z\| > r$. We choose a $p \in N$ such that $|p(y) - p(z)| > r\|p\|$. Since $p(y - x) \leqslant r\|p\|$, $|p(y)| \leqslant r\|p\|$, we have $0, p(x) \in D(p(y), r\|p\|)$ and $p(z) \notin D(p(y), r\|p\|)$. This implies $p(z) \notin [0; p(x)]$.

2.5. Note. This proposition can be used to determine norm intervals of X if $\mathrm{ex}\, D(X')$ is known:

(a) $\mathrm{ex}\, D((CK)') = \{\pm\delta_k \mid k \in K\}$ (K as in 1.4(c)). This yields a new proof for 1.4(c).

(b) More generally, if X is a function module, then

$$\mathrm{ex}\, D(X') = \{f \circ \pi_k \mid k \in K,\ X_k \neq 0,\ f \in \mathrm{ex}\, D((X_k)')\}$$

([8]). Using this fact, 1.6 is a corollary of 2.4. We preferred, however, to give a more elementary proof of 1.6 (the results of Cunningham and Roy will be proved independently in Corollary 4.4 below).

(c) For $X = l^1$,

$$x = (x_n) \in X,\ [0; x] = \Big\{(y_n) \mid (y_n) \in X,\ \sum \varepsilon_i y_i \in \Big[0; \sum \varepsilon_i x_i\Big] \text{ for every sequence } (\varepsilon_i) \text{ in } \{-1, 1\}\Big\}.$$

2.6. Proposition. (i) *Let $T: X \to Y$ be a linear continuous operator. T is an* nir-*operator iff $p \circ T$ is an* nir-*functional for every* nir-*functional p in Y' iff $p \circ T$ is an* nir-*functional for every $p \in \mathrm{ex}\, D(Y')$.*

(ii) *Multiples aT and compositions $S \circ T$ of* nir-*operators are also* nir-*operators.*

(iii) *The set of* nir-*operators in $[X, Y]$ is closed with respect to the weak operator topology. In particular, the set of* nir-*functionals on X is weak*-closed in X'.*

Proof. (i): This follows at once from 2.4.

(ii): This is obvious.

(iii): Let $(T_\alpha)_{\alpha \in I}$ be a net in $[X, Y]$, $T_\alpha \to T \in [X, Y]$ (convergence with respect to the weak operator topology). For $p \in \mathrm{ex}\, D(Y')$, $p \circ T_\alpha \to p \circ T$ in (X', weak*-topology). By 2.6(ii) the $p \circ T_\alpha$ are nir-functionals. Thus, for $x \in X$ and $z \in [0; x]$, $p \circ T_\alpha z \in [0; p \circ T_\alpha x]$ so that $p \circ Tz \in [0; p \circ Tx]$. It follows that T is also an nir-operator.

3. The structure of the set of nir-functionals. Let X be a real normed linear space. The results of Section 2 imply that every functional in the weak*-closure of $\boldsymbol{R}(\mathrm{ex}\, D(X'))$ is an nir-functional. Thus there are even three-dimensional spaces which admit nir-functionals that are not multiples of extreme functionals. However, it is possible to derive a Krein–Milman type theorem for nir-functionals (Theorem 3.6) by which the set of nir-functionals can be obtained from the extreme functionals.

3.1. Definition. Let X be a real normed linear space.

(i) $\mathrm{nir}(x) := \{q \mid q \in X',\ q([0; x]) \subset [0; q(x)]\}$ for $x \in X$.

(ii) $X'_{\mathrm{nir}} := \bigcap_{x \in X} \mathrm{nir}(x) = \{q \mid q \in X',\ q \text{ is an nir-functional}\}$.

3.2. Proposition. (i) $\mathrm{nir}(x) = X'$ *iff* $[0; x] = \mathrm{co}\{0, x\}$.

(ii) *X has small norm intervals iff every functional on X is an* nir-*functional.*

(iii) *If the multiples of the extreme functionals on X are weak*-dense in X', then X has small norm intervals (the converse is not true; cf.* 3.7(c)).

Proof. (i): Suppose that $\mathrm{nir}(x) = X'$ and that $z \in [0; x]$ (we will assume that $x \neq 0$). It follows that $q(z) = 0$ whenever $q(x) = 0$ (all $q \in X'$) so that $z = ax$ for some $a \in \boldsymbol{R}$. $z \in [0; x]$ implies that $a \in [0, 1]$. The converse implication is obvious.

(ii): This follows from (i).

(iii): This is a consequence of 2.6(ii), (iii), 2.3(e) and (ii) of this proposition.

The following definition will be needed to formulate the Krein–Milman type Theorem 3.6:

3.3. Definition. Let E be a real locally convex Hausdorff space,

$A \subset E$. For $f \in E'$, $a, b \in \boldsymbol{R}$, let $H_f[a, b] := \{x \mid x \in E,\ f(x) \in [a; b]\}$ (i.e. the set between the two hyperplanes $\{f = a\}$ and $\{f = b\}$).

The *linear convex hull*, $\widetilde{\mathrm{co}}\,A$, is defined by

$$\bigcap \{H_f[a, b] \mid f \in E',\ a, b \in \boldsymbol{R},\ A \subset H_f[a, b]\}.$$

A is called *linearly convex* if $A = \widetilde{\mathrm{co}}\,A$.

3.4. EXAMPLES. (a) $\widetilde{\mathrm{co}}\,A$ is a closed convex set which contains $\overline{\mathrm{co}}\,A$.

(b) If A is a bounded set, then $\widetilde{\mathrm{co}}\,A = \overline{\mathrm{co}}\,A$.

(c) A routine argument shows that $\widetilde{\mathrm{co}}\,A$ is always linearly convex.

(d) It can be shown that for finite-dimensional spaces the linearly convex sets are just the sums of a subspace and a compact convex set.

3.5. DEFINITION. Let X be a real normed linear space, $x \in X$, X' provided with the weak*-topology.

(i) $\mathrm{nir}^+(x) := \overline{\mathrm{co}}\,\boldsymbol{R}^+\{p \mid p \in \mathrm{ex}\,D(X'),\ p(x) \geqslant 0\}$.

(ii) $\mathrm{nir}^*(x) := \mathrm{nir}^+(x) \cup \mathrm{nir}^+(-x)$.

(iii) $\mathbf{nir}^*(x) := \big[\boldsymbol{R}\widetilde{\mathrm{co}}\big(\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}\big)\big]^-$ for $x \neq 0$, $\mathbf{nir}^*(0) := X'$.

3.6. THEOREM. (i) $\mathrm{nir}^*(x) \subset \mathbf{nir}^*(x) = \mathrm{nir}(x)$ *for every* $x \in X$.

(ii) *If* $\inf\{|p(x)| \mid p \in \mathrm{ex}\,D(X')\} > 0$, *then* $\mathrm{nir}^*(x) = \mathbf{nir}^*(x)$ $(x \in X)$.

(iii) $\bigcap_{x \in X} \mathrm{nir}^*(x) \subset \bigcap_{x \in X} \mathbf{nir}^*(x) = X'_{\mathrm{nir}}$.

Proof. (i) (we will assume that $x \neq 0$):

$\mathrm{nir}^*(x) \subset \mathrm{nir}(x)$: A direct computation shows that $\mathrm{co}\big(\boldsymbol{R}^+\{p \mid p \in \mathrm{ex}\,D(X'),\ p(x) \geqslant 0\}\big) \subset \mathrm{nir}(x)$. Since $\mathrm{nir}(x)$ is weak*-closed, this implies that $\mathrm{nir}^+(x) \subset \mathrm{nir}(x)$, Similarly one shows that $\mathrm{nir}^+(-x) \subset \mathrm{nir}(x)$.

$\mathbf{nir}^*(x) \subset \mathrm{nir}(x)$: Since $\boldsymbol{R}\,\mathrm{nir}(x) \subset \mathrm{nir}(x)$ and $\mathrm{nir}(x)$ is weak*-closed, it is sufficient to show that

$$\widetilde{\mathrm{co}}\big(\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}\big) \subset \mathrm{nir}(x).$$

Let q_0 be an element of $\widetilde{\mathrm{co}}\big(\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}\big)$, $z \in [0; x]$. We will prove that

$$\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\} \subset H_z[0, 1]$$

so that $q_0 \in H_z[0, 1]$, i.e.

$$q_0(z) \in [0; 1] = [0; q_0(x)]$$

(we have identified z with the evaluation functional $q \mapsto q(z)$ on X'; note that $q_0(x) = 1$ since $\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\} \subset H_x[1, 1]$). Since

$$\mathrm{nir}_0^+(x) := \mathrm{co}\boldsymbol{R}^+\{p \mid p \in \mathrm{ex}\,D(X'),\ p(x) \geqslant 0\}$$

is a cone, we have $\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\} = \big(\mathrm{nir}_0^+(x) \cap \{q \mid q(x) = 1\}\big)^-$ so

that it is sufficient to show that

$$\mathrm{nir}_0^+(x) \cap \{q \mid q(x) = 1\} \subset H_z[0, 1].$$

Let $\sum_{i=1}^{n} \lambda_i p_i$ be an element of this set, i.e. $\lambda_i \geqslant 0$, $p_i \in \mathrm{ex}\,D(X')$, $p_i(x) \geqslant 0$ $(i = 1, \ldots, n)$, $\sum \lambda_i p_i(x) = 1$. Since $p_i \in \mathrm{nir}(x)$ for all i, it follows that $p_i(z) \in [0; p_i(x)] = [0, p_i(x)]$ so that $0 \leqslant \sum \lambda_i p_i(z) \leqslant \sum \lambda_i p_i(x) = 1$, i.e. $\sum \lambda_i p_i \in H_z[0, 1]$. "$\mathrm{nir}(x) \subset \mathbf{nir}^*(x)$": Let q_0 be an element of $\mathrm{nir}(x)$. Since $\boldsymbol{R}\,\mathbf{nir}^*(x) \subset \mathbf{nir}^*(x)$, we have only to consider the cases $q_0(x) = 1$ and $q_0(x) = 0$.

First case: $q_0(x) = 1$: We will show that

$$q_0 \in \mathbf{nir}_0^*(x) := \widetilde{\mathrm{co}}\big(\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}\big) \subset \mathbf{nir}^*(x).$$

Suppose that $q_0 \notin \mathbf{nir}_0^*(x)$. By definition, there is a w in X such that $q_0 \notin H_w[a, b]$ for suitable $a, b \in \boldsymbol{R}$, $\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\} \subset H_w[a, b]$. We may assume that $a < b$ so that, with $z := (w - ax)/(b - a)$, $q_0 \notin H_w[a, b] = H_z[0, 1]$ (i.e. $q_0(z) \notin [0; q_0(x)]$). We claim that $z \in [0; x]$ in contradiction to $q_0 \in \mathrm{nir}(x)$. By 2.4 we have to show that $p(z) \in [0; p(x)]$ for every $p \in \mathrm{ex}\,D(X')$. If $p(x) > 0$, then $p/p(x) \in \mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}$, i.e. $0 \leqslant p(z) \leqslant p(x)$. If $p(x) = 0$, we choose a $p_0 \in \mathrm{ex}\,D(X')$ such that $p_0(x) > 0$. For every $n \in \boldsymbol{N}$, $(p_0 + np)/p_0(x) \in \mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}$ so that $0 \leqslant p_0(z) + np(z) \leqslant p_0(x)$. This implies that $p(z) = 0 \in [0; p(x)]$. The case $p(x) < 0$ can be reduced to the case $p(x) > 0$.

Second case: $q_0(x) = 0$: We choose a $p_0 \in \mathrm{ex}\,D(X')$ such that $p_0(x) > 0$. It is clear that $(p_0 + nq_0)/p_0(x) \in \mathrm{nir}(x)$ for every $n \in \boldsymbol{N}$ so that, by the first part of the proof, $(p_0 + nq_0)/p_0(x) \in \mathbf{nir}^*(x)$. Therefore $q_0 + (1/n)p_0 \in \mathbf{nir}^*(x)$ for every n and consequently $q_0 \in (\mathbf{nir}^*(x))^- = \mathbf{nir}^*(x)$.

(ii): Suppose that $|p(x)| \geqslant \varepsilon > 0$ for every $p \in \mathrm{ex}\,D(X')$. It follows that $\mathrm{nir}_0^+(x) \cap \{q \mid q(x) = 1\}$ is contained in the ball with radius $1/\varepsilon$ so that

$$\begin{aligned}\widetilde{\mathrm{co}}\big(\mathrm{nir}_0^+(x) \cap \{q \mid q(x) = 1\}\big) &= \overline{\mathrm{co}}\big(\mathrm{nir}_0^+(x) \cap \{q \mid q(x) = 1\}\big)\\ &= \overline{\mathrm{co}}\big(\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}\big)\\ &= \mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}.\end{aligned}$$

This yields $\boldsymbol{R}^+\widetilde{\mathrm{co}}\big(\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}\big) \subset \mathrm{nir}^+(x)$. Similarly one shows that $\boldsymbol{R}^-\widetilde{\mathrm{co}}\big(\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}\big) = \boldsymbol{R}^+\widetilde{\mathrm{co}}\big(\mathrm{nir}^+(-x) \cap \{q \mid q(-x) = 1\}\big) \subset \mathrm{nir}^+(-x)$ so that $\boldsymbol{R}\widetilde{\mathrm{co}}\big(\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}\big) \subset \mathrm{nir}^*(x)$. Since $\mathrm{nir}^*(x)$ is closed, this implies that $\mathbf{nir}^*(x) \subset \mathrm{nir}^*(x)$.

(iii): This follows from (i).

3.7. APPLICATIONS. (a) Suppose that there exists an element x of X such that $\inf\{|p(x)| \mid p \in \mathrm{ex}\,D(X')\} > 0$. Then $X'_{\mathrm{nir}} \subsetneqq X'$.

(b) If X is a (necessarily finite dimensional) space for which $\operatorname{ex} D(X')$ is finite, then $X'_{\mathrm{nir}} = \boldsymbol{R}\operatorname{ex} D(X')$.

(c) There exists a three-dimensional space such that $X' = X'_{\mathrm{nir}}$ for which $(\boldsymbol{R}\operatorname{ex} D(X'))^-$ is a proper subset of X'.

(d) If K is as in 1.4(d), then the elements in $(AK)'_{\mathrm{nir}}$ are of the form $a\delta_k$ with $a \in \boldsymbol{R}$ and $k \in K$. Consequently in order to determine $(AK)'_{\mathrm{nir}}$ it is sufficient to determine $K_{\mathrm{nir}} := \{k \mid \delta_k$ is an nir-functional on $AK\}$. It is clear that K_{nir} is a closed subset of K which contains the extreme points of K. There seems to be no simple way to construct K_{nir} from $\operatorname{ex} K$.

Proof. (a): It is clear that $\mathrm{nir}^*(x) \subsetneqq X'$ in this case (note that $\mathrm{nir}^+(x) \cap \{q \mid q(x) = 1\}$ is a bounded set). 3.6(ii) and (iii) imply that $X'_{\mathrm{nir}} \subsetneqq X'$.

(b): Let q_0 be an element of $X' \setminus (\boldsymbol{R}\operatorname{ex} D(X'))$. It is an easy exercise to establish the existence of an $x_0 \in X$ such that $q_0(x_0) = 0$, $p(x_0) \neq 0$ for every $p \in \operatorname{ex} D(X')$. It follows that $q_0 \notin \mathrm{nir}(x_0) \supset X'_{\mathrm{nir}}$. The reverse inclusion is always valid.

(c): Consider in $\boldsymbol{R}^3$ the "barrel" $B := \{(a, b, c) \mid a^2+b^2+c^2 \leqslant 1, |c| \leqslant 1/2\}$. Let X be that three-dimensional normed space for which B is the unit ball of X'. It is easy to see that $\mathrm{nir}^*(x) = X'$ for every $x \in X$ so that $X'_{\mathrm{nir}} = X'$ by 3.6.

(d): An easy calculation shows that $\mathrm{nir}^*(1) = \boldsymbol{R}\{\delta_k \mid k \in K\}$ so that, by 3.6(ii) and (iii), $(AK)'_{\mathrm{nir}} \subset \boldsymbol{R}\{\delta_k \mid k \in K\}$.

4. nir-Operators on function modules.

4.1. Definition. Let X be a function module as in 1.5, Y a real normed linear space, $T\colon X \to Y$ a linear continuous operator. We say that T is a *component operator* iff there are a $k \in K$ and an operator $T_k\colon X_k \to Y$ such that $T = T_k \circ \pi_k$ ($\pi_k\colon X \to X_k$ denotes the projection from X onto X_k). For $Y = \boldsymbol{R}$ T will be called a *component functional*.

4.2. Proposition. (i) *The following are equivalent*:

(a) *T is a component operator.*

(b) *There is a $k \in K$ such that $x(k) = 0$ implies $Tx = 0$ (all $x \in X$).*

(c) *$p \circ T$ is a component functional for every $p \in Y'$.*

(ii)

$$C_k := \{q \mid q \text{ is a component functional on } X,\ q = q_k \circ \pi_k \text{ for some } q_k \in X'_k\}$$

is a weak-closed subspace of X' (all $k \in K$). If, for $q_1, q_2 \in X'$, $q := (1/2)(q_1+q_2)$ belongs to C_k, then $q_1, q_2 \in C_k$ provided that $\|q_1\|, \|q_2\| \leqslant \|q\|$.*

(iii) *If T is a nonzero component operator on X, then k and T_k are uniquely determined for T.*

Proof. (i): (a) $\Rightarrow$ (c): This is obviously valid.

(c) $\Rightarrow$ (b): Let D_T be the set $\{p \mid p \in Y',\ p \circ T \neq 0\}$ (we may assume that $T \neq 0$ so that $D_T \neq \emptyset$). By definition there are, for every $p \in D_T$, a $k_p \in K$ and an $f_p \in (X_{k_p})'$ such that $p \circ T = f_p \circ \pi_{k_p}$. We claim that there is a $k \in K$ such that $k = k_p$ for every $p \in D_T$. Suppose that there are $p_1, p_2 \in D_T$ such that $k_{p_1} \neq k_{p_2}$. Since $p_i \circ T \neq 0$, there are $x_i \in X$ such that $p_i \circ T(x_i) = 1$ ($i = 1, 2$). With $x_0 := h_1 x_1 + h_2 x_2$ (h_1 and h_2 suitable functions in CK such that $h_1(k_1) = h_2(k_2) = 1$, $h_1 h_2 = 0$) we have $(p_1+p_2) \circ T x_0 = 2$ so that $p_1+p_2 \in D_T$; it follows that $(p_1+p_2) \circ T = f_{p_1+p_2} \circ \pi_{k_0}$ ($k_0 := k_{p_1+p_2}$). Suppose that $k_0 \neq k_{p_1}$. We choose a function $h \in CK$ for which $h(k_{p_1}) = 1$, $h(k_0) = h(k_{p_2}) = 0$. Then, with $x := hx_1$, we get $(p_1+p_2) \circ Tx = 1 \neq 0 = f_{p_1+p_2} \circ \pi_{k_0}(x)$. This contradiction proves that $k_0 = k_{p_1}$. Similarly one shows that $k_0 = k_{p_2}$ so that $k_{p_1} = k_{p_2}$ in contrast to our hypothesis.

If $x(k) = 0$, then $p \circ T(x) = f_p(x(k)) = 0$ for every $p \in D_T$ (and therefore for every $p \in X'$), i.e. $T(x) = 0$.

(b) $\Rightarrow$ (a): Define $T_k\colon X_k \to Y$ by $T_k(x(k)) := Tx$. T_k is well-defined by hypothesis (cf. also 1.5(iii)) and obviously linear and continuous (note that, for $x_k \in X_k$, there is an $x \in X$ such that $x(k) = x_k$ and $\|x\| < (1+\varepsilon)\|x_k\|$ ($\varepsilon > 0$ arbitrary); this follows at once from 1.5). We have $T = T_k \circ \pi_k$ by definition.

(ii): The first part follows at once from (i)(b). Suppose that $q := (1/2)(q_1+q_2) \in C_k$, $1 = \|q\| \geqslant \|q_1\|, \|q_2\|$. We will show that (i)(b) is satisfied for q_1 and q_2.

Let x be an element of X which vanishes in a neighbourhood U of k, $\|x\| \leqslant 1$. For $\varepsilon > 0$ we choose an $x_\varepsilon \in X$ such that $1 \geqslant q(x_\varepsilon) \geqslant 1-\varepsilon$, $x_\varepsilon(l) = 0$ for $l \notin U$, $\|x_\varepsilon\| \leqslant 1$ ($x_\varepsilon = h\tilde{x}_\varepsilon$, where h is in CK with $h(k) = 1$, $\operatorname{supp} h \subset U$, and $\tilde{x}_\varepsilon$ is a normalized vector in X such that $1 \geqslant q(\tilde{x}_\varepsilon) \geqslant 1-\varepsilon$; note that q is a component functional).

$q(x_\varepsilon) \geqslant 1-\varepsilon$ implies that $q_1(x_\varepsilon)$, $q_2(x_\varepsilon) \geqslant 1-2\varepsilon$. On the other hand, $\|x_\varepsilon \pm x\| \leqslant 1$ so that $q_1(x_\varepsilon \pm x)$, $q_2(x_\varepsilon \pm x) \leqslant 1$. This yields $|q_1(x)|$, $|q_2(x)| \leqslant 2\varepsilon$ so that $q_1(x) = q_2(x) = 0$. Now let x be an arbitrary vector in X such that $x(k) = 0$ (w.l.o.g. we may assume that $\|x\| \leqslant 1$). For $\varepsilon > 0$ there is a neighbourhood U of k such that $\|x(l)\| \leqslant \varepsilon$ for l in U. Let h be a continuous function on K such that $0 \leqslant h \leqslant 1$, $\operatorname{supp} h \subset U$, $h = 1$ in a neighbourhood of k contained in U. $(1-h)x$ vanishes in a neighbourhood of k so that $q_i((1-h)x) = 0$ ($i = 1, 2$). Since $\|x-(1-h)x\| \leqslant \varepsilon$, this yields $|q_1(x)|$, $|q_2(x)| \leqslant \varepsilon$ so that $q_1(x) = q_2(x) = 0$. (i)(b) implies that $q_1, q_2 \in C_k$.

(iii): Suppose that $T = T_{k_1} \circ \pi_{k_1} = \tilde{T}_{k_2} \circ \pi_{k_2}$ with $k_1 \neq k_2$. Choose an $x \in X$ such that $Tx \neq 0$ and a function $h \in CK$ such that $h(k_1) = 1$, $h(k_2) = 0$. It follows that

$$T_{k_1}\circ\pi_{k_1}(hx) = T_{k_1}\circ\pi_{k_1}(x) = Tx \neq 0$$
$$= \tilde{T}_{k_2}\circ\pi_{k_2}(hx) = T(hx)$$
$$= T_{k_1}\circ\pi_{k_1}(hx).$$

This contradiction shows that $k_1 = k_2$. $T_{k_1} = \tilde{T}_{k_2}$ is a consequence of 1.5(iii).

4.3. THEOREM. (i) *Every* nir-*functional on a function module is a component functional.*

(ii) *If Y has small norm intervals, then every* nir-*operator from a function module X to Y is a component operator.*

Proof. (i): Let $q\colon X \to \boldsymbol{R}$ be an (w.l.o.g. nonzero) nir-functional. We will proceed as follows:

1. For every $x \in X$ such that $q(x) \neq 0$ there is a $k_x \in K$ such that $q(hx) \neq 0$ whenever $h \in CK$, $h \geqslant 0$, $h(k_x) = 1$.

2. k_x does not depend on x (the common value will be denoted by k_0).

3. $q(x) = 0$ whenever x vanishes in a neighbourhood of k_0.

4. $q(x) = 0$ whenever $x(k_0) = 0$ (so that, by 4.2(i), q is a component functional).

1. Suppose that, for every $k \in K$, there is a $h_k \in CK$, $h_k \geqslant 0$, $h_k(k) = 1$ such that $q(h_k x) = 0$. Since K is compact, there are $k_1, \ldots, k_n \in K$, $m > 0$ such that $h := m(h_{k_1} + \ldots + h_{k_n}) \geqslant \mathbf{1}$. We have $q(hx) = 0$ and, by 1.6, $x \in [0; hx]$. Since q is an nir-functional this implies that $q(x) = 0$, a contradiction.

2. Suppose that there are $x_1, x_2 \in X$, $q(x_1) \neq 0 \neq q(x_2)$, $k_1 := k_{x_1} \neq k_2 := k_{x_2}$. We choose functions $h_1, h_2 \in CK$, $h_1(k_1) = h_2(k_2) = 1$, $h_1, h_2 \geqslant 0$, $h_1 h_2 = 0$. For a suitable constant $a \in \boldsymbol{R}$ we have $q(h_1 x_1 + a h_2 x_2) = 0$. Since $h_1 h_2 = 0$, $h_1 x_1 \in [0; h_1 x_1 + a h_2 x_2]$ so that $q(h_1 x_1) = 0$ which contradicts the construction of k_1.

3. Let x be an element of X such that $q(x) \neq 0$. For any neighbourhood U of k_0 there is a function $h \in CK$, $h \geqslant 0$, $h(k_0) = 1$, $\operatorname{supp} h \subset U$. Since $q(hx) \neq 0$, x cannot vanish identically on U.

4. 1.5 (i) and (ii) easily imply that $x = \lim h_n x$ for suitable $h_n \in CK$ where h_n vanishes in a neighbourhood of k_0 (cf. also the proof of 4.2 (ii)).

(ii) This follows from the first part of this proof, 4.2 (i)(c), 3.2 (ii), and 2.6 (ii).

Note. (1) It is clear that $q_k\circ\pi_k \in X'_{\text{nir}}$ for every $q_k \in (X_k)'_{\text{nir}}$ (2.3(b), 2.6(ii)). One might suspect that, for $q \in X'_{\text{nir}}$, $q = q_k\circ\pi_k$ as in 4.3(i), q_k must be in $(X_k)'_{\text{nir}}$. The following example shows that this must not be the case: Let $K := \{1, 2, \ldots, \infty\}$ $(= \alpha\boldsymbol{N})$, $X_k := l_2^1$ for $k \in K$. Define X by $\{(a_n, b_n)_{n\in K} \mid a_n \xrightarrow[n\to\infty]{} 0,\ b_n \xrightarrow[n\to\infty]{} b_\infty,\ a_\infty \in \boldsymbol{R}\}$. X is a function module over K. We consider the functional $q\colon (a_n, b_n)_{n\in K} \mapsto b_\infty$. An easy computation shows that $q \in X'_{\text{nir}}$ (in fact, q is the weak*-limit of the extreme functionals $p_m\colon (a_n, b_n)_{n\in K} \mapsto a_m + b_m$, $m \in \boldsymbol{N}$). In this case we have $q = q_\infty\circ\pi_\infty$, where $q_\infty\colon l_2^1 \to \boldsymbol{R}$ is the functional $(a, b) \mapsto b$ which does not respect norm intervals. (A similar function module has been considered by R. Evans to show that the component spaces of the function module representation of a Banach space must not have trivial M-structure.) In addition this is an example of a function module whose norm intervals do not exhaust the norm intervals of the components: for $k = \infty$ and $x = \big((0, 1), (0, 1), \ldots, (1, 1)\big)$ we have $[0; x]_k := \{y(k) \mid y \in [0; x]\} \subsetneqq [0; x(k)]$.

(2) The condition concerning the intervals of Y in 4.3(ii) is essential: the identity from CK to CK is an nir-operator but not a component operator (K a compact Hausdorff space which contains more than one point).

4.4. COROLLARY ([8]). *Let X be a function module as in* 1.5.

(i) *For $p \in \operatorname{ex} D(X')$ there are a $k \in K$ such that $X_k \neq 0$ and a $p \in \operatorname{ex} D(X'_k)$ such that $p = p_k\circ\pi_k$.*

(ii) *Conversely, for k and p_k as in* (i), *$p := p_k\circ\pi_k$ is an extreme functional on X.*

Proof. (i): $p = p_k\circ\pi_k$ by 4.3(i) and 2.3(e). $X_k \neq 0$ since $p \neq 0$. p_k is in the unit ball of X'_k: $\|p\| = \|p_k\| = 1$ (cf. the proof of 4.2(i) (b) $\Rightarrow$ (a)). $p_k = (1/2)(p_k^1 + p_k^2)$ (with $p_k^1, p_k^2 \in D(X'_k)$) implies that $p = (1/2)(p_k^1\circ\pi_k + p_k^2\circ\pi_k)$ with $p_k^1\circ\pi_k, p_k^2\circ\pi_k \in D(X')$ so that $p = p_k^1\circ\pi_k = p_k^2\circ\pi_k$, i.e. $p_k = p_k^1 = p_k^2$.

(ii): Let $p := p_k\circ\pi_k$. We have $\|p\| = 1$ since $\|p_k\| = 1$. If $p_1, p_2 \in D(X')$ satisfy $p = (1/2)(p_1 + p_2)$, then $p_i = p_k^i\circ\pi_k$ for suitable $p_k^i \in D(X'_k)$ by 4.2(ii) ($i = 1, 2$; note that $\|p_k^i\| = \|p_i\| \leqslant 1$). It follows that $p_k = (1/2)(p_k^1 + p_k^2)$ so that $p_k = p_k^1 = p_k^2$. This implies $p = p_1 = p_2$.

4.5. COROLLARY (Stone). *Let L be a locally compact Hausdorff space. The extreme functionals on $C_0 L$ (the space of real-valued continuous functions on L which vanish at infinity) are precisely the maps $\pm\delta_l$, $l \in L$.*

Proof. $C_0 L$ is a function module over $K = \alpha L$ with $X_l := \boldsymbol{R}$ for $l \in L$ and $X_\infty := 0$.

4.6. COROLLARY. *Let X be a real normed linear space, K a compact Hausdorff space for which the isolated points are a dense subset. Then every extreme point T in the unit ball of $[X, CK]$ is an* nir-*operator.*

Proof. Let k be an isolated point of K. It is easy to see that $\pm\delta_k\circ T$ is an extreme point of $D(X')$ and consequently an nir-functional. Since the isolated points are dense in K, the associated functionals $\pm\delta_k$ are dense in $\operatorname{ex} D\big((CK)'\big)$. It follows that $\pm\delta_k\circ T$ is an nir-functional for every $k \in K$ i.e. T is an nir-operator by 2.6(i).

Note. It has been shown by Sharir (Israel J. Math. 26 (1977)) that there are compact Hausdorff spaces K, L, an extreme operator $T\colon CK \to CL$, and an $l \in L$ such that $\delta_l\circ T \notin \bigcup_{k\in K} \boldsymbol{R}\delta_k$ (we are grateful to the referee for pointing out this reference to us). 2.6(i) and 4.3 imply that T is not an nir-operator. Thus extreme operators between CK-spaces need not be nir-operators.

If one is only interested in an extreme operator which is not norm interval respecting for the case of arbitrary Banach spaces it sufficies to consider two-dimensional spaces: Let Y be a two-dimensional Banach space such that there are two extreme points p, q in the unit ball for which (1) $[0;p] = \mathrm{co}\{0, p\}$, (2) p and q are linearly independent. Then, with $X := l_2^1$, let $T: X \to Y$ be defined by $(a, b) \mapsto ap+bq$. T corresponds, in $[X, Y] \cong Y \times Y$, to the extreme point (p, q) but T is not an nir-operator: The interval $[0;(0,1)]$ in l_2^1 spans l_2^1 so that the range of nir-operators in $[X, Y]$ which map $(1, 0)$ on p must be contained in the span of $[0;p]$ which is one-dimensional.

4.7. Theorem. *Let X and Y be function modules with base spaces K and L and component spaces $(X_k)_{k\in K}$ and $(Y_l)_{l\in L}$, respectively. Suppose that the component spaces Y_l all have small norm intervals. Then, for every* nir-*operator* $T: X \to Y$, *there are*

a subset L_0 of L;

a map $t: L_0 \to K$;

operators $T_{t(l)}: X_{t(l)} \to Y_l$ (all $l \in L_0$)

such that

$$(Tx)(l) = \begin{cases} 0 & \text{if} \quad l \notin L_0, \\ T_{t(l)}\big(x(t(l))\big) & \text{if} \quad l \notin L_0 \end{cases} \quad (\text{all } x \in X,\ l \in L).$$

Notes. (1) 4.3(ii) may be thought of as a special case of this theorem (consider Y in a trivial way as a function module over a base space containing only one point).

(2) The conditions on Y of the theorem are in particular satisfied for the class of square Banach spaces considered by Cunningham and Roy ([8]). Square Banach spaces can be regarded as function modules where the component spaces are at most one-dimensional. For example, every separable G-space is a square Banach space. Note that 4.3 and 4.4 imply that $X'_{\mathrm{nir}} = \mathbf{R}\mathrm{ex}D(X')$ for every square Banach space X.

Proof. Let $L_0 := \{l \mid l \in L, \pi_l \circ T \neq 0\}$. For $l \in L_0$, $\pi_l \circ T$ is an nir-operator, i.e. there are a $k \in K$ and an operator $T_k: X_k \to Y_l$ such that $\pi_l \circ T = T_k \circ \pi_k$ (4.3(ii)). k is uniquely determined for l by 4.2(iii). We will write $t(l) := k$. It is obvious that L_0, t, $(T_{t(l)})_{l\in L_0}$ have the claimed properties.

4.8. Corollary. *Let X be a Banach space, Y a Banach space having small norm intervals, K and L compact Hausdorff spaces, $T: C(K, X) \to C(L, Y)$ an* nir-*operator ($C(K, X)$ denotes the space of X-valued continuous functions on K, provided with the supremum norm). Then there are*

an open subset L_0 of L;

a continuous map $t: L_0 \to K$;

a continuous map $u: L \to [X, Y]_{\mathrm{nir}}$ with norm-bounded range which vanishes on $L \setminus L_0$

such that

$$(Tx)(l) = (T_{L_0,t,u}x)(l) := \begin{cases} 0 & l \notin L_0, \\ u(l)[x(t(l))] & l \in L_0 \end{cases}$$

for $x \in C(K, X)$, $l \in L$ ($[X, Y]_{\mathrm{nir}}$ denotes the set of nir-*operators from X to Y, provided with the strong operator topology).*

Conversely, if L_0, t, u are given as above, then $T_{L_0,t,u}$ is an nir-*operator.*

Proof. We regard $C(K, X)$ (resp. $C(L, Y)$) as a function module with base space K (resp. L) and component spaces X (resp. Y) at every point of the base space. We define L_0 and t as in 4.7, and $u: L \to [X, Y]$ is the mapping $l \mapsto T_{t(l)}$ (for $l \in L_0$; for $l \in L \setminus L_0$ we define $u(l) := 0$). We have to show that

1. L_0 is an open subset of L;

2. t is continuous;

3. u is continuous and normbounded;

4. every $u(l)$ is an nir-operator.

1. $L \setminus L_0 = \bigcap \{(Tx)^{-1}(0) \mid x \in C(K, X)\}$.

2. Let U be a neighbourhood of $t(l_0)$ ($l_0 \in L_0$). By definition, there is an x in $C(K, X)$ such that $(Tx)(l_0) \neq 0$. Replacing (if necessary) x by hx for a suitable $h \in CK$ we may assume that $x(k) = 0$ for $k \notin U$ (note that $(Tx)(l_0) = T(hx)(l_0)$ whenever $h(t(l_0)) = 1$). By continuity, there is a neighbourhood V of l_0 such that $(Tx)(l) \neq 0$ for $l \in V$. It follows that $T_{t(l)}\big(x(t(l))\big) \neq 0$ for $l \in V$, i.e. $t(l) \in U$ for these l.

3. For $z \in X$, let z be the constant function on K which assumes the value z at every point. We have $u(l)(z) = (Tz)(l)$ for every $l \in L$ so that $l \mapsto u(l)(z)$ is a continuous map. This proves that u is continuous. The $u(l)$, $l \in L$, are bounded in norm by $\|T\|$.

4. For $z, w \in X$, $z \in [0;w]$, we have $z \in [0;w]$ so that $Tz \in [0;Tw]$. 1.6 implies that $u(l)(z) = (Tz)(l) \in [0;(Tw)(l)] = [0;u(l)(w)]$ for every $l \in L$. It follows that $u(l)$ is an nir-operator (all $l \in L$).

Conversely, suppose that L_0, t, u are given. It is clear that $T_{L_0,t,u}$ is a linear continuous map from $C(K, X)$ into the space of bounded Y-valued functions on L (supremum norm). For $h \in CK$ and $z \in X$ the function $l \mapsto h(t(l))u(l)(z)$, i.e. the function $T_{L_0,t,u}(hz)$, is continuous by hypothesis. Since $\{hz \mid h \in CK, z \in X\}$ is total in $C(K, X)$ it follows that the range of $T_{L_0,t,u}$ is in fact contained in $C(L, Y)$.

It remains to show that $T_{L_0,t,u}$ is an nir-operator. For $x_0 \in C(K, X)$,

$x \in [0; x_0]$ we have $x(k) \in [0; x_0(k)]$ for all $k \in K$. In particular $u(l)x(t(l)) \in [0; u(l)x_0(t(l))]$ for $l \in L_0$ so that $T_{L_0,t,u}x \in [0; T_{L_0,t,u}x_0]$ by 1.6.

4.9. Corollary. *Let X, Y be real Banach spaces having small norm intervals, K and L compact Hausdorff spaces. Then, for every isometric isomorphism $T: C(K, X) \to C(L, Y)$ there are*

a homeomorphism $t: L \to K$;

a continuous map u from L into the space of isometric isomorphisms from X to Y (provided with the strong operator topology) such that $(Tx)(l) = u(l)x(t(l))$ for every $x \in C(K, X)$, $l \in L$.

Proof. $T = T_{L_0,t,u}$ by 4.8 (and 2.3(c)). Since T is surjective, t is defined on all of L, i.e. $(Tx)(l) = T_{t,u}x(l) := u(l)x(t(l))$ for $x \in C(K, X)$ and $l \in L$. Considering T^{-1} instead of T we get $T^{-1} = T_{t',u'}$ as well (where $t': K \to L$ and $u': K \to [Y, X]_{\mathrm{nir}}$ are continuous maps). $T_{t',u'} \circ T_{t,u}$ and $T_{t,u} \circ T_{t',u'}$ are the identity maps which easily implies that $t' \circ t = \mathrm{Id}_L$, $t \circ t' = \mathrm{Id}_K$, $u(l) \circ u'(t(l)) = \mathrm{Id}_Y$, $u'(k) \circ u(t'(k)) = \mathrm{Id}_X$ (all $l \in L$, $k \in K$). Since the $u(l)$ (resp. the $u'(k)$) are bounded in norm by $\|T\| = 1$ ($\|T^{-1}\| = 1$) it follows that these mappings are in fact isometric isomorphisms.

4.10. Corollary. *Let X be a Banach space having small norm intervals. Then X has the Banach–Stone property, i.e. the existence of an isometric isomorphism between $C(K, X)$ and $C(L, X)$ (K and L compact Hausdorff spaces) implies that K and L are homeomorphic.*

Note. Since $Z(X)$ is one-dimensional in this case (1.7), the corollary is also a consequence of [3], Theorem 3.1.

4.11. Corollary (Banach–Stone). *Let K and L be compact Hausdorff spaces, $T: CK \to CL$ an isometric isomorphism. Then there are a homeomorphism $t: L \to K$ and a function $h \in CL$ with $|h(l)| = 1$ for every $l \in L$ such that $(Tf)(l) = h(l)f(t(l))$ for every $f \in CK$, $l \in L$.*

4.12. Corollary. *Let X be a Banach space having small norm intervals, K a compact Hausdorff space, T an M-bounded operator from $C(K, X)$ to $C(K, X)$. Then there is a function $h_0 \in CK$ such that $T = M_{h_0}$ ($M_{h_0}x := h_0x$ for $x \in C(K, X)$).*

Proof. By 2.3(d) and 4.8, $T = T_{L_0,t,u}$ for suitable L_0, t, u. Since the collection of M-bounded operators on a Banach space is a commutative Banach algebra, T commutes with all multiplication operators M_h, $h \in CK$ (these operators are obviously M-bounded). This implies that $h(l)(Tx)(l) = h(t(l))(Tx)(l)$ for every $l \in L_0$ and every $h \in CK$ (all $x \in C(K, X)$) so that $t(l) = l$ for these l. It follows that $(Tx)(l) = (T_u x)(l) := u(l)(x(l))$ for $x \in C(K, X)$, $l \in K$. $u(l)$ is an M-bounded operator (we only have to apply the condition for M-boundedness of T to the subspace of constant functions $\{z \mid z \in X\}$) so that, by 1.7, $u(l) = h_0(l)\mathrm{Id}_X$ for a suitable $h_0(l) \in \mathbf{R}$ (all $l \in K$). Let h_0 be the function $l \mapsto h_0(l)$. h_0 is continuous, since $h_0 z = Tz$ is a continuous function for every $z \in X$. It is clear that $T = M_{h_0}$.

4.13. Corollary. (i) *Let K be a compact Hausdorff space. Then, for every nonzero multiplicative functional q on CK there is a $k \in K$ such that $q = \delta_k$.*

(ii) *Let K and L be compact Hausdorff spaces such that the isolated points of L are dense in L, $T: CK \to CL$ a linear continuous operator, $\|T\| \leqslant 1$. Then the following are equivalent:*

(a) *T is an extreme point of the unit ball of $[CK, CL]$;*

(b) *T is an* nir-operator *and, with $T = T_{L_0,t,u}$ as in 4.8 (we will regard u as continuous function from L to $\mathbf{R}$), we have $|u(l)| = 1$ for every $l \in L$ (so that, in particular, $L_0 = L$ in this case);*

(c) *T is essentially mutliplicative, i.e. $T(fg) = (T\mathbf{1})(Tf)(Tg)$ for $f, g \in CK$, and $|(T\mathbf{1})(l)| = 1$ for every $l \in L$;*

(d) *$\delta_l \circ T$ is an extreme functional on CK for every isolated point l of L.*

Note. Since for $K = L = \beta N$ the conditions of (ii) are satisfied, the results of Kim ([9]) concerning the extreme operators in $[l^\infty, l^\infty]$ are contained as a special case. Note, however, that in order to avoid notational complications we restricted ourselves to real spaces (the results in [9] are valid for the complex case as well).

Proof. (i): We claim that q is an nir-functional. Let $f \in CK$, $g \in [0; f]$ be given such that $q(f) \neq 0$. $g \in [0; f]$ implies that $0 \leqslant gf \leqslant f^2$ so that since multiplicative functionals on CK are monotone, $0 \leqslant q(g)q(f) \leqslant (q(f))^2$. Since $q(f) \neq 0$, it follows that $q(g) \in [0; q(f)]$, i.e. q is an nir-functional (2.2(ii)). For suitable $a \in \mathbf{R}$ and $k \in K$ we have $q(f) = af(k)$ (all $f \in CK$) by 4.3(i). Since $q \neq 0$ we have $q(\mathbf{1}) = 1$ so that $a = 1$.

(ii): (a) $\Rightarrow$ (b): T is an nir-operator by 4.6 so that $T = T_{L_0,t,u}$ for suitable L_0, t, u. $\|T\| = 1$ implies that $|u(l)| \leqslant 1$ for every $l \in L$. Suppose that $|u(l_0)| < 1$ for some $l_0 \in L_0$. We write $u = (1/2)(u_1 + u_2)$ with $u_1, u_2 \in CL$, $\|u_1\|, \|u_2\| \leqslant 1$, $u_1 \neq u \neq u_2$, $u_1(l) = u_2(l) = 0$ for $l \notin L_0$. It follows that $T = (1/2)(T_{L_0,t,u_1} + T_{L_0,t,u_2})$ with $T_{L_0,t,u_1} \neq T \neq T_{L_0,t,u_2}$, a contradiction. We thus have $|u(l)| = 1$ for $l \in L_0$ so that L_0 must be clopen in L. Let H be the operator $f \mapsto f(k_0)h$ ($k_0 \in K$ an arbitrary point, $h =$ the characteristic function of $L \setminus L_0$). We have $T = (1/2)((T+H) + (T-H))$ with $\|T \pm H\| \leqslant 1$ so that $H = 0$. It follows that $L = L_0$.

(b) $\Rightarrow$ (c): This is obvious (note that $T\mathbf{1} = u$).

(c) $\Rightarrow$ (d): $(1/(T\mathbf{1})(l))\,\delta_l \circ T = \delta_k$ for some $k \in K$ by (i). Thus $\delta_l \circ T \in \{\delta_k, -\delta_k\} \subset \operatorname{ex} D((CK)')$ (4.5).

(d) $\Rightarrow$ (a): Suppose that $T = (1/2)(T_1 + T_2)$, where $T_1, T_2 \in D([CK, CL])$. By hypothesis, we have $(T_1 f)(l) = (Tf)(l) = (T_2 f)(l)$ for $f \in CK$ and every isolated point of L. Since these points are dense in L it follows that $T = T_1 = T_2$.

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An approximation problem in $L^p([0, 2\pi])$, $2 < p < \infty$

by

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Abstract. We treat the remaining case ($2 < p < \infty$) of an approximation problem earlier considered by Kahane and Rider.

For $n \in Z$, let $e_n(x)$ be the exponential e^{inx}, and for $f \in L^1([0, 2\pi])$ define the Fourier coefficient $\hat{f}(n)$ to be $(2\pi)^{-1} \int_0^{2\pi} f(x) e_{-n}(x)\,dx$. Then the Fourier series of f is $\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$.

We consider the following question. If $f \in L^p$ $(= L^p([0, 2\pi]))$, is f the limit in the L^p norm of trigonometric polynomials q such that $\hat{q}(n) = \hat{q}(m)$ whenever $\hat{f}(n) = \hat{f}(m)$? For $p = 1$ a negative answer follows from results of Kahane [2], while a construction of Rider [4] gives the negative answer for $1 < p < 2$. For $p = 2$ the question is of course trivial. The purpose of this note is to give a negative answer for $2 < p < \infty$. Our method follows the broad outline of that of Kahane, but the details are quite different. We mention that this problem is closely related to a question about closed convolution subalgebras of L^p —see [5], [2], [4], [1].

THEOREM. *Fix p with $2 < p < \infty$. There exists a collection $\{E_j\}_{j=1}^{\infty}$ of pairwise disjoint finite subsets of Z and a function $f \in L^p$ such that $\hat{f}$ is constant on each E_j, $\hat{f} = 0$ off of $\bigcup_{j=1}^{\infty} E_j$, and such that f is not approximable in L^p by polynomials of the form $\sum_j b_j \sum_{n \in E_j} e_n$.*

Proof. In the following, C will denote a positive constant independent of k but which may increase from line to line. Let r be an even integer such that

$$r(1/2 - 1/p) > 1.$$

Let $n_0 = 0$, $m_0 = 1$, $n_k = (k!)^r$, $m_k = (k+2)^{r/2} n_k$ $(k = 1, 2, \ldots)$. Let $p_0 = 0$ and let $\{p_k\}_{k=1}^{\infty}$ be a sequence of positive integers which increases

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